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THE USE OF SOURCE AND SINK CONCEPTS IN THE CALCULATION OF WING CHARACTERISTICS AT SUPERSONIC SPEEDS

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The calculation of wing characteristics within the line... is performed by a superposition of known solutions of a mor... elementary nature. In incompressible flow the use of sourc... and vortex solutions has proven very useful, perhaps because ... elementary solutions themselves have been easy to visualize; certa..., the idea of building up a body of revolution by a continuous distribution of sources and sinks is a natural one. At supersonic speeds the use of sources, doublets, and vortices can lead to many simplifications and give the student a physical picture of what is occurring in the flow. Von Kármán and Moore (reference 1) were first to introduce source and sink concepts to supersonic aerodynamics when they calculated the flow about bodies of revolution by an axial distribution of sources. In 1935, at the Volta Congress, Von Kármán (reference 2) suggested the use of surface source distributions in the calculation of wing characteristics and thus laid the ground work for much of the present work.

Unfortunately, the spherical symmetry of incompressible source flow is lost as the velocity of the stream becomes greater than the speed of sound, as may be seen by comparison of the potential function ϕ of a source in incompressible flow and supersonic flow:

$$\phi_{M=0} = \frac{K}{\sqrt{x^2 + y^2 + z^2}} \tag{1}$$

and

$$\phi_{M>1} = \frac{K}{\sqrt{x^2 - (M^2 - 1)(y^2 + z^2)}} \tag{2}$$

where K is the strength of the source and M is Mach number of the undisturbed stream. Prandtl (reference 3) has given a very good derivation of the potential function of a supersonic source system that uses a superposition of sources fixed in the fluid but varying in strength with time. Consider a body moving at supersonic speed through a compressible fluid originally at rest. The motion produced must satisfy a differential equation which, upon restriction to motions that are small compared with the velocity of sound, becomes the wave equation in three dimensions, well-known in mathematical physics. That is,

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{3}$$

where c is the speed of sound.

The solution of this equation representing a fixed source of fluid is also known and more complicated solutions may be built up by distributing sources in the fluid. In order to get to the solution for a body or disturbance moving through the fluid, Prandtl assumes the x-axis or flight axis to be covered with sources, each source being fixed and having strength vary with time. (See fig. 1.) As a given disturbance moving along the z-axis reaches any source, that source starts to flow, flows according to a common law, and ceases to flow when the disturbance has passed. The potential at any point (x,y,z) in the fluid is then made up of the contribution from each source in accordance with the known expression for the potential function of the source flow. This expression is given in figure 1 for a source located at a point on the x-axis and is as follows:

$$d\phi = -\frac{f\left(t - \tau - \frac{R}{c}\right)}{R}\,dx'$$

where t is the time, τ is the time at which the source started to flow, c is the speed of sound, R is the distance between the source and field point, and f is the law which governs the strength of the source flow. As no disturbance can be produced at the point (x,y,z) until the sound or pressure wave reaches the field point, the potential is only affected by those sources, the initial waves of which have already reached the field point at the time t. It should be noticed that lines from the initial point of the disturbance tangent to the wave fronts from the sources form the Mach cone or region of influence of the disturbance. The potential will then be expressed as an integral and will be a function of the space coordinates and time. If, however, the observation point is allowed to move along at the same speed as the disturbance, the time element disappears, and finally an integral expression is obtained for a system of sources moving at a supersonic Mach number M. The following expression was used by Von Kármán and Moore (reference 1) in computing the flow over slender bodies of revolution at supersonic speeds:

$$\phi = -2\int_{a_1}^{a_2}\frac{f(x_1)\,dx_1}{\sqrt{(x - x_1)^2 - (M^2 - 1)(y^2 + z^2)}} \tag{4}$$

In this equation, the field point is at (x,y,z) and the position on the axis x_1 may be considered to be the location of a supersonic point source having as its potential field the expression of equation (2).

For wing problems, however, lifting surfaces, and therefore surface source distributions, are of interest. The potential at any point in the space will then be made up of the contributions from sources distributed over a region of the xy-plane and may be written as a double summation or integral:

$$\phi_S = -\int_{b_1}^{b_2}\int_{a_1}^{a_2} \frac{g(x_1,y_1)dx_1dy_1}{\sqrt{(x - x_1)^2 - (M^2 - 1)\left[(y - y_1)^2 + z^2\right]}} \tag{5}$$

In this expression the quantity $g(x_1,y_1)$ is the expression describing the source intensity at the point (x_1,y_1). Care must be taken in choosing the limits in the integral so that the area of integration includes only those sources which can effect the field point, that is, the sources which contain the field point in their Mach cones.

