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SHUTTLE PROGRAM

LYAPUNOV STABILITY AND ITS APPLICATION TO SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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1.0 INTRODUCTION

Two of the more influential factors in the study of numerical integration of a system of differential equations are mathematical (Lyapunov) stability and numerical stability. Some of the confusion surrounding these concepts seems to result from certain similarities; specifically, factors such as (1) both types of stability are related to the behavior of the eigenvalues of the system under analysis, (2) similar terminology, (3) interrelationships between the concepts, and (4) instability of either type frequently results in an unacceptable loss of accuracy of numerical approximations.

Lyapunov stability is concerned with the qualitative behavior of the solutions of a vector differential system

$$\dot{\underline{y}} = \underline{f}(t,\underline{y})$$

in some neighborhood of a specific solution $\underline{y}_0(t)$ of the same system. Appropriate behavior on some infinite interval $t_0 \leq t < \infty$ will result in $\underline{y}_0(t)$ being classified as stable, unstable, asymptotically stable, or one of several additional categories. Lyapunov stability is a global property of a particular solution (determined by the nature of its relationship to solutions in its immediate vicinity).

A useful technique in the study of the stability of a particular solution $\underline{y}_0(t)$ is to linearize the function $\underline{f}(t,\underline{y})$ about the solution $\underline{y}_0(t)$; i.e., to translate the origin to $\underline{y}_0(t)$. If the nonlinear perturbation is admissible in some sense, then the stability nature of the linear system can be transferred to the solution $\underline{y}_0(t)$. The eigenvalues of the linear system are characteristic for the classification of its solutions (with regard to Lyapunov stability).

This report provides an outline and a brief introduction to some of the concepts and implications of Lyapunov stability theory. Various aspects of the theory are illustrated by the inclusion of eight examples, including the Cartesian coordinate equations of the two-body problem, linear and nonlinear (Van der Pol's equation) oscillatory systems, and the linearized Kustaanheimo-Stiefel element equations for the unperturbed two-body problem. Lyapunov's direct method was not considered, nor was an attempt made to present the most relaxed version of a particular theorem. For additional information at the introductory level, see references 1 and 2. At the advanced level, references 3 and 4 provide an extensive survey of the field.

It is anticipated that this report will provide contrast to a second report on numerical stability.

2.0 LYAPUNOV STABILITY AND ITS GEOMETRY

Consider the concept of Lyapunov stability for arbitrary systems

$$\dot{\underline{y}} = \underline{f}\,(t,\underline{y}) \tag{1}$$

where

$$\underline{y} = (y_1(t), \ldots, y_n(t))^T$$

$$\dot{\underline{y}} \equiv \frac{d\underline{y}}{dt}$$

and

$$\underline{f}(t,\underline{y}) = (f_1(t,\underline{y}), \ldots, f_n(t,\underline{y}))^T$$

It is assumed that $\underline{f}$ is defined and continuous in some tube in $t \times R^n$ space.

$$T = \{(t,\underline{y}) : t_o \leq t < \infty,\ ||\underline{y}|| < a_o\}$$

where a_o is a given positive constant, t_o is an initial time, and $||\ ||$ denotes the euclidean norm

$$||\underline{y}|| = \sqrt{\sum_{i=1}^{n} {y_i}^2} \tag{2}$$

or some equivalent norm such as

$$||\underline{y}|| = \max_{i=1,\ldots,n} \{|y_i|\} \tag{3}$$

The continuity condition on $\underline{f}$ ensures the existence of solutions to initial value problems associated with equation (1). In many cases, $a_o = \infty$ for the tube T.

Let $\underline{y}_o(t) \equiv \underline{y}(t;t_o,\underline{y}_o)$ denote some solution to the initial value problem

$$\left.\begin{aligned}\dot{\underline{y}} &= \underline{f}\,(t,\underline{y})\\ \underline{y}(t_o) &= \underline{y}_o\end{aligned}\right\} \tag{4}$$

then

Definition 1. A solution $\underline{y}_o(t)$ is said to be stable in the sense of Lyapunov if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\underline{y}_1$ is any solution of equation (1) with $\underline{y}_1(t_o) = \underline{y}_1$, and the number

$$||\underline{y}_1 - \underline{y}_o|| < \delta \tag{S1}$$

then

$$||\underline{y}_1(t;t_o,\underline{y}_1) - \underline{y}_o(t;t_o,\underline{y}_o)|| < \varepsilon \tag{S2}$$

for all t, $t \geq t_o$. If $\underline{y}_o(t)$ is not stable, then it is unstable. If $\underline{y}_o(t)$ is stable, and if the additional condition

$$\lim_{t\to\infty}||\underline{y}_1(t;t_o,\underline{y}_1) - \underline{y}_o(t;t_o,\underline{y}_o)|| = 0 \tag{S3}$$

is valid, then $\underline{y}_o(t)$ is asymptotically stable.

Discussion. In general, stability or asymptotic stability is a property of a single solution $\underline{y}_o(t;t_o,\underline{y}_o)$ and requires a certain behavior over an infinite time interval $[t_o,\infty)$. In geometric terms, consider a tube about the solution $\underline{y}_o(t;t_o,\underline{y}_o)$ of radius $\varepsilon > 0$ (ε arbitrary).

Figure 1 represents a Lyapunov tube about a vector $\underline{y}_o(t;t_o,\underline{y}_o)$ in $t \times R^2$.

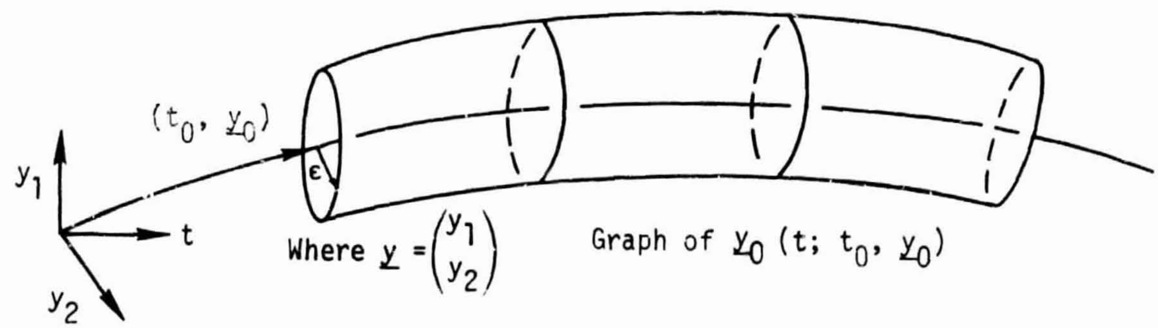


Figure 1. - A Lyapunov tube.

