# UNSTEADY LIFT IN HIGH-SPEED FLIGHT

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The problems discussed in this paper involve the initial time history of forces on a two-dimensional flat-plate wing section. The first of these problems is the calculation of the transient pressure on the flat plate starting from rest and continuing at a constant speed and angle of attack. This can correspond physically to a sudden angle-of-attack change. The second problem is the calculation of the transient pressure on the flat plate entering a sharp-edge gust. For large values of time, both of these solutions approach the more familiar steady-state value of the lift on the plate at a given angle of attack.

The problem involving the angle-of-attack change is the same as that studied by Wagner (reference 1) for the case of incompressible flow. In the present paper, this well-known solution is extended to include the effects of both subsonic and supersonic Mach numbers. This study shows that the effect of Mach number on the load distribution is entirely different at the beginning of the motion than at the end of the motion when the Prandtl-Glauert or Ackeret formula applies.

Previous studies have been made in the field of high-speed unsteady lift by Garrick and Rubinow (reference 2), and by Chang (reference 3).

The method of solution employed in this paper differs from that used by Garrick and Rubinow in their study of flutter in that emphasis is placed on the development of lift following a sudden unit change in angle of attack rather than on the lift of a harmonically oscillating wing. This unit-angle-of-attack method was used by Heaslet and Lomax (reference 4) and proceeds as follows: First, the basic partial differential equation is obtained and simplified to its linearized form; then a solution for a sudden "unit" displacement is found in terms of the pressure distribution; finally, since the basic equation has been linearized, these solutions for the unit displacement are superimposed with the result that the pressures on a flat plate undergoing any arbitrary motion can be found. The usefulness of the result is greatly increased by its ready adaptation to the operational methods used in similar problems by Jones (reference 5) and presented in detail by Churchill (reference 6).

The solutions which this type of analysis yields is given the name "indicial solutions." As an example, consider that the unit displacement is the angle of attack of the wing. This means by definition that $\alpha$ is zero for negative values of time and equal to unity for all positive values of time. The load distribution resulting from such a unit displacement is called the indicial angle-of-attack load. Similarly, the integrated value of this loading is called the indicial angle-of-attack lift.

Proceeding now in the manner which has been outlined, the basic partial differential equation is obtained. This governing equation used in the study of unsteady lift problems comes from a combination of the equations of motion, continuity, and state. The approximations used in reducing these equations to the linearized form suitable for analysis are simply that the induced velocities are small enough to be neglected in comparison with the free-stream velocity and that the velocity gradients are all of similar magnitude. These assumptions in simplifying the partial differential equation are consistent with those of thin airfoil theory in simplifying the boundary conditions - namely, that the boundary conditions be specified in the plane $Z = 0$ and that the tangent of the angle of attack be replaced by the angle. Such approximations result in an indeterminate error in the induced velocities of the solutions, so that for terms like the lift, velocity of sound, and entropy which can be expanded in terms of, say, the induced velocity $u$, only the lowest nonzero power of $u$ should be used in the expansion.

The resulting linearized partial differential equation is the same as that studied by the various authors mentioned. The two-dimensional form of this equation for the perturbation velocity potential in terms of the space coordinates $X'$, $Z'$, the time $t'$ and the free-stream velocity of sound $a_o$ and Mach number $M_o$ can be written as follows:

$$\left(1 - M_o^2\right) \frac{\partial^2 \Phi}{\partial X'^2} + \frac{\partial^2 \Phi}{\partial Z'^2} - \frac{2M_o}{a_o} \frac{\partial^2 \Phi}{\partial X' \partial t'} - \frac{1}{a_o^2} \frac{\partial^2 \Phi}{\partial t'^2} = 0 \ . \ . \ . \qquad (1)$$

In such a form, it is rather formidable, but by use of the transformation

$$\left.\begin{aligned} X &= X' - M_o a_o t' \\ Z &= Z' \\ t &= a_o t' \end{aligned}\right\} \qquad (2)$$

it can be reduced to

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial X^2} - \frac{\partial^2 \Phi}{\partial Z^2} = 0 \qquad (3)$$

the normalized form of the wave equation.