The velocity component normal to the xy-plane containing the sources is obtained by differentiating the potential function with respect to z. Puckett (reference 4) and others have shown that this normal velocity at a point on the surface $z = 0$ is affected only by the sources in the immediate vicinity of the point and is given by the expression

$$\underset{\pm z \to 0}{v_z} = \frac{\partial\phi}{\partial z} = \pm\pi g(x,y) \tag{6}$$

It is seen that the normal velocity is discontinuous at the surface and of magnitude proportional to the local-source strength. This result is not surprising if it is remembered that the source solutions used in the previous derivation produced a radial flow and therefore could not produce a velocity normal to the plane containing the source except directly at the source itself.

At the surface, the slope of the streamlines is given to a first approximation by

$$\frac{v_z}{V_0} = \tan\theta \approx \theta \tag{7}$$

The source distribution over the surface then produces a splitting of the flow streamlines and is therefore capable of representing the effect of thickness on a wing plan form. For example, on a wing of given plan form the surface slopes are known and, hence, from equations (6) and (7) the source distribution is known. The potential function for the wing can therefore be calculated by inserting for $g(x_1,y_1)$ in equation (5) the expression obtained from equations (6) and (7) to obtain

$$\phi_S - = \frac{V_0}{\pi}\int_{b_1}^{b_2}\int_{a_1}^{a_2}\frac{\theta\left(x_1,y_1\right)dx_1dy_1}{\sqrt{(x-x_1)^2-(M^2-1)\left[(y-y_1)^2+z^2\right]}} \tag{8}$$

Once the potential function is known, it is possible to obtain the velocities by differentiation. The method for calculating the pressure distribution and drag of symmetrical wings at zero angle of attack is thus direct and involves only the solution of definite integrals. This method has numerous applications in calculating the drag of supersonic wings. Rectangular wings, triangular wings, and tapered and untapered sweptback wings having various airfoil sections have been calculated and are available.

As has been seen, a source distribution produces a parting of the flow and hence an identical flow pattern above and below the xy-plane. Sources alone, therefore, will not produce a lifting force on a body.

What is needed is a potential function which is discontinuous at the plane of the wing and which will, therefore, produce a difference of pressure or lift on the wing. If the vertical velocity produced by a source distribution is used as a new potential function, this type of potential function can be obtained because, as demonstrated previously, the normal velocity is discontinuous at the plane of the sources. The new potential function, which is the derivative of a source potential, is called a doublet potential because it can be formed by an operation on a double sheet of positive and negative sources. The doublet potential may be written now as

$$\phi_D = \frac{\partial\phi_S}{\partial z} = \frac{\partial}{\partial z}\left\{-\int_{b_1}^{b_2}\int_{a_1}^{a_2}\frac{k(x_1,y_1)dx_1dy_1}{\sqrt{(x-x_1)^2-(M^2-1)\left[(y-y_1)^2+z^2\right]}}\right\} \tag{9}$$

and by referring again to equation (6) it is seen that the potential at the surface $z = 0$ is

$$\phi_{D \atop \pm z \to 0} = \pm \pi k(x,y) \tag{10}$$

where $k(x,y)$ is the doublet- or source-distribution function. At the surface, the x-, y-, and z-component velocities may be written as follows:

$$v_{x \atop \pm z \to 0} = \frac{\partial \phi}{\partial x} = \pm \pi \frac{\partial k}{\partial x} \tag{11}$$

$$v_{y \atop \pm z \to 0} = \frac{\partial \phi}{\partial y} = \pm \pi \frac{\partial k}{\partial y} \tag{12}$$

$$v_z = \frac{\partial \phi}{\partial z} = \frac{\partial^2}{\partial z^2} \left\{ -\int_{b_1}^{b_2} \int_{a_1}^{a_2} \frac{k(x_1,y_1)dx_1 dy_1}{\sqrt{(x - x_1)^2 - (M^2 - 1)\left[(y - y_1)^2 + z^2\right]}} \right\} \tag{13}$$

Inasmuch as the pressure is proportional to the x-component velocity, it can be seen that there is a difference in pressure or lift wherever $\frac{\partial k}{\partial x}$ exists. The z-component of the disturbance velocity is found to be continuous at the surface and the streamlines above and below the surface are therefore deflected in the same direction. The doublet distribution is thus capable of representing the effects of camber and angle of attack of wings.