If $\underline{y}_o(t;t_o,\underline{y}_o)$ is stable, then there exists a smaller neighborhood of radius δ of $\underline{y}_o$ such that any solutions $\underline{y}_1(t;t_o,\underline{y}_1)$ starting in that smaller neighborhood will never stray farther than ε distance from $\underline{y}_o(t;t_o,\underline{y}_o)$ (i.e., solutions starting in the δ disk never leave the ε tube). (See figure 2.) Asymptotic stability merely ensures that all solutions starting in some δ neighborhood of $\underline{y}_o$ tend to the solution $\underline{y}_o(t;t_o,\underline{y}_o)$ as $t \to \infty$.

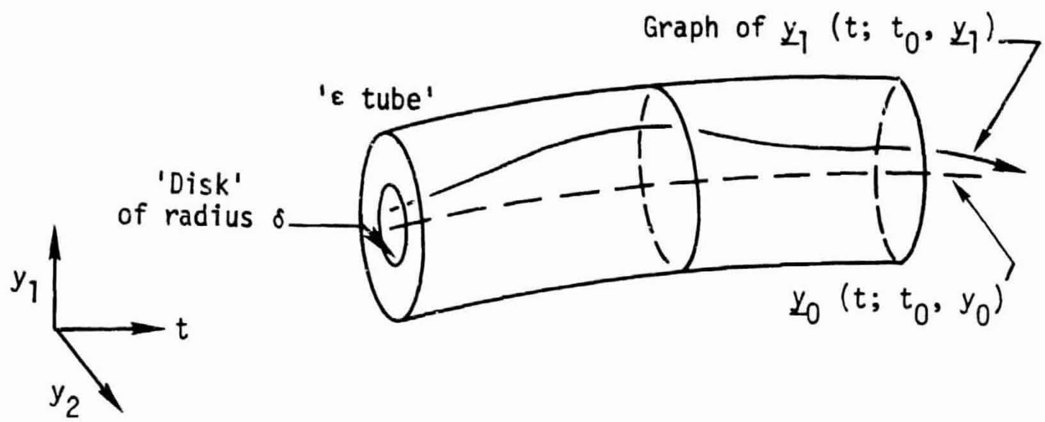


Figure 2.- A second trajectory within a Lyapunov tube.

The first example will establish that circular trajectories of the Cartesian coordinate system formulation of the two-body problem are unstable.

<u>Example 1.</u> For the two-body problem with masses m and M subject to the universal law of gravitation $\frac{k^2mM}{r^2}$ where $\underline{r} = (y_1,y_2,y_3)^T$, $r = |\underline{r}|$ (all functions of time t), the resulting differential equation is

$$\ddot{\underline{r}} + \frac{\mu}{r^3}\,\underline{r} = 0 \tag{E1.1}$$

where

$$\mu = k^2\,(M + m) = \text{gravitational parameter}$$

As a first-order vector system, equation (E1.1) can be expressed as

$$\begin{pmatrix} \dot{\underline{r}} \\ \\ \underline{v} \end{pmatrix} \equiv \begin{pmatrix} \dot{\underline{r}} \\ \\ \dot{\underline{v}} \end{pmatrix} = \begin{pmatrix} \underline{v} \\ \\ -\dfrac{\mu}{r^3}\underline{r} \end{pmatrix} = \begin{pmatrix} 0_{3x3} & I_{3x3} \\ \\ -\dfrac{\mu}{r^3} I_{3x3} & 0_{3x3} \end{pmatrix} \begin{pmatrix} \underline{r} \\ \\ \underline{v} \end{pmatrix} \qquad \text{(E1.2)}$$

where

$$\underline{v} = \dot{\underline{r}} = (\dot{y}_1(t), \dot{y}_2(t), \dot{y}_3(t))^T$$

0_{3x3} is the 3 by 3 zero matrix, and I_{3x3} is the 3 by 3 identity matrix. System equation (E1.2) is clearly nonlinear, and a well-posed initial value problem would require initial positions and initial velocities; i.e.,

$$\begin{pmatrix} \dot{\underline{r}} \\ \\ \underline{v} \end{pmatrix} = \begin{pmatrix} \underline{v} \\ \\ -\dfrac{\mu}{r^3}\underline{r} \end{pmatrix}, \quad \begin{pmatrix} \underline{r} \\ \\ \underline{v} \end{pmatrix}(t_o) = \begin{pmatrix} \underline{r}_o \\ \\ \underline{v}_o \end{pmatrix}, \qquad \text{(E1.3)}$$

where $(\underline{r}_o, \underline{v}_o)^T$ are given. Because motion occurs in a plane, it can be assumed (without loss of generality) that $y_3 \equiv 0$, and the motion is in the (y_1, y_2) plane. Equation (E1.1) implies

$$\ddot{y}_1 + \frac{\mu}{r^3} y_1 = 0$$

$$\ddot{y}_2 + \frac{\mu}{r^3} y_2 = 0 \qquad \text{(E1.4)}$$

where

$$r = \sqrt{y_1{}^2 + y_2{}^2}$$

Next, change to polar coordinates by setting

$$y_1 = r \cos \phi$$

$$y_2 = r \sin \phi$$

where r, ϕ are functions of t. This leads to the well-known and equivalent system

$$\ddot{r} - r\dot{\phi}^2 + \frac{\mu}{r} = 0 \qquad \text{(E1.5)}$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0$$

