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INTEGRATING MATRICES FOR ARBITRARILY
SPACED GRID POINTS

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INTEGRATING MATRICES FOR ARBITRARILY SPACED GRID POINTS

By

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SUMMARY

Integrating matrices form the basis of an efficient numerical procedure for solving differential equations associated with rotating beam configurations. By expressing the equations of motion in matrix notation, utilizing the integrating matrix as an operator, and applying the boundary conditions, the spatial dependence is removed from the governing partial differential equations and the resulting ordinary differential equations can be cast into standard eigenvalue form. Previous derivations of integrating matrices are limited to the case of equally spaced grid points and approximation by interpolating polynomials. The restriction to equally spaced grid points may not be appropriate for some beam configurations. This report derives integrating matrices for arbitrarily spaced grid points using either interpolating or least-squares fit orthogonal polynomials. Several previously unnoticed features of the equally spaced grid case are also discussed.

INTRODUCTION

The equations of motion governing the vibrations and aeroelastic stability of such rotating structures as helicopter rotor blades and propeller blades have no closed form solution, and approximate methods of solution such as asymptotic techniques, Galerkin's method, or direct numerical integration must be employed. A numerical procedure based on the use of integrating matrices (refs. 1, 2) has been employed to solve for the vibrations and stability of a wide variety of rotating beam configurations (refs. 3, 4). The integrating matrix provides a means

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for numerically integrating a function that is expressed in terms of the values of the function at increments of the independent variable. By expressing the equations of motion in matrix notation, utilizing the integrating matrix as an operator, and applying the boundary conditions, the spatial dependence is removed from the governing partial differential equations and the resulting ordinary differential equations can be cast into standard eigenvalue form. Solutions can now be determined by standard methods. Historically, integrating matrices have been derived for equal increments of the independent variable by expressing the integrand as a polynomial in the form of Newton's forward-difference interpolation formula. However, many practical beam configurations have spanwise variations in the sectional properties which require the use of an integrating matrix which can treat unequal increments. Thus, a need exists for integrating matrices which can accommodate arbitrary increments in the independent variable.

This report documents the derivation of integrating matrices which are valid for arbitrary increments of the independent variable. To set notation, the section titled "Existing Theory for Equally Spaced Grid Points" begins with a brief review of existing theory for equal increments in the independent variable. The integrating matrix is derived here through use of an interpolation polynomial. In addition to the basic derivation, this section discusses several previously unnoticed features of this approach. Integrating matrices for arbitrary increments of the independent variable are derived under "Theory for Grids with Arbitrary Spacing." Methods based on both interpolation and least-squares polynomial fitting are discussed and related. Finally, to test accuracy, the last section of this report ("Verification of Accuracy Using Integrating Matrices based on Grids with Unequal Spacing") describes the application of the generalized integrating matrices to several problems whose solutions are known. A number of sample matrices are given in Appendices A and B.

EXISTING THEORY FOR EQUALLY SPACED GRID POINTS

The integrating matrix method is a technique for obtaining approximations to the integral of a continuous function on a finite interval. To illustrate the basic ideas in the method, and to develop notation, this section will briefly review the existing theory for equally spaced grid points.

Let $f(x)$ be a continuous function on an interval $[a, b]$. Suppose the values of $f(x)$ are known at the $N+1$ equally spaced grid points

$$x_0 = a \;, \quad x_i = x_0 + ih \qquad (i = 0, 1, \ldots, N) \tag{1}$$

where $h = (b - a)/N$ and $x_N = b$. For convenience, we also define

$$f_i = f(x_i) \tag{2}$$

Information on the function at the grid points may now be used to approximate the function on each of the N subintervals $[x_j, x_{j+1}]$ $(j = 0, \ldots, N-1)$ in the interval $[a, b]$. Consider, for example, the subinterval $[x_0, x_1]$. Assume that $f(x)$ can be approximated by an nth degree polynomial $(n \le N)$ on each subinterval and write $f(x) \simeq g_0(p)$ where $g_0(p)$ is Newton's nth degree forward-difference interpolation polynomial

$$g_0(p) = f_0 + p\Delta f_0 + \frac{1}{2!} p(p-1)\Delta^2 f_0 + \frac{1}{3!} p(p-1)(p-2)\Delta^3 f_0$$
$$+ \ldots + \frac{1}{n!} p(p-1) \ldots (p-n+1)\Delta^n f_0 \tag{3}$$