In problems in which the camber or angle of attack is given and the pressure distribution is required, the doublet-distribution method is rather difficult to use because the doublet distribution function is not known but is beneath the integral signs, as in equation (13). The camber or angle of attack can give the value of the vertical velocity v_z, but the function $k(x_1,y_1)$, the doublet distribution, is not known. This type of equation, called an integral equation, is often quite difficult to evaluate.

When the pressure distribution is given, the potential on the surface and, therefore, the doublet distribution can be calculated directly. The camber can then be obtained by calculating the vertical velocity at each point. The difficulty of solving an integral equation is therefore not met in this case.

In order to illustrate briefly the manner in which camber lines may be calculated, consider a uniformly loaded triangular wing. The pressure coefficient may be prescribed as a constant A and, therefore, from the pressure equation, v_x is constant; that is,

$$\frac{\Delta p}{q} = -\frac{2v_x}{V_0} = A \tag{14}$$

Inasmuch as the potential is the integral of the velocity v_x by definition, the potential on the surface becomes

$$\phi_{z=0} = \int v_x \, dx = -\frac{AV_0}{2}\left[x + F(y)\right] \tag{15}$$

At the leading edge, however, $\phi = 0$ and, therefore, the unknown function $F(y)$ may be evaluated as follows:

$$x = |y| \tan \Lambda \tag{16}$$

hence

$$F(y) = -|y| \tan \Lambda \tag{17}$$

where Λ is angle of sweepback.

Since the surface potential distribution and doublet distribution are equal except for a constant factor π, the potential of the complete flow is given by the doublet distribution as

$$\phi = \frac{\partial}{\partial z}\left\{\int_{b_1}^{b_2}\int_{a_1}^{a_2} \frac{AV_0}{2\pi} \frac{\left(x_1 - |y_1| \tan \Lambda\right) dx_1 dy_1}{\sqrt{(x - x_1)^2 - (M^2 - 1)\left[(y - y_1)^2 + z^2\right]}}\right\} \tag{18}$$

This integral can now be evaluated and, from the potential function obtained, the vertical velocity at the surface may be computed; thus the camber lines can be found. The camber on such a wing has been given already by Jones in reference 5.

In the solution of the integrals used in this paper the order of integration and differentiation is found to be quite important. The solutions are difficult because the elementary solution is not defined at its Mach cone. Hadamard (reference 6) has treated such problems, however, and Heaslet and Lomax (reference 7) have applied his procedure to the solution of wing problems.

The foregoing method of doublet distributions can be tied in with the familiar vortex-theory concept of incompressible flow. Consider a three-dimensional wing represented by a certain doublet or surface-potential distribution. The circulation Γ about a chordwise strip can be computed in the usual manner by integrating the velocities along a line directly above and below the surface. The circulation thus becomes

$$\Gamma = \int_{\mathrm{L.E.}}^{\mathrm{T.E.}} \left(v_{x_u} - v_{x_l}\right) dx = \phi_u - \phi_l \tag{19}$$

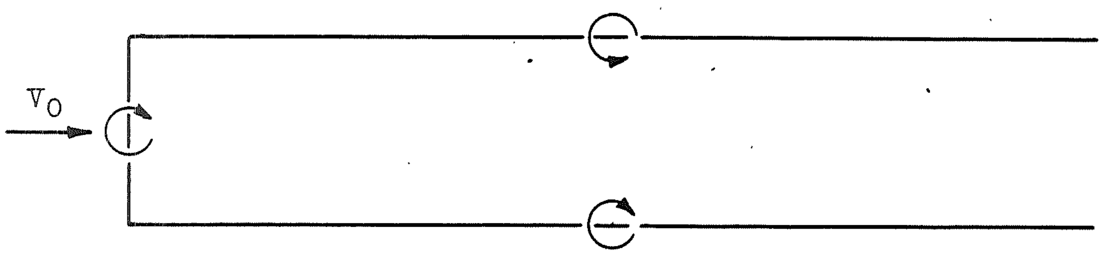
The integration is complete at the trailing edge because the x-component-velocity difference vanishes at this point. The potential difference in the wake becomes, then, only a function of y. The pressure on the strip is given by the relation

$$\Delta p = -\rho V_0 v_x \tag{20}$$

The lift on the strip of width dy is then

$$\frac{dL}{dy} = \int_{\mathrm{L.E.}}^{\mathrm{T.E.}} (\Delta p_l - \Delta p_u) dx = \rho V_0 \Gamma \tag{21}$$

This equation shows that the Joukowski hypothesis is valid also in the supersonic range of flight and the familiar concepts of vortex flow must apply. It will be interesting to show how a horseshoe vortex can be found by a doublet distribution. In the sketch, consider the area in the xy-plane enclosed by the straight lines to be uniformly covered by doublets.



