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RESEARCH IN NAVIGATION AND
OPTIMIZATION FOR SPACE TRAJECTORIES

October 1, 1975 through September 30, 1979

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FINAL REPORT

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Lyndon B. Johnson Space Center
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ANALYTICAL MECHANICS ASSOCIATES, INC.
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SUMMARY

This report is a compilation of the research in orbit determination, navigation and optimization in space trajectories carried out by Analytical Mechanics Associates, Inc. under contract to the Lyndon B. Johnson Space Center, covering the period October 1, 1975 through September 30, 1979.

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INTRODUCTION

Analytical Mechanics Associates, Inc., under contract to the Lyndon B. Johnson Space Center, acted in the capacity of consultants in the areas of orbit determination, navigation, optimization techniques and trajectory design for manned space flights. In this capacity, several reports were generated and are included in the text of this final report.

(1) Initial Cartesian Coordinates for Rapid Precision Orbit Prediction

This report contains the equations of motion for a variation of parameters precision trajectory prediction which is free of round-off error over long time periods and permits large time steps for rapid computation.

(2) Curvilinear Projection Developments

This report reviews various methods for accelerating convergence in optimization methods using search routines by applying curvilinear projection ideas.

(3) Perturbation-Magnitude Control For Difference-Quotient Estimation of Derivatives

This report develops estimates for choosing perturbation step-sizes used in difference-quotient approximations of derivatives for optimization programs.

(4) Determination of the Accelerometer Bias For In-Orbit Shuttle Trajectories

This report develops a closed form solution for estimating accelerometer bias when using "delta V" accelerometer output for on-board computation of position and velocity.

INITIAL CARTESIAN COORDINATES
for
RAPID PRECISION ORBIT PREDICTION

S. Pines

prepared for

Lyndon B. Johnson Space Center
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ANALYTICAL MECHANICS ASSOCIATES, INC.
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SUMMARY

This report developes the initial cartesian coordinates of the classical two body problem as perturbation parameters for use in rapid and precise computation of satellite coordinates in the presence of complex perturbations. The method is competitive with the KS theory in that it is numerically stable over long integration periods, permitting large integrator step sizes (of the order of 1/50 of the orbital period). Moreover, it requires only 7 first order differential equations and requires no transformation to utilize existing cartesian coordinate perturbation computing machine routines in the integration of the differential equations of motion.

I. INTRODUCTION

The numerical integration of a differential equation suffers from two major sources of numerical error. First, an improper choice of the integration scheme may produce spurious solutions in which the numerical error increases exponentially. Such errors are due to the characteristic roots of a class of linear differential equations with constant coefficients and are associated with the difference equations formulae used in the integration scheme. Thus, these errors are independent of the actual differential equations whose solution is required and may be controlled, or eliminated, by the proper choice of integration scheme and integration step size. References [1, 2, and 3] contain discussions of these effects. The second error source is due to the differential equation itself and is discussed in Reference [1]. This error growth comes about from the nature of the solution of the variational equations. If the homogeneous variational differential equation contains only bounded solutions, the numerical solution of the differential equation will be stable and integration can be accurately carried out over long time periods.

For the differential equations of the satellite numerical instability arises from the existence of mixed secular terms in the variational equation of the classical two body problem. Reference [4] contains a good illustration of this effect. The KS (Reference [5]) eliminates the mixed secular terms in the equations of motion by reducing the two body problem to the problem of the linear oscillation. However, the computing cost is somewhat increased by requiring a transformation between the cartesian space and the regularized variables. Moreover, nine differential equations are required in place of the conventional six. The method outlined in this report eliminates the mixed secular terms in the variational equations and retains the cartesian coordinate frame and time as the independent variable. Only seven differential equations are required and some computing time and data storage may be saved in comparison with the KS theory for the same accuracy.

II. THE INITIAL CONDITION CARTESIAN ELEMENTS

The equation of motion of a satellite in the inertial reference frame of the earth is given by

$$\frac{d^2}{dt^2} R = \frac{-\mu}{r^3} R + F \tag{1}$$

where F represents the pertubation forces other than the central attraction of the earth. We seek to represent the instantaneous position and velocity of the satellite as an osculating ellipse. Thus, if $R(t)$ and $\dot{R}(t)$ are the position and velocity we wish to describe, the solution is required in the form,

$$R(t) = f(\theta)\, R_o(t) + g(\theta)\, \dot{R}_o(t) \tag{2a}$$

and

$$\dot{R}(t) = \frac{\partial f(\theta)}{\partial t} R_o(t) + \frac{\partial g(\theta)}{\partial t} \dot{R}_o(t) \tag{2b}$$

where

$$f(\theta),\ g(\theta),\ f_t(\theta),\ g_t(\theta)$$

are the classical f and g functions of the two body problem given in References [6, 7, and 8].

$$
\begin{aligned}
f(\theta) &= 1 - \frac{a(1 - \cos\theta)}{r_o} \\
g(\theta) &= \frac{r_o\sqrt{a}\,\sin\theta}{\sqrt{\mu}} + \frac{d_o\, a(1 - \cos\theta)}{\mu} \\
f_t(\theta) &= \frac{-\sqrt{\mu a}\,\sin\theta}{r\, r_o} \\
g_t(\theta) &= 1 - \frac{a(1 - \cos\theta)}{r}
\end{aligned}
\tag{3}
$$

where

$$\frac{1}{a} = \frac{2}{r_o} - \frac{\dot{R}_o \cdot \dot{R}_o}{\mu} = \frac{2}{r} - \frac{\dot{R} \cdot \dot{R}}{\mu}$$

$$r_o = |R_o|$$

$$d_o = R_o \cdot \dot{R}_o \qquad (3a)\ \text{cont.}$$

$$r = |R| = a(1 - \cos\theta) + r_o \cos\theta + \frac{d_o}{\sqrt{\mu}} \sqrt{a} \sin\theta$$

and θ is the difference in eccentric anomaly

$$\theta = E - E_o \qquad (3b)$$

Given any function $h(R_o, \dot{R}_o, t)$ we define the total time derivative of h to be

$$\frac{d}{dt} h = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial R_o} \cdot \frac{\partial R_o}{\partial t} + \frac{\partial h}{\partial \dot{R}_o} \cdot \frac{\partial \dot{R}_o}{\partial t} \qquad (4)$$