The wave equation, of course, has been extensively studied, since it is one of the most important equations of mathematical physics. Most of this study, however, has been directed toward problems in which the boundary conditions are given for $t = 0$. (See, for instance, reference 7.) In such cases both the function and the initial values of its derivatives must be specified in order that a unique solution can be obtained. This is the so-called Cauchy problem. In unsteady lift problems, on the other hand, the boundary values are known only in the plane $Z = 0$; that is, the slope of the lifting surface is specified for all values of time. This difference in orientation completely changes the nature of the problem. And, in fact, it can be shown that when boundary values are specified for $Z = 0$, a unique solution can be found, just as in Laplace's equation, by specifying only the derivative of the function. This same situation – that of having the data given in another than the classical $t = 0$ plane – arose in the study of three-dimensional supersonic lifting-surface problems and a rather complete discussion of it in that connection is given in reference 8.

This analogue with the supersonic lifting-surface problem can be quite useful in establishing the data necessary for the unsteady lift cases. In order to construct this analogy, however, it is first necessary to discuss the boundary values for some typical unsteady lift problems. Consider a wing at an angle-of-attack $\alpha$ starting suddenly from rest at $t = 0$. (See fig. 1.) In the X,t plane, this wing would sweep out a region shown as the shaded area in figure 1(a). It is to be remembered from the transformation given as equation (2) that the coordinate $t$ represents true time $t'$ except for the stretching factor $a_o$, and, for a given $t$, a change in the coordinate $X$ represents a change in the true distance $X'$. Thus at $t = 0$, the shaded area extends from $X = 0$ one chord length back to $X = c_o$. At some later time, the wing will have moved in the negative X-direction to a new position such as the point A in figure 1(a). The trailing edge at such a time remains one chord length behind at the point C. The dash lines in the figure represent the traces of the characteristic cones. Physically, these lines represent the foremost and rearmost positions to which a pressure disturbance starting at their apex can travel in a given time. Thus a disturbance starting on the leading edge at $t = 0$ can be felt only between the points B and D when the wing has traveled so that its leading edge is at A. Hence, for a wing traveling at supersonic speeds, point A will fall to the left of the characteristic line and for a wing traveling at subsonic speeds it will fall to the right.

If the wing is to attain its unit angle of attack without rotation, then the boundary values are such that the vertical induced velocity is a constant over the entire shaded region of figure 1(a), and the loading is zero over the rest of the plane $z = 0$.

Compare these boundary conditions with those for a supersonic three-dimensional, flat-plate, lifting-surface problem. If the shaded region is thought of as a wing plan form, the problems are identical. The characteristic cones represent the familiar Mach cones; and since the dash lines have a $45^{\circ}$ slope, the equivalent Mach number is $\sqrt{2}$. The wing in unsteady lift flying at supersonic speeds has for its equivalent a three-dimensional lifting surface with supersonic edges. The solution to such a lifting-surface problem is relatively simple to find. On the other hand, the wing in unsteady lift flying at subsonic speeds has for its equivalent a three-dimensional lifting surface with some subsonic edges. Although this complicates the problem considerably, still by methods such as those introduced by Evvard (reference 9), solutions can be found.

Another type of boundary-value problem is shown in figure 1(b). This figure represents a wing at zero angle of attack traveling at supersonic speeds entering a sharp-edge gust the front of which is situated along the line $X = 0$. The vertical induced velocity over the shaded region is a constant, the value of which is equal and opposite to the gust intensity, and is zero in the unshaded region between the t-axis and the trailing-edge trace. Over the rest of the plane, the loading must be zero. Again by constructing the analogue with the three-dimensional lifting-surface problem a solution can easily be obtained.

The solution for the load coefficient for a wing starting from rest and continuing at supersonic speeds - that is, the indicial load coefficient for $\alpha$ - is shown in figure 2(a). At $t = 0$ the wing suddenly attains an angle of attack (without rotation). Immediately the load coefficient jumps to a constant value of magnitude $4\alpha/M_o$ over the entire chord. At a subsequent time this value moves off the chord with the trailing characteristic trace, the apex of which is at the origin. Between the traces of the characteristic cone a transition occurs, the load falling below the value $4\alpha/M_o$ and then rising to the higher value $4\alpha\Big/\sqrt{{M_o}^2 - 1}$. This latter value is the familiar steady-state two-dimensional value for load commonly called the Ackeret loading. As the leading characteristic trace leaves the wing, the Ackeret loading covers the chord and the wing has reached its steady-state value.

