N O T I C E

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Coupled Rotor and Fuselage Equations of Motion

William Warmbrodt

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William Warmbrodt, Ames Research Center, Moffett Field, California

NASA

National Aeronautics and
Space Administration

Ames Research Center
Moffett Field, California 94035

SYMBOLS

a	rotor blade section two-dimensional lift-curve slope
$\underset{\sim}{a}_{Pk}$	inertial acceleration vector of a point on the kth blade with respect to the undisturbed hub location
A	blade cross section area
b	blade half-chord
c	blade chord
$c_d,\ c_\ell,\ c_{m_{ac}}$	blade cross section profile drag, lift, and moment coefficients
$C(k)$	Theodorsen's lift deficiency function
D_{Sk}	magnitude of distributed aerodynamic profile drag on kth blade
e	offset between axis of rotation and blade root
E	Young's modulus of blade
F_X, F_Y, F_Z	components of $\underset{\sim}{P}_T$
g	gravitational constant
g_{SF}, g_{SL}, g_{ST}	structural damping coefficients in flap, lag and torsion, respectively
GJ	blade torsional rigidity
I_F	tensor of fuselage inertias
I_X, I_Y, I_Z, I_{XZ}	inertias of rigid body fuselage
I_{YY}	$\iint_A \xi^2\, dy_o dz_o$
I_{ZZ}	$\iint_A \eta^2\, dy_o dz_o$
k	subscript identifying the kth blade
K_β, K_ζ, K_ϕ	blade root flap, lag, and torsion spring stiffnesses, respectively

ℓ length of blade capable of elastic deformation, $\ell = R - e$

L_S, L_{US} steady and unsteady components of airfoil lift

m mass per unit length of blade

M rotor blade section Mach number

M_F mass of fuselage

M_R mass of rotor

M_S, M_{US} steady and unsteady component of airfoil aerodynamic moment

M_{tip} blade tip Mach number, $M_{tip} = \Omega R/\text{(speed of sound)}$

N number of blades composing rotor

M_X, M_Y, M_X blade cross section elastic moments expressed in 4k – coordinate system

$\underset{\sim}{P}_k, \underset{\sim}{q}_k$ kth blade distributed force and moment vector

$\underset{\sim}{P}_{Ak}, \underset{\sim}{q}_{Ak}$ kth blade distributed aerodynamic force and moment vector

$\underset{\sim}{P}_{Dk}, \underset{\sim}{q}_{Dk}$ kth blade distributed damping force and moment vector

$\underset{\sim}{P}_{Ik}, \underset{\sim}{q}_{Ik}$ kth blade distributed inertia force and moment vector

$\underset{\sim}{P}_F, \underset{\sim}{Q}_F$ fuselage inertia force and moment vector acting at fuselage center of mass

$\underset{\sim}{P}_G, \underset{\sim}{Q}_G$ fuselage gravitational force and moment vector acting at hub

$\underset{\sim}{P}_H, \underset{\sim}{Q}_H$ fuselage inertia force and moment vector acting at hub

$\underset{\sim}{P}_R, \underset{\sim}{Q}_R$ rotor-induced force and moment vector acting at hub

$\underset{\sim}{P}_T, \underset{\sim}{Q}_T$ generalized fuselage force and moment vector acting at hub

$\underset{\sim}{P}_{2k}, \underset{\sim}{Q}_{2k}$ kth blade root force and moment vector

Q_X, Q_Y, Q_Z components of $\underset{\sim}{Q}_T$

$\underset{\sim}{r}_{ECk}$ position vector of deformed kth blade cross section

         elastic center with respect to undisturbed hub location

$\underset{\sim}{r}_F$ position vector of displaced fuselage center of

         mass with respect to displaced hub

$\underset{\sim}{r}_h$ hub translation vector

$\underset{\sim}{r}_H$ position vector of hub with respect to fuselage center of

         mass (no fuselage motion)

$\underset{\sim}{r}_{Pk}$ position vector of point on deformed blade cross section

         with respect to undisturbed hub location

r_X, r_Y, r_Z rigid body displacement components of hub

R radius of blade

$\underset{\sim}{R}_F$ vector of fuselage translations

R_X, R_Y, R_Z fuselage rigid body translations

t time

T_k tensile force in kth blade

u_k, v_k, w_k elastic displacement in axial, lead-lag, and flap

         directions of kth blade, respectively

U_N relative velocity of air normal to blade cross section

         chord

U_{Rk} magnitude of resultant relative inplane air velocity

         vector of kth blade cross section

U'_{Rk} expression for U_{Rk} used in unsteady airload calculation

$\underset{\sim}{V}_{ECk}$	inertial velocity of deformed kth blade's cross section elastic center
$\underset{\cdot\cdot}{V}_k$	resultant relative air velocity of deformed kth blade cross section
$\underset{\sim}{V}_{Ak}$	free stream air flow vector
$\underset{\sim}{V}_F$	perturbation velocity of fuselage center of mass
V_Y, V_Z	blade cross section elastic shears expressed in 4k – coordinate system
V_{5Yk}, V_{5Zk}	kth blade cross section inplane relative air velocity components
x_A	blade cross section offset between elastic center and aerodynamic center
$x_{A_{RF}}$	effective value of x_A within reversed flow region
x_I	blade cross section offset between elastic center and center of gravity
x_T	blade cross section offset between elastic center and tension center
x_F, z_F	components of fuselage center of mass position vector with respect to hub
x_k	kth blade radial coordinate along elastic axis
y_{Ok}, z_{Ok}	blade cross section coordinates
$\dot{\alpha}_P$	pitch rate of airfoil cross section for airloads calculation
α	effective blade section angle of attack
α_R	angle of forward tilt of rotor plane in forward flight
β_P	blade precone angle

ϵ	order of magnitude associated with elastic blade slopes
η_B, ξ_B	blade principal axes coordinates
θ_{Gk}	total geometric pitch angle of blade cross section chordline with respect to hub plane
$\theta_0, \theta_{1C}, \theta_{1S}$	collective and cyclic pitch settings
$\theta_X, \theta_Y, \theta_Z$	fuselage rigid body rotations
λ_k	inflow ratio, $(V_F \sin\alpha_R + \nu_k)/\Omega R$
μ	advance ratio, $V_F \cos\alpha_R / \Omega R$
ν_k	induced inflow velocity of kth blade
ρ	mass density of blade material
ρ_A	density of air
ϕ_{ik}	inflow angle of kth blade
ϕ_k	elastic rotation of kth blade cross section
ψ_k	kth blade azimuth angle
$\underset{\sim}{\omega}_k$	angular velocity vector of kth rotor blade with respect to undisturbed hub
$\underset{\sim}{\omega}_F$	angular velocity vector of fuselage
Ω	constant rotor speed of rotation

Special Symbols

$$(\cdot) = \frac{d}{dt} (\quad) = \Omega \frac{d}{d\psi_k} (\quad)$$

$$(\quad)_{,x} = \frac{d}{dx_k}(\quad)$$

$$\overline{(\quad)} = \frac{(\quad)}{R}$$

v

COUPLED ROTOR AND FUSELAGE EQUATIONS OF MOTION

William Warmbrodt
Ames Research Center

SUMMARY

The governing equations of motion of a helicopter rotor coupled to a rigid body fuselage are derived. A consistent formulation is used to derive non-linear periodic coefficient equations of motion which can be used to study coupled rotor/fuselage dynamics in forward flight. Rotor/fuselage coupling is documented and the importance of an ordering scheme in deriving nonlinear equations of motion is reviewed. The nature of the final equations and the use of multiblade coordinates are briefly discussed.

I. INTRODUCTION

Helicopter rotors are complex aeroelastic systems where structural, inertia and aerodynamic forces interact to determine rotor behavior. To mathematically model this problem, a consistent formulation of the governing equations of motion must be used. Additionally, recent research has indicated that rotary wing aeroelasticity is inherently nonlinear.[1] Consequently this aeroelastic problem requires a careful derivation of the dynamic equations of equilibrium so that moderate blade deflections based upon the assumption of small strains and finite slopes are properly incorporated in the rotor math model.

Mathematical models of single isolated helicopter blades are well developed.[2,3] The results of these investigations have shown that the effect of nonlinear terms on blade aeroelastic stability is important. However,

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there has been relatively little emphasis on developing consistently derived nonlinear math models detailing coupled rotor/fuselage dynamics.

Considerable work has been done to analytically model coupled rotor/fuselage systems. These models include large general purpose helicopter analyses[4-7] and math models which study aeromechanical phenomenon for a particular rotor configuration[8-10]. The first group of analyses utilize a general rotor model and system degrees of freedom. Reference 11 presents a review of such comprehensive helicopter analyses and the level of technology used in the math modeling procedure. Despite the sophistication of these models, there has been little effort to derive consistent nonlinear equations of motion. The second group of analyses limit attention to a given rotor configuration and flight regime or restrict fuselage motion.

This report will present the derivation of a set of equations which model the basic dynamics of a coupled hingeless rotor/fuselage system. The math model is not intended to represent all the very complicated interactions the aircraft experiences in flight. Rather, this study is an attempt to mathematically model the complex dynamic interaction between the rotor and fuselage. This is performed by deriving a set of nonlinear equations of motion while clearly documenting the rotor/fuselage matching procedure.

II.   FORMULATION OF THE PROBLEM

A.   General

The configuration considered in this study is shown in Figs. 1 and 2. The rotor has N blades with distributed mass and stiffness properties. Each blade can have out-of-plane bending, w_k , inplane bending, v_k , and

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torsion, ϕ_k , deflections. Rotor geometry is shown in Figs. 3 and 4.

The helicopter fuselage is modeled as a rigid body with three transla-tional and three rotational degrees of freedom. The fuselage center of mass is assumed to be located in the vertical centerline plane of the fuselage.

B. Basic Assumptions

The following assumptions are used in deriving the equations of motion.

1) The blades are connected to the hub at an offset distance e from the axis of rotation.

2) The blade feathering axis is preconed by an angle β_p. The blade has no torque offset and no angle of sweep or droop.

3) The blade cross section is symmetric about the major principal axis.

4) The blade cross section can have four distinct points: center of gravity, aerodynamic center, tension center, and elastic center. The elastic axis is assumed to be a straight line and coincident with the feathering axis.

5) The blades can have distributed pretwist about the undeformed elastic axis.

6) The blades are assumed to be composed of an isotropic linear elastic homogeneous material.

7) Each blade can bend in two mutually perpendicular directions normal to the elastic axis and torsionally twist. Moderate deflections are assumed resulting in small strains and finite rotations.

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8) During deformation, the blade cross sections remain plane and normal to the elastic axis (Euler-Bernoulli hypothesis).

9) Two-dimensional quasi-steady aerodynamic loads are used. Apparent mass, compressibility, and stall effects are included; dynamic stall effects are neglected.

10) The rotor is assumed to operate at a constant speed of rotation.

11) All structural damping forces are assumed to be of a viscous type.

12) The rotor shaft is assumed rigid. No control system flexibility or engine dynamics are modeled.

13) The fuselage is modeled as a rigid body with total mass M_F and moments of inertia I_X, I_Y, I_Z and product of inertia I_{ZX}

$(I_{ZY} = I_{XY} = 0)$.

14) The fuselage center of mass is located in the vertical centerline plane of the fuselage.

15) The fuselage has translational degrees of freedom R_X, R_Y, R_Z, in three mutually perpendicular directions and three rotational degrees of freedom θ_X, θ_Y, θ_Z.

16) The helicopter is assumed to be in equilibrium forward flight.

Additional assumptions used in the derivation are given in the text.

C. Ordering Scheme

In the process of deriving the set of governing nonlinear partial differntial equations, a considerable number of terms which may be small and negligible are encountered. To neglect higher order terms in a systematic manner, an ordering scheme is used[1]. All quantities are assigned an order

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of magnitude in terms of a nondimensional quantity, ε, which represents typical elastic blade slopes. The ordering scheme is then used with the assumption that terms of $O(\varepsilon^2)$ are negligible when compared to terms of order one, i.e, $1 + O(\varepsilon^2) \cong 1$.

When modeling the dynamics of coupled rotor/support systems, the magnitude assignment of fuselage motion is very important. The assigned order of magnitude of fuselage motions must allow the math model to adequately represent actual system behavior. However, the ordering scheme should be such to restrict the total number of terms appearing in the final equations to those which are important. This will not only allow the solution of the final equations to be less cumbersome, but also yield greater physical insight into the nature of the problem. Consequently, it is assumed in this study that fuselage rotations and perturbation translations are of order $\varepsilon^{3/2}$. This will allow the model to be valid for a helicopter in equilibrium flight undergoing small fuselage motions. Moderate blade deformation is retained. Additionally, the analysis will remain general to adequately model the complex dynamics between the rotor and the fuselage. On the other hand, this assignment restricts the tremendous number of fuselage-dependent rotor forcing terms. If fuselage rotations and nondimensional displacements are assigned a lower order of magnitude, i.e., $O(\varepsilon)$, the expressions for the rotor/fuselage matching conditions are practically unmanageable. The present assignment of $O(\varepsilon^{3/2})$ will restrict the final equations of motion to a length where physical understanding is still possible.

The assigned orders of magnitude of the quantities used in this study are

$$a = O(\tfrac{1}{\varepsilon})$$

$$\frac{x_E}{R}, \frac{z_F}{R}, \mu, c_\ell, \frac{\rho_a a b R}{m} = O(1)$$

$$\frac{\partial}{\partial x_k}, \frac{\partial}{\partial t}, \frac{\partial}{\partial \psi}, \cos \psi_k, \sin \psi_k = O(1)$$

$$\frac{v_k}{R}, \frac{w_k}{R}, \phi_k, \frac{e}{R}, \frac{b}{R}, \lambda_k, \theta_{Gk}, \alpha_R, c_{m_{ac}} = O(\varepsilon)$$

$$\frac{R_x}{R}, \frac{R_y}{R}, \frac{R_z}{R}, \theta_{Fx}, \theta_{FY}, \theta_{FZ}, c_d = O(\varepsilon^{3/2})$$

$$\frac{u_k}{R}, \frac{x_A}{R}, \frac{x_I}{R}, \frac{x_T}{R} = O(\varepsilon^2)$$

The systematic application of this ordering scheme in the derivation procedure will yield a consistent set of nonlinear equations of motion.

III. DERIVATION OF GOVERNING EQUATIONS

A. General

The math model is composed of an N-bladed elastic rotor and a rigid body fuselage. Each blade will have governing partial differential equations of motion for the flap, lag, and torsion deflections. The fuselage motion will be governed by the three force and three moment equations of system equilibrium.

The math model of this study considers each rotor blade individually and the equations for the flap, lead-lag, and torsion of each blade are derived. Rigid body fuselage motions are included in the calculation of

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the blade loads. Consequently, the equations of motion for each rotor blade are coupled to the fuselage through the forcing terms in the loading expressions.

Overall system equilibrium is enforced by requiring force and moment balance at the rotor hub. These six equations of equilibrium are the governing fuselage equations of motion.

B. Coordinate Systems

The coordinate systems used in the derivation procedure are shown in Figs. 1-7. A brief description of these coordinate systems is given in Table 1. The Euler transformation matrix between each system is given in Appendix A. The unit coordinate vectors of the jth system in the X_j, Y_j, Z_j axe directions are given by e_{jX}, e_{jY}, e_{jZ}, respectively.

C. Blade Equations of Motion

The governing equations of motion for the kth blade of an N-bladed rotor are taken from Ref. 12. It is important to note that Ref. 12 neglects the influence of axial forces on the torsional rigidity of the rotor blade as being very small (compared to the bending stiffness). In addition, for rotor blades where cross sectional warping due to torsion may be significant, a more appropriate elasticity theory should be used to derive the blade math model. For the present analysis, this model is assumed to be sufficiently accurate.

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The kth blade equations of equilibrium written in the undeformed coordinate directions, under the assumptions of Sec. II. B, are given by

$$T_{k,x} + (v_{k,x} \, \mathcal{g}_{3Zk})_{,x} - (w_{k,x} \, \mathcal{g}_{3Yk})_{,x} + \mathcal{P}_{3Xk} = 0 \qquad (1)$$

$$-[E(I_{ZZ} \cos^2\Theta_{Gk} + I_{YY} \sin^2\Theta_{Gk})v_{k,xx}$$
$$+ E(I_{ZZ} - I_{YY})(\phi_k w_{k,xx} \cos 2\Theta_{Gk} + (w_{k,xx} - 2\phi_k v_{k,xx}) \cdot$$
$$\cdot \sin\Theta_{Gk} \cos\Theta_{Gk}) - T_k x_T (\cos\Theta_{Gk} - \phi_k \sin\Theta_{Gk})]_{,xx}$$
$$- (GJ \phi_{k,x} w_{k,xx})_{,x} + (v_{k,x} T_k)_{,x}$$
$$+ (w_{k,x} \, \mathcal{g}_{3Xk})_{,x} - \mathcal{g}_{3Zk,x} + \mathcal{P}_{3Yk} = 0 \qquad (2)$$

$$-[E(I_{ZZ} - I_{YY})\sin\Theta_{Gk} \cos\Theta_{Gk}(v_{k,xx} + 2\phi_k w_{k,xx})$$
$$+ E(I_{ZZ} \sin^2\Theta_{Gk} + I_{YY} \cos^2\Theta_{Gk})w_{k,xx}$$
$$+ E(I_{ZZ} - I_{YY})\phi_k v_{k,xx} \cos 2\Theta_{Gk})$$
$$- T_k x_T (\sin\Theta_{Gk} + \phi_k \cos\Theta_{Gk})]_{,xx}$$
$$+ (GJ \phi_{k,x} v_{k,xx})_{,x} + (w_{k,x} T_k)_{,x}$$
$$- (v_{k,x} \, \mathcal{g}_{3Xk})_{,x} + \mathcal{g}_{3Yk,x} + \mathcal{P}_{3Zk} = 0 \qquad (3)$$

$$\left[GJ(\phi_{k,x} + v_{k,xx} w_{k,x}) \right]_{,x} + E(I_{ZZ} - I_{YY}) \left[(v_{k,xx}^2 \right.$$

$$\left. - w_{k,xx}^2) \sin\theta_{Gk} \cos\theta_{Gk} - v_{k,xx} w_{k,xx} \cos 2\theta_{Gk} \right]$$

$$+ T_k x_T (w_{k,xx} \cos\theta_{Gk} - v_{k,xx} \sin\theta_{Gk})$$

$$+ q_{3Xk} + v_{k,x} q_{3Yk} + w_{k,x} q_{3Zk} = 0$$

$$(4)$$

These represent the equations of equilibrium for the kth blade axial direction and the lead-lag, flap and torsion deflections, respectively.

The distributed loading vectors which appear in Eqs. (1)-(4) may be written as the sum of inertia, aerodynamic and damping contributions.

$$\underset{\sim}{P}_{3k} = \underset{\sim}{P}_{3Ik} + \underset{\sim}{P}_{3Ak} + \underset{\sim}{P}_{3Dk} \qquad (5)$$

$$\underset{\sim}{q}_{3k} = \underset{\sim}{q}_{3Ik} + \underset{\sim}{q}_{3Ak} + \underset{\sim}{q}_{3Dk} \qquad (6)$$

The detailed derivations of $\underset{\sim}{P}_{3k}$ and $\underset{\sim}{q}_{3k}$ are presented below.

C.1. Inertia Loads

The distributed inertia loads are determined by calculating the inertial acceleration of a general point P on the deformed blade's cross section

$$\underset{\sim}{a}_{Pk} = \underset{\sim}{\ddot{r}}_{Pk} + 2(\underset{\sim}{\omega}_k \times \underset{\sim}{\dot{r}}_{Pk}) + (\underset{\sim}{\dot{\omega}}_k \times \underset{\sim}{r}_{Pk}) + \underset{\sim}{\omega}_k \times (\underset{\sim}{\omega}_k \times \underset{\sim}{r}_{Pk}) \qquad (7)$$

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where the position vector r_{Pk} is written with respect to the undisturbed hub location.

The position vector is obtained by first considering fuselage motion. The fuselage center of mass undergoes small perturbation translational motion given by

$$\underset{\sim}{R}_F = R_X \hat{e}_{FX} + R_Y \hat{e}_{FY} + R_Z \hat{e}_{RZ} \tag{8}$$

Additionally, the fuselage is assumed to undergo rigid body rotation by experiencing a yaw-pitch-roll sequence of Euler angle rotations, θ_Z, θ_Y, θ_X, respectively.