For circular orbits, r is constant; hence, $\dot{r} = \ddot{r} = 0$. Consider two orbits of fixed radii r_1 and r_2, then from equation (E1.5)

$$\dot{\phi}_i^2 = \frac{\mu}{r_i^3}$$

or

$$\dot{\phi}_i = \frac{\sqrt{\mu}}{r_i^{3/2}}$$

hence

$$\phi_i = \frac{\sqrt{\mu}\ t}{r_i^{3/2}}, \quad i = 1,2$$

Thus, two solutions of the vector formulation of equation (E1.4) are given by

$$\underline{\eta}_1 = r_1 \left(\cos \frac{\sqrt{\mu}\ t}{r_1^{3/2}}, \ \sin \frac{\sqrt{\mu}\ t}{r_1^{3/2}} \right)$$

and

$$\underline{\eta}_2 = r_2 \left(\cos \frac{\sqrt{\mu}\ t}{r_2{}^{3/2}} \ , \ \sin \frac{\sqrt{\mu}\ t}{r_2{}^{3/2}} \right)$$

Place r_1 in $\underline{\eta}_1$, but consider $\underline{\eta}_2$ to vary as a function of r_2. In order to show instability of $\underline{\eta}_1$, it is required to show that there is some $\varepsilon > 0$ such that for every $\delta > 0$ there exist solutions $\underline{\eta}_2$ for which

$$||\underline{\eta}_1(t;t_o,r_1) - \underline{\eta}_2(t;t_o,r_2)|| > \varepsilon \tag{E1.6}$$

for arbitrarily large values of t, and $||r_1 - r_2|| < \delta$. That is, every δ disk about r_1 contains solutions (starting in that disk) that escape the Lyapunov tube for arbitrary large t. An easy computation shows that

$$||\underline{\eta}_1 - \underline{\eta}_2|| = \sqrt{2}\ |\ 1 - r_1 r_2 \cos\ (r_1{}^{-3/2} - r_2{}^{-3/2})\ \sqrt{\mu}\ t\ |$$

but if $r_1 \neq r_2$, then there exists a sequence of t values

$$\{t_n\}_{n=1}^{\infty} \text{ with } \lim_{n\to\infty} t_n = \infty$$

such that

$$\cos\ \left[(r_1{}^{-3/2} - r_2{}^{-3/2})\ \sqrt{\mu}\ t_n \right] = 0$$

in fact, set

$$(r_1{}^{-3/2} - r_2{}^{-3/2})\ \sqrt{\mu}\ t_n = \frac{\pi}{2} + n\pi,\ n = 0,1,2,\ldots$$

Thus

$$||\underline{\eta}_1(t_n) - \underline{\eta}_2(t_n)|| = \sqrt{2},\ n = 1,2,\ldots$$

and therefore, if $\varepsilon < \sqrt{2}$ is considered, the instability of the circular trajectories has been established.

3.0 LINEAR SYSTEMS

Linear systems comprise the most important subset of the differential system equation (1), especially because the linearization of equation (1) about a particular solution is an important tool in the analysis of the original system. Consider the linear system

$$\dot{\underline{y}} = A(t)\underline{y} \tag{5}$$

where $A(t)$ is a continuous n by n matrix in the interval $[t_o,\infty)$. The characteristic polynomial of equation (5) is the polynomial of degree n in λ,

$$|A(t) - \lambda I| = 0 \tag{6}$$

where $|\ |$ denotes the determinant of $A(t) - \lambda I$.

If $A(t)$ is a constant matrix, then the constant coefficient system

$$\dot{\underline{y}} = A\underline{y} \tag{7}$$

has a stable characteristic polynomial if every eigenvalue λ has $Re(\lambda) < 0$ where Re denotes the real part of the possible complex number λ.

A very useful result for arbitrary linear systems is the next theorem, which is proved in reference 1.

Theorem 1: A necessary and sufficient condition for stability of every solution of the linear system equation (5) is that all solutions of equation (5) be bounded.

A frequently occurring case for orbital mechanics is where the linear system equation (5) has multiple, purely imaginary eigenvalues $\lambda = \pm i\beta$ (usually occurring an even number of times), or if zero is a multiple eigenvalue. This case can be either stable or unstable depending on the nature of the solutions. One procedure is to find a fundamental matrix $\underline{\phi}(t)$ of equation (5) and apply Theorem 1. A fundamental matrix is defined as an n by n matrix such that

$$\dot{\overline{\underline{\phi}}}(t) = A(t)\ \overline{\underline{\phi}}(t)$$

and

$$\det \overline{\underline{\phi}}(t) \neq 0, \quad t_o \leq t < \infty$$

(i.e., it contains n linearly independent (vector) solutions of equation (5)).

Example 2 (unstable). The system

$$\dot{\underline{y}} = A\underline{y}, \text{ where } A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \text{(E2.1)}$$

has the characteristic equation $|A - \lambda I| = \lambda^4 = 0$ and, hence, $\lambda = 0$ is an eigenvalue of multiplicity 4. To determine the fundamental matrix for this system, eigenvectors $\underline{\xi} = (c_1, c_2, c_3, c_4)^T$ associated with $\lambda = 0$ are sought. In this case, the system of four equations

$$A\underline{\xi} = \lambda\underline{\xi}$$

has three linearly independent solutions

$$\underline{\xi}_1 = (1,0,0,0)^T$$

$$\underline{\xi}_2 = (0,1,0,0)^T$$

and

$$\underline{\xi}_3 = (0,0,0,1)^T$$

Consequently, $\underline{\xi}_i e^{\lambda t} \equiv \underline{\xi}_i$ are three linearly independent solutions of equation (E2.1). To find a fourth, linearly independent solution of $\dot{\underline{y}} = A\underline{y}$, $\underline{\xi}_4$, which satisfies $(A - \lambda I)^2 \underline{\xi}_4 = \underline{0}$, but does not satisfy $(A - \lambda I)\underline{\xi}_4 = \underline{0}$, is sought. Such a vector is $\underline{\xi}_4 = (0,0,1,0)^T$, and the corresponding solution is

$$e^{\lambda t}\left(\underline{\xi}_4 + t(A - \lambda I)\,\underline{\xi}_4\right) = (-t,t,1,-t)^T$$

(This is easily verified by substitution into the equation.) A fundamental matrix for this system is

$$\bar{\Phi}(t) = e^{\lambda t} \begin{pmatrix} 1 & 0 & 0 & -t \\ 0 & 1 & 0 & t \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -t \end{pmatrix} \tag{E2.2}$$

where

$$e^{\lambda t} \equiv 1$$

The fourth column of $\bar{\Phi}(t)$ is a solution of equation (E2.1) and is unbounded because

$$||(-t,t,1,-t)^T|| = \sqrt{3t^2 + 1}$$

Thus, Theorem 1 verifies the conclusion that equation (E2.1) is unstable. For further details on the generation of the solutions of linear systems with multiple eigenvalues, see reference 2, section 3.10.