The first term represents the change in h when the initial conditions are considered constant in time, and the sum of the second and third term are the perturbation derivative of h in which only the variations in the initial condition parameters are permitted. Thus,

$$h_\tau = \frac{\partial h}{\partial R_o} \cdot \frac{\partial R_o}{\partial t} + \frac{\partial h}{\partial \dot{R}_o} \cdot \frac{\partial \dot{R}_o}{\partial t} \qquad (4a)$$

For $R(t)$, $\dot{R}(t)$ defined in Eq. (2a) and 2b) to be a solution of Eq. (1), we require

$$\frac{d}{d\tau} R(t) = 0 \qquad (5a)$$

and

$$\frac{d}{d\tau} \dot{R}(t) = F \qquad (5b)$$

Equations (5a) and (5b) provide us with the necessary tools to obtain the perturbation derivatives of $R_o(t)$ and $\dot{R}_o(t)$.

Since we have

$$f(\theta)\, g_t(\theta) - g(\theta)\, f_t(\theta) = 1 \qquad (6)$$

Equations (2a) and (2b) may be solved for $R_o(t)$ and $\dot{R}_o(t)$ in terms of $R(t)$ and $\dot{R}(t)$.

$$R_o(t) = g_t(\theta)\, R(t) - g(t)\dot{R}(t) \tag{7a}$$

$$\dot{R}_o(t) = -f_t(\theta) R(t) + f(\theta)\dot{R}(t) \tag{7b}$$

In order to utilize Equations (5a) and (5b) to obtain the perturbation differential equations for the vectors $R_o(t)$ and $\dot{R}_o(t)$ it is necessary to express the f and g functions in terms of the vectors $R(t)$ and $\dot{R}(t)$. We have,

$$\frac{1}{a} = \frac{2}{r} - \frac{\dot{R}\cdot\dot{R}}{\mu}$$

$$r_o = a(1-\cos\theta) + r\cos\theta - \frac{R\cdot R\sqrt{a}\ (\sin\theta)}{\sqrt{\mu}} \tag{8}$$

$$g(\theta) = \frac{r\sqrt{a}}{\sqrt{\mu}}\sin\theta - \frac{R\cdot\dot{R}\ a(1-\cos\theta)}{\mu}$$

To eliminate the mixed secular terms in the variational equations we follow the recommendation of Reference [9] and set the perturbation derivative of the change in eccentric anomaly, θ, to zero. Thus,

$$\frac{d}{d\tau}\theta = 0 \tag{9}$$

This ensures that the perturbation differential equation will contain period terms and not mixed secular terms. Differentiating Equations (7a) and (7b), we have

$$\frac{d}{dt}R_o(t) = g_{t\tau}R(t) - g_\tau\dot{R}(t) - g\,F \tag{10a}$$

$$\frac{d}{dt}\dot{R}_o(t) = -f_{t\tau}R(t) + f_\tau\dot{R}(t) + f\,F \tag{10b}$$

Eliminating $R(t)$ and $\dot{R}(t)$ in Equations (10a) and (10b) we have,

$$\frac{d}{dt} R_o(t) = (g_{t\tau} f - f_t g_\tau) R_o(t) + (g g_{t\tau} - g_\tau g_t) \dot{R}_o(t) - g F \quad (11a)$$

$$\frac{d}{dt} \dot{R}_o(t) = (f_\tau f_t - f f_{t\tau}) R_o(t) + (f_\tau g_t - f_{t\tau} g) \dot{R}_o(t) + f F \quad (11b)$$

where

$$\begin{aligned}
f_\tau &= \frac{a}{r_o}(1 - \cos\theta)\left(\frac{(r_o)_\tau}{r_o} - \frac{a_\tau}{a}\right) \\
g_\tau &= \left(\frac{r \sin\theta}{2\sqrt{\mu a}} - \frac{d(1-\cos\theta)}{\mu}\right) a_\tau - \frac{a(1-\cos\theta)}{\mu} d_\tau \\
f_{t\tau} &= f_t\left(\frac{a_\tau}{2a} - \frac{(r_o)_\tau}{r_o}\right) \\
g_{t\tau} &= \cdot \frac{(1-\cos\theta)}{r} a_\tau \\
a_\tau &= 2a^2 \dot{R}(t) \cdot F \\
d &= R(t) \cdot \dot{R}(t) \\
d_\tau &= R(t) \cdot F \\
(r_o)_\tau &= \left(1 - \cos\theta + \frac{d \sin\theta}{2\sqrt{\mu a}}\right) a_\tau - \frac{\sqrt{a}\,\sin\theta}{\sqrt{\mu}} d_\tau
\end{aligned} \quad (12)$$

Since all the functions are given in terms of θ and not the time, t, it is necessary to have the value of θ as a function of the independent variable t. Let β be given by

$$\beta = \theta\sqrt{a} \quad (13)$$

Then, the total derivative of β with respect to time is given by,

$$\frac{d}{dt}\beta = \theta_t \sqrt{a} + \frac{\theta}{2\sqrt{a}} a_\tau \quad (14)$$

Since

$$\theta_t = \sqrt{\frac{\mu}{a}}\ \frac{1}{r} \tag{15}$$

we have,

$$\frac{d}{dt}\ \beta = \frac{\sqrt{\mu}}{r} + \frac{\theta \cdot a_T}{2\sqrt{a}} \tag{16}$$

Integrating Eq. (16), we obtain $\beta(t)$. The required $\theta(t)$ is given by,

$$\theta(t) = \frac{\beta(t)}{\sqrt{a(t)}} \tag{17}$$

The seven differential equations that are to be integrated are Equations (11a), (11b) and (16).

To obtain the perturbation vector $F(t, R, \dot{R})$, we compute $R(t)$ and $\dot{R}(t)$ from Equations (2a) and (2b) (given $R_o(t)$, $\dot{R}_o(t)$ and $\theta(t)$) and employ existing perturbation routines for F as a function of t, $R(t)$, and $\dot{R}(t)$.