Fuselage motion produces hub displacement $\underset{\sim}{r}_h$. As shown in Fig. 1, the hub is assumed to be located from the fuselage center of mass by

$$\underset{\sim}{r}_H = -x_F \hat{e}_{FX} + z_F \hat{e}_{FZ} \tag{9}$$

Consequently as the fuselage experiences rigid body motion, the hub is displaced by

$$\begin{aligned}
\underset{\sim}{r}_h &= \left\{ R_X + z_F(\theta_Y + \theta_X \theta_Z) \right\} \hat{e}_{RX} \\
&\quad + \left\{ R_Y - x_F \theta_Z - z_F(\theta_X - \theta_Y \theta_Z) \right\} \hat{e}_{RY} \\
&\quad + \left\{ R_Z + x_F \theta_Y \right\} \hat{e}_{RZ} \\
&= r_X \hat{e}_{RX} + r_Y \hat{e}_{RY} + r_Z \hat{e}_{RZ} \tag{10}
\end{aligned}$$

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Hence the position vector $\underset{\sim}{r}_{Pk}$, Eq. (7), written from the undisturbed hub position to the point P on the deformed blade cross section, is

$$
\begin{aligned}
\underset{\sim}{r}_{Pk} =\ & r_X\,\hat{e}_{RX} + r_Y\,\hat{e}_{RY} + r_Z\,\hat{e}_{RZ} \\
& + e\,\hat{e}_{2Xk} + (x_k + u_k)\,\hat{e}_{3Xk} + v_k\,\hat{e}_{3Yk} \\
& + w_k\,\hat{e}_{3Zk} + y_{0k}\,\hat{e}_{4Yk} \\
& + z_{0k}\,\hat{e}_{4Zk} \tag{11}
\end{aligned}
$$

Rewriting Eq. (11) in the 2k-coordinate system using the transformations of Appendix A yields

$$
\begin{aligned}
\underset{\sim}{r}_{2Pk} =\ & \{\, r_X \cos\psi_k + r_Y \sin\psi_k + e + x_k \\
& + u_k - \beta_P w_k + y_{0k}(-v_{k,x} - \phi_k(w_{k,x} + \beta_P)) \\
& + z_{0k}(-w_{k,x} - \beta_P + \phi_k v_{k,x})\,\}\,\hat{e}_{2Xk} \\
& + \{\, r_Y \cos\psi_k - r_X \sin\psi_k + v_k + y_{0k} \\
& - z_{0k}(\phi_k + v_{k,x} w_{k,x})\,\}\,\hat{e}_{2Yk} \\
& + \{\, r_Z + w_k + \beta_P(x_k + u_k) \\
& + y_{0k}(-\beta_P v_{k,x} - w_{k,x}\phi_k\beta_P + \phi_k) \\
& + z_{0k}(1 - w_{k,x}\beta_P + \phi_k v_{k,x}\beta_P)\,\}\,\hat{e}_{2Zk} \tag{12}
\end{aligned}
$$

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The angular velocity vector $\omega_{\sim k}$, Eq. (7), includes the fuselage rigid body rotations plus rotor rotation.

$$\omega_{\sim k} = \dot{\Theta}_X \, \hat{e}_{RX} + \dot{\Theta}_Y \, \hat{e}_{RY} + \dot{\Theta}_Z \, \hat{e}_{RZ} + \Omega \, \hat{e}_{1X} \tag{13}$$

Rewriting Eq. (13) in the 2k-coordinate system yields

$$\begin{aligned}
\omega_{\sim 2k} = &\left\{ (\dot{\Theta}_X + \dot{\Theta}_Y \Theta_Z - \dot{\Theta}_Z \Theta_Y) \cos \psi_k \right.\\
&\left. + (-\dot{\Theta}_X \Theta_Z + \dot{\Theta}_Y + \dot{\Theta}_Z \Theta_X) \sin \psi_k \right\} \hat{e}_{2Xk}\\
&+ \left\{ (-\dot{\Theta}_X - \dot{\Theta}_Y \Theta_Z + \dot{\Theta}_Z \Theta_Y) \sin \psi_k \right.\\
&\left. + (-\dot{\Theta}_X \Theta_Z + \dot{\Theta}_Y + \dot{\Theta}_Z \Theta_X) \cos \psi_k \right\} \hat{e}_{2Yk}\\
&+ \left\{ \Omega + \dot{\Theta}_Z \right\} \hat{e}_{2Zk} \tag{14}
\end{aligned}$$

Substituting Eqs. (12) and (14) into Eq. (7), performing the differentiations and the vector cross products, the inertial acceleration of a general point P on the deformed blade cross section expressed in the 2k-coordinate system is obtained, $a_{\sim 2Pk}$. Rewriting in the 3k-coordinate system yields $a_{\sim 3Pk}$, the inertial acceleration vector in the undeformed blade coordinate directions. Applying D'Alembert's principle and integrating the differential force acting on a differential element of mass over the blades cross section yields $p_{\sim 3Ik}$.

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$$\underset{\sim}{P}_{3Ik} = -\iint_A m\, \underset{\sim}{a}_{3Pk}\, dA \qquad\qquad (15)$$

Introducing the identities of Appendix B, Eqs. (B-1)-(B-9), and applying the ordering scheme, the final distributed inertia force vector of the kth blade is given by

$$\underset{\sim}{P}_{3Ik} = m\Omega^2 \left\{ -\ddot{r}_X \cos\psi_k - \ddot{r}_Y \sin\psi_k + 2\dot{v}_k \right.$$
$$\left. + (e+x_k) + 2\dot{\theta}_Z (e+x_k) \right\} \hat{e}_{3Xk}$$

$$+ m\Omega^2 \left\{ \cos\psi_k \left[-\ddot{r}_Y + 2\dot{w}_k \dot{\theta}_X + \ddot{\theta}_X (w_k + \beta_P(e+x_k)) \right] \right.$$
$$+ \sin\psi_k \left[\ddot{r}_X + 2\dot{w}_k \dot{\theta}_Y + \ddot{\theta}_Y (w_k + \beta_P(e+x_k)) \right]$$
$$- \ddot{v}_k - 2\dot{u}_k + 2\beta_P \dot{w}_k - (e+x_k)\ddot{\theta}_Z + v_k$$
$$\left. + 2\dot{\theta}_Z v_k + x_I \cos\theta_{Gk} \right\} \hat{e}_{3Yk}$$

$$+ m\Omega^2 \left\{ \cos\psi_k \left[-2\dot{v}_k - v_k \ddot{\theta}_X - 2\dot{\theta}_Y v_k + \beta_P \ddot{r}_X \right. \right.$$
$$\left. - (2\dot{\theta}_X - \ddot{\theta}_Y)(e+x_k) \right] + \sin\psi_k \left[-2\dot{\theta}_Y \dot{v}_k \right.$$
$$\left. - v_k \ddot{\theta}_Y + 2 v_k \dot{\theta}_X + \beta_P \ddot{r}_Y - (2\dot{\theta}_Y + \ddot{\theta}_X)(e+x_k) \right]$$
$$- \ddot{r}_Z - \ddot{w}_k - 2\dot{v}_k \beta_P - 2\beta_P \dot{\theta}_Z (e+x_k)$$
$$\left. - \beta_P(e+x_k) - x_I \ddot{\theta}_{Gk} \cos\theta_{Gk} \right\} \hat{e}_{3Zk}$$

$$= P_{3IXk} \hat{e}_{3Xk} + P_{3IYk} \hat{e}_{3Yk} + P_{3IZk} \hat{e}_{3Zk}$$

$$\qquad\qquad (16)$$

The distributed inertia moment is determined by taking the moment of the differential forces acting on the blade cross section about the blade cross section elastic center and integrating over the blade cross section.

$$\underset{\sim}{g}_{Ik} = -\iint_A \rho \left[(y_{0k} \hat{e}_{4Yk} + z_{0k} \hat{e}_{4Zk}) \times \underset{\sim}{a}_{3Pk} \right] dA \tag{17}$$

Evaluating Eq. (17), introducing the identities of Appendix B, Eqs. (B-8)-(B-15), and applying the ordering scheme yields the final distributed inertia moment vector of the kth blade.

$$\underset{\sim}{g}_{3Ik} = \Omega^2 \Big\{ m x_I \cos\theta_{Gk} \big[(e+x_k)(\ddot{\theta}_Y - 2\dot{\theta}_X)\cos\psi_k$$

$$- (e+x_k)(2\dot{\theta}_Y + \ddot{\theta}_X)\sin\psi_k - \ddot{r}_Z - \ddot{w}_k - 2\beta_P \dot{v}_k$$

$$- \beta_P (e+x_k) + \phi_k(\ddot{v}_k - v_k) \big] + m x_I \sin\theta_{Gk} \big[\ddot{v}_k$$

$$- v_k \big] - I_P (\ddot{\phi}_k + \ddot{\theta}_{Gk} + \ddot{\theta}_X \cos\psi_k + \ddot{\theta}_Y \sin\psi_k)$$

$$+ I_{mB3} \big[\cos^2\theta_{Gk} \langle \beta v_{k,x} + 2(\theta_X \sin\psi - \theta_Y \cos\psi_k) \rangle$$

$$- (1 - 2\dot{v}_{k,x})\cos\theta_{Gk}\sin\theta_{Gk} - \phi_k \cos 2\theta_{Gk}(1$$

$$+ 2\dot{v}_{k,x}) \big] + I_{mB2} \big[\cos^2\theta_{Gk} \langle 2v_{k,x}(\dot{\phi}_k + \dot{\theta}_{Gk})$$

$$+ v_{k,x}(w_{k,x} - \ddot{w}_{k,x}) - \ddot{v}_{k,x} w_{k,x} - 2\dot{v}_{k,x}\dot{w}_{k,x}$$

$$- 2\dot{w}_{k,x} \rangle + \phi_k \cos 2\theta_{Gk}(1 + 2v_{k,x}) +$$

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$$+ \cos\theta_{Gk}\sin\theta_{Gk}\left(1 + 2\dot{v}_{k,x}\right)\Big]\Big\}\,\hat{e}_{3Xk}'$$

$$+ \Omega^2\Big\{ m x_I \cos\theta_{Gk}\Big[\,(e+x_k)(\phi_k - v_{k,x}\beta_P)$$

$$+ 2\phi_k\dot{v}_k - v_{k,x}\ddot{w}_k\Big] + m x_I \sin\theta_{Gk}\Big[2\dot{v}_k + (e+x_k)\Big]$$

$$- I_P v_{k,x}\left(\ddot{\phi}_k + \ddot{\theta}_{Gk}\right) + I_{mB3}\Big[\,\phi_k\left(\ddot{v}_{k,x} - v_{k,x}\right)\cos2\theta_{Gk}$$

$$+ \left(\ddot{v}_{k,x} - v_{k,x}\right)\cos\theta_{Gk}\sin\theta_{Gk}\Big]$$

$$+ I_{mB2}\Big[\,\phi_k\cos\theta_{Gk}\left(v_{k,x} - \ddot{v}_{k,x}\right) + \cos^2\theta_{Gk}\big\langle \ddot{w}_{k,x}$$

$$- \left(w_{k,x} + \beta_P\right) - 2\left(\dot{\theta}_{Gk}v_{k,x} + \dot{\phi}_k\dot{v}_{k,x}\right) - 2\left(\dot{\phi}_k + \dot{\theta}_{Gk}\right)$$

$$- 2\left(\dot{v}_{k,x}w_{k,x} + v_{k,x}\dot{w}_{k,x}\right) - 2\dot{\theta}_{Gk}\dot{\theta}_Z$$

$$- \ddot{\theta}_Y\cos\psi_k + \ddot{\theta}_X\sin\psi_k\big\rangle + \cos\theta_{Gk}\sin\theta_{Gk}\left(v_{k,x}\right.$$

$$\left. - \ddot{v}_{k,x}\right)\Big]\Big\}\,\hat{e}_{3Yk}$$

$$+ \Omega^2\Big\{ m x_I \cos\theta_{Gk}\Big[\,\ddot{r}_X\cos\psi_k + \ddot{r}_Y\sin\psi_k$$

$$- 2\dot{v}_k - (e+x_k)(1 - 2\dot{\theta}_z)\Big] + \phi_k(e+x_k)\,m x_I\sin\theta_{Gk}$$

$$- I_P w_{k,x}\left(\ddot{\phi}_k + \ddot{\theta}_{Gk}\right) + I_{mB3}\Big[\,\phi_k\left(\beta_P - \ddot{w}_{k,x}\right)\cos2\theta_{Gk}$$

$$+ \cos^2\theta_{Gk}\big\langle -\ddot{v}_{k,x} - 2\dot{\phi}_k\dot{w}_{k,x} - 2\dot{v}_{k,x}v_{k,x}$$

$$- 2\dot{\theta}_{Gk}(\dot{w}_{k'x} - \phi_k) - 2\dot{v}_{k'x}v_{k'x} + 2\dot{\theta}_Y(\dot{\theta}_{Gk}$$

$$+ \phi_k)\cos\psi_k - 2\dot{\theta}_x(\dot{\theta}_{Gk} + \dot{\phi}_k)\sin\psi_k \rangle$$

$$+ \cos\theta_{Gk}\sin\theta_{Gk}\langle \beta_P - \ddot{w}_{k'x} + 2(\dot{\theta}_{Gk} + \dot{\phi}_k)\rangle] +$$

$$+ I_{mB2}[\ \phi_k(\ddot{w}_{k'x} - \beta_P)\cos\theta_{Gk}$$

$$+ \cos^2\theta_{Gk}\langle -2\phi_k(\dot{\phi}_k + \dot{\theta}_{Gk}) - 2w_{k'x}\dot{w}_{k'x}\rangle$$

$$+ \cos\theta_{Gk}\sin\theta_{Gk}\langle \ddot{w}_{k'x} - 2(\dot{\theta}_{Gk} + \dot{\phi}_k)$$

$$+ \beta_P\rangle]\}\ \hat{e}_{3zk}$$

$$= q_{I3xk}\hat{e}_{3xk} + q_{I3Yk}\hat{e}_{3Yk} + q_{I3zk}\hat{e}_{3zk}$$

$$(18)$$

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C.2 Aerodynamic Loads

The determination of the aerodynamic loads on the rotor requires the resultant relative velocity between the blade and air. This may be written as

$$\underset{\sim}{V}_k = \underset{\sim}{V}_{Ak} - \underset{\sim}{v}_{ECk} \tag{19}$$

where $\underset{\sim}{V}_{Ak}$ is the free stream air flow and $\underset{\sim}{v}_{ECk}$ is the velocity of the deformed blade's cross section elastic center. The free stream air flow includes the effect of forward flight and the induced inflow, v_k, which has radial and azimuthal dependence. The vector $\underset{\sim}{V}_{Ak}$ is given by

$$\underset{\sim}{V}_{Ak} = V_F \cos \alpha_R \, \hat{e}_{RX} - V_F \sin \alpha_R \, \hat{e}_{RZ} - v_k \, \hat{e}_{2Ik} \tag{20}$$

Rewriting Eq. (20) in the deformed blade's 5k-coordinate directions yields $\underset{\sim}{V}_{A5k}$.

The velocity of the deformed blade's cross section elastic center, $\underset{\sim}{v}_{ECk}$, is given by

$$\underset{\sim}{v}_{ECk} = \underset{\sim}{\dot{r}}_{ECk} + (\underset{\sim}{\omega}_k \times \underset{\sim}{r}_{ECk}) \tag{21}$$

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where r_{ECk} is given by Eq. (12) with $y_{0k} = z_{0k} = 0$ and ω_k is given by Eq. (14). Evaluating Eq. (20), rewriting in the 5k-coordinate system, and substituting it along with V_{A5k} into Eq. (19) yields the resultant relative velocity V_{5k}. Since two-dimensional aerodynamic theory is used, only the velocity components in the plane of the deformed blade cross section are required. The final expressions are

$$
\begin{aligned}
V_{5Yk} = \Omega R \Big\{ &\cos\psi_k \left[-\mu\theta_Z - \mu v_{k,x} - \dot{\bar{r}}_Y \right. \\
&\left. -\mu \tan\alpha_R \theta_X \right] + \sin\psi_k \left[-\mu - \mu\tan\alpha_R \theta_Y \right. \\
&\left. + \dot{r}_Z \right] - \dot{\bar{v}}_k - (\bar{e}+\bar{x}_k) - (\bar{e}+\bar{x}_k)\dot{\theta}_Z \Big\} \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
V_{5Zk} = \Omega R \Big\{ &\cos\psi_k \left[-\mu(\beta_P + w_{k,x}) - \bar{v}_k \dot{\theta}_X \right. \\
&\left. + (\bar{e}+\bar{x}_k)\dot{\theta}_Y + \dot{\bar{r}}_X(w_{k,x} + \beta_P) \right] \\
&+ \sin\psi_k \left[\mu v_{k,x} w_{k,x} - \bar{v}_k \dot{\theta}_Y - (\bar{e}+\bar{x}_k)\dot{\theta}_X \right. \\
&\left. + \dot{\bar{r}}_Y (w_{k,x} + \beta_P) \right] + \mu\theta_Y - \lambda_k - \dot{\bar{r}}_Z \\
&- \dot{\bar{w}}_k - v_k(w_{k,x} + \beta_P) + (\bar{e}+\bar{x}_k)v_{k,x}w_{k,x} \Big\}
\end{aligned}
$$

$$(23)$$

The airfoil configuration for the calculation of the steady aerodynamic loads is shown in Fig. 8. The steady aerodynamic lift, drag, and moment per unit length acting on the kth blade are given by

$$L_{Sk} = \rho_A b c_\ell (\alpha_k, m_k) U_{Rk}^2 \tag{24}$$

$$D_{Sk} = \rho_A b c_d (\alpha_k, m_k) U_{Rk}^2 \tag{25}$$

$$M_{Sk} = x_A L_{Sk} + 2 \rho_A b^2 c_{m_{ac}} (\alpha_k, m_k) U_{Rk}^2 \tag{26}$$

where

$$U_{Rk}^2 = V_{5Yk}^2 + V_{5Zk}^2 \cong \bar{V}_{5Yk}^2 \tag{27}$$

$$\alpha_k = \Theta_{Gk} + \phi_k - \tan^{-1}\left(\frac{V_{5Zk}}{V_{5Yk}}\right)$$

$$\cong \Theta_{Gk} + \phi_k - \left(\frac{V_{5Zk}}{V_{5Yk}}\right) \tag{28}$$

$$m_k = m_{tip} U_{Rk} \cong -m_{tip} V_{5Yk} \tag{29}$$

The unsteady lift and moment, L_{US} and M_{US}, are obtained from two-dimensional incompressible thin airfoil theory, Ref. 13. The assumption of quasi-steady airloads, $C(k) = 1$, is made. The influence of unsteady aerodynamics on rotor blade behavior was first studied in Refs. 14 and 15. It was shown that a quasi-steady airload assumption is usually conservative when determining rotor blade stability. The airfoil environment for the unsteady aerodynamic load calculation is shown in Fig. 9. The unsteady lift, L_{USk}, and moment, M_{USk}, are given by

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$$L_{USk} = \rho_A a b \left\{ \frac{b}{2} \dot{U}_{Nk} - \frac{b}{2} \left(x_A - \frac{b}{2} \right) \ddot{\alpha}_{Pk} \right.$$
$$\left. + U'_{Rk} \dot{\alpha}_{Pk} \left(b - x_A \right) \right\} \tag{30}$$

$$m_{USk} = \rho_A a b \left\{ \frac{b}{2} \left(x_A - \frac{b}{2} \right) \dot{U}_{Nk} + \frac{b}{2} \ddot{\alpha}_{Pk} \left(x_A^2 \right. \right.$$
$$\left. - x_A b + \frac{3b^2}{8} \right) + U'_{Rk} \dot{\alpha}_{Pk} \left[x_A b - x_A^2 - \left(\frac{b}{2} \right)^2 \right] \bigg\}$$
$$\tag{31}$$

Since the unsteady lift and moment are determined using a linear theory, only linear terms in perturbation motion are retained in the unsteady load expression. Consequently, L_{USk} and M_{USk} are evaluated using

$$U_{Nk} = V_{SZk} - V_{SYk} \left(\theta_{Gk} + \phi_k \right)$$
$$\cong \Omega R \left\{ \cos \psi_k \left[-\mu \left(\beta_P + w_{k,x} \right) + \left(\bar{e} + \bar{x}_k \right) \dot{\theta}_Y \right] \right.$$
$$+ \sin \psi_k \left[-\left(\bar{e} + \bar{x}_k \right) \dot{\theta}_X + \mu \left(\theta_{Gk} + \phi_k \right) \right]$$
$$\left. + \mu \theta_Y - \lambda_k - \dot{\bar{r}}_z - \dot{\bar{w}}_k + \left(\bar{e} + \bar{x}_k \right) \left(\theta_{Gk} + \phi_k \right) \right\}$$
$$\tag{32}$$

$$U'_{Rk} = -V_{SYk} \sqrt{1 + \left(\frac{V_{SZk}}{V_{SYk}} \right)^2} \cong -V_{SYk}$$
$$= \Omega R \left\{ \mu \sin \psi_k + \left(\bar{e} + \bar{x}_k \right) \right\} \tag{33}$$

$$\dot{\alpha}_{Pk} = \ddot{\theta}_X \cos \psi_k + \dot{\theta}_Y \sin \psi_k + \dot{\theta}_{Gk} + \dot{\phi}_k$$
$$+ \Omega \left(w_{k,x} + \beta_P \right) \tag{34}$$

Note that although a linear (valid up to $0(\epsilon)$) unsteady aero-dynamic theory is used, terms linear in body motion ($0(\epsilon^{3/2})$) are retained for completeness.