Example 3 (stable). Consider the two oscillatory systems

$$\left.\begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 &= \varepsilon_1 \, (x_1, \dot{x}_1, x_2, \dot{x}_2) \\ \ddot{x}_2 + \omega_2^2 x_2 &= \varepsilon_2 \, (x_1, \dot{x}_1, x_2, \dot{x}_2) \end{aligned}\right\} \tag{E3.1}$$

where ω_1 and ω_2 are constants. Equation (E3.1) can be expressed as a system of first-order equations by the usual substitutions

$$\left.\begin{aligned} \dot{x}_1 &= p_1 \\ \dot{p}_1 &= -\omega_1^2 x_1 + \varepsilon_1(x_1, p_1, x_2, p_2) \\ \dot{x}_2 &= p_2 \\ \dot{p}_2 &= -\omega_2^2 x_2 + \varepsilon_2(x_1, p_1, x_2, p_2) \end{aligned}\right\} \tag{E3.2}$$

or in matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_1 \\ 0 \\ \varepsilon_2 \end{pmatrix} \qquad \text{(E3.3)}$$

Then, if

$$\underline{x} = (x_1, p_1, x_2, p_2)^T$$

$$\underline{\varepsilon} = (0, \varepsilon_1, 0, \varepsilon_2)^T$$

and A denotes the matrix in equation (E3.3), then system equation (E3.3) can be written.

$$\dot{\underline{x}} = A\underline{x} + \underline{\varepsilon} \qquad \text{(E3.4)}$$

For the present, $\underline{\varepsilon}$ is suppressed by setting it equal to zero. The resulting linear system

$$\dot{\underline{x}} = A\underline{x} \qquad \text{(E3.5)}$$

has characteristic equation

$$|A - \lambda I| = (\lambda^2 + \omega_1^2)(\lambda^2 + \omega_2^2) = 0$$

and eigenvalues

$$\lambda = \pm i\omega_1$$

$$\lambda = \pm i\omega_2$$

If $\omega_1 \neq \omega_2$, then it is well known (see Theorem 3) that system equation (E3.5) is stable; i.e., every solution of equation (E3.5) is stable. Consider the case where $\omega_1 = \omega_2$ so that $\pm i\omega_1$ are eigenvalues of multiplicity 2.

Proceeding as in equation (E2.1), linearly independent eigenvectors $\underline{\xi} = (c_1, c_2, c_3, c_4)^T$ corresponding to $\lambda = i\omega_1$ are sought, such that

$$A\underline{\xi} = i\omega_1\underline{\xi} \tag{E3.6}$$

Solving equation (E3.6) for $\underline{\xi}$, it is found that $\underline{\xi}_1 = (1, i\omega_1, 0, 0)^T$ and $\underline{\xi}_2 = (0, 0, 1, i\omega_1)^T$ are linearly independent solutions of the algebraic system equation (E3.6), and the corresponding solutions to equation (E3.5) are given by

$$\underline{x}_1 = e^{i\omega_1 t}\,(1, i\omega_1, 0, 0)^T$$

and

$$\underline{x}_2 = e^{i\omega_2 t}\,(0, 0, 1, i\omega_1)^T$$

Similarly, with $\lambda = -i\omega_1$, solve

$$A\underline{\xi} = -i\omega_1\underline{\xi}$$

to find solutions

$$\underline{x}_3 = e^{-i\omega_1 t}\,(1, -i\omega_1, 0, 0)^T$$

and

$$\underline{x}_4 = e^{-i\omega_1 t}\,(0, 0, 1, -i\omega_1)^T$$

A fundamental matrix is

$$\underline{\Phi}(t) = (\underline{x}_1(t),\ \underline{x}_2(t),\ \underline{x}_3(t),\ \underline{x}_4(t))$$

Moreover, because each solution $\underline{x}_i(t)$, $i = 1,2,3,4$ is bounded, it is concluded that the linear system equation (E3.5) is stable.

Example 4 (unstable). Reference 5 considers the Kustaanheimo-Stiefel (KS) element formulation of the two-body problem without perturbations. The linearized equations are

$$\overset{*}{\delta \underline{y}} = A\delta \underline{y} \tag{E4.1}$$

where

$$A = \begin{pmatrix} & \Theta_{9 \times 10} & & \\ 0 \; . \; . \; . \; 0 & -\dfrac{3\mu}{8\omega^4} & 0 \end{pmatrix}, \quad \overset{*}{\delta \underline{y}} \equiv \frac{d\delta \underline{y}}{dE}$$

where E is the eccentric anomaly used as the independent variable and

$$\delta \underline{y} = (\delta\alpha_1, \delta\alpha_2, \delta\alpha_3, \delta\alpha_4, \delta\beta_1, \delta\beta_2, \delta\beta_3, \delta\beta_4, \delta\omega, \delta\tau)^T$$

In this case, $|A - \lambda I| = \lambda^{10}$ so that $\lambda = 0$ is an eigenvalue of multiplicity 10. Solutions to equation (E4.1) can be generated by first finding eigenvectors $\underline{\xi} = (c_1, c_2, \ldots, c_9, c_{10})^T$ such that

$$A\underline{\xi} = \lambda\underline{\xi}$$

which, in this case, reduces to

$$-\frac{3\mu}{8\omega^4} c_9 = 0$$

Thus, $c_9 = 0$ (as $-\dfrac{3\mu}{8\omega^4} \neq 0$) and nine linearly independent eigenvectors are given by $\underline{\xi}_i = (0, \ldots, 0, 1, 0, \ldots 0)^T$ where 1 is in the i^{th} position, and zero otherwise for $i = 1, \ldots, 10$; $i \neq 9$. The tenth eigenvector (denoted $\underline{\xi}_9$) can be found using the technique of the previous examples, and is $\underline{\xi}_9 = (0, \ldots, 0, 1, 0)^T$. The corresponding solution to equation (E4.1) is given by

$$\delta \underline{y}_9 = \underline{\xi}_9 + E\,(A - \lambda I)\,\underline{\xi}_9 = \left(0, \ldots, 0, 1, -\frac{3\mu E}{8\omega^4}\right)^T$$

The other nine linearly independent solutions to equation (E4.1) are given by

$$\underline{\delta y}_i = \underline{\xi}_i,\ i = 1,\ \ldots,\ 10;\ i \neq 9$$

Because $\underline{\delta y}_9$ is unbounded as a function of E, it is concluded that equation (E4.1) is unstable (in the sense of Lyapunov).