III. A COMMENT

It is to be noted that the equations outlined in this report are valid only for satellites. Thus, for parabolic and hyperbolic orbits, a universal formulation of these equations are required which are outlined in Reference [4].

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CURVILINEAR PROJECTION DEVELOPMENTS

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Contract NAS9-14818

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CURVILINEAR PROJECTION DEVELOPMENTS*

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ABSTRACT

Gradient Projection is a powerful algorithm for minimization of a function subject to constraints (Refs. 1-5), at its best when the constraint functions are linear or nearly so. Constraint nonlinearities hamper projection computations, often requiring termination of a one-dimensional search in the projected negative gradient direction short of the 1-D minimum sought, on account of build-up of constraint violations. The constraints must then be restored before another projection cycle, at a certain computational expense. The restoration steps taken in the process of following nonlinear constraint surfaces can be used as a guide to the construction of a curve which more nearly follows the constraints than does the straight line in the projected gradient direction. This scheme, termed "curvilinear projection", was explored in Refs. 7 and 8. The study presently reported carries out some computational experiments using a related version of the technique. Some other details of projection computations which turn out to be practically important are taken up: rules for updating the variable metric in projection when early termination of the 1-D search on constraint violation occurs; active-constraint logic for screening inequalities that makes use of the Kuhn-Tucker necessary conditions. Computational comparisons on simple problems are presented.

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INTRODUCTION

Original and variable-metric versions of gradient projection algorithms for constrained minimization of a function are reported in Refs. 1-5. The present paper presents some recent improvements, and further investigations of a curved-search feature explored in Refs. 7 and 8 which affords improved constraint following.

The resurgence of interest in projection, on the part of the present writers, came with a surprise in the results of a comparison involving a seemingly slight modification of the Kelley-Speyer projection algorithm (Ref. 3). The modification was a provision for early updating of the variable metric whenever a screening test is passed. A notable convergence improvement was realized, resulting in the projection algorithm, which had been carried along merely for comparison, outperforming a more complex algorithm utilizing linear and quadratic penalties.

The algorithm will first be reviewed in its original equality-constraint version, then the updating rule just mentioned taken up. The restoration of constraints and the handling of inequality constraints will be discussed. Attention will then turn to the use of search along a curve, proposed in Refs. 7 and 8 with the idea of staying closer to constraint surfaces. Some computational experiments will then be described.

VARIABLE-METRIC PROJECTION

The projection version of the Davidon-Fletcher-Powell algorithm (Ref. 6) described in the following is essentially the algorithm of Ref. 3; some details are different, however, and the differences important computationally. The process begins with constraint restoration, usually requiring several cycles; then optimization cycles alternate with restorations, which sometimes require more than one cycle. The present section will deal with optimization cycles, the following one with restorations.

A function $f(x)$ (x an n-vector) is to be minimized subject to m equality constraints

$$g_j = 0 \qquad j = 1, ---, m \tag{1}$$

The process of Ref. 3 employs the formulas

$$\Delta x = -\alpha H(f_x + g_x \lambda) \tag{2}$$

$$\lambda = -(g_x^T H g_x)^{-1} g_x^T H f_x \tag{3}$$

A one-dimensional search on the scalar α is then carried out to minimize the function $f + g\lambda$. Appropriate penalty terms arrest the one-dimensional search whenever equality constraint violations much exceed $c_j|\bar{g}_j|$ in magnitude (Ref. 5). Projection cycles employ a DFP H-matrix, separate from that used in restoration cycles, updated according to

$$H + \Delta H = H + \frac{\Delta x \, \Delta x^T}{\Delta x^T(\Delta f_x + \Delta g_x \lambda)} - \frac{H(\Delta f_x + \Delta g_x \lambda)(\Delta f_x + \Delta g_x \lambda)^T H}{(\Delta f_x + \Delta g_x \lambda)^T H(\Delta f_x + \Delta g_x \lambda)} \tag{4}$$

The update is performed only if

$$\Delta x^T(\Delta f_x + \Delta g_x \lambda) > 0 \tag{5}$$

which assures positive definiteness of the updated H. This represents a departure from earlier versions of the algorithm (Refs. 3, 5) in which termination of one-dimensional search on a minimum of $f + g\lambda$ was required before updating of H was permitted, i.e., updating was deferred until the vicinity of the constrained minimum had been reached.

CONSTRAINT RESTORATION PHASE

The initial nulling out of constraint functions often proves more challenging than subsequent restorations in that the constraint violations to be dealt with are ordinarily larger in magnitude. For this purpose, minimization of a function $\hat{f}$ is employed:

$$\hat{f} = \frac{1}{2}\sum_{j=1}^{m} k_j g_j^2 + \frac{k_o}{2}(f_o - f)^2 h(f - f_o) \tag{6}$$

This is a weighted sum of squares of the constraint functions plus a term intended to counter gross increases in f. The term corresponds to penalty-function treatment of an inequality $f_o - f \geq 0$. Here h is the heaviside unit step function. The k_j are determined from

$$k_j = \frac{\bar{k}}{m} \frac{\sum_{i=1}^{m} |g_{i_x}|^2}{|g_{j_x}|^2} \qquad j = 1, 2, --, m \tag{7}$$

where $\bar{k}$ is input. This choice would make equal the contribution of each equality constraint to the second directional derivative of (6) in its own gradient direction at $g_j = 0$, if the constraints were linear. The constraint $f_o - f \geq 0$ is included quadratic-penalty-wise in (6) only during the first restoration sequence, with a coefficient k_o taken as 1/10 the smallest of the k_j calculated from (7). The constant f_o is estimated as the initial value of $f + g\,\lambda$.