The resulting distributed aerodynamic loads written in the 5k-coordinate system are given by

$$
\begin{aligned}
\underset{\sim}{p}_{A5k} &= \left\{ -L_{Sk} \sin\left(\tan^{-1}\left(\frac{V_{5Zk}}{V_{5Yk}}\right)\right) \right. \\
&\quad \left. - D_{Sk} \cos\left(\tan^{-1}\left(\frac{V_{5Zk}}{V_{5Yk}}\right)\right)\right\} \hat{e}_{5Yk} \\
&\quad + \left\{ L_{Sk} \cos\left(\tan^{-1}\left(\frac{V_{5Zk}}{V_{5Yk}}\right)\right) + L_{uSk} \right. \\
&\quad \left. - D_{Sk} \sin\left(\tan^{-1}\left(\frac{V_{5Zk}}{V_{5Yk}}\right)\right)\right\} \hat{e}_{5Zk} \\
&\cong \left\{ -L_{Sk}\left(\frac{V_{5Zk}}{V_{5Yk}}\right) - D_{Sk}\right\} \hat{e}_{5Yk} \\
&\quad + \left\{ L_{Sk} + L_{uSk}\right\} \hat{e}_{5Zk} \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
\underset{\sim}{q}_{A5k} &= \left\{ m_{Sk} + m_{uSk}\right\} \hat{e}_{5Xk} \\
&= q_{A5Xk}\, \hat{e}_{5Xk} \qquad (36)
\end{aligned}
$$

Rewriting Eqs. (35) and (36) in the undeformed blade coordinate directions and neglecting the axial force component yields

$$
\underset{\sim}{p}_{A3k} = \left\{ -L_{Sk}\left(\frac{V_{5Zk}}{V_{5Yk}}\right) - (L_{Sk} + L_{USk})v_{k,x}w_{k,x} - D_{Sk} \right\} \hat{e}_{3Yk}
$$
$$
+ \left\{ L_{Sk} + L_{USk} \right\} \hat{e}_{3Zk} \tag{37}
$$

$$
\underset{\sim}{q}_{A3k} = q_{A5Xk} \hat{e}_{3Xk} + q_{A5Xk} v_{k,x} \hat{e}_{3Yk}
$$
$$
+ q_{A5Xk} w_{k,x} \hat{e}_{3Zk} \tag{38}
$$

Substituting the velocity expressions into the steady and unsteady load expressions, evaluating, $\underset{\sim}{p}_{A3k}$ and $\underset{\sim}{q}_{A3k}$ and applying the ordering scheme yields

$$
\underset{\sim}{p}_{A3k} = -\rho_A b \Omega^2 R^2
$$
$$
\times \left\{ c_\ell < \cos^2 \psi_k \left[\mu^2 \theta_Z (\beta_P + w_{k,x}) \right. \right.
$$
$$
\left. + \mu^2 v_{k,x}(\beta_P + w_{k,x}) - \mu v_{k,x}\ddot{\theta}_Y (\bar{e} + \bar{x}_k) + \right.
$$

$$+ \mu \dot{\bar{r}}_Y (\beta_P + W_{k,x})] + \sin^2 \psi_k [\mu \dot{\theta}_X (\bar{e} + \bar{x}_k)$$

$$- \mu \beta_P \dot{\bar{r}}_Y - \mu^2 V_{k,x} W_{k,x}] + \cos \psi_k \sin \psi_k [-\mu \beta_P \dot{\bar{r}}_X$$

$$+ \mu^2 (\beta_P + W_{k,x}) + \mu (\bar{e} + \bar{x}_k)(\dot{\theta}_X V_{k,x} - \dot{\theta}_Y)]$$

$$+ \cos \psi_k [\mu \theta_Z (\lambda_k + \dot{\bar{w}}_k) + \mu V_{k,x} (-\mu \theta_Y$$

$$+ \lambda_k + \dot{\bar{r}}_Z + \dot{\bar{w}}_k) + \dot{\bar{r}}_Y (\lambda_k + \dot{\bar{w}}_k) + (\dot{\bar{v}}_k$$

$$+ (\bar{e} + \bar{x}_k)(1 + \dot{\theta}_Z)) \mu (W_{k,x} + \beta_P) - \dot{\theta}_Y (\bar{e} + \bar{x}_k)(\dot{\bar{v}}_k$$

$$+ (\bar{e} + \bar{x}_k))] + \sin \psi_k [(\bar{e} + \bar{x}_k) \dot{\theta}_X (\dot{\bar{v}}_k + (\bar{e} + \bar{x}_k))$$

$$- V_{k,x} W_{k,x} (\bar{e} + \bar{x}_k)((\bar{e} + \bar{x}_k) - 2\mu) + \mu (\lambda_k$$

$$- \mu \theta_Y + \dot{\bar{r}}_Z + \dot{\bar{w}}_k) - \dot{\bar{r}}_Z (\lambda_k + \dot{\bar{w}}_k)]$$

$$+ \dot{\bar{v}}_k (-\mu \theta_Y + \lambda_k + \dot{\bar{w}}_k + \dot{\bar{r}}_Z) - (\bar{e} + \bar{x}_k)(\mu \theta_Y$$

$$+ (\bar{e} + x_k) V_{k,x} W_{k,x} - \lambda_k - \dot{\bar{r}}_Z - \dot{\bar{w}}_k - (\lambda_k + \dot{\bar{w}}_k) \dot{\theta}_Z \rangle$$

$$+ c_d \langle \mu^2 \sin^2 \psi_k + 2\mu^2 V_{k,x} \cos \psi_k \sin \psi_k$$

$$- 2\mu (\bar{e} + \bar{x}_k) V_{k,x} \cos \psi_k - 2\mu (\dot{\bar{v}}_k + (\bar{e} + \bar{x}_k)) \sin \psi_k$$

$$+ (\bar{e} + \bar{x}_k)^2 + 2\dot{\bar{v}}_k (\bar{e} + \bar{x}_k) \rangle \} \hat{e}_{3Yk}$$

$$+ \rho_A b \Omega^2 R^2$$

$$\times \{ c_\ell \langle (\mu^2 + 2\mu \dot{\bar{r}}_X) \sin^2 \psi_k$$

$$+ \cos \psi_k \sin \psi_k [2\mu (\mu (\theta_Z + V_{k,x}) - \dot{\bar{r}}_Y)] +$$

$$+ \cos \psi_k \left[2(\bar{e}+\bar{x}_k)(\dot{\bar{r}}_Y - \mu(\theta_Z + V_{k,x})) \right]$$

$$+ \sin \psi_k \left[-2\mu(\dot{V}_k + (\bar{e}+\bar{x}_k)(1+\dot{\theta}_Z)) \right.$$

$$\left. + 2\dot{\bar{r}}_x(\bar{e}+\bar{x}_k) \right] + (\bar{e}+\bar{x}_k)^2(1+2\dot{\theta}_Z)$$

$$+ 2\dot{V}_k(\bar{e}+\bar{x}_k) \rangle$$

$$+ a \langle \mu \dot{\theta}_Y (\bar{b}-\bar{x}_A)\sin^2\psi_k$$

$$+ \mu \dot{\theta}_x (\bar{b}-\bar{x}_A)\cos\psi_k \sin\psi_k + \cos\psi_k \left[\frac{\bar{b}}{2}(-\mu \dot{w}_{k,x} \right.$$

$$\left. + (\bar{e}+\bar{x}_k)(\ddot{\theta}_Y - \dot{\theta}_x) + \mu(\dot{\theta}_{Gk} + \dot{\phi}_k)) \right]$$

$$+ \sin\psi_k \left[\mu(\tfrac{3}{2}\bar{b}-x_A)(w_{k,x} + \beta_P + \dot{\theta}_{Gk} + \dot{\phi}_k) \right.$$

$$\left. - \frac{\bar{b}}{2}(\bar{e}+\bar{x}_k)(\dot{\theta}_Y + \ddot{\theta}_x) \right] + (\bar{b}-\bar{x}_A)(\bar{e}+\bar{x}_k)(\dot{\theta}_x \cos\psi_k$$

$$+ \dot{\theta}_Y \sin\psi_k + w_{k,x} + \beta_P + \dot{\theta}_{Gk} + \dot{\phi}_k) + \frac{\bar{b}}{2}(\mu\dot{\theta}_Y$$

$$- \lambda_k - \dot{\bar{r}}_Z - \dot{\bar{w}}_k) \rangle \} \hat{e}_{3Zk}$$

$$= p_{A3Yk}\,\hat{e}_{3Yk} + p_{A3Zk}\,\hat{e}_{3Zk} \qquad (39)$$

$$q_{A5xk} = \rho_A b \Omega^2 R^3$$

$$\times \{ \langle c_\ell \bar{x}_A + \frac{\bar{b}}{2} c_{mac} \rangle$$

$$\times \langle (2\mu\dot{\bar{r}}_x + \mu^2)\sin^2\psi_k + 2\mu(\mu(\theta_Z+V_{k,x})$$

$$- \dot{\bar{r}}_Y)\cos\psi_k \sin\psi_k + 2(\bar{e}+\bar{x}_k)(\dot{\bar{r}}_Y - \mu(\theta_Z +$$

$$+ v_{k,x})) \cos \psi_k + \sin \psi_k [-2\mu((\bar{e}+\bar{x}_k)(1+\dot{\theta}_z)$$

$$+ \dot{\bar{v}}_k) - 2\dot{\bar{r}}_x(\bar{e}+\bar{x}_k)] + (\bar{e}+\bar{x}_k)^2(1+2\dot{\theta}_z)$$

$$+ 2\dot{\bar{v}}_k(\bar{e}+\bar{x}_k)\rangle$$

$$+ a\langle \mu\dot{\theta}_Y(\bar{x}_A\bar{b} - \bar{x}_A^2 - \frac{\bar{b}^2}{4})\sin^2\psi_k$$

$$+ \mu\dot{\theta}_X(x_A\bar{b} - \bar{x}_A^2 - \frac{\bar{b}^2}{4})\cos\psi_k\sin\psi_k$$

$$+ \cos\psi_k[\frac{\bar{b}}{2}(\bar{x}_A - \frac{\bar{b}}{2})(-\mu w_{k,x} + (\bar{e}+\bar{x}_k)(\ddot{\theta}_Y - \dot{\theta}_X)$$

$$+ \mu(\theta_{Gk} + \phi_k))] + \sin\psi_k[\mu(w_{k,x} + \beta_P$$

$$+ \dot{\theta}_{Gk} + \dot{\phi}_k)(\bar{x}_A\bar{b} - \bar{x}_A^2 - \frac{\bar{b}^2}{4}) + \frac{\bar{b}}{2}(\bar{x}_A - \frac{\bar{b}}{2})(\mu(w_{k,x}$$

$$+ \beta_P + \dot{\theta}_{Gk} + \dot{\phi}_k) - (\bar{e}+\bar{x}_k)(\ddot{\theta}_Y + \ddot{\theta}_X)]$$

$$+ \frac{\bar{b}}{2}(\bar{x}_A - \frac{\bar{b}}{2})(\mu\dot{\theta}_Y - \dot{\lambda}_k - \ddot{\bar{r}}_z - \ddot{\bar{w}}_k + (\bar{e}+\bar{x}_k)(\dot{\theta}_{Gk}$$

$$+ \dot{\phi}_k) + (\bar{x}_A\bar{b} - \bar{x}_A^2 - \frac{\bar{b}^2}{4})(\dot{\theta}_X\cos\psi_k + \dot{\theta}_Y\sin\psi_k + w_{k,x} + \beta_P$$

$$+ \dot{\theta}_{Gk} + \dot{\phi}_k)(\bar{e}+\bar{x}_k)\rangle\} = q_{A3Xk} \qquad (40)$$

$$q_{A3Yk} = v_{k,x} q_{A5Xk} \qquad (41)$$

$$q_{A3Zk} = w_{k,x} q_{A5Xk} \qquad (42)$$

Several comments should be made concerning the aerodynamic loads presented in Eqs. (39)-(42). Apparent mass terms, stall, and compressibility effects are included in the analysis. The aeroelastic phenomenon of dynamic stall, although neglected in the present analysis, may be incorporated in a semi-empirical manner. Including these effects will yield a more realistic rotor model. Several such approaches are available in the literature for aeroelastic response calculations[17,18]. Also the aerodynamic loads as presented herein are valid only in the normal flow region. The inculsion of the effects of the reversed flow region on the retreating blade can be incorporated in the model by first determining the region of reversed flow for the given flight condition. Re-calculating the airload expressions within this region using

$$x_{A_{RF}} = x_A - \frac{c}{2} \tag{43}$$

$$\alpha_{RF} = - \langle \Theta_{Gk} + \phi_k - (\frac{V_{Szk}}{V_{SYk}})) \tag{44}$$

will model the influence of reverse flow.

In Eqs. (39)-(42), the induced inflow field has not been specified. Typically it will have radial and azimuthal dependence. The high frequency and nonuniform nature of the induced inflow field can be incorporated using an analytical wake modeling program[19]. On the other hand, if one is primarily concerned with low frequency variation, a dynamic inflow model such as the one presented in Ref. 20 may be used.

C.3 Damping Loads

The blade structural damping loads are assumed to be of viscous nature (assumption 11). The distributed structural damping forces and moments written in the undeformed blade coordinate directions are

$$\underset{\sim}{P}_{D3k} = -g_{SL} \dot{v}_k \, \hat{e}_{3Yk} - g_{SF} \dot{w}_k \, \hat{e}_{3Zk} \quad (45)$$

$$\underset{\sim}{q}_{D3k} = -g_{ST} \dot{\phi}_k \, \hat{e}_{3Xk} \quad (46)$$

respectively.

C.4 Final Blade Equations of Motion

The governing nonlinear partial differential equations of motion for the kth blade of an N-bladed rotor are obtained by substituting the inertia, aerodynamic, and damping loads into the equations of equilibrium, Eqs. (1)-(4). After some algebraic mainpulation and a final application of the ordering scheme, the equations for the lead-lag, flap and torsion deflections are obtained. These equations are presented in Appendix C.

The boundary conditions which the blade deflections must satisfy depend on the type of rotor. For a hingeless rotor, the boundary conditions are

(a) At the root ($x_{3k} = 0$):

$$u = v = w = v_{,x} = w_{,x} = 0$$

$$K_\phi \phi = GJ\phi_{,x}$$

(b) At the tip $(x_{3k} = \ell)$

$$M_X = M_Y = M_Z = T = V_Y = V_Z = 0$$

From Ref. 10, the blade cross section elastic moments and shears
are given by:

$$M_X = GJ(\phi_{,x} + v_{,xx}w_{,x}) \tag{47}$$

$$
\begin{aligned}
M_Y = &-E(I_{ZZ} - I_{YY})\sin\theta_G \cos\theta_G (v_{,xx} + \phi w_{,xx}) \\
&- E(I_{ZZ}\sin^2\theta_G + I_{YY}\cos^2\theta_G)(w_{,xx} - \phi v_{,xx}) \\
&+ Tx_T \sin\theta_G \tag{48}
\end{aligned}
$$

$$
\begin{aligned}
M_Z = &E(I_{ZZ}\cos^2\theta_G + I_{YY}\sin^2\theta_G)(v_{,xx} + \phi w_{,xx}) \\
&+ E(I_{ZZ} - I_{YY})\sin\theta_G \cos\theta_G (w_{,xx} + \phi v_{,xx}) \\
&- Tx_T \cos\theta_G \tag{49}
\end{aligned}
$$

$$
\begin{aligned}
V_Y = &-\big[E(I_{ZZ} - I_{YY})\sin\theta_G \cos\theta_G (w_{,xx} - \phi v_{,xx}) \\
&+ E(I_{ZZ}\cos^2\theta_G + I_{YY}\sin^2\theta_G)(v_{,xx} + \phi w_{,xx}) \\
&- Tx_T \cos\theta_G\big]_{,x} - Tx_T \phi_{,x}\sin\theta_G - GJ\phi_{,x}w_{,xx}^+
\end{aligned}
$$

$$+ E(I_{ZZ}-I_{YY})\sin\theta_G\cos\theta_G V_{,xx}\phi_{,x} + E(I_{ZZ}\sin^2\theta_G$$

$$+ I_{YY}\cos^2\theta_G)W_{,xx}\phi_{,x} - q_{3Z} + W_{,x}q_{3X} + \phi_k q_{3Y} \quad (50)$$

$$V_Z = - E(I_{ZZ}\cos^2\theta_G + I_{YY}\sin^2\theta_G)V_{,xx}\phi_{,x}$$

$$- E(I_{ZZ}-I_{YY})\sin\theta_G\cos\theta_G W_{,xx}\phi_{,x}$$

$$+ T_{X_T}\phi_{,x}\cos\theta_G + GJ\phi_{,x}V_{,xx}$$

$$- \left[E(I_{ZZ}-I_{YY})\sin\theta_G\cos\theta_G (V_{,xx}+\phi W_{,xx}) \right.$$

$$+ E(I_{ZZ}\sin^2\theta_G + I_{YY}\cos^2\theta_G)(W_{,xx}-\phi V_{,xx})$$

$$\left. - T_{X_T}\sin\theta_G\right]_{,x} + q_{3Y} - V_{,x}q_{3X} - W_{,x}q_{3Z}$$

$$(51)$$

For an articulated rotor, the natural boundary conditions at the
blade tip remain the same. The blade root boundary conditions are

(a) At the root ($x_{3k} = 0$):

$$u = v = w = 0 \quad ; \quad K_\phi\phi = GJ(\phi_{,x} + V_{,xx}W_{,x})$$

$$m_Y = K_\beta W_{,x} \quad ; \quad m_Z = K_S V_{,x}$$

where M_Y and M_Z are given above.

D. Fuselage Equations of Motion

The governing fuselage equations of motion are determined by
balancing rotor- and fuselage-induced forces and moments at the hub.
These matching conditions between the N-bladed rotor and the fuselage ensure
dynamic equilibrium of the overall system.