The surface potential in this region will also be constant as discussed previously. The circulation computed along any circuit enclosing the boundary of the region will then be a constant which is proportional to the strength of the doublet distribution. The doublet distribution is therefore seen to be equivalent to a single vortex line along the boundary of the region considered. This result is directly analogous to that found in incompressible flow; however, the induced velocity fields in the two cases are different except at a large distance behind the lifting line where they become the same. Any wing and its wake may be represented by a vortex distribution in the usual manner, although for supersonic wing problems the lifting-line theory is unsatisfactory because of the discontinuities present in the solutions which make the assumptions used in lifting-line theory very poor indeed. The alternative is, of course, to proceed to a surface distribution of vortices or doublets in which case the objections are overcome. Calculations of this type have been performed by Schlichting. (See reference 8.) At a great distance behind the wing, the change in potential with respect to the flight direction approaches zero and the differential equation of motion

$$(M^2 - 1)\frac{\partial^2 \phi}{dx^2} = \frac{\partial^2 \phi}{dy^2} + \frac{\partial^2 \phi}{dz^2}$$

approaches Laplace's equation in two dimensions involving only the cross-flow velocities. The downwash produced by the wing is then seen to be affected by only the trailing vortex system as in incompressible flow. (See reference 9.) It might be supposed that the induced drag of the wing could be calculated from the energy in the wake; however, an additional amount of energy due to lift is found to be transported to infinity by the sound waves produced at the wing. (See reference 10.) The calculation of the induced drag for certain cases requires the use of second-order terms, which were originally dropped in the analysis. The use of such terms is, therefore quite a controversial subject. Available information on the subject, however, indicates that a procedure in which these terms are included is at least qualitatively correct.

Puckett and others have demonstrated that under certain conditions the flow over lifting surfaces may be obtained by source distributions instead of doublet distributions which involve integral equations. Such conditions arise when the flow component normal to a leading edge is supersonic; then, the two sides become independent. This fact is illustrated by the first sketch of figure 2. The region of influence of a disturbance on one side of the triangular wing does not intersect the leading edge; hence, the effect of the disturbance is not felt on the opposite side. The pressure distribution on the surface for the case in which the opposite surface forms a wedge is the same as that for the case in which the opposite side is coincident with the original surface. The potential can therefore be found for both cases by the source-distribution method.

The second sketch in figure 2 indicates that when the flow component normal to the leading edge is subsonic, the two surfaces are not independent, and the source distribution method cannot be used without further consideration. Evvard (reference 11) has found a very ingenious way to extend the source-distribution method for calculating lift to those cases in which the leading-edge components of velocity become subsonic. It is assumed that the wing leading edge is extended until the normal component of flow is supersonic and the two sides become independent. The problem is then to determine the proper source distribution over the wing extension that makes the potential difference in this region zero. In figure 2 the region on the wing extension affecting the field point shown is labeled A. Evvard discovered that the effect of the proper source distribution over this area was equal but opposite in sign to the effect produced by the sources in region B. The potential at the field point can therefore be calculated by performing an integration of the sources over region C. Problems involving the use of two interacting wing extensions are no longer as simple but can still be done. The aforementioned method is indeed of great utility as it is very simple to apply and obviates the necessity for solving integral equations.

The theoretical work discussed herein must be carefully checked experimentally before it can be trusted to any great extent. At this time it is too early to set down the limits of applicability, such as the Mach number range, maximum angle of attack, or thickness ratios; however, the fact that the theory is of great value cannot be questioned inasmuch as the results provide analytic expressions from which trends and the effects of variation of parameters may be found.