The conclusions of the preceding examples could have been brought forth much faster by discussion of the rank of the matrix $(A - \lambda I)$, which indicates the number of linearly independent solutions of the linear system

$$(A - \lambda I)\ \underline{\xi} = \underline{0}$$

however, the constructive approach seemed more appropriate. Because of the importance of the situation encountered in these examples, the following remark is stated.

Remark. Suppose that in a real, linear system $\dot{\underline{y}} = A\underline{y}$, for every eigenvalue λ such that $\mathrm{Re}\lambda = 0$ and λ has multiplicity m, the equation

$$A\underline{\xi} = \lambda\underline{\xi}$$

has m linearly independent solutions $\{\underline{\xi}_i\}_{i=1}$ then $\dot{\underline{y}} = A\underline{y}$ will be stable or unstable in accordance with whether the remaining solutions (associated with the other eigenvalues) are bounded or unbounded. This is a consequence of the fact that the solutions $\underline{y}_i = e^{\lambda t}\underline{\xi}_i$, $i = 1, \ldots, m$ are bounded.

A second point to be aware of is that it suffices to study only the trivial, or zero, solution of linear systems, because by linearity

$$\underline{y}_1(t;t_o,\underline{y}_1) - \underline{y}_o(t;t_o,\underline{y}_o) = \underline{\eta}(t;t_o,\underline{y}_1 - \underline{y}_o)$$

is a solution, and conditions (S2) and (S1) reduce to

$$||\underline{\eta}(t;t_o,\underline{y}_1 - \underline{y}_o)|| < \epsilon,\ t \geq t_o$$

whenever

$$||\underline{y}_1 - \underline{y}_o|| < \delta$$

Another implication of this second point is that constant coefficient linear systems can be classified as stable, asymptotically stable, or unstable because whatever is true for the zero solution is true for all solutions. This is in contrast to nonlinear systems where specific solutions must be concentrated on; i.e., boundedness and stability are independent concepts, in general.

Theorem 2: If the characteristic polynomial (eq. (6)) of the linear system (eq. (7)) is stable, then every solution of the linear system (eq. (7)) is asymptotically stable. (See reference 1 for verification.)

The word stable is unfortunately associated with the characteristic polynomial, because it implies more than that for the solutions. A final result in this direction is as follows:

Theorem 3: If the roots of the characteristic polynomial (eq. (6)) of the linear system (eq. (7)) are $\{\lambda_i\}_{i=1}^{n}$ (not necessarily distinct), then every solution of the linear system is stable if (1) $Re(\lambda_i) < 0$ for all eigenvalues λ_i with $Re(\lambda_i) \neq 0$, and (2) if there exist roots λ_i with $Re(\lambda_i) = 0$, then λ_i is a simple root (i.e., has multiplicity 1).

Remark. Consider the constant coefficient system $\dot{\underline{y}} = A\underline{y}$. Suppose A is unstable because of the existence of an eigenvalue λ with $Re\lambda > 0$. If there exists a nonsingular matrix M that diagonalizes A; i.e.,

$$M^{-1}AM = D = \text{diag}\ (\lambda_1, \ldots, \lambda_n)$$

then it is well known that

$$|A - \lambda I| = |D - \lambda I|$$

and consequently the transformed system $\dot{\underline{x}} = D\underline{x}$ is unstable where $\underline{x} = M^{-1}\underline{y}$. This implies that if instability due to $Re\lambda > 0$ in a system $\dot{\underline{y}} = A\underline{y}$ is eliminated, it will be necessary to use a change of independent variables or some transformation other than a similarity transform.

3.1 PERTURBED LINEAR SYSTEMS

Several of the preceding theorems were applicable to the constant coefficient linear system $\dot{\underline{y}} = A\underline{y}$, with the exception of the general nonautonomous linear system $\dot{\underline{y}} = A(t)\ \underline{y}$. Perturbations of asymptotically stable (or stable) constant coefficient linear systems can inherit stability provided the perturbations are small in some normal sense. Specifically, the linear systems under consideration are

$$\dot{\underline{y}} = (A + C(t))\underline{y} \tag{8}$$

where $C(t) = (C_{ij}(t))_{i,j=1}^{n}$ is a continuous n by n matrix on $[t_o,\infty)$. In what follows,

$$||C(t)|| = (\sum_i \sum_j c_{ij}^2(t))^{1/2}$$

or some other finite dimensional norm such as

$$||C(t)|| = \max_{i,j} (\sup_{t>t_o} |C_{ij}(t)|)$$

provided such maximum exists.

Theorem 4: If the characteristic polynomial of $\dot{\underline{y}} = A\underline{y}$ is stable (i.e., $\dot{\underline{y}} = A\underline{y}$ is asymptotically stable) and

$$\int_{t_o}^{\infty} ||C(t)||dt < \infty \tag{9}$$

(i.e., the integral exists), then all solutions of equation (8) are asymptotically stable. (See reference 1 for verification.)

Corollary 1 (to the proof of Theorem 4). If there exists a sufficiently small constant $\xi > 0$ such that

$$||C(t)|| < \xi, \text{ t included in the interval } [t_o,\infty) \tag{10}$$

then the conclusion of Theorem 4 is valid. The exact nature of ξ is discussed in reference 1. Observe that inequality (eq. (10)) does not imply inequality (eq. (9)).

Corollary 2. If all the solutions of $\dot{\underline{y}} = A\underline{y}$ are stable and condition (eq. (9)) holds, then all the solutions of the perturbed system (eq. (8)) are stable. (See reference 1 for verification.)

4.0 PERTURBED NONLINEAR SYSTEMS

Analysis of the stability properties of solutions of nonlinear systems can often be achieved by considering the nonlinear system

$$\dot{\underline{y}} = \underline{f}(t,\underline{y})$$

as a nonlinear perturbation of a linear system; i.e.,

$$\dot{\underline{y}} = A\underline{y} + \underline{P}(t,\underline{y}) \tag{11}$$

This situation occurs when attempting to analyze a particular solution of system equation (1) by linearizing that system about the particular solution. Because of the importance of this process, details are provided below.