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Using the rigid body model of the fuselage and invoking D'Alembert's principle, the inertia force vector $\underset{\sim}{P}_F$, acting at the fuselage center of mass is given by

$$\underset{\sim}{P}_F = -\frac{d}{dt}(M_F \underset{\sim}{V}_F)$$
$$= -M_F(\ddot{R}_X \hat{e}_{FX} + \ddot{R}_Y \hat{e}_{FY} + \ddot{R}_Z \hat{e}_{FZ})$$

$$(52)$$

Note that Eq. (52) is restricted (assumption 16) to flight conditions where the fuselage F-coordinate system is an inertial reference frame. The inertia moment vector, $\underset{\sim}{Q}_F$, obtained from D'Alembert's principle and acting at the fuselage center of mass is given by

$$\underset{\sim}{Q}_F = -\frac{d}{dt}(I_F \underset{\sim}{\omega}_F)$$
$$= -\{\ddot{\theta}_X(I_X - 2I_{ZX}\theta_Y) + I_{ZX}(\ddot{\theta}_X$$
$$-\theta_X\ddot{\theta}_Y - 3\dot{\theta}_Y\dot{\theta}_X) + \theta_Z\ddot{\theta}_Y(I_Y - I_X)$$
$$+\theta_Y\ddot{\theta}_Z(I_X - I_Z) + \dot{\theta}_Y\dot{\theta}_Z(I_Y - I_Z)\}\hat{e}_{FX}$$
$$-\{I_Y\ddot{\theta}_Y + I_{ZX}(\theta_Z\ddot{\theta}_Z - \theta_X\ddot{\theta}_X$$
$$+\dot{\theta}_Z^2 - \dot{\theta}_X^2) + \theta_Z\ddot{\theta}_X(I_Y - I_X)$$
$$+\theta_X\ddot{\theta}_Z(I_Z - I_Y) + \dot{\theta}_X\dot{\theta}_Z(I_Z - I_X)\}\hat{e}_{FY}$$

$$-\{ I_Z \ddot{\Theta}_Z + \Theta_Y \ddot{\Theta}_X (I_X - I_Z) + \Theta_X \Theta_Y (I_Z$$

$$-I_Y) + \dot{\Theta}_X \dot{\Theta}_Y (I_X - I_Y) + I_{ZX} (\ddot{\Theta}_X$$

$$+ \Theta_Z \ddot{\Theta}_Y + 3 \dot{\Theta}_Y \dot{\Theta}_Z + 2 \Theta_Y \ddot{\Theta}_Z)\} \, \hat{e}_{FZ}$$

$$(53)$$

The fuselage-induced inertia force acting at the displaced hub is $\underset{\sim}{P}_H = \underset{\sim}{P}_F$, Eq. (52). The fuselage-induced inertia moment acting at the hub, $\underset{\sim}{Q}_H$, is given by

$$\underset{\sim}{Q}_H = \underset{\sim}{Q}_F + (\underset{\sim}{r}_F \times \underset{\sim}{P}_F)$$

$$= -\{ \ddot{\Theta}_X (I_X - 2 I_{ZX} \Theta_Y) + I_{ZX} (\ddot{\Theta}_X - \Theta_X \ddot{\Theta}_Y$$

$$- 3 \dot{\Theta}_Y \dot{\Theta}_X) + \Theta_Z \ddot{\Theta}_Y (I_Y - I_X) + \Theta_Y \ddot{\Theta}_Z (I_X - I_Z)$$

$$+ \dot{\Theta}_Y \dot{\Theta}_Z (I_Y - I_Z) + M_F z_F (\ddot{R}_Y + \ddot{R}_Z \Theta_X)$$

$$+ M_F x_F (\ddot{R}_Z \Theta_Z + \ddot{R}_Y \Theta_Y)\} \, \hat{e}_{RX}$$

$$- \{ I_Y \ddot{\Theta}_Y + I_{ZX} (\Theta_Z \ddot{\Theta}_Z - \Theta_X \ddot{\Theta}_X + \dot{\Theta}_Z^2 - \dot{\Theta}_X^2)$$

$$+ \Theta_Z \ddot{\Theta}_X (I_Y - I_X) + \Theta_X \ddot{\Theta}_Z (I_Z - I_Y)$$

$$+ \dot{\Theta}_X \dot{\Theta}_Z (I_Z - I_X) - M_F x_F (\ddot{R}_Z + \ddot{R}_X \Theta_Y)$$

$$- M_F z_F (\ddot{R}_X - \ddot{R}_Z \Theta_Y)\} \, \hat{e}_{RY}$$

$$-\{ I_Z \ddot{\theta}_Z + I_{ZX}(\ddot{\theta}_X + \theta_Z \ddot{\theta}_Y + 3\dot{\theta}_Y \dot{\theta}_Z$$

$$+ 2\theta_Y \ddot{\theta}_Z) + \theta_Y \ddot{\theta}_X (I_X - I_Z) + \theta_X \ddot{\theta}_Y (I_Z - I_Y)$$

$$+ \dot{\theta}_X \dot{\theta}_Y (I_X - I_Y) + M_F x_F (\ddot{R}_Y - \ddot{R}_X \theta_Z)$$

$$- M_F z_F (\ddot{R}_X \theta_X + \ddot{R}_Y \theta_Y)\} \, \hat{e}_{RX}$$

$$(54)$$

where, from Eq. (10),

$$\vec{r}_F \;=\; x_F \, \hat{e}_{1X} \;-\; z_F \, \hat{e}_{1Z} \qquad (55)$$

The gravity loads acting at the hub, assuming the helicopter is in level horizontal flight, are given by

$$\underset{\sim}{P}_G = -(M_F + M_R) g \sin\alpha_R \, \hat{e}_{RX}$$

$$- (M_F + M_R) g \cos\alpha_R \, \hat{e}_{RZ} \qquad (56)$$

$$\underset{\sim}{Q}_G = - M_F g \cos\alpha_R (x_F + z_F \theta_X) \hat{e}_{RX}$$

$$+ M_F g < \cos\alpha_R (x_F - z_F \theta_Y)$$

$$+ \sin\alpha_R (z_F + x_F \theta_Y) > \hat{e}_{RY}$$

$$+ M_F g \sin\alpha_R (x_F \theta_Z + z_F \theta_X) \hat{e}_{RZ} \qquad (57)$$

Gravity loads are considered to primarily influence the helicopter trim problem. The rotor blade gravity loads are considered to be negligible compared to the blade inertia and aerodynamic loads and have been neglected in Sec. III. C.

Other forces and moments acting on the fuselage and induced at the hub may be expressed in general form as

$$\underset{\sim}{P}_T = F_X \hat{e}_{RX} + F_Y \hat{e}_{RY} + F_Z \hat{e}_{RZ} \tag{58}$$

$$\underset{\sim}{Q}_T = Q_X \hat{e}_{RX} + Q_Y \hat{e}_{RY} + Q_Z \hat{e}_{RZ} \tag{59}$$

These loads can represent airframe aerodynamic forces, including aerodynamic interference effects and tail rotor influence, or the effect of landing skids when in ground contact. This latter model could allow for the possibility of using this math model to investigate the dynamic phenomenon of ground resonance.[21]

The final ingredient of the matching conditions are the rotor forces and moments induced at the hub, $\underset{\sim}{P}_R$ and $\underset{\sim}{Q}_R$, respectively. The distributed loads acting on the kth blade expressed in the undeformed 3k-coordinate system are given by Eqs. (5) and (6). The individual components of these vectors were derived and presented in Secs. III. C.1 - III. C.3. Rewriting these loads in the rotating 2k hub plane system, $\underset{\sim}{p}_{2k}$, and $\underset{\sim}{q}_{2k}$, the three forces and moments induced at the blade root of the kth blade are

$$P_{\sim 2k} = \int_0^\ell p_{\sim 2k} \, dx_k \qquad (60)$$

$$Q_{\sim 2k} = \int_0^\ell \left[q_{\sim 2k} + \left(r_{\sim Hk} \times p_{\sim 2k} \right) \right] dx_k \qquad (61)$$

where

$$r_{\sim Hk} = e\,\hat{e}_{2xk} + x_k \hat{e}_{3xk} + v_k \hat{e}_{3Yk} + w_k \hat{e}_{3zk}$$
$$(62)$$

Transforming the forces and moments into the nonrotating rotor coordinate system, $P_{\sim Rk}$, and $Q_{\sim Rk}$, and performing a summation over all N blades of the rotor yields the final expressions for the rotor-induced forces and moments at the hub.

$$P_{\sim R} = \sum_{k=1}^{N} P_{Rk} \qquad (63)$$

$$Q_{\sim R} = \sum_{k=1}^{N} Q_{Rk} \qquad (64)$$

The final fuselage equations of motion are given by writing force and moment balance at the hub due to fuselage- and rotor-induced loads.

$$P_{\sim H} + P_{\sim G} + P_{\sim T} + P_{\sim R} = 0 \qquad (65)$$

$$Q_{\sim H} + Q_{\sim G} + Q_{\sim T} + Q_{\sim R} = 0 \qquad (66)$$

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These equations, even after application of the ordering scheme, are very lengthy and reflect the complexity of modeling coupled rotor/fuselage dynamics. The equations themselves are given in Appendix D.

IV. ROTOR/FUSELAGE SYSTEM
EQUATIONS OF MOTION

The final governing equations of motion include the three equations governing lead-lag, flap, and torsion deflections for each blade of the N-bladed rotor, Eqs. (C-2)-(C-4). Consequently there are a total of 3N rotor equations of motion. Fuselage motion is governed by Eqs. (D-1) - (D-6). Note that in the fuselage equations of motion, Eq. (11) was used to express the fuselage translational degrees of freedom in terms of hub translations and rotations.

$$R_X = r_X - z_F (\theta_Y + \theta_X \theta_Z) \tag{67}$$

$$R_Y = r_Y + x_F \theta_Z + z_F (\theta_X - \theta_Y \theta_Z) \tag{68}$$

$$R_Z = r_Z - x_F \theta_Y \tag{69}$$

Therefore the system motions are r_X, r_Y, r_Z, θ_X, θ_Y, θ_Z, v_k, w_k and ϕ_k, $k = 1, N$.

As written, the final equations are in second order partial differential form. The spatial dependence may be removed by expressing the blade deflections as summations of elastic mode shapes and, by Galerkin's

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method, integrating the rotor equations to eliminate the spatial dependence. The selection of blade mode shapes will depend on the rotor model, hingeless or articulated. The number and type of mode shapes, used in each deflection representation, will directly effect the size of the final system of equations and the ability to determine system aeroelastic behavior.

Several comments can be made about the nature of these second order nonlinear periodic coefficient equations. As derived, the rotor equations of motion are written in a rotating reference system whereas the matching conditions between the rotor and the fuselage are written in a nonrotating system. Although this may cause some difficulty when visualizing results, this procedure allows for greater physical insight into the physics of the problem.

It is possible to rewrite the rotor equations of motion in terms of nonrotating rotor degrees of freedom using multiblade coordinates[22]. This transformation will replace some of the periodic coefficient terms in the final equations with constant coefficient terms. For some cases (hover, $N \geq 3$) this procedure will remove all periodic coefficient terms from the rotor equations of motion. However this is not the case for a helicopter in forward flight. Thus to be able to model an N-bladed rotor in forward flight, the rotor equations of motion will include periodic coefficients. Even after applying a multiblade coordinate transformation, it is possible to use a constant coefficient approximation for the remaining periodic coefficients. Although this approximation is good for many practical cases, it is sensitive to advance ratio, operating state, and system component frequencies[23].

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The equations of motion presented herein can be used to study both aeroelastic stability and response of coupled rotor and fuselage systems in hover or forward flight. There is no restriction on the number of rotor blades. Additionally, due to the consistent formulation of the complete problem, these equations can be used to represent single isolated rotor blade behavior. By suppressing the hub motions in Eqs. (C-1)-(C-4), governing lead-lag, flap, and torsion equations of motion are obtained.

V. CONCLUSIONS

This paper presents the formulation of the governing equations of motion of a coupled rotor and fuselage helicopter math model. The derivation procedure yields a consistently derived set of second order nonlinear periodic coefficient partial differential equations. These equations may be used to study coupled rotor/fuselage dynamics in hover or forward flight.

VI. ACKNOWLEDGMENT

References

1. Friedmann, P.: "Recent Developments in Rotary-Wing Aeroelasticity," Journal of Aircraft, Vol. 14, No. 11, Nov. 1977, pp. 1027-1041.

2. Hodges, D. H. and Ormiston, R. A.: "Stability of Elastic Bending and Torsion of Uniform Cantilever Rotor Blades in Hover with Structural Coupling," NASA TN D-8192, April 1976.

3. Shamie, J. and Friedmann, P.: "Effect of Moderate Deflections on the Aeroelastic Stability of a Rotor Blade in Forward Flight," Proceedings of the Third European Rotorcraft and Power Lift Aircraft Forum, Aix-en-Provence, France, Sept. 1977, pp. 24.1-24.37.

4. Anderson, W. D., Connor, F., Kretsinger, P., and Reaser, J. S.: "REXOR Rotorcraft Simulation Model," USAAMRDL TR 76-28, July 1976.

5. Johnson, W.: "Aeroelastic Analysis for Rotorcraft in Flight or in a Wind Tunnel," NASA TN D-8515, July 1977.

6. Johnston, R. A., and Cassarino, S. J.: "Aeroelastic Rotor Stability Analysis," USAARMDL TR 75-40, January 1976.

7. McLarty, T. T., Van Gaasbeek, J. R., and Hsieh, P. Y.: "Rotorcraft Flight Simulation with Coupled Rotor Aeroelastic Stability Analysis," USAAMRDL TR 76-41, May 1977.

38

8. Ormiston, R. A.: "Aeromechanical Stability of Soft Inplane
    Hingeless Rotor Helicopters," Proceedings of the Third European
    Rotorcraft and Power Lift Aircraft Forum, Aix-en-Provence,
    France, Sept. 1977, pp. 25.1-25.22.

9. Hodges, D. H.: "Aeromechnical Stability of Helicopters with a
    Bearingless Main Rotor-Part I: Equations of Motion," NASA
    TM-78459, February 1978.

10. Warmbrodt, W. and Friedmann, P.: "Formulation of the Aeroelastic
    Stability and Response Problem of Coupled Rotor/Support Systems,"
    Proceedings of the AIAA/ASME/ASCE/AHS 20th Structures, Structural
    Dynamics and Material Conference, St. Louis, Missouri, April 1979,
    AIAA Paper No. 79-0732.

11. Johnson, W.: "Comprehensive Helicopter Analyses: A State-of-
    the-Art Review," NASA TM 78539, November 1978.

12. Rosen A., and Friedmann, P.: "Nonlinear Equations of Equilibrium
    for Elastic Helicopter or Wind Turbine Blades Undergoing Moderate
    Deformation," DOE/NASA/3082-78/1, December 1978 (also available
    as NASA CR-159478).

13. Bisplinghoff, R. I., Ashley, H., and Halfman, R. L.: Aeroelasti-
    city, Addison- Wesley Publ. Co., 1955.

14. Miller, R. H. and Ellis, C. W.: "Helicopter Blade Vibration
    and Flutter," Journal of the American Helicopter Society,
    Vol. 1, No. 3, pp. 19-38, July 1956.

15. Daughaday, H., DuWaldt, F., and Gates, C.: "Investigation of Helicopter Rotor Blade Flutter and Load Amplification Problems," Journal of the American Helicopter Society, Vol. 2, No. 3, pp. 27-45, July 1957.

16. Ham, N. D. and Garelick, M. S.: "Dynamic Stall Considerations in Helicopter Rotors," Journal of the American Helicopter Society, Vol. 13, No. 2, pp. 49-55, 1968.

17. Carta, F. O., Commerford, G. L., and Carlson, R. G.: "Determination of Airfoil and Rotor Blade Dynamic Stall Response," Journal of the American Helicopter Society, Vol. 18, No. 2, pp. 31-39, April 1973.

18. Beddoes, T. S.: "A Synthesis of Unsteady Aerodynamic Effects Including Stall Hysteresis," Vertica, Vol. 1., pp. 113-123, 1976.

19. Landgrebe, A. J., Moffitt, R. C., and Clark, D. R.: "Aerodynamic Technology for Advanced Rotorcraft - Part II," Journal of the American Helicopter Society, Vol. 22, No. 3, pp. 2-9, July 1977.

20. Peters, D. A.: "Hingeless Rotor Frequency Response with Unsteady Inflow," Rotorcraft Dynamics, NASA SP-352, 1974.

21. Lytwyn, R. T., Miao, W., Woitsch, W.: "Airborne and Ground Resonance of Hingeless Rotors," Journal of the American Helicopter Society, Vol. 16, No. 2, pp. 2-9, 1971.

22. Hohenemser, K. H. and Yin, S.: "Some Applications of the Method of Multiblade Coordinates," Journal of the American Helicopter Society, Vol. 17, No. 3, pp. 3-12, 1972.

23. Gaonkar, G. H. and Peters, D. A.: "Flap-Lag Stability with Dynamic Inflow by the Method of Multiblade Coordinates," Proceedings of the AIAA/ASME/ASCE/AHS 20th Structures, Structural Dynamics, and Materials Conference, St. Louis, April 4-6, 1979, AIAA Paper No. 79-0729.

TABLE 1

DESCRIPTION OF COORDINATE SYSTEMS

SYSTEM	DESCRIPTION
F	Fuselage axes. Origin fixed at undisturbed fuselage center of mass
R	Rotor axes. Nonrotating. Origin fixed at undisturbed hub location. Directions coincident with F-system.
1	Nonrotating rotor system with origin fixed in hub. Directions coincident with deformed R-system directions.
2k	Rotates with kth blade. Origin fixed in hub.
3k	Rotates with origin fixed in kth blade's pitch bearing. Preconed by $-\beta_p \hat{e}_{2xk}$. System yields undeformed blade coordinate directions.
4k	Origin fixed in deformed kth blade cross section at elastic center. System undergoes rigid body translation and rotation from 3k-system.
5k	Origin fixed in deformed blade at elastic center. Rotated from 4k-system by removing elastic torsion rotation.
V	Nonrotating. Origin fixed in hub. X_V axis defines direction of oncoming air velocity due to forward flight of aircraft.

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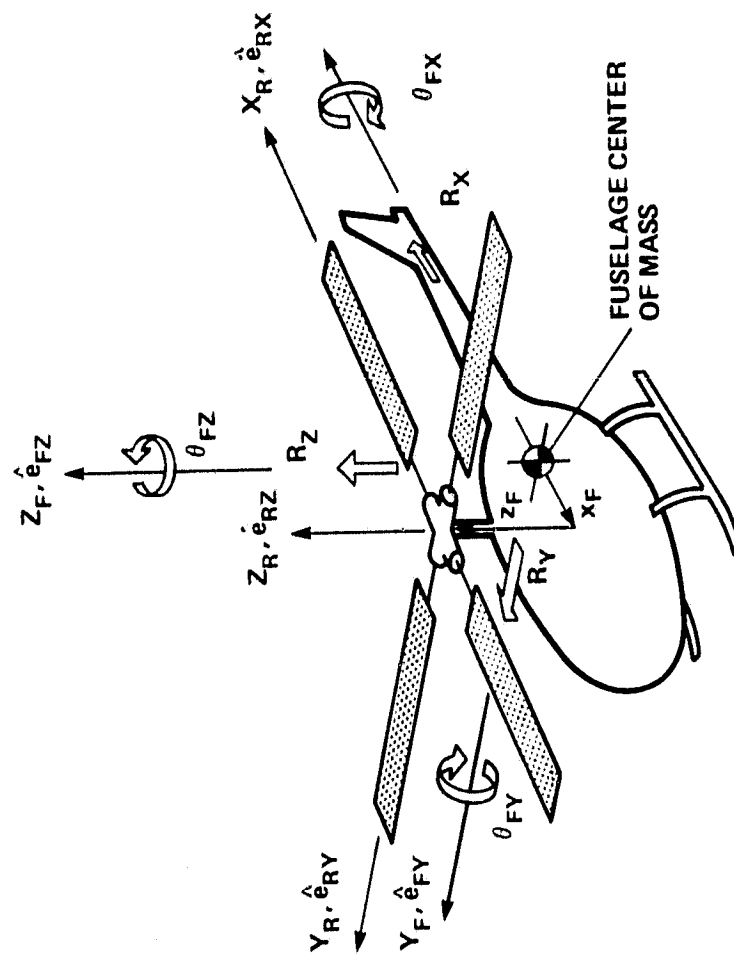
Fig. 1 Rotor/fuselage model
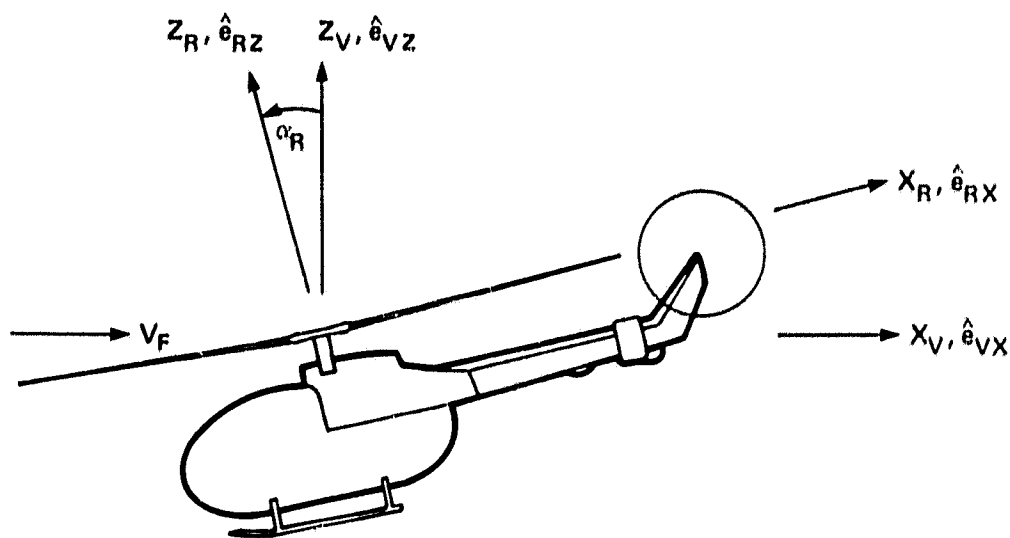
Fig. 2   Helicopter coordinate system
in forward flight
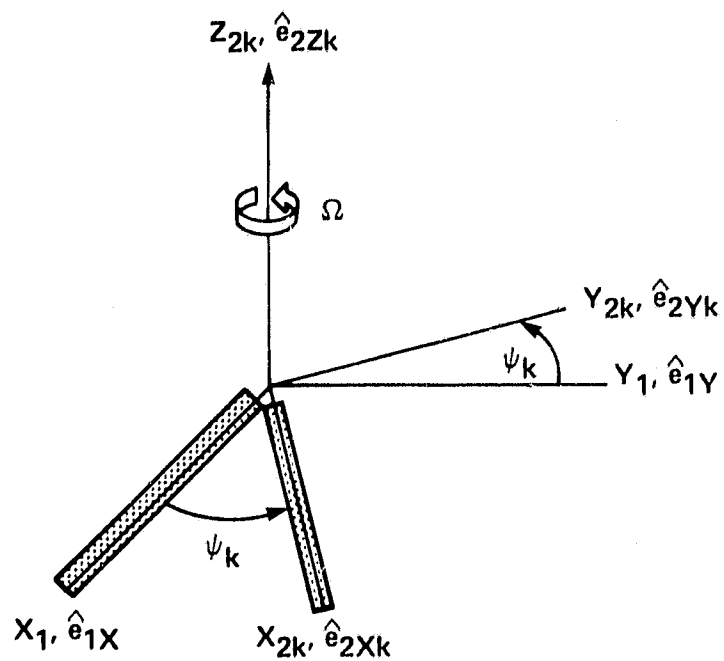
Fig. 3   Rotor blade coordinate systems