In this formula

$$p = \frac{x - x_0}{h} \tag{4}$$

while

$$\Delta^k f_0 = (E - 1)^k f_0 \quad \text{witl} \quad E^j f_0 = f_j, \ (j = 1, \ldots, n) \tag{5}$$

i.e., E is a shift operator.

As $dx = hdp$, $f(x) \simeq g_0(p)$ implies that for the integral of $f(x)$ over the interval $[x_0, x_1]$

$$\int_{x_0}^{x_1} f(x)dx \simeq h \int_0^1 g_0(p)dp \tag{6}$$

Integration of equation (3) with respect to p and considerable manipulation now give the approximate result

$$\int_{x_0}^{x_1} f(x)dx \simeq h[a_{00}f_0 + a_{01}f_1 + \ldots + a_{0n}f_n] \tag{7}$$

For example, if n = 3, the approximation to the integral in equation (7) is

$$\int_{x_0}^{x_1} f(x)dx \simeq \frac{h}{24}(9f_0 + 19f_1 - 5f_2 + f_3) \tag{8}$$

Approximations to the integral of $f(x)$ on each of the subintervals $[x_j, x_{j+1}]$ $(0 \le j \le N-1)$ may be similarly obtained. Using the appropriate set of $n+1$ grid points, say $x_m, x_{m+1}, \ldots, x_{m+n}$, which includes x_j and x_{j+1}, $f(x)$ is approximated by an nth degree Newton's forward-difference interpolation polynomial $g_j(p)$. This polynomial is then integrated with respect to p from j to $j+1$ and the result manipulated into the form of a linear combination of the function values $f_m, \ldots, f_{m+n}$

$$\int_{x_j}^{x_{j+1}} f(x)dx \approx h \int_{j}^{j+1} g_j(p)dp = h[a_{jm}f_m + a_{jm+1}f_{m+1}$$

$$+ \dots + a_{jm+n}f_{m+n}] \qquad (9)$$

For example, if $n = 3$, approximations to the integrals of $f(x)$ over the subintervals $[x_1,x_2]$, ..., $[x_{N-1}, x_N]$ are

$$\int_{x_1}^{x_2} f(x)dx \approx \frac{h}{24}(-f_0 + 13f_1 + 13f_2 - f_3)$$

$$\int_{x_2}^{x_3} f(x)dx \approx \frac{h}{24}(-f_1 + 13f_2 + 13f_3 - f_4)$$

$$\vdots$$

$$\int_{x_{N-2}}^{x_{N-1}} f(x)dx \approx \frac{h}{24}(-f_{N-3} + 13f_{N-2} + 13f_{N-1} - f_N)$$

$$\int_{x_{N-1}}^{x_N} f(x)dx \approx \frac{h}{24}(f_{N-3} - 5f_{N-2} + 19f_{N-1} + 9f_N) \qquad (10)$$

Having obtained approximations to the integral of $f(x)$ over each of the N subintervals in $[a, b]$, an approximation to $\int_a^b f(x)dx$ itself is now easily obtained. We need only note that $\int_a^b f(x)dx$ can be written as the "collapsing sum"

$$\int_a^b f(x)\,dx = \left\{\int_{x_0}^{x_1} + \int_{x_1}^{x_2} + \ldots. + \int_{x_{N-1}}^{x_N}\right\} f(x)\,dx \tag{11}$$

and hence sum the approximations on the N subintervals.