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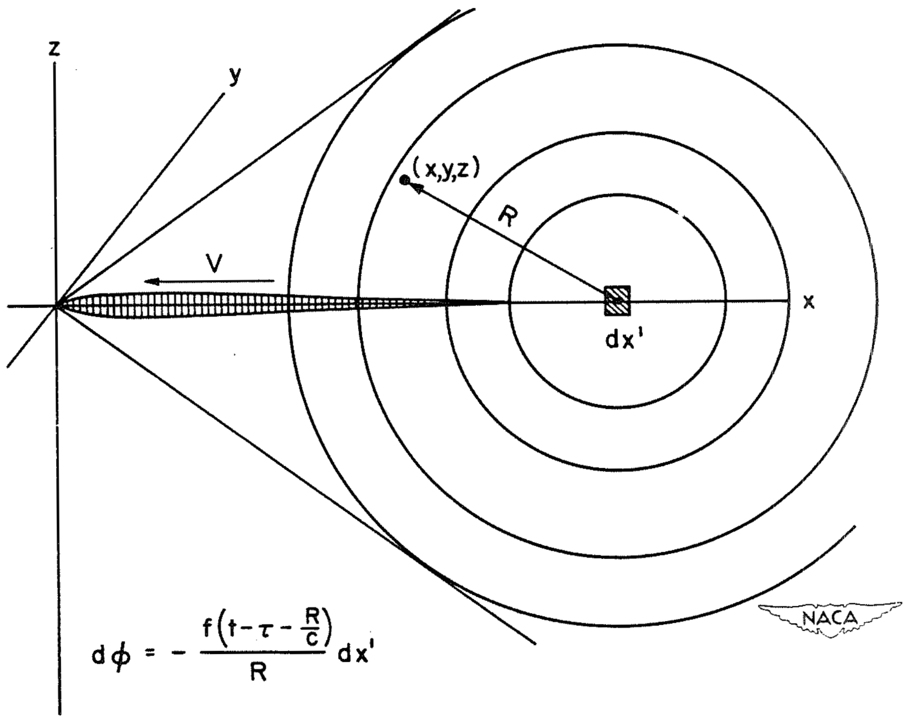

Figure 1.- Source system in motion.

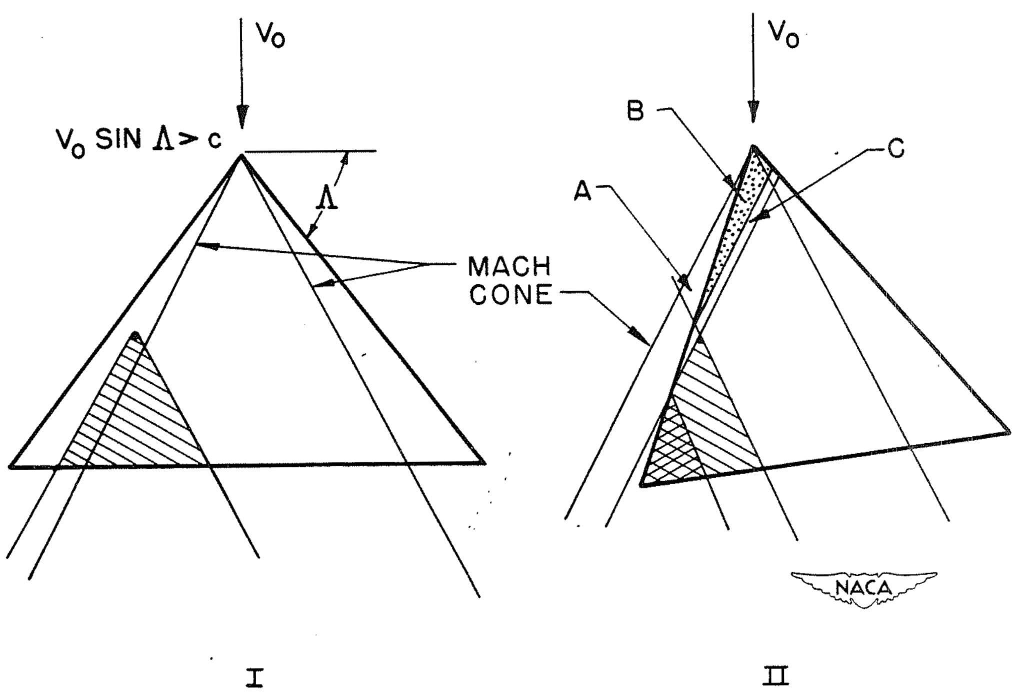

Figure 2.- Illustration of Evvard's method.