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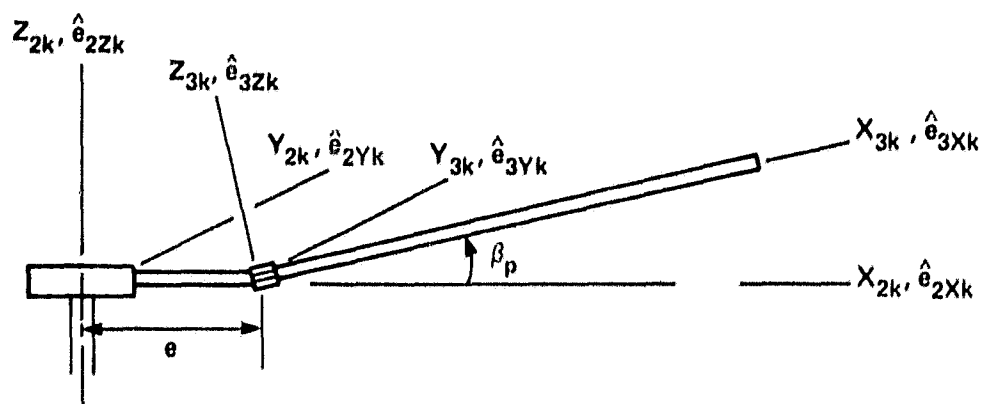
Fig. 4   Rotor blade coordinate systems
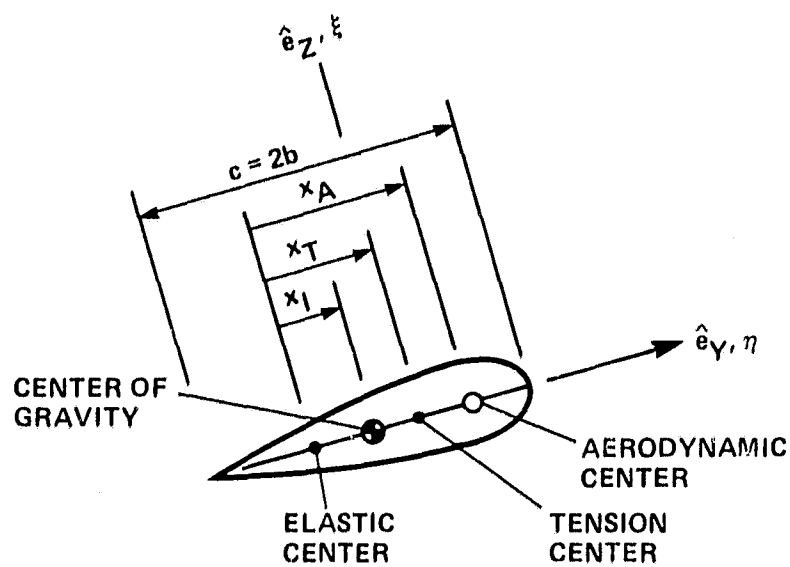
Fig. 5   Blade cross section configuration
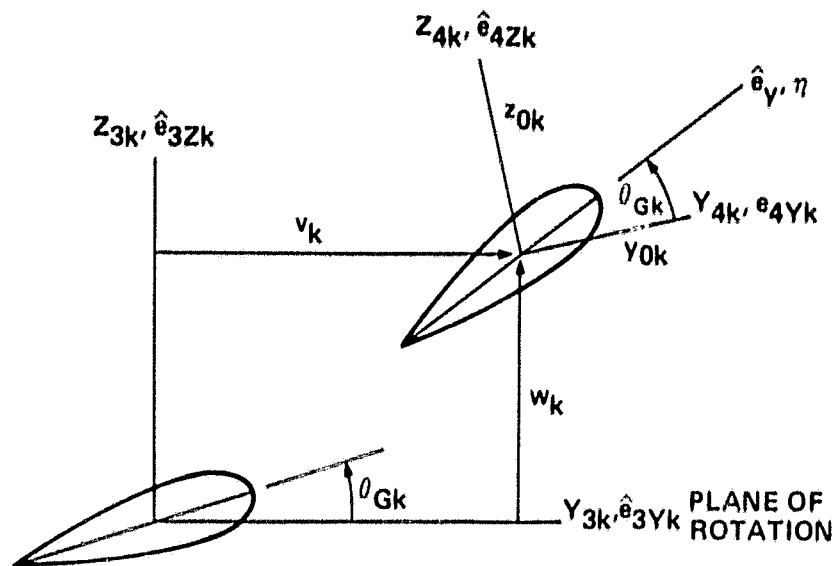
Fig. 6   Undeformed and deformed kth
          blade cross section
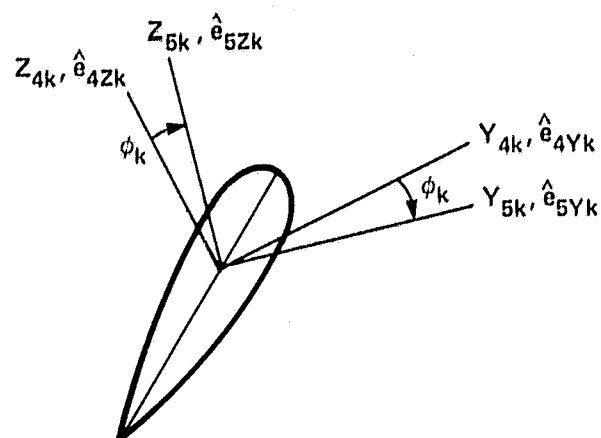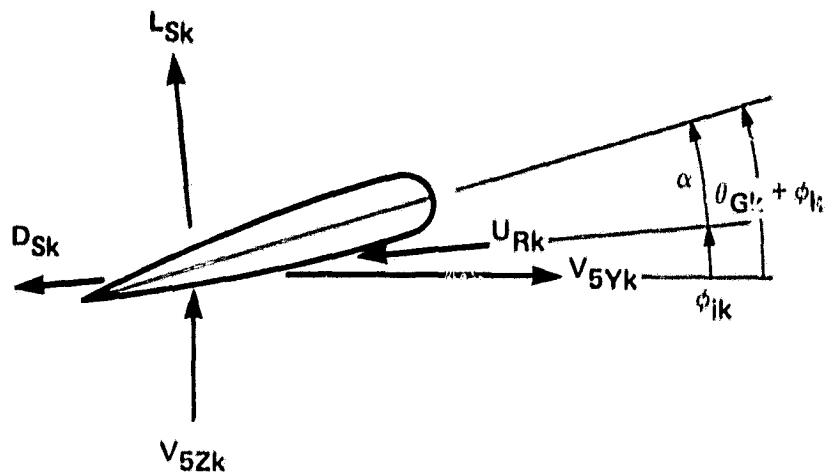

Fig. 7   Deformed blade coordinate system

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Fig. 8   Airfoil velocities and steady
         aerodynamic forces on deformed
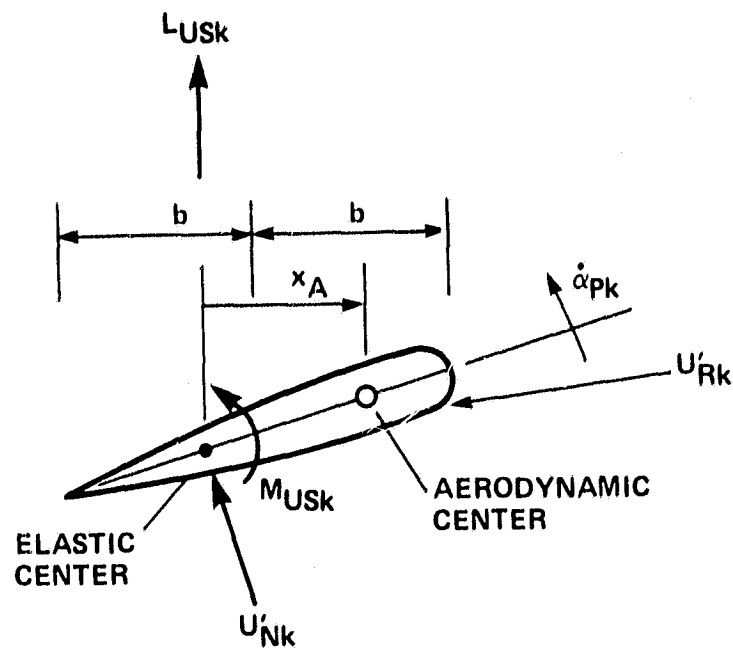   kth blade cross section

Fig. 9   Airfoil configuration for unsteady
         airloads calculation

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APPENDIX A .

The Euler transformation matrices used in the derivation of the equations motion are given below. For rotations of $0(\epsilon)$ and higher, the small angle assumption is used ($\cos \theta \approx 1$, $\sin \theta \approx \theta$).

$$\begin{Bmatrix} \hat{e}_{VXk} \\ \hat{e}_{VYk} \\ \hat{e}_{VZk} \end{Bmatrix} = \begin{bmatrix} \cos \alpha_R & 0 & -\sin \alpha_R \\ 0 & 1 & 0 \\ \sin \alpha_R & 0 & \cos \alpha_R \end{bmatrix} \begin{Bmatrix} \hat{e}_{RX} \\ \hat{e}_{RY} \\ \hat{e}_{RZ} \end{Bmatrix} \qquad (A.1)$$

$$\begin{Bmatrix} \hat{e}_{1X} \\ \hat{e}_{1Y} \\ \hat{e}_{1Z} \end{Bmatrix} = \begin{bmatrix} 1 & \theta_Z & -\theta_Y \\ -\theta_Z + \theta_Y \theta_X & 1 & \theta_X \\ \theta_Y + \theta_X \theta_Z & -\theta_X + \theta_Y \theta_Z & 1 \end{bmatrix} \begin{Bmatrix} \hat{e}_{RX} \\ \hat{e}_{RY} \\ \hat{e}_{RZ} \end{Bmatrix} \qquad (A.2)$$

$$\begin{Bmatrix} \hat{e}_{2Xk} \\ \hat{e}_{2Yk} \\ \hat{e}_{2Zk} \end{Bmatrix} = \begin{bmatrix} \cos \psi_k & \sin \psi_k & 0 \\ -\sin \psi_k & \cos \psi_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{e}_{1X} \\ \hat{e}_{1Y} \\ \hat{e}_{1Z} \end{Bmatrix} \qquad (A.3)$$

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$$\begin{Bmatrix} \hat{e}_{3Xk} \\ \hat{e}_{3Yk} \\ \hat{e}_{3Zk} \end{Bmatrix} = \begin{bmatrix} \cos\beta_P & 0 & \sin\beta_P \\ 0 & 1 & 0 \\ -\sin\beta_P & 0 & \cos\beta_P \end{bmatrix} \begin{Bmatrix} \hat{e}_{2Xk} \\ \hat{e}_{2Yk} \\ \hat{e}_{2Zk} \end{Bmatrix} \quad (A.4)$$

$$\begin{Bmatrix} \hat{e}_{4Xk} \\ \hat{e}_{4Yk} \\ \hat{e}_{4Zk} \end{Bmatrix} = \begin{bmatrix} 1 & v_{k,x} & w_{k,x} \\ -v_{k,x}-w_{k,x}\phi_k & 1 & \phi_k \\ -w_{k,x}+\phi_k v_{k,x} & -\phi_k-w_{k,x}v_{k,x} & 1 \end{bmatrix} \begin{Bmatrix} \hat{e}_{3Xk} \\ \hat{e}_{3Yk} \\ \hat{e}_{3Zk} \end{Bmatrix} \quad (A.5)$$

$$\begin{Bmatrix} \hat{e}_{5Xk} \\ \hat{e}_{5Yk} \\ \hat{e}_{5Zk} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\phi_k \\ 0 & \phi_k & 1 \end{bmatrix} \begin{Bmatrix} \hat{e}_{4Xk} \\ \hat{e}_{4Yk} \\ \hat{e}_{4Zk} \end{Bmatrix} \quad (A.6)$$

APPENDIX B

From basic definitions, Fig. 5, and the assumptions of section
II.B the following identities are used in the derivation.

$$y_{0k} = \eta \cos \theta_{Gk} - \xi \sin \theta_{Gk} \tag{B.1}$$

$$z_{0k} = \xi \sin \theta_{Gk} + \eta \cos \theta_{Gk} \tag{B.2}$$

$$\dot{y}_{0k} = -\dot{\theta}_{Gk} z_{0k} \tag{B.3}$$

$$\dot{z}_{0k} = \dot{\theta}_{Gk} y_{0k} \tag{B.4}$$

$$\ddot{y}_{0k} = -\ddot{\theta}_{Gk} z_{0k} - \dot{\theta}_{Gk}^2 y_{0k} \tag{B.5}$$

$$\ddot{z}_{0k} = \ddot{\theta}_{Gk} y_{0k} - \dot{\theta}_{Gk}^2 z_{0k} \tag{B.6}$$

$$\iint_A \rho \, dA = m \tag{B.7}$$

$$\iint_A \rho z_{0k} \, dA = m x_I \sin \theta_{Gk} \tag{B.8}$$

$$\iint_A \rho y_{0k} \, dA = m x_I \cos \theta_{Gk} \tag{B.9}$$

$$\iint_A \rho \eta^2 \, dA = I_{MB3} \tag{B.10}$$

$$\iint_A \rho \xi^2 \, dA = I_{MB2} \tag{B.11}$$

$$\iint_A \rho \eta \xi \, dA = 0 \tag{B.12}$$

$$\iint_A \rho y_{0k}^2 \, dA = I_{MB3} \cos^2 \theta_{Gk} + I_{MB2} \sin^2 \theta_{Gk} \tag{B.13}$$

$$\iint_A \rho y_{0k}^2 \, dA = I_{MB3} \sin^2 \theta_{Gk} + I_{MB2} \cos^2 \theta_{Gk} \quad (B.14)$$

$$\iint_A \rho y_{0k} z_{0k} \, dA = (I_{MB3} - I_{MB2}) \sin \theta_{Gk} \cos \theta_{Gk} \quad (B.15)$$

where

$$\theta_{Gk} = \theta_B(x_k) + \theta_0 + \theta_{1C} \cos \psi_k + \theta_{1S} \sin \psi_k$$

$$(B.16)$$

$$\dot{\theta}_{Gk} = \Omega \theta_{1S} \cos \psi_k - \Omega \theta_{1C} \sin \psi_k \quad (B.17)$$

$$\ddot{\theta}_{Gk} = -\Omega^2 \theta_{1C} \cos \psi_k - \Omega^2 \theta_{1S} \sin \psi_k \quad (B.18)$$

APPENDIX C

The governing nonlinear partial differential equations of motion for kth blade of an N-bladed rotor are presented below. They represent equations of equilibrium in the axial direction and the lead-lag, flap, and torsional deflections, respectively.

$$T_{k,x} + m\Omega^2 \left[-\ddot{r}_x \cos\psi_k - \ddot{r}_Y \sin\psi_k \right.$$
$$\left. + 2\dot{v}_k + (e+x_k)(1+2\dot{\theta}_z) \right] = 0 \qquad (C-1)$$

$$-\left[E(I_{zz}\cos^2\theta_{Gk} + I_{YY}\sin^2\theta_{Gk})v_{k,xx} \right.$$
$$+ E(I_{zz}-I_{YY})\phi_k w_{k,xx}\cos 2\theta_{Gk}$$
$$+ E(I_{zz}-I_{YY})(w_{k,xx} - 2\phi_k v_{k,xx})\sin\theta_{Gk}\cos\theta_{Gk}$$
$$\left. - C_1 x_T \cos\theta_{Gk} \right]_{,xx} - (GJ\phi_{k,x}w_{k,xx})_{,x}$$
$$+ (v_{k,x}T_k)_{,x} - \Omega^2 \left[-mx_I\cos\theta_{Gk}(e+x_k) \right.$$
$$\left. + 2\dot{\theta}_{Gk}(I_{mB3} - I_{mB2})\cos\theta_{Gk}\sin\theta_{Gk} \right]_{,x}$$
$$+ m\Omega^2 \left[\cos\psi_k(-\ddot{r}_Y + 2\dot{w}_k\dot{\theta}_X + \ddot{\theta}_X(w_k \right.$$
$$+ \beta_P(e+x_k)) + \sin\psi_k(\ddot{r}_X + 2\dot{w}_k\dot{\theta}_Y$$
$$+ \ddot{\theta}_Y(w_k + \beta_P(e+x_k)) - \ddot{v}_k - 2\ddot{u}_k + 2\beta_P\dot{w}_k$$
$$\left. + v_k - (e+x_k)\ddot{\theta}_z + 2\dot{\theta}_z v_k \right] - g_{SL}\dot{v}_k\Omega$$
$$- p_{A3Yk} = 0 \qquad (C-2)$$

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$$-\left[E(I_{ZZ}-I_{YY})\sin\theta_{Gk}\cos\theta_{Gk}(v_{k,xx}+2\phi_k w_{k,xx})\right.$$

$$+E(I_{ZZ}\sin^2\theta_{Gk}+I_{YY}\cos^2\theta_{Gk})w_{k,xx}$$

$$+E(I_{ZZ}-I_{YY})\phi_k v_{k,xx}\cos 2\theta_{Gk}$$

$$\left.-C_1 x_T \sin\theta_{Gk}\right]_{,xx} + (GJ\phi_{k,x}v_{k,xx})_{,x}$$

$$+ (w_{k,x}T_k)_{,x} + \Omega^2\left[m x_I(e+x_k)\sin\theta_{Gk}\right.$$

$$\left.- 2\dot{\theta}_{Gk}(I_{MB2}\cos^2\theta_{Gk}+I_{MB3}\sin^2\theta_{Gk})\right]_{,x}$$

$$+ m\Omega^2\left[\cos\psi_k\langle -2\dot{v}_k\dot{\theta}_X - v_k\ddot{\theta}_X - 2\dot{\theta}_Y v_k\right.$$

$$+ \beta_P\ddot{r}_X - (2\dot{\theta}_X-\ddot{\theta}_Y)(e+x_k)\rangle$$

$$+ \sin\psi_k\langle -2\dot{\theta}_Y\dot{v}_k - v_k\ddot{\theta}_Y + 2v_k\dot{\theta}_X + \beta_P\ddot{r}_Y$$

$$-(2\dot{\theta}_Y+\ddot{\theta}_X)(e+x_k)\rangle - \ddot{r}_Z - \ddot{w}_k - 2\beta_P(\dot{v}_k$$

$$\left.-\dot{\theta}_Z(e+x_k)) - \beta_P(e+x_k) - x_I\ddot{\theta}_{Gk}\cos\theta_{Gk}\right]$$

$$- g_{SF}\dot{w}_k\Omega + p_{A3\bar{z}k} = 0 \qquad\qquad (c\text{-}3)$$

$$\left[GJ(\phi_{k,x}+v_{k,xx}w_{,x})\right]_{,x} + C_1 x_T(w_{k,xx}\cos\theta_{Gk}$$

$$-v_{k,xx}\sin\theta_{Gk}) + E(I_{ZZ}-I_{YY})\langle -v_{k,xx}w_{k,xx}\cos 2\theta_{Gk}$$

$$+ (v_{k,xx}{}^2 - w_{k,xx}{}^2)\sin\theta_{Gk}\cos\theta_{Gk}\rangle \quad +$$

$$+ \, m\Omega^2 x_I \cos\theta_{Gk} \Big[\ddot{\cos}\psi_k (e+x_k)(\ddot{\theta}_Y - 2\dot{\theta}_X)$$

$$- (e+x_k)(2\dot{\theta}_Y + \ddot{\theta}_X)\sin\psi_k \; -\ddot{\ddot{r}}_z - \ddddot{w}_k$$

$$- 2\dot{v}_k(\beta_P + w_{k,x}) + \phi_k(\ddot{v}_k - v_k) + (e+x_k)(-\beta_P$$

$$- w_{k,x} + v_{k,x}\phi_k)\Big] + m\Omega^2 x_I \sin\theta_{Gk}\Big[\ddot{v}_k$$

$$- v_k + v_{k,x}(e+x_k)\Big] - \Omega^2 I_P (\ddot{\phi}_k + \ddot{\theta}_{Gk}$$

$$+ \ddot{\theta}_X \cos\psi_k + \ddot{\theta}_Y \sin\psi_k) + \Omega^2 I_{mB3}\Big[(\beta_P v_{k,x}$$

$$- w_{k,x}\ddot{v}_{k,x} - 2\dot{\theta}_Y \cos\psi_k + 2\dot{\theta}_X \sin\psi_k)\cos^2\theta_{Gk}$$

$$- (1 + 2\dot{v}_{k,x})\cos\theta_{Gk}\sin\theta_{Gk} - \phi_k\cos\theta_{Gk}(1$$

$$+ 2\dot{v}_{k,x})\Big] + \Omega^2 I_{mB2}\Big[\cos^2\theta_{Gk}(-\ddot{v}_{k,x}w_{k,x}$$

$$- v_{k,x}\beta_P - 2\dot{w}_{k,x}(1 + \dot{v}_{k,x})) + \cos\theta_{Gk}\sin\theta_{Gk}(1$$

$$+ 2\dot{v}_{k,x}) + \phi_k(1 + 2\dot{v}_{k,x})\cos 2\theta_{Gk}\Big]$$

$$- g_{ST}\dot{\phi}_k\Omega + q_{A3Xk} = 0$$

$$(C-4)$$

The elastic axial degree of freedom is defined by

$$u_k = -\frac{1}{2}\int_0^x (v_{k,x}^2 + w_{k,x}^2)\,dx \qquad (C-5)$$

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The aerodynamic quantities $p_{\underset{\sim}{A3Yk}}$, $p_{\underset{\sim}{A3Zk}}$, and $q_{\underset{\sim}{A3Xk}}$ are given by Eqs. (41) and (42), respectively. The coefficient appearing in Eqs. (C-2) – (C-4) is given by

$$C_1 = \Omega^2 \int_x^{\ell} m(e + x_k) \, dx_k \qquad (C-6)$$

Appendix D

The following equations represent force and moment equilibrium at the rotor hub.