Compare this loading with the indicial load coefficient due to angle of attack for a wing flying at subsonic speeds (fig. 2(b)). Again at $t = 0$, the incremental load jumps immediately to the constant value $4\alpha/M_o$ over the entire chord. However, in this subsonic case, the load near the trailing edge immediately falls so that for all values of time greater than zero the loading at the trailing edge is continuous and zero. That is to say, the Kutta-Joukowski condition is satisfied except at the instant $t = 0$. Subsequently, the load distribution approaches asymptotically the normal additional load distribution associated with steady two-dimensional subsonic theory — that is, very high values near the leading edge fall to zero at the trailing edge.

Figure 3 shows the indicial load coefficient due to angle of attack for a restrained wing flying at supersonic speeds and entering a sharp-edge gust. The gust is located in the region from $X = -\infty$ to $X = 0$. Initially the loading over the chord is zero. As the leading edge begins to penetrate the gust, however, in the region between the characteristic traces the load begins to rise from zero to the Ackeret loading, and again as the loading characteristic trace crosses the trailing edge the wing has reached its final steady-state Ackeret value.

Further discussion of these phenomena can best be completed by considering the integrated values of these loadings plotted against a variable representing time. Thus figure 4 shows a plot of lift-curve slopes against $s$, the number of half-chords traveled by the wing, for a wide range of Mach number. The curve for $M = 0$ — that is, the curve for the two-dimensional wing with incompressible flow — was first studied by Wagner. Since the starting value is always $4/M$, this incompressible flow value of $C_{L_\alpha}$ must initially jump to infinity. The infinite value, which results from the infinite acceleration imposed in the boundary conditions by the step function, lasts only for an infinitesimal time; however, $C_{L_\alpha}$ then falls to $\pi$ and rises gradually to the asymptotic value $2\pi$. The calculations for the curves in the region $0 < M_o < 1$ were completed only to the time necessary for a pressure signal to travel from the trailing to the leading edge. Further computations are possible but would have been more complex. On the other hand, the qualitative nature of the curves for larger values of time is fairly obvious. Thus at a Mach number of 0.8 the starting value is 5. The lift-curve slope falls linearly for the time required to travel about a half-chord length and then rises to approach asymptotically the value $2\pi / \sqrt{1 - M^2}$. The curve for a Mach number of 0.4 is similar. Such a behavior obviously invalidates the use of the Prandtl-Glauert Mach number correction to unsteady lift analysis. Between $t = 0$ and $t = \infty$ the correction factor must lie between $1/M$ and $1 / \sqrt{1 - M^2}$, but the exact value is quite complex. The variation of this transient $C_{L_\alpha}$ with Mach number is accentuated most sharply by considering the value at $M = 1$.

The curve in figure 4 for $M = 1$ presents the build-up of lift-curve slope for a wing starting from rest and traveling at the speed of sound. Since the Mach number is unity, the starting value of $C_{L_\alpha}$ is 4. The magnitude of $C_{L_\alpha}$ increases with time and is infinity at $s = \infty$. However, since the whole theory is based on the assumption that the induced velocities are small compared with the free-stream velocity, the number of half-chords traveled before the theory breaks down is severely limited. Just how much so depends, of course, on the angle-of-attack change chosen. Nevertheless, some insight into the nature of sonic flow has been gained.

Curves for supersonic Mach numbers of 1.2 and 1.4 are also plotted in figure 4. At a Mach number of 1.2 the variation of $C_{L_\alpha}$ with time approximates rather closely Wagner's curve for $M = 0$. In the supersonic case, however, $C_{L_\alpha}$ is initially constant at $4/M$ for about one-half chord length traveled and then rises and reaches its steady-state Ackeret value of $4/\sqrt{M^2 - 1}$ after a few chord lengths; whereas, in the subsonic case, the value is nowhere constant and approaches its steady-state value asymptotically. For higher supersonic Mach numbers, the magnitudes of the curves decrease and the difference between the starting and final values becomes less.

So far, the discussion has been limited to the indicial lift for a sudden angle-of-attack change. A comparison between this lift and that developed by a supersonic wing entering a sharp-edge gust is given in figure 5. The curves shown are both for a Mach number of 1.2, the dash curve representing the change in lift coefficient for the wing entering the gust. The principal difference between the curves is in the initial value, the gust curve starting at zero and the angle-of-attack curve starting at $4/M$. After about 12 chord lengths, however, the curves are identical, since both have assumed the Ackeret value.

A simple but important dynamic maneuver can be studied by means of the two indicial lift functions, the one for $\alpha$ and the other for a gust. This maneuver is the response of an unrestrained airfoil to a gust when the effects of pitching are neglected. Such a maneuver for sharp-edge gusts has been studied in reference 4.