Linearization. Let $\underline{y}_o(t)$ be a given particular solution of the nonlinear system equation (1). Study the stability of $\underline{y}_o(t)$. Let $\underline{y} = \underline{y}(t)$ be any other solution of equation (1), then using a vector valued Taylor series expansion of $\underline{f}$ (expanded about $\underline{y}_o(t)$), the result is

$$\underline{f}(t,\underline{y}(t)) = \underline{f}(t,\underline{y}_o(t)) + \left[\frac{d\underline{f}}{d\underline{y}}(t,\underline{y}_o(t))\right](\underline{y} - \underline{y}_o(t)) +$$

$$\left(\sum_{m=2}^{\infty} \frac{((\underline{y} - \underline{y}_o)\cdot\underline{\nabla})^m}{m!}\right)\underline{f}(t,\underline{y}_o(t)) \tag{12}$$

where $\underline{\nabla} = \left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right)^T$, $\underline{f}(t,\underline{y}_o) = (f_1(t,\underline{y}_o), \ldots, f_n(t,\underline{y}_o))^T$ and $((\underline{y} - \underline{y}_o)\cdot\underline{\nabla})^m$ is a symbolic operator in which

$$(\underline{y} - \underline{y}_o)\cdot\underline{\nabla} \tag{13}$$

is performed and then raised algebraically to the m^{th} power. At this point, it is applied to the components of $\underline{f}$ and evaluated at $(t,\underline{y}_o(t))$. Next, observe that

$$\underline{f}(t,\underline{y}(t)) = \dot{\underline{y}}(t), \quad \underline{f}(t,\underline{y}_o(t)) = \dot{\underline{y}}_o(t)$$

which implies that equation (12) can be written as

$$\frac{d}{dt}(\underline{y} - \underline{y}_o) = \left[\frac{d\underline{f}}{d\underline{y}}(t,\underline{y}_o)\right](\underline{y} - \underline{y}_o) + \left(\sum_{m=2}^{\infty} \frac{((\underline{y} - \underline{y}_o)\cdot\underline{\nabla})^m}{m!}\right)\underline{f}(t,\underline{y}_o)$$

Set

$$\underline{u} = \underline{y}(t) - \underline{y}_o(t) \tag{14}$$

in the last equation to obtain

$$\dot{\underline{u}} = \left[\frac{\partial\underline{f}}{\partial\underline{y}}(t,\underline{y}_o(t))\right]\underline{u} + \left(\sum_{m=2}^{\infty} \frac{(\underline{u}\cdot\underline{\nabla})^m}{m!}\right)\underline{f}(t,\underline{y}_o(t)) \tag{15}$$

or,

$$\dot{\underline{u}} = A\underline{u} + \underline{p}(t,u) \tag{16}$$

where

$$A = \left[\frac{\partial\underline{f}}{\partial\underline{y}}(t,\underline{y}_o(t))\right] = \text{the Jacobian of } \underline{f} \text{ at } \underline{y}_o(t) \tag{17}$$

and

$$\underline{p}(t,\underline{u}) = \left(\sum_{m=2}^{\infty} \frac{(\underline{u}\cdot\underline{\nabla})^m}{m!}\right)\underline{f}(t,\underline{y}_o(t)) \tag{18}$$

The linear part of equation (16), which is obtained by suppressing $\underline{p}(t,\underline{\mu})$, is called the linearized system associated with equation (1), and is often written

$$\dot{\delta \underline{y}} = \left[\frac{\partial \underline{f}}{\partial \underline{y}}\right]_{(t, y_o(t))} \delta \underline{y} \tag{19}$$

In some instances, a linearization about initial conditions $(t_o, \underline{y}_o)$ is useful, and in this case the result is

$$\dot{\delta \underline{y}} = \left[\frac{\partial \underline{f}}{\partial \underline{y}}\right]_{(t_o, \underline{y}_o(t))} \delta \underline{y} \tag{20}$$

Some of the most important cases of linearization occur for autonomous systems

$$\dot{\underline{y}} = \underline{f}(\underline{y}) \tag{21}$$

The preceding formulas are modified accordingly. The following examples will illustrate most of the ideas of this section.

Example. Consider the nonlinear equation from reference 3.

$$\dot{y} = 1 - y^2, \; y(t_o) = y_o \tag{E5.1}$$

The real solutions of equation (E5.1) are given by

$$y = \tanh\,(t - t_o + k)$$

where

$$k = \text{arc tanh}\,(y_o), \; -1 < y_o < 1 \tag{E5.2}$$

It is easy to show directly that the solution $y_o = -1$ is unstable as $t \to \infty$, whereas $y_1 = 1$ is stable. The objective is to relate equations (1), (16), and (19) for this example. Here,

$$f(t,y) = 1 - y^2$$

$$\frac{\partial f}{\partial y}\,(t,y) = -2y$$

and if an expansion occurs about $y_o = -1$,

$$A = \left[\frac{\partial f}{\partial y}\right]_{y_o} = 2 \qquad \text{(E5.3)}$$

Moreover, regarding $p(t,u)$, the following occurs.

$$\frac{(\underline{u}\cdot\underline{\nabla})^m}{m!}\,\underline{f}(\underline{y}_o(t)) = \begin{cases} -u^2, & \text{if } m = 2 \\ 0, & \text{if } m \geq 3 \end{cases} \qquad \text{(E5.4)}$$