The integrating matrix representation for the approximation to the integral of $f(x)$ uses the above information but puts it in a compact matrix form suitable for matrix manipulation Let the superscript T on a vector denote the transpose, and define the $N+1$ dimensional column vectors $\{f\}$ and $\{F\}$ by

$$\{f\} = \left(f_0,\ f_1,\ \ldots\ ,\ f_N\right)^T \tag{12}$$

and

$$\{F\} = \left(0,\ \int_{x_0}^{x_1} f(x)\,dx,\ \int_{x_1}^{x_2} f(x)\,dx,\ \ldots\ ,\ \int_{x_{N-1}}^{x_N} f(x)\,dx\right)^T \tag{13}$$

Then, relations (7) and (9) for all $j = 0, \ldots, N-1$ can be consolidated into a single matrix equation

$$\{F\} = [A_n]\{f\} \tag{14}$$

where $[A_n]$ is an $N+1 \times N+1$ matrix. As the first element of $\{F\}$ is 0, the first row of $[A_n]$ contains all zeros. Each of the remaining N rows of $[A_n]$ contains a group of $n+1$ nonzero elements. For example, by equation (7), the second row of $[A_n]$ is

$$a_{00}\ a_{01}\ a_{02}\ \ldots\ a_{0n}\ 0\ \ldots.\ 0$$

Several sample matrices $[A_n]$ are shown in Appendix A. Finally, in accord with the collapsing sum [eq. (11)], the integrating matrix $[I_n]$ is obtained by left-multiplying both sides of relation (14) by the $N+1 \times N+1$

summing matrix

$$[B] = \begin{bmatrix} 1 & & & & & 0 \\ \vdots & \cdot & & & & \\ \vdots & & \cdot & & & \\ \vdots & & & \cdot & & \\ 1 & \cdot & \cdot & \cdot & \vdots & \vdots & 1 \end{bmatrix} \tag{15}$$

i.e. $b_{ij} = 1$ if $i \geq j$ but $b_{ij} = 0$ if $i < j$. If $\{\hat{F}\}$ is the N+1 dimensional column vector

$$\{\hat{F}\} = \left(0, \int_{x_0}^{x_1} f(x)dx, \int_{x_0}^{x_2} f(x)dx, \ldots, \int_{x_0}^{x_N} f(x)dx\right)^T \tag{16}$$

we now obtain the integrating matrix relation

$$\{\hat{F}\} = [I_n]\{f\} \tag{17}$$

with

$$[I_n] = [B][A_n] \tag{18}$$

As [B] is known a priori, the derivation of the integrating matrix $[I_n]$ is equivalent to the derivation of the matrix $[A_n]$ in equation (14). We also note that while $[A_n]$ depends on the number and spacing of the grid points in the interval of interest and on the degree of the approximating polynomials, it does <u>not</u> depend on the values f_i that the function takes at the grid points. This separation of grid dependence and function dependence is one of the major strengths of the integrating matrix.

As described above, the theory developed in reference 2 is based on the use of a Newton's forward-difference interpolation polynomial to approximate the function on each subinterval. However, as the dimensionless variable p is based on the uniform spacing h, it is impossible to generalize this derivation directly to grid points with unequal spacing.

Two subtle aspects of the above derivation of the integrating matrix for equal spacing have apparently not been recognized. One point involves the use of Newton's forward-difference interpolation formula rather than a Lagrange interpolation formula. The former formula is equivalent to consolidating powers of x with coefficients that involve combinations of the functional values, i.e.

$$f(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n \tag{19}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_n$ are linear combinations of the f_i. A large amount of manipulation may be required to turn equation (19) into an explicit linear combination of the functional values, which is the form that is really required to derive the integrating matrix. By contrast, the Lagrange interpolation formula is already in the proper form without manipulation, i.e. functional values are isolated and coefficients involve linear combinations of powers of x. For example, the Lagrange interpolation polynomial of degree n on the $n+1$ points $x_0, x_1, \ldots x_n$ is

$$F(x) = f_0 \ell_0(x) + f_1 \ell_1(x) + \ldots + f_n \ell_n(x) \tag{20}$$

where

$$\ell_j(x) = \frac{(x-x_0) \ldots (x-x_{j-1})(x-x_{j+1}) \ldots (x-x_n)}{(x_j-x_0) \ldots (x_j-x_{j-1})(x_j-x_{j+1}) \ldots (x_j-x_n)} \tag{21}$$