The study of the unsteady lift problem is far from being complete. It is believed that the effect of the indicial lift on the downwash at the vertical tail plane at supersonic speeds has not been touched, nor has the effect of gusts on wings traveling at high subsonic speeds. Furthermore, the methods described in this paper might be used as another approach to the study of compressible flutter problems for Mach numbers less than 1 – especially the problem of aileron flutter. Another type of research, the study of which has just been started at the Ames Aeronautical Laboratory, is that of the two-dimensional wing accelerating through the speed of sound. The results already presented for the indicial lift around a Mach number of 1 indicate that further research along these lines might produce worthwhile results.

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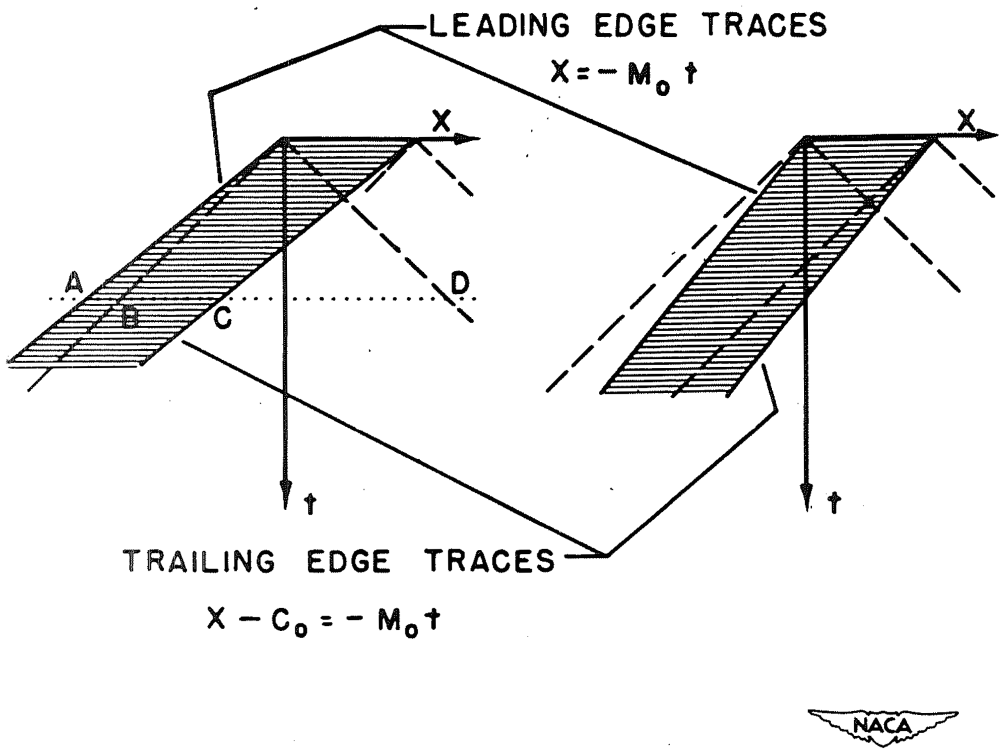
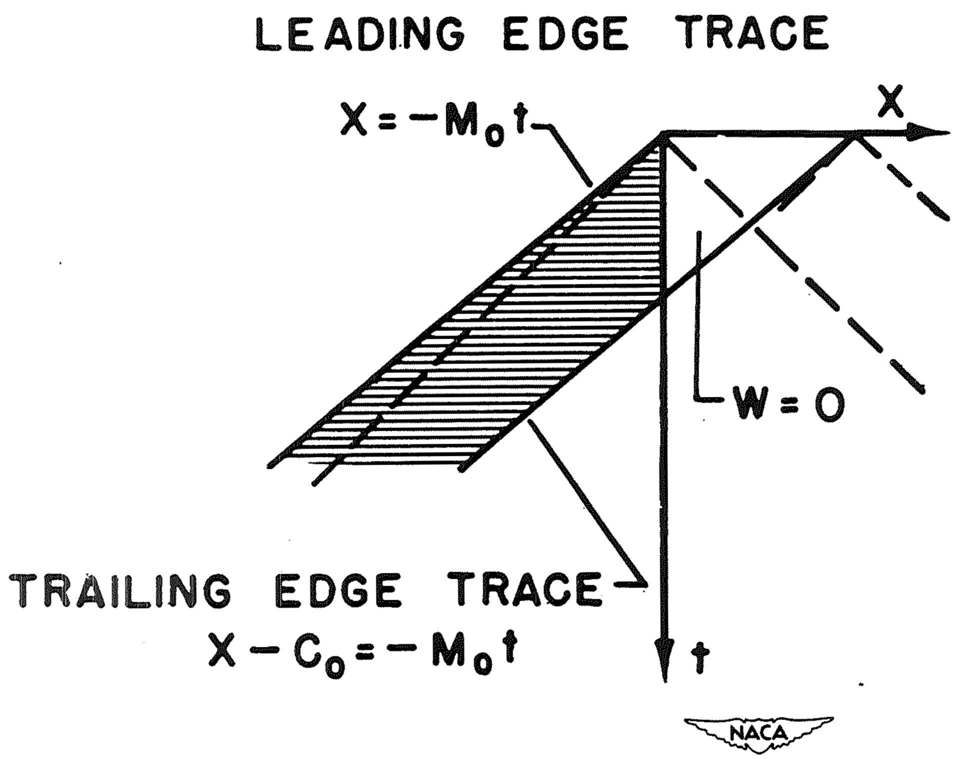