Consequently, equation (16) equals

$$\dot{u} = 2u - u^2 \qquad \text{(E5.5)}$$

where

$$u = y - y_o(t) = y + 1 \qquad \text{(E5.6)}$$

The linearized equation (19) is

$$\delta\dot{y} = 2\delta y \qquad \text{(E5.7)}$$

Assuming $0 < u < 2$ (for convenience), then equation (E5.5) can be solved, and

$$u(t) = \frac{2ce^{2t}}{1 + ce^{2t}}, \quad c > 0 \qquad \text{(E5.8)}$$

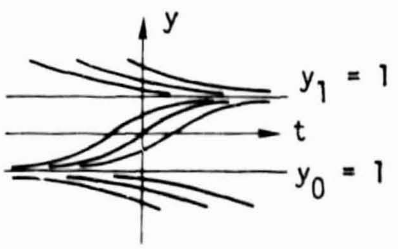
As noted by l'Hopital's rule that $u(t) \to 2$ as $t \to \infty$, thus because $u = y + 1$, this implies that $y(t) \to 1$ as $t \to \infty$. Thus, every solution starting near $y_o(t) = -1$, but greater than -1, approaches asymptotically $y_1 = 1$. Hence, $y_o = -1$ is unstable. Clearly, the linearized system equation (E5.7) has solutions

$$\delta y = ke^{2t} \qquad \text{(E5.9)}$$

Figures 3, 4, and 5 illustrate various solutions of systems equations (E5.1), (E5.5), and (E5.7), respectively. Figure 4 is the same as figure 3 but translated one unit upwards (in the y direction). Figure 5 (illustrating the linearized system) only fits the nonlinear system initially. In this case, the nonlinear part $-u^2$ results in the solutions being asymptotic to 2.

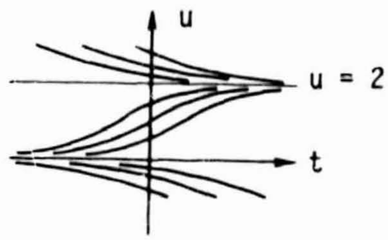
A similar expansion could be made about $y_1 = 1$. For this example, the instability of the linearized system correctly predicted the instability of the solution $y_o = -1$ of the nonlinear system. Two important remarks are in order.

<u>Remark 1</u>. The linearized system equation (19) will not always correctly predict the stability behavior of the nonlinear system equation (1). It is possible for equation (19) to be stable but for the nonlinear system equation (1) to have a solution $y_o(t)$, which is stable or unstable depending on the nature of $\underline{p}(t,\underline{u})$ in equation (16). Essentially, a stable situation in which there are multiple eigenvalues of linearized systems of the form $\lambda = \pm i\beta$ can become stable or unstable depending on the original nonlinear system. This situation is illustrated with a classic example in reference 2. (See equation (E6.1)).



$\dot{y} = 1 - y^2$

Figure 3.- Solutions of $\dot{y} = 1 - y^2$

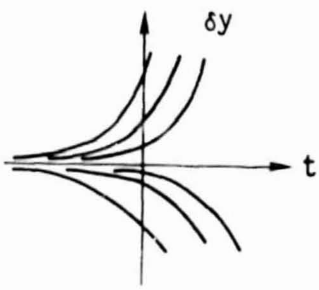


$\dot{u} = 2u - u^2$

where

$u = y - y_o = y + 1$

Figure 4.- Solutions of $\dot{u} = 2u - u^2$



$\dot{\delta y} = 2\delta y$

superimposed over solutions of equation (E5.5)

Figure 5.- Solutions of $\delta\dot{y} = 2\delta y$

Remark 2. When studying autonomous nonlinear systems and the stability of certain solutions, many times a change of dependent variables is made that essentially translates the system to the origin. This same result is achieved by linearization.

For example, the nonlinear system

$$\dot{y} = 1 - y^2$$

becomes

$$\dot{u} = 2u - u^2$$

under the change of variables $u = y + 1$.

Example 6. Consider the nonlinear system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 - y_1(y_1^2 + y_2^2) \\ -y_1 - y_2(y_1^2 + y_2^2) \end{pmatrix} \tag{E6.1}$$

which has $\underline{y}(t) = (y_1(t), y_2(t))^T \equiv \underline{0}$ as a solution. The linearized system is

$$\begin{pmatrix} \delta\dot{y}_1 \\ \delta\dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta y_1 \\ \delta y_2 \end{pmatrix} \tag{E6.2}$$

which has eigenvalues $\lambda = \pm i$. Thus, the linearized system is stable. Next, observe that

$$\frac{d}{dt}(y_1^2 + y_2^2) = -2\,(y_1^2 + y_2^2)^2 \tag{E6.3}$$

where equation (E6.1) is used to evaluate on the left $2(y_1\dot{y}_1 + y_2\dot{y}_2)$. Setting

$$r(t) = y_1^2 + y_2^2$$

equation (E6.3) equals

$$\dot{r}(t) = -2r^2(t)$$

which integrates to give

$$\frac{1}{r(t)} = 2t + c \tag{E6.4}$$

This last equation implies

$$y_1^2(t) + y_2^2(t) = \frac{c}{1 + 2ct} \tag{E6.5}$$

where

$$c = y_1^2(o) + y_2^2(o) > 0$$

(provided the initial value problem is nontrivial). Recalling that $||\underline{y}|| = \sqrt{y_1^2(t) + y_2^2(t)}$, equation (E6.5) implies that

$$\lim_{t\to\infty} ||\underline{y}|| = 0$$

where $\underline{y} = \underline{y}(t;t_o,\underline{y}_o)$ and $\underline{y}_o = (y_1(o),y_2(o))^T$. Thus, the result is that $\underline{0}$ is an asymptotically stable solution of equation (E6.1). Because, in addition,

$$||\underline{y}(t)|| \leq ||\underline{y}(o)||, \quad t \geq 0 \ .$$

Example 7. On the other hand, consider the system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 + y_1(y_1^2 + y_2^2) \\ -y_1 - y_2(y_1^2 + y_2^2) \end{pmatrix} \tag{E7.1}$$

The linearized system associated with equaticn (E7.1) is exactly the same as equation (E6.2), which, in turn, is stable. In this case

$$\frac{d}{dt}(y_1^2 + y_2^2) = 2\,(y_1^2 + y_2^2)^2 \qquad \text{(E7.2)}$$

which implies

$$y_1^2 + y_2^2 = \frac{c}{1 - 2ct} \qquad \text{(E7.3)}$$

where

$$c = y_1^2(o) + y_2^2(o)$$

The right side of equation (E7.3) has a vertical asymptote at $t = \frac{1}{2c} > 0$; i.e., the solution has finite escape time. Thus,

$$\lim_{t \to 1/2c} ||y(t;t_o,\underline{y}_o)|| = \infty$$

regardless of c. This implies that equation (E7.1) is unstable at the origin.