It is certainly somewhat easier to integrate equation (19) than equations (20) and (21); however, equation (20) is clearly in the precise form required by the integrating matrix derivation. Further, unlike Newton's difference interpolation formula, Lagrange interpolation is not intrinsically limited to equally spaced grid points.

A second important point involves the choice of a forward-difference formula. Rather than using forward differences, which involve starting at x_0 and moving to the right across the interval to x_N, one could equally well use backward differences, start at x_N, and move to the left toward x_0. The two choices are equivalent when the number of interpolation points $n+1$ is even as, away from the end points, the interval $[x_j, x_{j+1}]$ is centered in the interpolation interval $[x_\gamma, x_{\gamma+n}]$ where $\gamma = j - (n-1)/2$ (see fig. 1a). When $n+1$ is odd, however, two distinct choices are possible (see figs. 1b and 1c). Use or forward differences leads to a "left biasing" with one more interpolation grid point to the right of x_{j+1} than to the left of x_j (fig. 1b). Similarly, backward differences lead to a "right biasing" (fig. 1c). As the sample matrices in Appendix A for n = 4 and n = 6 indicate, the right and left biased matrices are related, but quite different. In particular, a different row "marches" across the center of the matrix $[A_n]$. Hence, even for equally spaced grid points, care must be taken when an odd number of interpolation grid points is used to approximate the function on each subinterval. No general rule can be formulated for preferring left to right biasing when $n+1$ is odd. The choice of which type of biasing, if any, to use must be made in the context of each individual application.

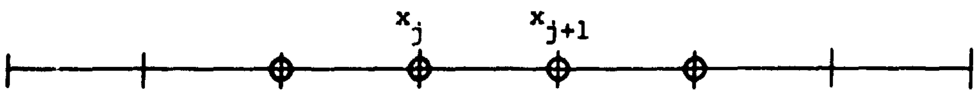
THEORY FOR GRIDS WITH ARBITRARY SPACING

Let $x_0, x_1, \ldots, x_N$ be $N+1$ grid points such that

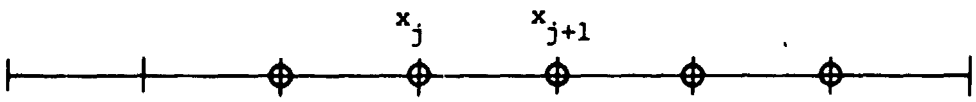
$$a = x_0 < x_1 < \ldots < x_N = b \tag{22}$$

Also, let $f(x)$ again be a continuous function on the interval $[a, b]$ whose values $f_i = f(x_i)$ are known at the grid points. In this section, we do not assume that the grid points are equally spaced in $[a, b]$.

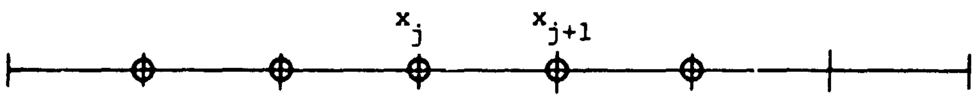
The key step in the derivation of the integrating matrix is use of the data at the grid points to obtain an appropriate approximation to the function $f(x)$ in each subinterval $[x_j, x_{j+1}]$ $(j = 0, \ldots, N-1)$. For arbitrary grid spacing, these approximations, and the corresponding integrating matrices, can be obtained in several ways. Two methods, interpolation and least-squares polynomial fitting, are discussed in this report.



(a) No biasing



(b) Left biasing



(c) Right biasing

Figure 1. Examples of grid points used to approximate the function on the interval $[x_j, x_{j+1}]$. Approximating points are marked with O. In (a), $n+1 = 4$, while in (b) and (c) $n+1 = 5$.