(a) Subsonic and supersonic wing receiving sudden angle-of-attack change at $t = 0$.

Figure 1.- Sketches showing different types of boundary conditions for two-dimensional unsteady lift problems.




(b) Supersonic wing entering sharp-edge gust at $X = 0$.

Figure 1.- Concluded.

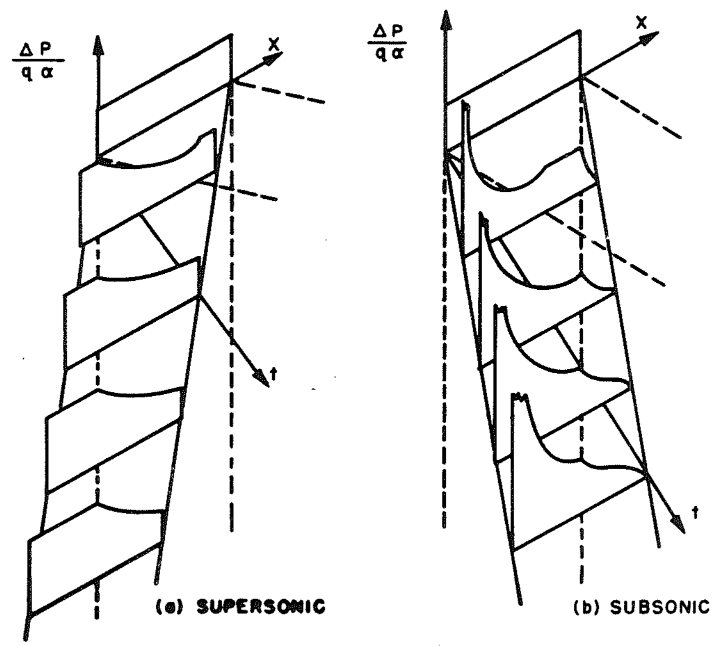



Figure 2.- Pressure distributions on wing receiving sudden angle-of-attack change at $t = 0$.

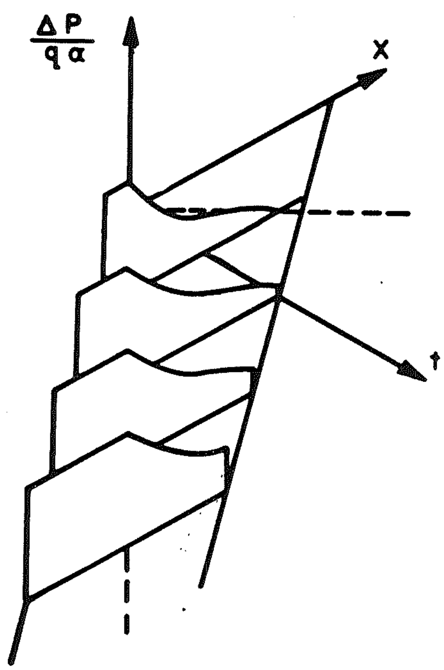



Figure 3.- Pressure distribution on supersonic wing entering sharp-edge gust at $X = 0$.