The metric employed in correction sequences may be denoted A (to distinguish it from H of the optimization cycles). It is adjusted approximately for changes in the k_j, one at a time, using

$$A + \Delta A = A - \left[\frac{\Delta k_j}{1 + \Delta k_j \, g_{j_x}^T \, A \, g_{j_x}} \right] A \, g_{j_x} \, g_{j_x}^T \, A \qquad (8)$$

This correction, from Ref. 9, is based on the idea that A approximates $\hat{f}_{xx}^{-1}$. The metric to start the first correction sequence is obtained as $A + \Delta A$ from (8), using $A = I$ and $\Delta k_j = k_j - 1$ (k_j from (7)). If n or more updates are completed in this sequence, the emerging DFP metric is carried over to the next: if not, the initial metric is carried over. In either case, adjustments for any changes in the k_j are performed via (8) before use. Negative increments Δk_j are limited in magnitude to insure that the denominator of the fraction in parenthesis does not nearly vanish.

The second and subsequent restoration sequences employ

$$\Delta x = -\alpha A \, g_x \, (g_x^T \, A \, g_x)^{-1} \, g \qquad (9)$$

together with a one-dimensional search versus α for a minimum of $\hat{f}$ given by

(6), but with the last term deleted. This correction scheme, with $\alpha = 1$ and without a search, was originally proposed by Rosen (Ref. 1); it effects restoration in a single step for linear g. The existence of the inverse in (9) (and in (3)) requires that the matrix g_x have rank m. This condition is met at the constrained minimum in the classical normal case in which the tangent-plane approximations to the constraints are well-defined and distinct. Note that there is no guarantee that (9) is a direction of descent for $\hat{f}$, with general k_j values; thus the one-dimensional search may fail and reversion to DFP minimization of $\hat{f}$ become necessary.

The magnitude of constraint violation upon which optimization cycles are terminated short of a one-dimensional minimum is $c_j \bar{g}_j$, where $\bar{g}_j$ is a preconceived tolerance and c_j, usually $\gg 1$, is a factor adjusted with the aim of just permitting restoration with a single cycle of (9), to within the tolerance. Use of a single c-factor for all constraints met with only limited success, so a c-vector was resorted to, the components adjusted adaptively if somewhat heuristically in the following way: c_j is increased 10% if a single restoration proves successful; it is halved if two restoration cycles are required; and it is cut to one-quarter if there are additional cycles.

TREATMENT OF INEQUALITIES

It is of interest to determine a minimum subject to a mix of equality and inequality constraints, the latter expressed by

$$g_j \geq 0 \qquad j = m+1, \text{---}, m+p \qquad (10)$$

During the initial correction sequence, these are handled penalty-function fashion (Ref. 2), the function $\hat{f}$ to be minimized given by

$$\hat{f} = \frac{1}{2}\sum_{j=1}^{m} k_j g_j^2 + \frac{1}{2}\sum_{j=m+1}^{m+p} k_j g_j^2 \, h(-g_j) + \frac{k_o}{2}(f_o - f)^2 \, h(f - f_o) \tag{11}$$

with the k_j determined as though all constraints were equalities:

$$k_j = \frac{\bar{k}}{(m+p)} \; \frac{\sum_{i=1}^{m+p} \left| g_{i_x} \right|^2}{\left| g_{j_x} \right|^2} \qquad j = 1, 2, --, m+p \tag{12}$$

The determination of the active constraint set for optimization and restoration cycles proceeds first by excluding those satisfied with a margin $g_j \geq \bar{g}_j$, where $\bar{g}_j > 0$ is a preset threshhold. Those candidate inequality constraints for which $g_j < \bar{g}_j$, are then screened further via the Kuhn-Tucker conditions $\lambda_i \leq 0$ (Refs. 10 and 11), using (3) first with all the candidates included, then successively with Kuhn-Tucker violators dropped, as many times as necessary, until all $\lambda_j \leq 0$ or all candidates are screened out. Inactive constraints are treated in penalty-function approximation.

The Kuhn-Tucker conditions employed apply to the problem of minimizing a linear approximation to the function f subject to linearized constraints and to a quadratic constraint on step size. They become identical to the Kuhn-Tucker conditions for the original problem when evaluated at the constrained minimum sought.

The Kuhn-Tucker screening has generally been found to be worth the computational expense in reducing tendencies of constraints to switch between active and inactive status from cycle to cycle. The present effort has proceeded on the assumption that vector-matrix operations are cheap computationally in relation to the cost of gradient and function samples; this is realistic for the trajectory optimization applications of particular interest to the writers.

CURVED SEARCH

Constraint nonlinearities hamper projection computations a great deal in applications work, often requiring termination of a one-dimensional search short of the 1-D minimum sought on account of constraint-violation build-up. It is of interest to deflect the search away from the straight line in the negative projected gradient direction so as to follow approximately the nonlinear constraint intersection, as proposed in Refs. 7 and 8. An improved version of the curved-search technique is given in the following.

It is assumed that at least one projection cycle has already been completed (the first is done with a linear search) and that the derivative of $f + g\lambda$ with respect to the step-size parameter α has been reduced in magnitude by no more than half, that the constraints have been restored by one or more correction cycles, and that there has been no change in the active constraint set.

A curvilinear-projection cycle proceeds by

$$\Delta x = \xi\alpha + \eta\alpha^2 \tag{13}$$

which replaces (11). Here

$$\xi = -H(f_x + g_x\lambda) \tag{14}$$

$$\eta = -\frac{\Delta\tilde{x} - \xi\tilde{\alpha}}{\tilde{\alpha}^2} \tag{15}$$

$$\tilde{\alpha} = -\frac{\sum_{i=1}^{n} \xi_i \Delta\tilde{x}_i}{\sum_{i=1}^{n} \xi_i^2} \tag{16}$$

The vector $\Delta\tilde{x}$ is the difference between x from the beginning of the preceding projection cycle to the present restored point, the beginning of the next. The scalar $\tilde{\alpha} < 0$ is such that the earlier point is regenerated when $\alpha = \tilde{\alpha}$ is introduced into (13). Thus (13) generates a parabola in x space which passes through both restored points and is tangent to the projected gradient vector at the later one.