$$-M_F\left[\ddot{r}_X - z_F(\ddot{\theta}_Y + \ddot{\theta}_X\theta_Z + 2\dot{\theta}_X\dot{\theta}_Z + \theta_X\ddot{\theta}_Z)\right]$$

$$-(M_F + M_R)g\sin\alpha_R + F_X + P_{RX} = 0$$

$$(D.1)$$

$$-M_F\left[\ddot{r}_Y + x_F\ddot{\theta}_Z + z_F(\ddot{\theta}_X - \ddot{\theta}_Y\theta_Z - 2\dot{\theta}_Y\dot{\theta}_Z - \theta_Y\ddot{\theta}_Z)\right]$$

$$+ F_Y + P_{RY} = 0 \qquad (D.2)$$

$$-M_F(\ddot{r}_Z - x_F\ddot{\theta}_Y) - (M_F + M_R)g\cos\alpha_R$$

$$+ F_Z + P_{RZ} = 0 \qquad (D.3)$$

$$-(I_X - 2I_{ZX}\theta_Y)\ddot{\theta}_X - I_{ZX}(\ddot{\theta}_X - \theta_X\ddot{\theta}_Y - 3\dot{\theta}_Y\dot{\theta}_X)$$

$$+ (I_X - I_Y)\theta_Z\ddot{\theta}_Y + (I_Z - I_X)\theta_Y\ddot{\theta}_Z$$

$$+ (I_Z - I_Y)\dot{\theta}_Y\dot{\theta}_Z - M_F z_F\left[\ddot{r}_Y + x_F(\ddot{\theta}_Z - \ddot{\theta}_Y\theta_X)\right.$$

$$\left. + z_F(\ddot{\theta}_X - \ddot{\theta}_Y\theta_Z - 2\dot{\theta}_Y\dot{\theta}_Z - \theta_Y\ddot{\theta}_Z) + \ddot{r}_Z\theta_X\right]$$

$$- M_F x_F\left[\ddot{r}_Z\theta_Z + \ddot{r}_Y\theta_Y + x_F(-\theta_Z\ddot{\theta}_Y + \theta_Y\ddot{\theta}_Z) + \right.$$

$$+ z_F \theta_Y \ddot{\theta}_X \big] \ - M_F g \cos\alpha_R (x_F \theta_Z + z_F \theta_X)$$

$$+ Q_X \ + \ Q_{RX} = 0 \qquad\qquad (D.4)$$

$$-I_Y \ddot{\theta}_Y \ - I_{ZX}(\theta_Z \ddot{\theta}_Z - \theta_X \ddot{\theta}_X + \dot{\theta}_Z^2 - \dot{\theta}_X^2) + \theta_Z \ddot{\theta}_X (I_X - I_Y)$$

$$+ (I_Y - I_Z)\theta_X \ddot{\theta}_Z \ + (I_X - I_Z)\dot{\theta}_X \dot{\theta}_Z + M_F x_F \big[\ddot{r}_Z$$

$$- (x_F + z_F \theta_Y)\ddot{\theta}_Y + \ddot{r}_X \theta_Y \big] + M_F z_F \big[\ddot{r}_X + x_F \theta_Y \ddot{\theta}_Y$$

$$- \ddot{r}_Z \theta_Y - z_F (\ddot{\theta}_Y + \ddot{\theta}_X \theta_Z + 2\dot{\theta}_X \dot{\theta}_Z + \theta_X \ddot{\theta}_Z) \big]$$

$$+ M_F g \big[\cos\alpha_R (x_F - z_F \theta_Y) + \sin\alpha_R (z_F + x_F \theta_Y) \big]$$

$$+ Q_Y \ + \ Q_{RY} = \ 0 \qquad\qquad (D.5)$$

$$-I_Z \ddot{\theta}_Z \ - I_{ZX}(\ddot{\theta}_X + \theta_Z \ddot{\theta}_Y + 3\dot{\theta}_Y \dot{\theta}_Z + 2\theta_Y \ddot{\theta}_Z) + \theta_Y \ddot{\theta}_X (I_Z - I_X)$$

$$+ (I_Y - I_Z)\theta_X \ddot{\theta}_Y \ + (I_Y - I_X)\dot{\theta}_X \dot{\theta}_Y - M_F x_F \big[\ddot{r}_Y - \ddot{r}_X \theta_Z$$

$$+ x_F \ddot{\theta}_Z \ + z_F (\ddot{\theta}_X - 2\dot{\theta}_Y \dot{\theta}_Z - \theta_Y \ddot{\theta}_Z) \big] + M_F z_F \big[\ddot{r}_X \theta_X$$

$$+ \ddot{r}_Y \theta_Y + x_F \ddot{\theta}_Z \theta_Y + z_F (-\ddot{\theta}_Y \theta_X + \theta_Y \ddot{\theta}_Z) \big]$$

$$+ M_F g \sin\alpha_R (x_F \theta_Z + z_F \theta_X) + Q_Z \ + Q_{RZ} = 0$$

$$\qquad\qquad (D.6)$$

The generalized fuselage forces and moments, F_j, Q_j, $j = X, Y, Z$, are left undefined. The rotor-induced forces and moments, P_{R_j}, Q_{R_j}, $j = X, Y, Z$, are defined in Appendix E.

The following are the rotor-induced forces and moments acting at the hub in the R-system directions.

$$P_{RX} = \Omega^2 R \sum_{k=1}^{N}$$

$$\times \int_0^{\ell} \Big\{ \rho_A \bar{b} \sum c_{\ell} < \sin^3\psi_k \Big[\mu(\bar{e}+\bar{x}_k)\dot{\theta}_X - \mu\beta_P\dot{\bar{r}}_Y$$

$$-\mu^2 V_{k,x} W_{k,x} + \mu^2\beta_P\theta_Z \Big] + \cos^2\psi_k \sin\psi_k \Big[\mu^2 W_{k,x}(V_{k,x}$$

$$+ 2\theta_Z) + \mu\dot{\bar{r}}_Y(3\beta_P + W_{k,x}) - \mu V_{k,x}(\bar{e}+\bar{x}_k)\dot{\theta}_Y$$

$$-\mu^2\beta_P V_{k,x} \Big] + \cos\psi_k \sin^2\psi_k \Big[\mu^2 W_{k,x} - \mu(\bar{e}+\bar{x}_k)\dot{\theta}_Y$$

$$-3\mu\beta_P\dot{\bar{r}}_X + \mu(\bar{e}+\bar{x}_k)\dot{\theta}_X V_{k,x} \Big]$$

$$+ \cos^2\psi_k \Big[2\beta_P(\bar{e}+\bar{x}_k)(\mu V_{k,x} - \dot{\bar{r}}_Y) + \mu(\bar{e}+\bar{x}_k)\theta_Z(W_{k,x}$$

$$+ 3\beta_P) \Big] + \sin^2\psi_k \Big[2\mu(\bar{e}+\bar{x}_k)V_{k,x}W_{k,x} + (\bar{e}+\bar{x}_k)^2\dot{\theta}_X$$

$$+ (\bar{e}+\bar{x}_k)\dot{\bar{V}}_k\dot{\theta}_X + \mu(\lambda_k + \dot{\bar{r}}_Z + \dot{\bar{W}}_k) - \dot{\bar{r}}_Z(\lambda_k$$

$$+ \dot{\bar{W}}_k) - 2\mu(\bar{e}+\bar{x}_k)\beta_P\theta_Z \Big] + \cos\psi_k \sin\psi_k \Big[\mu^2\theta_Y V_{k,x}$$

$$+ 2\mu\theta_Z(\lambda_k + \dot{\bar{W}}_k) + \mu V_{k,x}(\lambda_k + \dot{\bar{r}}_Z + \dot{\bar{W}}_k)$$

$$+ \dot{\bar{r}}_Y(\lambda_k + \dot{\bar{W}}_k) + 3\mu\beta_P(\dot{\bar{V}}_k + (\bar{e}+\bar{x}_k)(1 + \dot{\theta}_Z))$$

$$+ \mu W_{k,x}(\dot{\bar{V}}_k + (\bar{e}+\bar{x}_k)(1 + \dot{\theta}_Z)) - (\bar{e}+\bar{x}_k)^2\dot{\theta}_Y$$

$$- (\bar{e}+\bar{x}_k)\dot{\bar{V}}_k\dot{\theta}_Y + 2\dot{\bar{r}}_X\beta_P(\bar{e}+\bar{x}_k) \Big]$$

$$+ \cos\psi_k \Big[-\beta_P(\bar{e}+\bar{x}_k)(2\dot{\bar{V}}_k + (\bar{e}+\bar{x}_k)(1 + 2\dot{\theta}_Z)) +$$

$$+ (\bar{e}+\bar{x}_k)\theta_Z(\lambda_k + \dot{\bar{w}}_k) - 2\mu(\bar{e}+\bar{x}_k)v_{k,x}\theta_Y]$$

$$+ \sin\psi_k[-3\mu\theta_Y(\dot{\bar{v}}_k + (\bar{e}+\bar{x}_k)) + \dot{\bar{v}}_k(\lambda_k + \dot{\bar{r}}_Z + \dot{\bar{w}}_k)$$

$$- v_{k,x}w_{k,x}(\bar{e}+\bar{x}_k)^2 + (\bar{e}+\bar{x}_k)(\lambda_k + \dot{\bar{r}}_Z + \dot{\bar{w}}_k)$$

$$+ (\bar{e}+\bar{x}_k)\dot{\theta}_Z(\lambda_k + \dot{\bar{w}}_k) + \beta_P\theta_Z(\bar{e}+\bar{x}_k)^2]$$

$$+ (\bar{e}+\bar{x}_k)\theta_Y(2\dot{\bar{v}}_k + (\bar{e}+\bar{x}_k)))\rangle$$

$$+ a\langle \cos^2\psi_k\sin\psi_k[-\mu\dot{\theta}_X\beta_P(\bar{b}-\bar{x}_A)]$$

$$+ \sin^2\psi_k\cos\psi_k[-\mu\beta_P\dot{\theta}_Y(\bar{b}-\bar{x}_A)]$$

$$+ \cos^2\psi_k[\frac{\bar{b}}{2}\beta_P(\mu\dot{\bar{w}}_{k,x} - (\bar{e}+\bar{x}_k)\ddot{\theta}_Y$$

$$+ (\bar{e}+\bar{x}_k)\dot{\theta}_X - \mu(\theta_{Gk}+\phi_k)) - \beta_P\dot{\theta}_X(\bar{e}+\bar{x}_k)(\bar{b}-\bar{x}_A)]$$

$$+ \cos\psi_k\sin\psi_k[-\mu\beta_P(w_{k,x}+\beta_P)(\frac{3}{2}\bar{b}-\bar{x}_A)$$

$$- \mu(\dot{\theta}_{Gk}+\dot{\phi}_k)(\frac{3}{2}\bar{b}-\bar{x}_A)\beta_P + (\bar{e}+\bar{x}_k)\ddot{\theta}_X\beta_P\frac{\bar{b}}{2}$$

$$- (\bar{e}+\bar{x}_k)\dot{\theta}_Y\beta_P(\frac{\bar{b}}{2}-x_A)] + \cos\psi_k[-\beta_P(w_{,x}+\beta_P)(\bar{e}$$

$$+ \bar{x}_k)(b-\bar{x}_A) - \beta_P(\dot{\theta}_{Gk}+\dot{\phi}_k)(\bar{e}+\bar{x}_k)(\frac{3}{2}\bar{b}-\bar{x}_A)$$

$$- \frac{\bar{b}}{2}\beta_P(\mu\dot{\theta}_Y-\lambda_k-\ddot{\bar{r}}_Z-\ddot{\bar{w}}_k) + \mu\theta_Y\frac{\bar{b}}{2}(\theta_{Gk}+\phi_k-\dot{w}_{k,x})]$$

$$+ \sin\psi_k[\mu\theta_Y(\frac{3}{2}\bar{b}-\bar{x}_A)(w_{k,x}+\beta_P+\dot{\theta}_{Gk}+\dot{\phi}_k)]$$

$$+ \theta_Y(\bar{e}+\bar{x}_k)(w_{k,x}+\beta_P)(b-\bar{x}_A) + \theta_Y(\bar{e}+\bar{x}_k)(\dot{\theta}_{Gk}+\dot{\phi}_k)(\frac{3}{2}\bar{b}$$

$$- \bar{x}_A) - \theta_Y(\lambda_k+\ddot{\bar{w}}_k)\frac{\bar{b}}{2}\rangle \quad +$$

$$+ c_d \Big\langle \mu^2 \sin^3 \psi_k - 2\mu^2 v_{k,x} \cos \psi_k \sin^2 \psi_k$$

$$-2\mu (\bar{e}+\bar{x}_k) v_{k,x} \sin \psi_k \cos \psi_k - 2\mu (c\bar{e}+\bar{x}_k)$$

$$+ \dot{v}_k) \sin^2 \psi_k + \sin \psi_k [(\bar{e}+\bar{x}_k)^2 + 2\dot{v}_k (\bar{e}+\bar{x}_k)] \Big\rangle \Big\}$$

$$+ m \Big\langle \cos \psi_k [2\dot{v}_k (1+\dot{\theta}_Z) + \ddot{v}_k \theta_Z + \bar{v}_k (\ddot{\theta}_Z - \theta_Z)]$$

$$+ \sin \psi_k [-2\dot{v}_k \theta_Z + \ddot{v}_k - \bar{v}_k - 2\dot{\theta}_Z \bar{v}_k + 2\ddot{u}_k - \bar{x}_I \cos \theta_{6k}$$

$$- 2\beta_P \ddot{w}_k] - \ddot{r}_X - \ddot{w}_k \theta_Y - 2\ddot{w}_k \dot{\theta}_Y - \bar{w}_k \ddot{\theta}_Y$$

$$- \beta_P \ddot{\theta}_Y (\bar{e}+\bar{x}_k) \Big\rangle \Big\} dx_k$$

$$(E.1)$$

$$P_{RY} = -\Omega^2 R \sum_{k=1}^{N}$$

$$\times \int_o^{\ell} \Big\{ \rho_A \bar{b} \sum c_\ell \Big\langle [-\mu^2 (\beta_P + w_{k,x})(\theta_Z + v_{k,x})$$

$$+ \mu v_{k,x} (\bar{e}+\bar{x}_k) \dot{\theta}_Y - \mu (\beta_P + w_{k,x}) \dot{r}_Y] \cos^3 \psi_k$$

$$- \beta_P (2\mu \dot{r}_X + \mu^2) \sin^3 \psi_k + \cos^2 \psi_k \sin \psi_k [\mu \beta_P \dot{r}_X$$

$$- \mu^2 (\beta_P + w_{k,x}) + \mu (\bar{e}+\bar{x}_k) \dot{\theta}_Y - \mu (\bar{e}+\bar{x}_k) \dot{\theta}_X v_{k,x}]$$

$$+ \cos \psi_k \sin \psi_k [-\mu \dot{\theta}_X (\bar{e}+\bar{x}_k) + \mu \beta_P \ddot{r}_Y + \mu^2 v_{k,x} w_{k,x}$$

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$$+ 2\mu\dot{\bar{r}}_Y\beta_P - 2\mu^2\beta_P(\theta_Z + v_{k_1x}) + \mu^2 W_{k_1x}\theta_Z\Big]$$

$$+ \cos^2\psi_k\Big[-\mu\theta_Z(\lambda_k + \dot{\bar{w}}_k) + \mu^2\theta_Y v_{k_1x}$$

$$-\mu v_{k_1x}(\lambda_k + \dot{\bar{r}}_Z + \dot{\bar{w}}_k) - \dot{\bar{r}}_Y(\lambda_k + \dot{\bar{w}}_k)$$

$$-\mu(\beta_P + w_{k_1x})(\dot{\bar{v}}_k + (\bar{e} + \bar{x}_k)(1 + \dot{\theta}_Z))$$

$$+ (\bar{e} + \bar{x}_k)\dot{\bar{v}}_k\dot{\theta}_Y + (\bar{e} + \bar{x}_k)^2\dot{\theta}_Y\Big]$$

$$+ \sin^2\psi_k\Big[2\mu\beta_P(\dot{\bar{v}}_k + (\bar{e} + \bar{x}_k)(1 + \dot{\theta}_Z)) - \mu^2\theta_X$$

$$+ 2\beta_P\dot{\bar{r}}_X(\bar{e} + \bar{x}_k) + \mu\theta_Z(\lambda_k + \dot{\bar{w}}_k)\Big]$$

$$+ \cos\psi_k\sin\psi_k\Big[-(\bar{e} + \bar{x}_k)(\dot{\bar{v}}_k\dot{\theta}_X + 2\mu v_{k_1x}w_{k_1x}$$

$$+ (\bar{e} + \bar{x}_k)\dot{\theta}_X + \mu^2\theta_Y - \mu(\lambda_k + \dot{\bar{r}}_Z + \dot{\bar{w}}_k)$$

$$+ \dot{\bar{r}}_Z(\lambda_k + \dot{\bar{w}}_k) + 2\beta_P(\bar{e} + \bar{x}_k)(\tfrac{5}{2}\mu\theta_Z - \dot{\bar{r}}_Y + \mu v_{k_1x})$$

$$+ \mu\theta_Z w_{k_1x}(\bar{e} + \bar{x}_k) - 2\mu^2 v_{k_1x}\theta_X\Big]$$

$$+ \cos\psi_k\Big[\dot{\bar{v}}_k(\mu\theta_Y - \lambda_k - \dot{\bar{r}}_Z - \dot{\bar{w}}_k) + v_{k_1x}w_{k_1x}(\bar{e} + \bar{x}_k)^2$$

$$+ (\bar{e} + \bar{x}_k)(\mu\theta_Y - \lambda_k - \dot{\bar{r}}_Z - \dot{\bar{w}}_k) - \dot{\theta}_Z(\bar{e} + \bar{x}_k)(\lambda_k + \dot{\bar{w}}_k)$$

$$- \beta_P\theta_Z(\bar{e} + \bar{x}_k)^2 + 2\mu\theta_X v_{k_1x}(\bar{e} + \bar{x}_k)\Big]$$

$$+ \sin\psi_k\Big[-2\beta_P\dot{\bar{v}}_k(\bar{e} + \bar{x}_k) - \beta_P(\bar{e} + \bar{x}_k)^2(1 - 2\dot{\theta}_Z)$$

$$+ (\bar{e} + \bar{x}_k)\theta_Z(\lambda_k + \dot{\bar{w}}_k) + 2\mu\dot{\bar{v}}_k\theta_X + 2\mu(\bar{e} + \bar{x}_k)\theta_X\Big]$$

$$- \theta_X(\bar{e} + \bar{x}_k)(2\dot{\bar{v}}_k + (\bar{e} + \bar{x}_k))\Big\rangle \quad +$$

$$+ a \Big\langle - \mu \beta_P \dot{\Theta}_Y (\bar{b} - \bar{x}_A) \sin^3 \psi_k$$

$$- \mu \beta_P \dot{\Theta}_X (\bar{b} - \bar{x}_A) \cos \psi_k \sin^2 \psi_k$$

$$+ \cos \psi_k \sin \psi_k \Big[- \frac{\bar{b}}{2} \beta_P (\mu (\Theta_{Gk} + \phi_k - \dot{w}_{k,x})$$

$$+ (\bar{e} + \bar{x}_k)(\ddot{\Theta}_Y - \dot{\Theta}_X)) - \beta_P \dot{\Theta}_X (\bar{e} + \bar{x}_k)(\bar{b} - \bar{x}_A) \Big]$$

$$+ \sin^2 \psi_k \Big[- \mu \beta_P (w_{k,x} + \beta_P + \dot{\Theta}_{Gk} + \dot{\phi}_k)(\tfrac{3}{2} \bar{b} - \bar{x}_A)$$