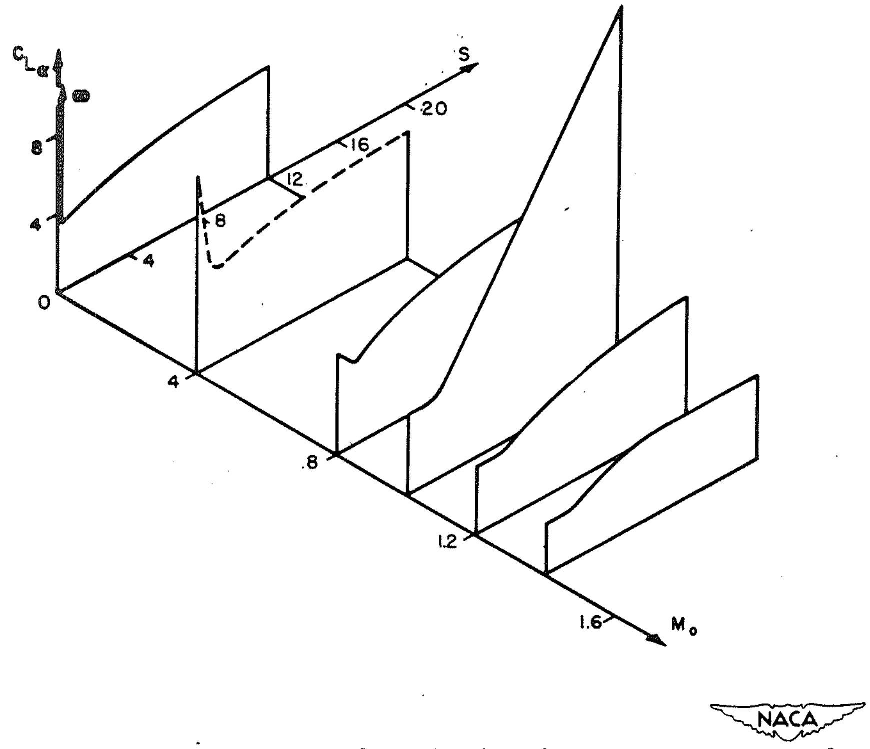



Figure 4.- Indicial lift-curve slope for Mach numbers between 0 and 1.4 shown to time required to travel 12 half-chords.

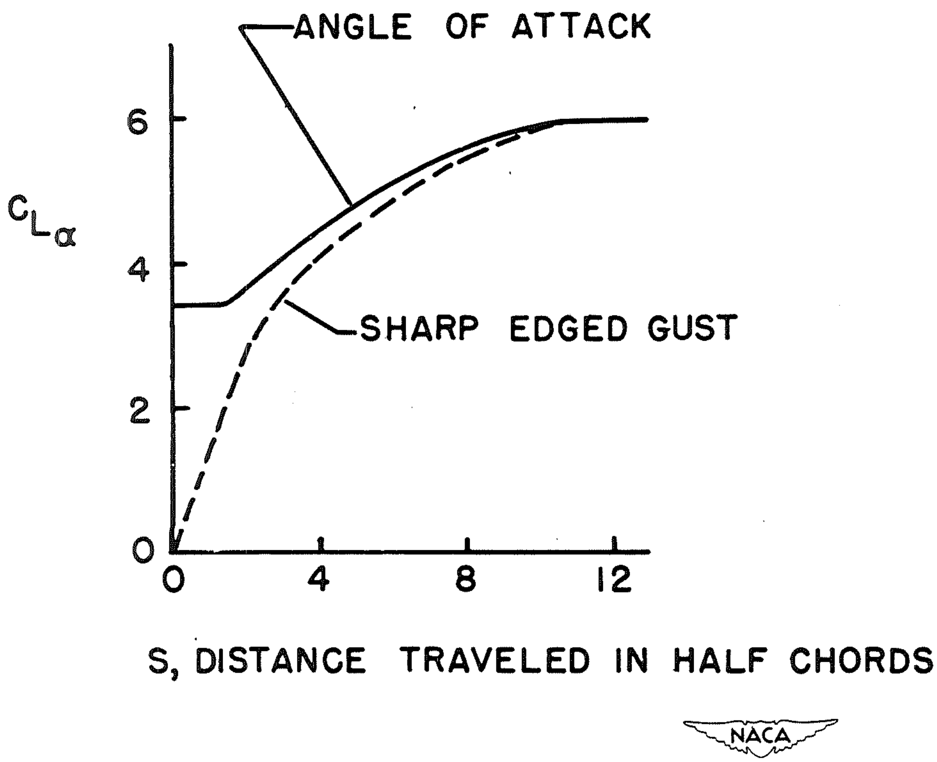



Figure 5.- Indicial lift-curve slopes for restrained wing.