The examples presented above indicate that stability of the linearized system will not necessarily be inherited by the nonlinear system. The following theorems indicate more positive results for

$$\dot{\underline{y}} = A\underline{y} + \underline{p}(t,\underline{y})$$

The following hypothesis regarding the system equation (11) is required:

a. $\underline{p}(t,\underline{y})$ is continuous in T, and

b. $\lim_{||\underline{y}|| \to 0} \frac{||\underline{p}(t,\underline{y})||}{||\underline{y}||} = 0$, uniform in t.

Condition (b) implies that $\underline{p}(t,\underline{o}) = \underline{o}$ so $\underline{y} = \underline{o}$ is a solution of equation (11).

Theorem 5. Suppose $|A - \lambda I|$ is stable (i.e., the linear system is asymptotically stable) and $\underline{p}(t,\underline{y})$ satisfies (a) and (b), then the solution $\underline{y} = \underline{o}$ of system equation (11) is also asymptotically stable. (See reference 1 for verification.)

A similar version for autonomous systems

$$\dot{\underline{y}} = A\underline{y} + \underline{p}(\underline{y}) \tag{22}$$

appears in reference 2, where the norm

$$||\underline{y}|| = \max_i |y_i|$$

is used.

Theorem 6. Suppose the vector valued function

$$\frac{\underline{p}(\underline{y})}{||\underline{y}||} \tag{23}$$

is a continuous function of $\underline{y}$ (i.e., the components y_i of $\underline{y}$ are continuous) and vanishes for $\underline{y} = \underline{0}$. Then

a. The solution $\underline{y} = \underline{0}$ of equation (22) is asymptotically stable if the linearized system $\delta\dot{\underline{y}} = A\delta\underline{y}$ is asymptotically stable.

b. The solution $\underline{y} = \underline{0}$ of equation (22) is unstable if there exists at least one eigenvalue λ of the characteristic polynomial $|A - \lambda I|$ such that $\mathrm{Re}\lambda > 0$.

c. There is no conclusion regarding the stability of $\underline{y} = \underline{0}$ if every eigenvalue λ of the characteristic polynomial $|A - \lambda I|$ has $\mathrm{Re}\lambda \leq 0$, and there is at least one eigenvalue with $\mathrm{Re}\lambda = 0$; i.e., there exists a purely imaginary eigenvalue of A.

Example 8. For the final example, consider the one-dimensional perturbed harmonic oscillator,

$$\ddot{y} + \varepsilon(1 - y^2)\dot{y} + y = 0 \tag{E8.1}$$

where ε is a small, positive parameter. This equation is equivalent to the system

$$\begin{pmatrix} \dot{y}_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\varepsilon \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon\, y_1^2 y_2 \end{pmatrix} \tag{E8.2}$$

where $y_1 = y$, $y_2 = \dot{y}$. The linearized system is

$$\begin{pmatrix} \delta \dot{y}_1 \\ \delta y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\varepsilon \end{pmatrix} \begin{pmatrix} \delta y_1 \\ \delta y_2 \end{pmatrix} \tag{E8.3}$$

which has eigenvalues (distinct for $0 < \varepsilon < 2$)

$$\lambda = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$$

Because $\operatorname{Re}\lambda < 0$, the linearized system is asymptotically stable to $(0,0)^T$. Moreover,

$$\lim_{\underline{y} \to \underline{0}} \frac{\underline{p}(\underline{y})}{||\underline{y}||} = \underline{0} \tag{E8.4}$$

where $\underline{p}(\underline{y}) = (0, \varepsilon y_1^2 y_2)^T$. Thus, by either of the preceding theorems, $(0,0)^T$ is an asymptotically stable solution of equation (E8.2).

With regard to equation (E8.4), it should be noted that it suffices to consider

$$\lim_{||\underline{y}|| \to 0} \frac{\varepsilon y_1^2 y_2}{\sqrt{y_1^2 + y_2^2}}$$

but

$$\frac{\varepsilon y_1^2 y_2}{\sqrt{y_1^2 + y_2^2}} = \frac{\varepsilon y_1^2 y_2 \sqrt{y_1^2 + y_2^2}}{y_1^2 + y_1^2} = \frac{\varepsilon y_2 \sqrt{y_1^2 + y_2^2}}{1 + \left(\frac{y_1}{y_2}\right)^2}$$

and

$$\frac{\varepsilon y_2 \sqrt{y_1^2 + y_2^2}}{1 + (y_1/y_2)^2} = \leq \varepsilon \, |y_2| \sqrt{y_1^2 + y_2^2} \tag{E8.5}$$

The inequality (eq. (E8.5)) implies the conclusion (eq. (E8.4)).

5.0 CONCLUSIONS

The study of the qualitative behavior of solutions of linear and nonlinear systems of differential equations is of major importance and has implications for the accuracy of numerical solutions. It is important for scientists and engineers involved in the analysis (numerical and analytical) and the solution of large scale systems to clearly understand the concept of stability both from the numerical and mathematical viewpoint. For various reasons, most publications emphasizing the mathematical and numerical stability theory of linear systems tend to focus on small systems; i.e., one or two first-order differential equations. This does not necessarily enhance understanding of the larger systems. In orbital mechanics, systems of 6 to 10 equations are more prevalent, and they frequently have high multiplicity eigenvalues, which are either zero or purely imaginary. In this report, the emphasis is on mathematical (Lyapunov) stability analyses for these higher dimension systems with eigenvalues of the form $i\beta$ where $i^2 = -1$ and β are real, nonnegative numbers. Eight examples are presented from orbital mechanics, mechanical, and electrical systems, etc., which illustrate the techniques and theory of Lyapunov stability.

A basic tool in the study of nonlinear systems of equations is to linearize the system and analyze the resulting linear system. Unfortunately, the stability properties of the linear system are not necessarily inherited by the original nonlinear system, and this is illustrated by example. Of more concern is that if the linearized system is only (marginally) stable (i.e., it is stable but has repeated eigenvalues of the form $i\beta$), or (marginally) unstable, then the nonlinear system may go either way. This is precisely the situation often encountered in orbital mechanics. This report attempts to emphasize understanding of Lyapunov stability by illustrating techniques and giving examples that occur in practice, and to stress the type of system for which analysis is more difficult.

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