The curved-search sequencing presently in use provides for a possible curved-search on all optimization cycles except the first, which uses a linear search. Subsequent optimization cycles use a curved search provided the H-update test (5) was met on the preceding cycle, none of the inequality constraints has changed status (from or to active), and the preceding one-dimensional search did not proceed more than halfway to a minimum, as measured in terms of the magnitude of the derivative of $f + g\lambda$ with respect to the step-size parameter α. Earlier exploratory computations were more cautious in the use of curved searches, and generally less successful. The curving steps do nothing beneficial for conjugacy in the subspace of the constraint intersection, but this is already a lost cause with DFP when full steps to 1-D minima are not being taken.

TEST PROBLEMS

The test problems employed for experimentation were:

$$f = x_1 + a_1 x_2^2 + a_2 x_1^3$$

$$g_1 = x_1 - b_1 x_2^2 - b_2 x_3^2 - b_3 x_2^4$$

$$g_2 = x_3 - c$$

The coefficients were

$$a_1 = -10^{-2} \quad , \quad a_2 = 10^3$$

$$b_1 = 1 \quad , \quad b_2 = 10^2 \quad , \quad b_3 = 10^{-1}$$

$$c = 10^{-1}$$

The starting point for the numerical computations to be presented was

$$x_1 = 10 \quad , \quad x_2 = 5 \quad , \quad x_3 = 10$$

Test Problem #1 had a single equality, $g_1 = 0$; #2 had two equalities, $g_1 = 0$; $g_2 = 0$; #3 one equality, $g_1 = 0$, and one inequality $g_2 \geq 0$; #4 two inequalities, $g_1 \geq 0$, $g_2 \geq 0$; #5 two inequalities with the second reversed, $g_1 \geq 0$, $-g_2 \geq 0$; #6 one equality, $g_1 = 0$, and one inequality, also reversed, $-g_2 \geq 0$.

COMPUTATIONAL COMPARISONS

The following results illustrate various features in the context of equality constraints.

NUMBER OF GRADIENT AND FUNCTION-SAMPLE COMPUTATIONS REQUIRED FOR VARIOUS TEST PROBLEMS

Test Problem	#1	#2
Original Kelley-Speyer (Ref. 3)	44 & 691	72 & 1022
Kelley-Speyer plus early H update	35 & 480	40 & 499
Improved restoration logic added	29 & 338	31 & 324
Curved search added	17 & 218	24 & 253

Linear versus curvilinear projection results are as follows for a larger set of test problems:

NUMBER OF GRADIENT AND FUNCTION-SAMPLE COMPUTATIONS REQUIRED FOR VARIOUS TEST PROBLEMS						
Test Problem	#1	#2	#3	#4	#5	#6
Kelley-Speyer improved, linear	29&338	31&324	27&304	27&310	27&324	27&329
Kelley-Speyer curvilinear	17&218	24&253	23&259	24&254	26&296	25&285

It is noted that the improvement provided by the curved-search feature is considerably smaller in problems which include inequalities.

CONCLUDING REMARKS

Several developments and refinements of variable-metric projection have been presented including a curved-search technique for nonlinear-constraint-surface following, improved means for control and correction of constraint violations, and screening criteria for active-constraint logic for use with inequalities. Projection schemes appear quite promising and worth further development and evaluation effort. Experience with a larger variety of problems applying the various features described is of interest.

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PERTURBATION-MAGNITUDE CONTROL

FOR DIFFERENCE-QUOTIENT ESTIMATION OF DERIVATIVES

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SUMMARY

A process for adjusting perturbation magnitude for accurate difference-quotient estimation of derivatives is described in the following. The process is intended to be carried out sequentially, alternating with iterations of a parameter-optimization algorithm. A more complex and computationally-expensive scheme for occasional auxiliary use is also described.

INTRODUCTION

It has been recognized for some time that accuracy of partial derivatives is important for convergence of variable-metric optimization processes (Refs. 1 and 2). An adjustment scheme, based upon central differences and truncation error, is described in Ref. 3. The adjustment process of the next section is similar in concept but focuses on agreement between forward and backward difference quotients.

The perturbation-control logic is being introduced into PEACE, NASA-JSC's trajectory-shaping computer program currently in use for space-shuttle applications.

A TECHNIQUE FOR THE SEQUENTIAL ADJUSTMENT OF PERTURBATION MAGNITUDE

First- and second-derivative estimates f_α and $f_{\alpha\alpha}$ are given by the usual central-difference formulas,

$$f_\alpha = \frac{f^+ - f^-}{2\,\delta\alpha} \quad , \quad f_{\alpha\alpha} = \frac{f^+ + f^- - 2f}{\delta\alpha^2}$$

Here $f = f(\alpha)$ is a function of a scalar parameter α, $f^+ \equiv f(\alpha + \delta\alpha)$, and $f^- \equiv f(\alpha - \delta\alpha)$. If one requires that the magnitude of the difference between forward- and backward-difference estimates of f_α be at most

$$\begin{aligned} a &\equiv \varepsilon\,|f_\alpha| && \text{if} \quad \varepsilon\,|f_\alpha| \geq a_L \\ &\equiv a_L && \text{if} \quad \varepsilon\,|f_\alpha| < a_L \end{aligned}$$

Then an analysis accounting for terms through second order in $\delta\alpha$ leads to

$$\Delta \equiv a/b$$

where

$$\begin{aligned} b &\equiv 2\,|f_{\alpha\alpha}| && \text{if} \quad 2\,|f_{\alpha\alpha}| \geq b_L \\ &\equiv b_L && \text{if} \quad 2\,|f_{\alpha\alpha}| < b_L \end{aligned}$$

for the largest magnitude perturbation which will hold the truncation (nonlinearity) error to the specified level.

Bounds are imposed upon the candidate perturbation magnitude,

$$\begin{aligned} \delta\alpha^* &\equiv \Delta_L && \text{if} \quad \Delta < \Delta_L \\ &\equiv \Delta && \text{if} \quad \Delta_L \le \Delta \le \Delta_U \\ &\equiv \Delta_U && \text{if} \quad \Delta_U < \Delta \end{aligned}$$

To relieve any tendency toward thrashing from cycle to cycle, the value used to generate the next derivative estimate is the geometric mean of the old and the new:

$$\delta\alpha^{++} = \sqrt{\delta\alpha \cdot \delta\alpha^*}$$

The idea of the scheme is to obtain equality of forward and backward differences to a specified number of significant figures, e.g., an ε of 10^{-4} corresponds to four-figure agreement. The bounds a_L, b_L, Δ_L, Δ_U are obvious safety devices.