Interpolation

Both the classical Lagrange interpolation formula and the more modern divided difference interpolation formulas are valid for arbitrarily spaced grid points. Indeed, the Newton's difference formulas are the specialization of divided differences to uniform grid spacings. Unfortunately, like the Newton's difference formulas, divided differences in effect consolidate powers of x. The resulting coefficients again involve linear combinations of the data and are not well suited to use in deriving the integrating matrix. For thi' reason, we shall restrict attention here to the more appropriate Lagrange interpolation formula.

The first step in approximating $f(x)$ by an nth degree interpolating polynomial on the interval $[x_j, x_{j+1}]$, $j = 0, \ldots, N-1$, is to determine the appropriate set of $n+1$ grid points

$$G_\gamma = \{x_\gamma, x_{\gamma+1}, \ldots, x_{\gamma+n}\} \tag{23}$$

to use in the Lagrange interpolation formula. The index γ in equation (23) will be a function of both j and n, and will further depend on whether $n+1$ is even, odd with right biasing required, or odd with left biasing required. For grids $x_0, \ldots, x_N$ with $n+1$ points, there will be $N - n + 1$ different sets G_γ of $n+1$ consecutive grid points.

Define $\gamma_1(j, n)$, $\gamma_2(j, n)$ and $\gamma_3(j, n)$ by the relations

$$\gamma_1(j, n) = j - \frac{(n - 1)}{2} \tag{24}$$

$$\gamma_2(j, n) = j - \frac{n}{2} \tag{25}$$

and

$$\gamma_3(j, n) = j - \frac{n}{2} + 1 \tag{26}$$

Then, with

$$\bar{\gamma}(j,\ n) = \begin{cases} \gamma_1 & \text{if } n+1 \text{ is even} \\ \gamma_2 & \text{if } n+1 \text{ is odd with right biasing} \\ \gamma_3 & \text{if } n+1 \text{ is odd with left biasing} \end{cases} \tag{27}$$

the appropriate definition of the interpolation set index $\gamma(j,n)$ in equation (23) is

$$\gamma(j,\ n) = \begin{cases} 0 & \text{if} \quad \bar{\gamma} \le 0 \\ \bar{\gamma} & \text{if} \quad 0 < \bar{\gamma} < N-n \\ N-n & \text{if} \quad \bar{\gamma} \ge N-n \end{cases} \tag{28}$$

Having properly defined the interpolation index γ, and hence the interpolation set G_γ, the nth degree Lagrange interpolation polynomial for $f(x)$ on the subinterval $[x_j,\ x_{j+1}]$, $j = 0, \ldots, N-1$, is of the form

$$f(x) \simeq f_\gamma \ell_\gamma^{(\gamma)}(x) + f_{\gamma+1}\ell_{\gamma+1}^{(\gamma)} + \ldots + f_{\gamma+n}\ell_{\gamma+n}^{(\gamma)}(x) \tag{29}$$

where the Lagrange coefficients $\ell_m^{(\gamma)}(x)$ are now

$$\ell_m^{(\gamma)}(x) = \frac{(x - x_\gamma) \ldots (x - x_{m-1})(x - x_{m+1}) \ldots (x - x_{\gamma+n})}{(x_m - x_\gamma) \ldots (x_m - x_{m-1})(x_m - x_{m+1}) \ldots (x_m - x_{\gamma+n})} \tag{30}$$

If we further define

$$F_m^{(j)} = \int_{x_j}^{x_{j+1}} \ell_m^{(\gamma)}(x)\,dx \tag{31}$$

then equation (30) immediately shows that the (j+2) row of the matrix $[A_n]$ for $j = 0, \ldots, N-1$ is

$$\underbrace{0 \ldots 0}_{\gamma \text{ zeros}} F_{\gamma}^{(j)} \; F_{\gamma+1}^{(j)} \ldots F_{\gamma+n}^{(j)} \underbrace{0 \ldots 0}_{N-n-\gamma \text{ zeros}} \tag{32}$$

As before, the first row of $[A_n]$ contains all zero elements.