$$+ \beta_P (\bar{e} + \bar{x}_k) \ddot{\Theta}_X \frac{\bar{b}}{2} - \beta_P \dot{\Theta}_Y (\bar{e} + \bar{x}_k)(\frac{\bar{b}}{2} - \bar{x}_A) \Big]$$

$$+ \cos \psi_k \Big[\frac{\bar{b}}{2} \Theta_X \mu (\dot{w}_{k,x} - \Theta_{Gk} - \phi_k) \Big]$$

$$+ \sin \psi_k \Big[- \beta_P (w_{k,x} + \beta_P)(\bar{e} + \bar{x}_k)(\bar{b} - \bar{x}_A)$$

$$- \beta_P (\dot{\Theta}_{Gk} + \dot{\phi}_k)(\bar{e} + \bar{x}_k)(\tfrac{3}{2} \bar{b} - \bar{x}_A) - \beta_P \frac{\bar{b}}{2} (\mu \dot{\Theta}_Y$$

$$- \dot{\lambda}_k - \ddot{\bar{r}}_z - \ddot{w}_k) - \mu \Theta_X (\tfrac{3}{2} \bar{b} - \bar{x}_A)(w_{k,x} + \beta_P + \dot{\Theta}_{Gk} + \dot{\phi}_k) \Big]$$

$$- \Theta_X (w_{k,x} + \beta_P)(\bar{e} + \bar{x}_k)(\bar{b} - \bar{x}_A)$$

$$- \Theta_X (\dot{\Theta}_{Gk} + \dot{\phi}_k)(\bar{e} + \bar{x}_k)(\tfrac{3}{2} \bar{b} - \bar{x}_A)$$

$$+ \frac{\bar{b}}{2} \Theta_X (\dot{\lambda}_k + \ddot{w}_k) \Big\rangle$$

$$+ c_d \Big\langle - \mu^2 \cos \psi_k \sin \psi_k + 2 \mu^2 v_{k,x} \cos^2 \psi_k \sin \psi_k$$

$$+ 2 \mu (\bar{e} + \bar{x}_k) v_{k,x} \cos^2 \psi_k$$

$$+ 2 \mu (\dot{\bar{v}}_k + (\bar{e} + \bar{x}_k)) \cos \psi_k \sin \psi_k$$

$$- (\bar{e} + \bar{x}_k)(2 \dot{\bar{v}}_k + (\bar{e} + \bar{x}_k)) \cos \psi_k \Big\rangle \Big\} +$$

$$+ m \left\langle \cos\psi_k \left[-\ddot{\bar{v}}_k - 2\dot{\bar{u}}_k + 2\beta_P \dot{\bar{w}}_k + \bar{v}_k + 2\dot{\theta}_z \bar{v}_k \right. \right.$$

$$\left. + \bar{x}_I \cos\theta_{Gk} + 2\dot{\bar{v}}_k \theta_z \right] + \sin\psi_k \left[2\dot{\bar{v}}_k + \ddot{\bar{v}}_k \theta_z \right.$$

$$\left. - \bar{v}_k \theta_z + 2\dot{\bar{v}}_k \dot{\theta}_z + \bar{v}_k \ddot{\theta}_z \right] - \ddot{\bar{r}}_Y + 2\dot{\bar{w}}_k \dot{\theta}_x$$

$$\left. + \bar{w}_k \ddot{\theta}_x + \theta_x \ddot{\bar{w}}_k + \beta_P \ddot{\theta}_x (\bar{e} + \bar{x}_k) \right\rangle \right\} dx$$

$$(E.2)$$

$$P_{RZ} = \Omega^2 R \sum_{k=1}^{N}$$

$$\times \int_0^\ell \left\{ \rho_A \bar{b} \left\{ c_\ell \left\langle \sin^2\psi_k \left[\mu^2 + 2\mu \dot{\bar{r}}_X \right] \right. \right. \right.$$

$$+ \cos\psi_k \sin\psi_k \left[2\mu^2 (\theta_z + v_{k,x}) - 2\mu \dot{\bar{r}}_Y \right]$$

$$+ \cos\psi_k \left[2\dot{\bar{r}}_Y (\bar{e} + \bar{x}_k) - 2\mu(\bar{e} + \bar{x}_k)(\theta_z + v_{k,x}) \right]$$

$$+ \sin\psi_k \left[-2\mu (\dot{\bar{v}}_k + (\bar{e} + \bar{x}_k)) - 2(\bar{e} + \bar{x}_k)(\dot{\theta}_z + \dot{\bar{r}}_X) \right]$$

$$+ (\bar{e} + \bar{x}_k)^2 (1 + 2\dot{\theta}_z) + 2\dot{\bar{v}}_k (\bar{e} + \bar{x}_k) \right\rangle$$

$$+ a \left\langle \mu \dot{\theta}_Y (\bar{b} - \bar{x}_A) \sin^2\psi_k \right.$$

$$+ \mu \dot{\theta}_x (\bar{b} - \bar{x}_A) \cos\psi_k \sin\psi_k + \cos\psi_k \left[\frac{\bar{b}}{2} (-\mu \dot{w}_{k,x} \right.$$

$$+ (\bar{e} + \bar{x}_k)\ddot{\theta}_Y + \mu(\theta_{Gk} + \phi_k)) + \dot{\theta}_x (\bar{e} + \bar{x}_k)(\frac{\bar{b}}{2} - \bar{x}_A) \right]$$

$$+ \sin\psi_k \left[\mu(w_{k,x} + \beta_P)(\frac{3}{2}\bar{b} - \bar{x}_A) + \mu(\dot{\theta}_{Gk} + \dot{\phi}_k)(\frac{3}{2}\bar{b} - \bar{x}_A) \right.$$

$$\left. - (\bar{e} + \bar{x}_k)\ddot{\theta}_x \frac{\bar{b}}{2} - (\bar{e} + \bar{x}_k)\dot{\theta}_Y (\frac{\bar{b}}{2} - \bar{x}_A) \right] +$$

$$+ (w_{k,x} + \beta_P)(\bar{e} + \bar{x}_k)(\bar{b} - \bar{x}_A) + (\dot{\theta}_{Gk} + \dot{\phi}_k)(\bar{e} + \bar{x}_k)(\tfrac{3}{2}\bar{b} - \bar{x}_A)$$

$$+ \frac{\bar{b}}{2}(\mu\dot{\theta}_Y - \dot{\lambda}_k - \ddot{\bar{r}}_Z - \ddot{\overline{w}}_k) \rangle \Big\}$$

$$+ m \langle \cos\psi_k [-2\theta_Y \dot{\bar{v}}_k + \theta_X(\bar{v}_k - \ddot{\bar{v}}_k) - \bar{v}_k\ddot{\theta}_X$$

$$- 2\dot{\bar{v}}_k \dot{\theta}_X - 2\dot{\theta}_Y \bar{v}_k] + \sin\psi_k [\theta_Y(\bar{v}_k - \ddot{\bar{v}}_k)$$

$$+ 2\dot{\bar{v}}_k(\theta_X - \dot{\theta}_Y) - \bar{v}_k(\ddot{\theta}_Y - 2\dot{\theta}_X] - \ddot{\bar{r}}_Z$$

$$- \ddot{\overline{w}}_k \rangle \Big\} dx_k$$

$$(E.3)$$

$$Q_{RX} = \Omega^2 R^2 \sum_{k=1}^{N}$$

$$\times \int_0^\ell \Big\{ \rho_A \bar{b} \sum c_\ell \langle \sin^3\psi_k [(2\mu\dot{\bar{r}}_X + \mu^2)(\bar{e} + \bar{x}_k)$$

$$- \mu^2 \bar{v}_k \theta_Z] + \cos^2\psi_k \sin\psi_k [\mu^2 \overline{w}_k(\beta_P + w_{k,x})$$

$$+ 2\mu^2 \bar{v}_k(\theta_Z + v_{k,x}) - \mu(\bar{e} + \bar{x}_k)\dot{\theta}_Y \overline{w}_k - 2\mu\bar{v}_k \dot{\bar{r}}_Y$$

$$+ \mu^2 \beta_P(w_{k,x} + \beta_P)(\bar{e} + \bar{x}_k) - \mu(\bar{e} + \bar{x}_k)^2 \beta_P \dot{\theta}_Y$$

$$+ 2\mu^2 v_{k,x}(\bar{e} + \bar{x}_k)\theta_Z] + \sin^2\psi_k \cos\psi_k [2\mu\dot{\bar{r}}_X \bar{v}_k$$

$$+ \mu^2 \bar{v}_k + \mu(\bar{e} + \bar{x}_k)\dot{\theta}_X \overline{w}_k + \mu\beta_P(\bar{e} + \bar{x}_k)^2 \dot{\theta}_X$$

$$+ 2(\mu^2(\theta_Z + v_{k,x}) - \mu\dot{\bar{r}}_Y)(\bar{e} + \bar{x}_k) + \mu^2\theta_Z(\bar{e} + \bar{x}_k)]$$

$$+ \cos^2\psi_k [\mu(\bar{e} + \bar{x}_k)\overline{w}_k(\beta_P + w_{k,x}) + \mu\beta_P^2(\bar{e} + \bar{x}_k)^2 +$$

$$- (\bar{e} + \bar{x}_k)^2 \dot{\theta}_Y \bar{w}_k + \mu (\bar{e} + \bar{x}_k)^2 \beta_P w_{k,x}$$

$$- (\bar{e} + \bar{x}_k)^3 \beta_P \dot{\theta}_Y + 2 (\bar{e} + \bar{x}_k) \dot{\bar{r}}_Y \bar{v}_k - 2\mu (\bar{e} + \bar{x}_k)^2 \theta_{\mathbb{Z}} v_{k,x}$$

$$- 2\mu (\bar{e} + \bar{x}_k)(\theta_{\mathbb{Z}} + v_{k,x}) \bar{v}_k] + \sin^2 \psi_k [-2\mu \dot{\bar{v}}_k (\bar{e} + \bar{x}_k)$$

$$- 2 (\bar{e} + \bar{x}_k)^2 (\mu + \mu \dot{\theta}_{\mathbb{Z}} + \dot{\bar{r}}_X) + 2\mu (\bar{e} + \bar{x}_k) \bar{v}_k \theta_{\mathbb{Z}}]$$

$$+ \cos \psi_k \sin \psi_k [(\bar{e} + \bar{x}_k)^2 (\dot{\theta}_X \bar{w}_k + 2\dot{\bar{r}}_Y - 4\mu \theta_{\mathbb{Z}} - 2\mu v_{k,x})$$

$$+ \mu \bar{w}_k (-\mu \theta_Y + \lambda_k + \dot{\bar{r}}_{\mathbb{Z}} + \dot{\bar{w}}_k) - 2 \bar{v}_k (\mu \dot{\bar{v}}_k$$

$$- \mu (\bar{e} + \bar{x}_k) \dot{\theta}_{\mathbb{Z}} - \dot{\bar{r}}_X (\bar{e} + \bar{x}_k) - \mu (\bar{e} + \bar{x}_k))$$

$$+ \beta_P (\bar{e} + \bar{x}_k)((\bar{e} + \bar{x}_k)^2 \dot{\theta}_X - 2\mu^2 \theta_Y + \mu (\lambda_k$$

$$+ \dot{\bar{r}}_{\mathbb{Z}} + \dot{\bar{w}}_k) - 2\mu \dot{\bar{v}}_k \theta_{\mathbb{Z}} (\bar{e} + \bar{x}_k) - \mu^2 w_{k,x} (\bar{e} + \bar{x}_k) \theta_Y]$$

$$+ \cos \psi_k [\bar{v}_k (\bar{e} + \bar{x}_k)((\bar{e} + \bar{x}_k)(1 + 2\dot{\theta}_{\mathbb{Z}}) + 2\bar{v}_k)$$

$$+ \bar{w}_k (\bar{e} + \bar{x}_k)(-\mu \theta_Y + \lambda_k + \dot{\bar{r}}_{\mathbb{Z}} + \dot{\bar{w}}_k)$$

$$+ (\bar{e} + \bar{x}_k)^2 \beta_P (\lambda_k + \dot{\bar{r}}_{\mathbb{Z}} + \dot{\bar{w}}_k) + (\bar{e} + \bar{x}_k)^3 \theta_{\mathbb{Z}}$$

$$+ (\bar{e} + \bar{x}_k)^2 (2\dot{\bar{v}}_k \theta_{\mathbb{Z}} - \mu \theta_Y (2\beta_P + w_{k,x}))]$$

$$+ \sin \psi_k [(\bar{e} + \bar{x}_k)^3 (1 + 2\dot{\theta}_{\mathbb{Z}}) + (\bar{e} + \bar{x}_k)^2 (2\dot{\bar{v}}_k$$

$$- \bar{v}_k \theta_{\mathbb{Z}}) - \mu (\lambda_k + \dot{\bar{w}}_k)(\bar{e} + \bar{x}_k) \theta_Y]$$

$$- (\bar{e} + \bar{x}_k)^2 \theta_Y (\lambda_k + \dot{\bar{w}}_k) >$$

$$+ a < \mu \dot{\theta}_Y (\bar{e} + \bar{x}_k)(\bar{b} - \bar{x}_A) \sin^3 \psi_k \quad +$$

$$+ \sin^2\psi_k \cos\psi_k \left[\mu \left(\bar{v}_k \dot{\theta}_Y + \dot{\theta}_X (\bar{e}+\bar{x}_k) \right) (\bar{b}-\bar{x}_A) \right]$$

$$+ \cos^2\psi_k \sin\psi_k \left[\mu \dot{\theta}_X \bar{v}_k (\bar{b}-\bar{x}_A) \right]$$

$$+ \cos^2\psi_k \left[\frac{\bar{b}}{2} \bar{v}_k \left(-\mu \dot{w}_{k,x} + (\bar{e}+\bar{x}_k) \ddot{\theta}_Y + \mu (\theta_{Gk} \right. \right.$$

$$\left. + \phi_k) \right) + \mu \theta_z (\bar{e}+\bar{x}_k) \frac{\bar{b}}{2} \left(\theta_{Gk} + \phi_k - \dot{w}_{k,x} \right)$$

$$+ \bar{v}_k \dot{\theta}_X (\bar{e}+\bar{x}_k) \left(\frac{\bar{b}}{2} - \bar{x}_A \right) \right]$$

$$+ \sin^2\psi_k \left[\mu (w_{k,x} + \beta_P) \left(\frac{3}{2}\bar{b} - \bar{x}_A \right) (\bar{e}+\bar{x}_k) \right.$$

$$+ \mu (\dot{\theta}_{Gk} + \dot{\phi}_k)(\bar{e}+\bar{x}_k) \left(\frac{3}{2}\bar{b} - \bar{x}_A \right) - (\bar{e}+\bar{x}_k)^2 \ddot{\theta}_X \frac{\bar{b}}{2}$$

$$+ \dot{\theta}_Y (\bar{e}+\bar{x}_k)^2 \left(\frac{\bar{b}}{2} - \bar{x}_A \right) \right] + \cos\psi_k \sin\psi_k \left[\frac{b}{2} (\bar{e}+\bar{x}_k)(-\mu \dot{w}_{k,x} \right.$$

$$+ (\bar{e}+\bar{x}_k) \ddot{\theta}_Y + \mu (\theta_{Gk} + \phi_k)) + \mu \bar{v}_k \left(\frac{3}{2}\bar{b} - \bar{x}_A \right) (\dot{\theta}_{Gk}$$

$$+ \dot{\phi}_k + w_{k,x} + \beta_P) - (\bar{e}+\bar{x}_k) \bar{v}_k \ddot{\theta}_X \frac{\bar{b}}{2}$$

$$+ \dot{\theta}_Y (\bar{e}+\bar{x}_k) \left(\frac{\bar{b}}{2} - \bar{x}_A \right) \bar{v}_k + \dot{\theta}_X (\bar{e}+\bar{x}_k)^2 \left(\frac{\bar{b}}{2} - \bar{x}_A \right) \right]$$

$$+ \cos\psi_k \left[(w_{k,x} + \beta_P)(\bar{e}+\bar{x}_k)(\bar{b}-\bar{x}_A) (\bar{v}_k + \theta_z (\bar{e}+\bar{x}_k)) \right.$$

$$+ (\dot{\theta}_{Gk} + \dot{\phi}_k)(\bar{e}+\bar{x}_k) \left(\frac{3}{2}\bar{b} - \bar{x}_A \right) (\bar{v}_k + \theta_z (\bar{e}+\bar{x}_k))$$

$$+ \frac{\bar{b}}{2} (\mu \dot{\theta}_Y - \dot{\lambda}_k - \ddot{r}_z - \ddot{w}_k) \bar{v}_k - \frac{\bar{b}}{2} \theta_z (\bar{e}+\bar{x}_k)(\dot{\lambda}_k$$

$$+ \ddot{w}_k) \right] + \left[(\bar{e}+\bar{x}_k)^2 ((w_{k,x} + \beta_P)(\bar{b}-\bar{x}_A) \right.$$

$$+ (\dot{\theta}_{Gk} + \dot{\phi}_k) \left(\frac{3}{2}\bar{b} - \bar{x}_A \right)) + \frac{\bar{b}}{2} (\bar{e}+\bar{x}_k) (\mu \theta_Y - \dot{\lambda}_k$$

$$\left. - \ddot{r}_z - \ddot{w}_k) \right] \sin\psi_k \bigg\rangle +$$

$$+ c_d \Big< \mu^2 (\overline{w}_k + \beta_p (\bar{e} + \bar{x}_k)) \sin^2 \psi_k \cos \psi_k$$

$$- 2\mu (\bar{e} + \bar{x}_k)(\overline{w}_k + \beta_p (\bar{e} + \bar{x}_k)) \cos \psi_k \sin \psi_k$$

$$+ (\bar{e} + \bar{x}_k)^2 (\overline{w}_k + \beta_p (\bar{e} + \bar{x}_k)) \cos \psi_k \Big>$$

$$+ \Big< c_\ell \bar{x}_A + \frac{\bar{b}}{2} c_{mac} \Big> \Big< \mu^2 \cos \psi_k \sin^2 \psi_k$$

$$- 2\mu (\bar{e} + \bar{x}_k) \sin \psi_k + (\bar{e} + \bar{x}_k)^2 \Big> \Big\}$$

$$+ m \Big< -\overline{v}_k (2\dot{\theta}_X - \ddot{\theta}_Y)(\bar{e} + \bar{x}_k) \cos^2 \psi_k$$

$$+ \sin^2 \psi_k \Big[(\bar{e} + \bar{x}_k)(\overline{v}_k (2\dot{\theta}_X - \ddot{\theta}_Y) - 2\dot{\overline{v}}_k \dot{\theta}_Y)$$

$$- (2\dot{\theta}_Y + \ddot{\theta}_X)(\bar{e} + \bar{x}_k)^2 \Big] + \cos \psi_k \sin \psi_k \Big[-2\dot{\overline{v}}_k \dot{\theta}_X$$

$$- 2\overline{v}_k (2\dot{\theta}_Y + \ddot{\theta}_X)(\bar{e} + \bar{x}_k) - (2\dot{\theta}_X - \ddot{\theta}_Y)(\bar{e} + \bar{x}_k)^2 \Big]$$

$$+ \cos \psi_k \Big[-\overline{v}_k (\ddot{\bar{r}}_Z + \ddot{\overline{w}}_k + \beta_p (\bar{e} + \bar{x}_k))$$

$$+ \overline{w}_k (\ddot{\overline{v}}_k + (\bar{e} + \bar{x}_k)\ddot{\theta}_Z - \overline{v}_k) + \ddot{\overline{v}}_k \beta_p (\bar{e} + \bar{x}_k)$$

$$- \theta_Z (\bar{e} + \bar{x}_k)(\ddot{\overline{w}}_k + \overline{w}_k) \Big] + \sin \psi_k \Big[-\overline{w}_k (2\dot{\overline{v}}_k$$

$$+ (\bar{e} + \bar{x}_k)(1 + 2\dot{\theta}_Z)) + (\bar{e} + \bar{x}_k)(\ddot{\bar{r}}_Z + \ddot{\overline{w}}_k$$

$$+ 2\beta_p (\dot{\overline{v}}_k + \dot{\theta}_Z (\bar{e} + \bar{x}_k)) \Big] - \ddot{\overline{v}}_k \theta_Y (\bar{e} + \bar{x}_k)$$

$$+ \ddot{\bar{r}}_Y (\beta_p (\bar{e} + \bar{x}_k) + \overline{w}_k) \Big> \Big\} \, dx_k$$

$$(E.4)$$

$$Q_{RY} = \Omega^2 R^2 \sum_{k=1}^{N}$$

$$\times \int_0^\ell \Big\{ \rho_A \bar{b} \sum c_\ell < \sin^3 \psi_k \Big[\mu(\bar{e}+\bar{x}_k)\dot{\theta}_X \overline{w}_k$$

$$+ \mu^2 (\bar{v}_k + (\bar{e}+\bar{x}_k)\theta_z) + 2\mu \bar{v}_k \dot{\bar{r}}_X \Big]$$