The adjustment process described generally works well if the function f is smooth and nonlinearity the dominant source of error, i.e., random errors are relatively small. The process should survive brief encounters even with such errors as jump discontinuities in f. However, if some algorithm is relentlessly driving α toward the site of a modeling weakness, such as one characterized by a jump in f or in its first derivative, any perturbation-control scheme will be hard pressed to cope.

The lower bound Δ_L should be chosen to insure against appreciable round-off error, but the choice is not an easy one to make a priori. A lower-bound adjustment process, based on the number of significant figures agreement between reference and perturbed values of f, will next be described.

The functions R^+ and R^- measure the agreement between reference and perturbed values of f (e.g., $R = 10^{-7}$ indicates seven-figure agreement):

$$R^+ \equiv \frac{|f^+ - f|}{\max(|f^+|, |f|)}$$

$$R^- \equiv \frac{|f^- - f|}{\max(|f^-|, |f|)}$$

With R_{MN} and R_{MX} defined by

$$R_{MN} \equiv \min(R^+, R^-)$$

$$R_{MX} \equiv \max(R^+, R^-)$$

an index of agreement between samples, ε_R, initially input, may be employed to determine the lower bound Δ_L, which is to be adjusted from cycle to cycle by a rule such as the following:

If $\varepsilon_R > R_{MN}$ and $R_{MX} > 1.1\, R_{MN}$, take

$$\varepsilon_R^* = 10\, \varepsilon_R$$

If $\varepsilon_R > R_{MN}$ and $R_{MX} \le 1.1\, R_{MN}$, take

$$\varepsilon_R^* = \frac{\varepsilon_R}{2}$$

If $\varepsilon_R \le R_{MN}$, take

$$\varepsilon_R^* = \varepsilon_R$$

The lower bound on perturbation magnitude which would correspond to the index ε_R^* is

$$\Delta_L^* = \frac{\varepsilon_R^*}{R_{MN}}\, \delta\alpha$$

where $\delta\alpha$ is the perturbation magnitude actually used in the determination of R_{MN} and ε_R^*. To avert undue downward fluctuation in the bound Δ_L, however, the geometric mean of Δ_L^* and the current Δ_L value

$$\Delta_L^{++} = \sqrt{\Delta_L \cdot \Delta_L^*}$$

is the updated value for use in the next cycle. The corresponding update for ε_R is

$$\varepsilon_R^{++} = \sqrt{\varepsilon_R \cdot \varepsilon_R^*}$$

The process just described adjusts the index of agreement ε_R and the perturbation bound Δ_L upward or downward whenever small-perturbation samples become available. It attributes any disagreement between forward and backward differences to random error, adjusting ε_R upward when agreement is poor, downward when agreement is good. The factors 1.1, 10, and .5 are rather arbitrary and intended merely to be suggestive.

An alternative scheme for the calculation of Δ_L^* and Δ_L^{++} will next be described. This is in process of evaluation at the present writing. The round-off error magnitude is estimated from the residuals of a least-squares fit in the vicinity of the minimum found in the preceding one-dimensional search. Of main interest for fitting are the samples in the band $k_{min} \pm \delta k$, where

$$\delta k \equiv \frac{\sum_{i=1}^{n} \left| f_{x_i} \right| \delta\alpha_i}{\sqrt{\sum_{i=1}^{n} f_{x_i}^2}}$$

corresponds to the perturbation magnitude employed in the most recent estimate of the gradient vector f_x. All of the samples in the band are used for the fit provided there are at least four in addition to the k_{min} sample. If not, more close samples are added to make a total of five. If any of these fall outside the band $k_{min} \pm 10\ \delta k$, the computation is abandoned and no update of Δ_L^* and Δ_L^{++} carried out.

After the least-squares fit has been completed, the residuals at k_{min} and the two closest points are examined. If these all have the same sign, the attempt to update Δ_L^* and Δ_L^{++} is abandoned, otherwise the average of the absolute values is adopted for ε_R. Note that ε_R in this mode is a scalar, applicable to adjustment of all components. The component-by-component update proceeds as follows: If $f_{\alpha\alpha} \geq 10^{-9}$, take

$$\Delta_L^* = \sqrt{\frac{40\ \varepsilon_R}{f_{\alpha\alpha}}}$$

otherwise do not update Δ_L for the particular component. Obtain Δ_L^{++} from the geometric mean:

$$\Delta_L^{++} = \sqrt{\Delta_L \Delta_L^*}$$

While the parameter α is a scalar in the preceding formulae, the process is intended to apply to each component of the parameter vector in turn. The idea of tailoring the choice of perturbation magnitude to the particular component is hardly earth-shaking but, in fact, it is not often carried out in practice.

A SEARCH TECHNIQUE FOR PERTURBATION-MAGNITUDE DETERMINATION

Applications of variable-metric optimization algorithms to complex models sometimes encounter convergence difficulties which are attributed to numerical errors, real or fancied. The analyst, in a moment of paranoia, suspects overlap between the regions of round-off and truncation, with no good compromise choice of perturbation magnitude for the generation of secant partials available. The situation may be thought serious enough to warrant a search, usually a tedious cut-and-try affair. The present section describes a mechanization of such a search, employing the one-dimensional minimization technique of Ref. 4.

A function Q may be defined which measures the error between forward and backward derivatives:

$$Q \equiv \frac{|f_\alpha^+ - f_\alpha^-|}{\max(|f_\alpha^+|, |f_\alpha^-|)}$$

Here the denominator-normalization choice between forward and backward derivative magnitudes is intended to avoid small-divisor difficulties. The proposed search is for a minimum of Q^2 subject to a round-off constraint on perturbation magnitude introduced via quadratic penalty. Thus

$$\min \left[Q^2 + k_R(\varepsilon_R - R_{MN})^2 \, h(\varepsilon_R - R_{MN}) \right]$$

where h, the Heaviside unit step function, is unity for argument ≥ 0, zero otherwise. The function R_{MN} is that of the previous section. The round-off constraint incorporated via penalty is $R_{MN} \geq \varepsilon_R$.