The derivation of the integrating matrix via Lagrange interpolation is quite straightforward. However, in practice, the actual computation of the nN+N quantities $F_m^{(j)}$ in the matrix $[A_n]$ for arbitrary grid spacings may be algebraically difficult. This is because the Lagrange coefficients $\ell_m^{(\gamma)}(x)$ in equations (30) and (31) involve a product of n factors rather than simply powers of x. Further, evaluation of the denominators in the $\ell_m^{(\gamma)}(x)$ requires computation of a large number of differences of the form $x_k - x_m$ ($m \neq k$ and $|k - m| \leq n$). Fortunately, these implementation problems are not associated with the next method to be considered. Further, the following method for deriving an integrating matrix includes interpolation as a special case.

Least-Squares Polynomial Fitting

In interpolation, the interpolating polynomial agrees exactly with the function at the n+1 interpolation grid points. However, in certain situations, interpolation may not make the best use of the data at the grid points. Suppose, for example, that the degree of reliability of the values assigned to the function at the grid points is not well established. For example, this might be the case if the data on the function comes from experimental measurements and small random measurement errors are suspected. In these circumstances, as pointed out in reference 5, " ... it is foolish (and, indeed, inherently dangerous) to attempt to determine a polynomial of high degree which fits the vagaries of such data exactly and hence, in all probability, is represented by a curve which oscillates violently about the curve which represents the true function." Fitting the data with a lower degree polynomial may be far more appropriate as the resulting approximation to the function will "smooth" out unrealistic oscillations.

In the present context, we will consider approximating $f(x)$ on each subinterval by a least-squares polynomial fit based on data at n+1 grid points with polynomials of degree k (where $k \leq n$). To briefly review least-square polynomial fits, let $p_0(x), p_1(x), \ldots, p_k(x)$ be a sequence of $k+1$ appropriately chosen polynomials of degrees 0, 1, 2, ... , k. Let $f_k(x)$ be the kth degree polynomial

$$f_k(x) = \sum_{m=0}^{k} \beta_m \, p_m(x) \tag{33}$$

Suppose we wish to approximate $f(x)$ on the interval $[a_0, a_n]$ containing the $n+1$ grid points $a_0, a_1, a_2, \ldots, a_n$. Then $f_k(x)$ is the required least-squares polynomial fit of degree k on this grid if the sum of the errors squared over the grid is a minimum, i.e.

$$\varepsilon = \sum_{\ell=0}^{n} \left[f(a_\ell) - \sum_{m=0}^{k} \beta_m p_m(a_\ell) \right]^2 = \text{minimum} \tag{34}$$

We note that if $k = n$, the interpolating polynomial on the grid $a_0, a_1, \ldots, a_n$ has the minimum value of ε as it agrees with $f(x)$ exactly at all grid points. Hence, interpolation can be considered a special case of least-square polynomial for which $k = n$.

The coefficients β_m in the kth degree least-squares fit $f_k(x)$ are solutions of the $k+1$ linear equations $\partial\varepsilon/\partial\beta_m = 0$ ($m = 0, \ldots, k$). Solution of these equations is considerably simplified if the polynomials $p_m(x)$ in $f_k(x)$ are chosen to be orthogonal on the grid points $a_0, \ldots, a_n$. In the present context, this means

$$\sum_{\ell=0}^{n} p_m(a_\ell) \; p_s(a_\ell) = 0 \qquad \text{for} \qquad m \neq s \tag{35}$$