$$+ \cos^2 \psi_k \sin \psi_k \Big[2\mu(\bar{e}+\bar{x}_k)(\dot{\bar{r}}_Y - \mu(\theta_z + v_{k,x}) \Big]$$

$$+ \sin^2 \psi_k \cos \psi_k \Big[-2\mu \dot{\bar{r}}_X (\bar{e}+\bar{x}_k) - \mu^2 (\bar{e}+\bar{x}_k)$$

$$+ \mu^2 \bar{v}_k \theta_z + 2\mu^2 \theta_z ((\bar{e}+\bar{x}_k)v_{k,x} + \bar{v}_k)$$

$$+ \mu^2 \overline{w}_k (\beta_P + w_{k,x}) - \mu \overline{w}_k \dot{\theta}_Y (\bar{e}+\bar{x}_k)$$

$$+ 2\mu \bar{v}_k (\mu v_{k,x} - \dot{\bar{r}}_Y) \Big] + \cos^2 \psi_k \Big[-2\dot{\bar{r}}_Y (\bar{e}+\bar{x}_k)^2$$

$$+ 2\mu(\bar{e}+\bar{x}_k)^2 (\theta_z + v_{k,x}) \Big] + \sin^2 \psi_k \Big[-2\dot{\bar{r}}_z \bar{v}_k (\bar{e}+\bar{x}_k)$$

$$-2\mu\bar{v}_k (\dot{\bar{v}}_k + (\bar{e}+\bar{x}_k)(1+2\dot{\theta}_z)) - 2\mu \dot{\bar{v}}_k \theta_z (\bar{e}+\bar{x}_k)$$

$$-2\mu(\bar{e}+\bar{x}_k)^2 \theta_z + (\bar{e}+\bar{x}_k)^2 \dot{\theta}_X (\mu\beta_P + \overline{w}_k)$$

$$+ \mu \overline{w}_k (-\mu\theta_Y + \lambda_k + \dot{\bar{r}}_z + \dot{\overline{w}}_k) \Big]$$

$$+ \cos \psi_k \sin \psi_k \Big[2\mu(\bar{e}+\bar{x}_k)(\dot{\bar{v}}_k + (\bar{e}+\bar{x}_k)(1+2\dot{\theta}_z))$$

$$+ 2\dot{\bar{r}}_X (\bar{e}+\bar{x}_k)^2 - 2\mu\theta_z \bar{v}_k (\bar{e}+\bar{x}_k) - \mu\beta_P (\bar{e}+\bar{x}_k)^2 \dot{\theta}_Y$$

$$+ \mu^2 (\beta_P + w_{k,x})(\bar{e}+\bar{x}_k)(\beta_P + \theta_X)$$

$$- (\bar{e}+\bar{x}_k)^2 (2\mu v_{k,x}\theta_z + \dot{\theta}_Y \overline{w}_k) + \mu \overline{w}_k (\bar{e}+\bar{x}_k)(\beta_P$$

$$+ w_{k,x}) + 2(\bar{e}+\bar{x}_k)\bar{v}_k (\dot{\bar{r}}_Y - \mu(\theta_z + v_{k,x})) \Big] +$$

$$+ \cos\psi_k \left[-(\bar{e}+\bar{x}_k)^3 (1 + 2\dot{\theta}_z + \beta_P \dot{\theta}_Y) \right.$$

$$+ (\bar{e}+\bar{x}_k)^2 (-2\dot{\bar{v}}_k + \mu\beta_P(\beta_P + w_{k,x} + \theta_x) + w_{k,x}\mu\theta_X$$

$$\left. + \bar{v}_k \theta_z) \right] + \sin\psi_k \left[\beta_P (\bar{e}+\bar{x}_k)^3 \dot{\theta}_X - \mu^2 \theta_Y \beta_P (\bar{e}+\bar{x}_k) \right.$$

$$+ \mu\beta_P(\bar{e}+\bar{x}_k)(\lambda_k + \dot{r}_z + \dot{\bar{w}}_k) + (\bar{e}+\bar{x}_k)\bar{v}_k(2\dot{\bar{v}}_k$$

$$+ (\bar{e}+\bar{x}_k)(1 + 2\dot{\theta}_z)) + (\bar{e}+\bar{x}_k)^2 \theta_z (2\dot{\bar{v}}_k + (\bar{e}+\bar{x}_k))$$

$$+ \mu(\bar{e}+\bar{x}_k)\theta_X(\lambda_k + \dot{\bar{w}}_k) - \mu(\bar{e}+\bar{x}_k)\bar{w}_k \theta_Y$$

$$\left. + \bar{w}_k(\bar{e}+\bar{x}_k)(\lambda_k + \dot{r}_z + \dot{\bar{w}}_k) \right]$$

$$+ \beta_P(\bar{e}+\bar{x}_k)^2(-\mu\theta_Y + \lambda_k + \dot{r}_z + \dot{\bar{w}}_k)$$

$$\left. + (\bar{e}+\bar{x}_k)^2 \theta_X (\lambda_k + \dot{\bar{w}}_k) \right\rangle$$

$$+ a \left\langle \mu\dot{\theta}_Y(\bar{b}-\bar{x}_A)\bar{v}_k \sin^3\psi_k \right.$$

$$+ \cos\psi_k \sin^2\psi_k \left[\mu(\bar{b}-\bar{x}_A)(\bar{v}_k\dot{\theta}_X - \dot{\theta}_Y(\bar{e}+\bar{x}_k)) \right]$$

$$+ \cos^2\psi_k \sin\psi_k \left[-\mu\dot{\theta}_X(\bar{e}+\bar{x}_k)(\bar{b}-\bar{x}_A) \right]$$

$$+ \cos^2\psi_k \left[-\frac{\bar{b}}{2}(\bar{e}+\bar{x}_k)(-\mu\dot{w}_{k,x} + (\bar{e}+\bar{x}_k)\ddot{\theta}_Y \right.$$

$$\left. + \mu(\theta_{Gk} + \phi_k)) - (\bar{e}+\bar{x}_k)^2 \dot{\theta}_X(\frac{\bar{b}}{2} - \bar{x}_A) \right]$$

$$+ \sin^2\psi_k \left[\mu(\bar{e}+\bar{x}_k)\theta_z(w_{k,x} + \beta_P)(\frac{3}{2}\bar{b} - \bar{x}_A) \right.$$

$$+ \mu(\dot{\theta}_{Gk} + \dot{\phi}_k)\theta_z(\bar{e}+\bar{x}_k)(\frac{3}{2}\bar{b} - \bar{x}_A) - (\bar{e}+\bar{x}_k)\ddot{\theta}_X \frac{\bar{b}}{2}\bar{v}_k$$

$$+ \mu(w_{k,x} + \beta_P + \dot{\theta}_{Gk} + \dot{\phi}_k)\bar{v}_k(\frac{3}{2}\bar{b} - \bar{x}_A) +$$

$$+ \dot{\theta}_Y \, (\bar{e}+\bar{x}_k)\left(\tfrac{\bar{b}}{2}-\bar{x}_A\right)\bar{v}_k\Big]$$

$$+ \cos\psi_k \sin\psi_k \Big[-\mu\,(\bar{e}+\bar{x}_k)\left(\tfrac{3}{2}\bar{b}-\bar{x}_A\right)(w_{k,x}+\beta_P+\dot{\theta}_{Gk}+\dot{\phi}_k)$$

$$+ (\bar{e}+\bar{x}_k)^2\,\tfrac{\bar{b}}{2}\,\ddot{\theta}_X + \mu\,(\bar{e}+\bar{x}_k)\,\theta_Z\,\tfrac{\bar{b}}{2}\,(\theta_{Gk}+\phi_k-\dot{w}_{k,x})$$

$$+ \bar{v}_k\,\tfrac{\bar{b}}{2}\,(-\mu\,w_{k,x} + (\bar{e}+\bar{x}_k)\ddot{\theta}_Y + \mu\,(\theta_{Gk}+\phi_k))$$

$$- (\bar{e}+\bar{x}_k)^2\,\dot{\theta}_Y\left(\tfrac{\bar{b}}{2}-\bar{x}_A\right) + (\bar{e}+\bar{x}_k)\bar{v}_k\,\dot{\theta}_X\left(\tfrac{\bar{b}}{2}-\bar{x}_A\right)\Big]$$

$$+ \cos\psi_k\Big[-(\bar{e}+\bar{x}_k)^2(w_{k,x}+\beta_P+\dot{\theta}_{Gk}+\dot{\phi}_k)\left(\tfrac{3}{2}\bar{b}-\bar{x}_A\right)$$

$$- \tfrac{\bar{b}}{2}\,(\mu\dot{\theta}_Y-\dot{\lambda}_k-\ddot{r}_Z-\ddot{w}_k)(\bar{e}+\bar{x}_k)\Big]$$

$$+ \sin\psi_k\Big[(\bar{e}+\bar{x}_k)^2\,\theta_Z\left(\tfrac{3}{2}\bar{b}-\bar{x}_A\right)(w_{k,x}+\beta_P+\dot{\theta}_{Gk}+\dot{\phi}_k)$$

$$- \tfrac{\bar{b}}{2}\,(\bar{e}+\bar{x}_k)\,\theta_Z\,(\dot{\lambda}_k+\ddot{w}_k) + (\bar{e}+\bar{x}_k)\bar{v}_k\,(\bar{b}-\bar{x}_A)(w_{k,x}$$

$$+ \beta_P) + (\bar{e}+\bar{x}_k)(\dot{\theta}_{Gk}+\dot{\phi}_k)\bar{v}_k\left(\tfrac{3}{2}\bar{b}-\bar{x}_A\right)$$

$$+ \tfrac{\bar{b}}{2}\,(\mu\dot{\theta}_Y-\dot{\lambda}_k-\ddot{r}_Z-\ddot{w}_k)\bar{v}_k\Big]\Big\rangle$$

$$+ c_d\Big\langle \mu^2\,\bar{w}_k\,\sin^3\psi_k + \sin^2\psi_k\Big[\mu^2\beta_P(\bar{e}+\bar{x}_k)$$

$$- 2\mu\,\bar{w}_k(\bar{e}+\bar{x}_k)\Big] + \sin\psi_k\Big[(\bar{e}+\bar{x}_k)^2(\bar{w}_k-2\mu\beta_P)\Big]$$

$$+ \beta_P(\bar{e}+\bar{x}_k)^3\Big\rangle$$

$$+ \Big\langle c_\ell\,\bar{x}_A + \tfrac{\bar{b}}{2}\,c_{mac}\Big\rangle\Big\langle \mu^2\sin^3\psi_k$$

$$- 2\mu\,(\bar{e}+\bar{x}_k)\sin^2\psi_k + (\bar{e}+\bar{x}_k)^2\sin\psi_k\Big\rangle$$

$$+ m\Big\langle \cos^2\psi_k\Big[(\bar{e}+\bar{x}_k)(\bar{v}_k(\ddot{\theta}_X+2\ddot{\theta}_Y)+2\dot{\bar{v}}_k\dot{\theta}_X +$$

$$+ (\bar{e}+\bar{x}_k)(2\dot{\theta}_X - \ddot{\theta}_Y))] - \bar{v}_k(\bar{e}+\bar{x}_k)(2\dot{\theta}_Y + \ddot{\theta}_X)\sin^2\psi_k$$

$$+ \cos\psi_k \sin\psi_k [(\bar{e}+\bar{x}_k)(2\dot{\theta}_Y\bar{v}_k + \bar{v}_k\ddot{\theta}_Y - 2\bar{v}_k\dot{\theta}_X$$

$$+ (2\dot{\theta}_Y + \ddot{\theta}_X)(\bar{e}+\bar{x}_k)) - \bar{v}_k(2\dot{\theta}_X - \ddot{\theta}_Y)(\bar{e}+\bar{x}_k)]$$

$$+ \cos\psi_k [\overline{w}_k(2\dot{\bar{v}}_k + (\bar{e}+\bar{x}_k)(1+2\dot{\theta}_z)$$

$$+ (\bar{e}+\bar{x}_k)(\ddot{\overline{w}}_k + 2\dot{\bar{v}}_k\beta_p)] + \sin\psi_k[\ddot{\bar{v}}_k\beta_p(\bar{e}+\bar{x}_k)$$

$$+ \overline{w}_k(\ddot{\bar{v}}_k - \bar{v}_k + (\bar{e}+\bar{x}_k)\ddot{\theta}_z) - \bar{v}_k(\ddot{\bar{r}}_z + \ddot{\overline{w}}_k$$

$$+ \beta_p(\bar{e}+\bar{x}_k)) - (\overline{w}_k + \ddot{\overline{w}}_k)\theta_z(\bar{e}+\bar{x}_k)]$$

$$- \overline{w}_k\ddot{\bar{r}}_X + (\bar{e}+\bar{x}_k)(\ddot{\bar{v}}_k\theta_X - \beta_p\ddot{\bar{r}}_X)\rangle \} dx_k$$

$$(E.5)$$

$$Q_{RZ} = \Omega^2 R^2 \sum_{k=1}^{N}$$

$$\times \int_0^\ell \{ \rho_A\bar{b} \xi c_\ell \langle \mu^2(\bar{v}_k\theta_X - (\bar{e}+\bar{x}_k)\theta_Y)\sin^3\psi_k$$

$$+ \cos\psi_k \sin^2\psi_k [-\mu^2(v_k\theta_Y + 2v_{k,x}(\bar{e}+\bar{x}_k)\theta_Y$$

$$+ (\bar{e}+\bar{x}_k)\theta_X] - 2\mu^2 v_{k,x}\theta_X(\bar{e}+\bar{x}_k)\cos^2\psi\sin\psi_k$$

$$+ \cos^2\psi_k [-(\bar{e}+\bar{x}_k)\mu^2\theta_z(\beta_p + w_{k,x})$$

$$- (\mu^2 v_{k,x}(\beta_p + w_{k,x}) - \mu v_{k,x}(\bar{e}+\bar{x}_k)\dot{\theta}_Y$$

$$+ \mu\dot{\bar{r}}_Y(\beta_p + w_{k,x}))(\bar{e}+\bar{x}_k) + 2\mu v_{k,x}\theta_X(\bar{e}+\bar{x}_k)^2] +$$

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$$+ \sin^2 \psi_k \left[-(\bar{e}+\bar{x}_k)(\mu(\bar{e}+\bar{x}_k)\dot{\theta}_X - \mu\beta_p \ddot{r}_Y \right.$$

$$- \mu^2 v_{k,x} w_{k,x} + 2\mu \bar{v}_k \theta_X - 2\mu \dot{v}_k \theta_Y)$$

$$+ 2\mu(\bar{e}+\bar{x}_k)^2 \theta_Y + \mu^2 \beta_p \bar{v}_k \Big]$$

$$+ \cos\psi_k \sin\psi_k \left[2\mu(\bar{e}+\bar{x}_k)\bar{v}_k \theta_Y \right.$$

$$+ 2\mu(\bar{e}+\bar{x}_k)^2 v_{k,x} \theta_Y - (\bar{e}+\bar{x}_k)(\mu^2(\beta_p + w_{k,x})$$

$$- \mu(\bar{e}+\bar{x}_k)\dot{\theta}_Y - \mu\beta_p \dot{r}_X + \mu(\bar{e}+\bar{x}_k)\dot{\theta}_X v_{k,x})$$

$$+ 2\mu\theta_X(\bar{e}+\bar{x}_k)(\dot{v}_k + (\bar{e}+\bar{x}_k)) \Big]$$

$$+ \cos\psi_k \left[-(\bar{e}+\bar{x}_k)^2(\theta_Y \bar{v}_k + (\bar{e}+\bar{x}_k)\theta_X - 2\dot{v}_k \theta_X) \right.$$

$$- (\bar{e}+\bar{x}_k)(\mu\theta_Z(\lambda_k + \dot{w}_k) + \mu v_{k,x}(-\mu\theta_Y + \lambda_k$$

$$+ \dot{r}_Z + \dot{w}_k) + \dot{r}_Y(\lambda_k + \dot{w}_k) + \mu(\beta_p + w_{k,x})(\dot{v}_k$$

$$+ (\bar{e}+\bar{x}_k)(1+\dot{\theta}_Z)) - (\bar{e}+\bar{x}_k)\dot{\theta}_Y(\dot{v}_k + (\bar{e}+\bar{x}_k))) \Big]$$

$$+ \sin\psi_k \left[-2\mu\bar{v}_k\beta_p(\bar{e}+\bar{x}_k) - (\bar{e}+\bar{x}_k)^2\theta_Y \right.$$

$$+ (\bar{e}+\bar{x}_k)^2(\bar{v}_k\theta_X - 2\dot{v}_k\theta_Y - \dot{v}_k\dot{\theta}_X)$$

$$- (\bar{e}+\bar{x}_k)((\bar{e}+\bar{x}_k)^2\dot{\theta}_X + 2\mu(\bar{e}+\bar{x}_k)v_{k,x}w_{k,x}$$

$$- \mu^2\theta_Y + \mu(\lambda_k + \dot{r}_Z + \dot{w}_k) - \lambda_k\dot{r}_Z$$

$$- \dot{r}_Z\dot{w}_k) \Big] - (\bar{e}+\bar{x}_k)((\dot{v}_k + (\bar{e}+\bar{x}_k))(-\mu\theta_Y + \lambda_k$$

$$+ \dot{r}_Z + \dot{w}_k) + (\bar{e}+\bar{x}_k)\dot{\theta}_Z(\lambda_k + \dot{w}_k) - v_{k,x}w_{k,x}(\bar{e}+\bar{x}_k)^2) +$$

$$+ \beta_P \bar{v}_k (\bar{e}+\bar{x}_k)^2 \rangle$$

$$+ a \langle \cos^2\psi_k [\mu \frac{\bar{b}}{2} \theta_X (\bar{e}+\bar{x}_k)(\dot{w}_{k,x} - \theta_{Gk} - \phi_k)]$$

$$+ \sin^2\psi_k [-\mu(\bar{e}+\bar{x}_k)\theta_Y (\tfrac{3}{2}\bar{b} - \bar{x}_A)(w_{k,x} + \beta_P$$

$$+ \dot{\theta}_{Gk} + \dot{\phi}_k)] + \cos\psi_k \sin\psi_k [-\mu(\bar{e}+\bar{x}_k)\theta_X (\tfrac{3}{2}\ddot{b}$$

$$-\bar{x}_A)(w_{k,x} + \beta_P + \dot{\theta}_{Gk} + \dot{\phi}_k) + \mu\theta_Y(\bar{e}+\bar{x}_k)\tfrac{\bar{b}}{2}(\dot{w}_{k,x}$$

$$- \theta_{Gk} - \phi_k)] + \cos\psi_k [\theta_X(\bar{e}+\bar{x}_k)\tfrac{\bar{b}}{2}(\dot{\lambda}_k + \dddot{w}_k)$$

$$- \theta_X(\bar{e}+\bar{x}_k)^2 (\tfrac{3}{2}\bar{b} - \bar{x}_A)(w_{k,x} + \beta_P + \dot{\theta}_{Gk} + \dot{\phi}_k)]$$

$$+ \sin\psi_k [-\theta_Y(\bar{e}+\bar{x}_k)^2 (\tfrac{3}{2}\bar{b} - \bar{x}_A)(w_{k,x} + \beta_P + \dot{\theta}_{Gk} + \dot{\phi}_k)$$

$$+ \theta_Y \tfrac{\bar{b}}{2}(\bar{e}+\bar{x}_k)(\dot{\lambda}_k + \dddot{w}_k)] \rangle$$

$$+ c_d \langle -\mu^2 (\bar{e}+\bar{x}_k)\sin^2\psi_k - 2\mu^2 v_{k,x}(\bar{e}+\bar{x}_k)\cos\psi_k \sin\psi_k$$

$$+ 2\mu(\bar{e}+\bar{x}_k)^2 v_{k,x}\cos\psi_k + 2\mu(\bar{e}+\bar{x}_k)((\bar{e}+\bar{x}_k)$$

$$+ \dot{\bar{v}}_k)\sin\psi_k - (\bar{e}+\bar{x}_k)^3 - 2\dot{\bar{v}}_k(\bar{e}+\bar{x}_k)^2 \rangle \}$$

$$+ m \langle \cos\psi_k [(\bar{e}+\bar{x}_k)(\theta_X(\bar{w}_k + \dddot{w}_k) + 2\dot{\bar{w}}_k\dot{\theta}_X$$

$$+ \bar{w}_k\ddot{\theta}_X) + \bar{v}_k\ddot{r}_X] + \sin\psi_k [(\bar{e}+\bar{x}_k)(2\dot{\bar{w}}_k\dot{\theta}_Y$$

$$+ \bar{w}_k(\theta_Y + \ddot{\theta}_Y)) + \bar{v}_k\ddot{r}_Y] + (\bar{e}+\bar{x}_k)(-\ddot{\bar{v}}_k$$

$$- 2\dot{\bar{u}}_k + 2\beta_P\bar{w}_k - (\bar{e}+\bar{x}_k)\ddot{\theta}_z) - 2\bar{v}_k\dot{\bar{v}}_k \rangle \} dx_k$$

$$(E.6)$$