A choice of agreement-index ε_R is no easier to make than in the case of the sequential-adjustment process. Values obtained in the course of sequential-adjustment cycles may be employed, or, in difficult cases, a range of values investigated.

CONCLUDING REMARKS

The two processes described herein have been given only limited trials and require further attention. An initial application of the adjustment scheme indicates, a bit surprisingly, that it may turn out to have particular merit as a "debugging" tool. The first full-scale application presently planned is to the accelerated-gradient trajectory shaping program PEACE at NASA-JSC.

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DETERMINATION OF THE ACCELEROMETER BIAS FOR IN-ORBIT SHUTTLE TRAJECTORIES

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SUMMARY

This report yields a closed form deterministic solution of the accelerometer bias for in-orbit shuttle trajectories given the difference between the ground tracking solution of the trajectory and the on board estimate of the trajectory using platform accelerometers.

INTRODUCTION

Due to the presence of small unpredictable accelerations acting on the shuttle in orbit it may be necessary to use the stable platform accelerometers to integrate the vehicle state. In this mode the most significant error will be the accelerometer bias. A method is provided whereby these biases may be estimated by using the difference between the vehicle inertial position vectors as measured by ground tracking and that obtained by numerical integration of the on board accelerometers.

SOLUTION OF THE ACCELEROMETER BIAS IN ORBIT

Let the inertial equations of motion of the orbiting vehicle as measured by the ground tracking system be

$$\ddot{R}_1 = -\mu \frac{R_1}{r_1^3} + F_1 \tag{1}$$

Let the inertial equations of motion of the orbiting vehicle as measured by the on board accelerometers be given by

$$\ddot{R}_2 = -\mu \frac{R_2}{r_2^3} + F_1 + B \tag{2}$$

where F_1 is the true specific force acting and B an inertial vector of the small accelerometer bias error. The difference between the inertial vector positions is assumed small compared to either R_1 or R_2.

That is,

$$\left|R_1 - R_2\right| \Big/ \left|R_1\right| >> .001$$

and

$$\left|R_1 - R_2\right| \Big/ \left|R_2\right| >> .001 \tag{3}$$

Under these conditions the difference in the orbiting states is given by

$$\begin{Bmatrix} R_2 - R_1 \\ \dot{R}_2 - \dot{R}_1 \end{Bmatrix}_t = \phi(t, t_o) \begin{Bmatrix} R_2 - R_1 \\ \dot{R}_2 - \dot{R}_1 \end{Bmatrix}_{t_o} + \int_{t_o}^{t} \phi(t, \tau) \begin{Bmatrix} O \\ B \end{Bmatrix} d\tau \tag{4}$$

The constant accelerometer bias vector $\left\{ B \right\}$ is given by

$$\left\{ B \right\} = (C) \left[\left\{ R_2 - R_1 \right\}_t - \left(\frac{\partial R_1(t)}{\partial R_1(t_o)} \right) \left\{ R_2 - R_1 \right\}_{t_o} - \left(\frac{\partial R_1(t)}{\partial \dot{R}(t_o)} \right) \left\{ \dot{R}_2 - \dot{R}_1 \right\}_{t_o} \right] \tag{5}$$

The 3 matrices (C) , $\left(\dfrac{\partial R_1(t)}{\partial R(t_o)}\right)$, $\left(\dfrac{\partial R_1(t)}{\partial \dot{R}_1(t_o)}\right)$

are given by

$$(C) = c_1 R_1(t) R_1^T(t) + c_2 R_1(t) \dot{R}_1(t) + c_3 \dot{R}_1(t) R_1^T(t) + c_4 \dot{R}_1(t) \dot{R}_1^T(t) + c_5 I(3) \tag{6}$$

$$\left(\frac{\partial R_1(t)}{\partial R_1(t_o)}\right) = a_1 R_1(t_o) R_1^T(t_o) + a_2 R_1(t_o) \dot{R}^T(t_o) + a_3 \dot{R}_1(t_o) R_1^T(t_o) + a_4 \dot{R}_1(t_o) \dot{R}^T(t_o) + a_5 I(3)$$

$$\left(\frac{\partial R_1(t)}{\partial \dot{R}_1(t_o)}\right) = b_1 R_1(t_o) R_1^T(t_o) + b_2 R_1(t_c) \dot{R}_1^T(t_o) + b_3 \dot{R}_1(t_o) R^T(t_o) + b_4 \dot{R}_1(t_o) \dot{R}_1^T(t_o) + b_5 I(3)$$

The matrix I (3) is the 3 x 3 unit matrix

$$I(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

The scalar coefficients a_i, b_i and c_i are given by

$$a_1 = \frac{G_2}{r_o^3} - \frac{1}{r_o}\left(\frac{G_1}{r}\left(\frac{p_1}{r_o^3} - \frac{G_1}{r_o}\right) - \frac{(2 G_4 - \beta G_3)}{r_o^3}\right)$$

$$a_2 = \frac{G_1 G_2}{r r_o \sqrt{\mu}} \tag{8}$$

$$a_3 = -\frac{G_2}{r\sqrt{\mu}}\left(\frac{p_1}{r_o^3} - \frac{G_1}{r_o}\right) + \frac{3\,G_5 - \beta\,G_4}{\sqrt{\mu}\;r_o^3}$$

(8 con't.)

$$a_4 = \frac{G_2^{\,2}}{r\,\mu}$$

$$a_5 = 1 - \frac{G_2}{r_o}$$

where

$$p_1 = 3\,G_5 - \beta G_4 + r_o\,(G_3 - \beta\,G_2) + \frac{d_o}{\sqrt{\mu}}\,(2\,G_4 - \beta G_3)$$

$$r_o = \left| R_1\,(t_o)\right|$$

$$r = \left| R_1\,(t)\right|$$

$$d_o = R_1\,(t_o)\cdot \dot{R}_1\,(t_o)$$

$$d = R_1\,(t)\cdot \dot{R}_1\,(t)$$

$$v_o^{\,2} = \dot{R}_1\,(t_o)\cdot \dot{R}_1\,(t_o)$$

$$v^2 = \dot{R}_1\,(t)\cdot \dot{R}_1\,(t) \qquad (8a)$$

$$\beta = \frac{\sqrt{\mu}\,(t - t_o)}{a} - \frac{d}{\sqrt{\mu}} + \frac{d_o}{\sqrt{\mu}}$$

$$\frac{1}{a} = \frac{1}{2}\left(\frac{2}{r_o} - \frac{v_o^{\,2}}{\mu} + \frac{2}{r} - \frac{v^2}{\mu}\right)$$

$$G_i = \beta^i \left(\sum_{k=0}^{\infty} \frac{\beta^{2k} (-1)^k}{(2k+i)!} \left(\frac{1}{a} \right)^k \right) \qquad \text{(8a con't.)}$$

$$b_1 = \frac{G_1 G_2}{r r_o \sqrt{\mu}}$$

$$b_2 = - \frac{G_1 p_1}{\mu \, r r_o} + \frac{2 \, G_4 - \beta \, G_3}{\mu \, r_o} \qquad \text{(8b)}$$

$$b_3 = \frac{G_2^{\,2}}{r \mu}$$

$$b_4 = - \frac{G_2 \, p_1}{r \mu^{3/2}} - \frac{3 \, G_5 - \beta \, G_4}{\mu^{3/2}}$$

$$b_5 = \frac{r_o \, G_1}{\sqrt{\mu}} + \frac{d_o \, G_2}{\mu}$$

$$c_1 = \Big((d \, h_1 + v^2 h_3) \, h_3 - (d \, h_2 + v^2 \, h_4 + h_5) \, h_1 \Big) \Big/ \Delta h_5$$

$$c_2 = \Big((h_2 \, r^2 + h_4 d) \;\; h_1 - (h_1 \, r^2 + h_3 \, d + h_5) \, h_3 \Big) \Big/ \Delta h_5$$

$$c_3 = \Big((d \, h_1 + v^2 \, h_3) \, h_4 - (d \, h_2 + v^2 \, h_4 + h_5) \, h_2 \Big) \Big/ \Delta h_5 \qquad \text{(8c)}$$

$$c_4 = \Big((h_2 \, r^2 + h_4 \, d) \, h_2 - (h_1 \, r^2 + h_3 \, d + h_5) \, h_4 \Big) \Big/ \Delta h_5$$

$$c_5 = 1/h_5$$

where

$$h_1 = - \frac{G_2^{\,2}}{2 \, r \mu^{\frac{1}{2}}} \qquad \text{(8d)}$$

$$h_2 = \frac{1}{\mu}\left(-2\,G_5 + \frac{G_o G_5}{2} + \frac{\beta G_4}{2} + \frac{\beta^3}{12}\,G_2\right)$$

$$h_3 = -\frac{1}{r\,\mu^{3/2}}\left\{\begin{array}{l} G_2\,(3\,G_5 - \beta G_4) - G_3\,(2\,G_4 - \beta G_3) - 2\,G_3 G_4 \\ + 3\,(2\,G_7 - 3/2\,\beta\,G_6 + \frac{1}{2}\,G_o\,G_5 + \frac{1}{12}\,\beta^3 G_4 + \frac{1}{48}\,\beta^5 G_2) \end{array}\right\} \quad (8d)$$

$$- \frac{1}{\mu^{3/2}}\left(G_2\,(G_3 - \beta G_2) + G_2 G_3 + 2\,G_5 - \frac{1}{2}\,G_o G_5 - \frac{\beta}{2}\,G_4 - \frac{\beta}{12}\,G_2\right)$$

$$h_4 = \frac{d}{\mu^2}\left\{\begin{array}{l} G_2\,(3\,G_5 - \beta\,G_4) - G_3\,(2\,G_4 - \beta\,G_3) - 2\,G_3\,G_4 \\ + 3\,(2\,G_7 - 3/2\,\beta\,G_6 + \frac{1}{2}\,G_o\,G_5 + \frac{1}{12}\beta^3\,G_4 + \frac{1}{48}\,\beta^5 G_2) \end{array}\right\}$$

$$- \frac{r}{\mu^{3/2}}\left(G_1\,(3\,G_5 - \beta\,G_4) - G_2\,(2\,G_4 - \beta\,G_3)\right)$$

$$h_5 = -\frac{r}{\mu}\left(\frac{G_2^{\,2}}{2} + \frac{r}{2}\,G_1^{\,2} - \frac{d}{2\sqrt{\mu}}\,(G_3 G_o + \beta G_2)\right)$$

$$+ \frac{d}{\mu^{3/2}}\left\{\begin{array}{l} -2\,G_5 + \frac{1}{2}\,G_o\,G_5 + \frac{\beta}{2}\,G_4 + \frac{\beta^3}{12}\,G_2 \\ + r\,(G_1 G_2 - \frac{1}{2}\,G_o\,G_3 - \frac{1}{2}\,\beta G_2) \\ - \frac{d}{2\sqrt{\mu}}\,G_2^{\,2} \end{array}\right\}$$

$$\Delta = (h_1 h_4 - h_2 h_3)\,(r^2 v^2 - d^2) + h_5\,(h_1 r^2 + h_3 d + h_2 d + h_4 v^2 + h_5)$$

Since the vector bias, $\{B\}$, is considered to be constant over an estimate period, it is recommended that several estimates of $\{B\}$ be obtained using the same initial epoch (t_o , $R_1\ (t_o)$, $\dot{R}_1\ (t_o)$, $R_2\ (t_o)$, $\dot{R}_2\ (t_o)$) and several different later times, t . The recommended value of $\left\{ B \right\}$ will be given by

$$\left\{ \hat{B} \right\} = \frac{1}{N} \sum_{i=1}^{N} \left\{ B (t_i) \right\} \qquad (9)$$