Orthogonal polynomials on a discrete grid may be defined recursively. Let the orthogonal polynomials on the grid be denoted by $P_j(x)$ and let

$$T_j = \sum_{i=0}^{n} a_i \left[P_j(a_i)\right]^2 \tag{36}$$

$$S_j = \sum_{i=0}^{n} \left[P_j(a_i)\right]^2 \tag{37}$$

$$\mu_{j+1} = T_j/S_j \qquad \text{and} \qquad \nu_j = S_j/S_{j-1} \tag{38}$$

Then,

$$P_0(x) = 1 \tag{39}$$

$$P_1(x) = x - \mu_1 \tag{40}$$

and

$$P_{j+1}(x) = (x - \mu_{j+1})P_j(x) - \nu_j\, P_{j-1}(x) \tag{41}$$

The coefficients β_m in the least-squares orthogonal polynomial fit of degree $k \leq n$ on the $n+1$ grid points $a_0, \ldots, a_n$ are now

$$\beta_m = \frac{\sum_{i=0}^{n} y_i\, P_m(a_i)}{\sum_{i=0}^{n} \left[P_m(a_i)\right]^2} \tag{42}$$

where $y_i = f(a_i)$.

Equations (33) and (42) as written involve linear combinations of the grid data in each coefficient β_m and so must be modified for the present purposes. However, the required modifications to isolate grid data are quite straightforward. We obtain the basic relation

$$f_k(x) = \sum_{i=0}^{n} y_i \, q_i(x) \tag{43}$$

where the coefficients $q_i(x)$ are given by

$$q_i(x) = \sum_{i=0}^{k} \frac{P_j(a_i) \; P_j(x)}{\left(\sum_{\ell=0}^{n} \left[P_j(a_\ell) \right]^2 \right)} \tag{44}$$

Equations (43) and (44) may now be used to approximate the integral of $f(x)$ on any interval contained in $[a_0, a_n]$. We note that the only integrals contained in such approximations will be integrals of the orthogonal polynomials $P_j(x)$ $(j = 0, \ldots, k)$ which are quite easy to evaluate. Hence, the implementation difficulties associated with Lagrange interpolation will not be present when the least-squares fit is used.

The above discussion gives the least-squares orthogonal polynomial fit to $f(x)$ based on data at the grid points $a_0, a_1, \ldots, a_n$. We now wish to apply these results to approximate the integral of $f(x)$ on the subintervals $[x_j, x_{j+1}]$ $(j = 0, \ldots, N-1)$. For this purpose, let the set G_γ be as defined in equations (23) through (28) and let

$$a_0 = x_\gamma, \; a_1 = x_{\gamma+1}, \; \ldots, \; a_n = x_{\gamma+1} \tag{45}$$

We also denote by $P_m^{(\gamma)}(x)$ $(m = 0, \ldots, k)$ the orthogonal polynomials of degree less than or equal to k on the grid G_γ. As the grid data

y_i is now, by equation (45), $y_i = f(x_{\gamma+1}) = f_{\gamma+i}$, by equations (43) and (44) the appropriate fit for $f(x)$ on the interval $[x_j, x_{j+1}]$ is now

$$f(x) \simeq f_k^{(\gamma)}(x) = \sum_{i=0}^{n} f_{\gamma+i}\, q_i^{(\gamma)}(x) \tag{46}$$

where

$$q_i^{(\gamma)}(x) = \sum_{m=0}^{k} \frac{P_m^{(\gamma)}(x_{\gamma+1})\, P_m^{(\gamma)}(x)}{\left\{\sum_{\ell=0}^{n} \left[P_m^{(\gamma)}(x_{\gamma+\ell})\right]^2\right\}} \tag{47}$$

Thus, if

$$Q_{\gamma+i}^{(j)} = \int_{x_j}^{x_{j+1}} q_i^{(\gamma)}(x)\,dx = \sum_{m=0}^{k} \frac{P_m^{(\gamma)}(x_{\gamma+i}) \displaystyle\int_{x_j}^{x_{j+1}} P_m^{(\gamma)}(x)\,dx}{\left\{\sum_{\ell=0}^{m} \left[P_m^{(\gamma)}(x_{\gamma+\ell})\right]^2\right\}} \tag{48}$$

equation (46) immediately gives for the (j+2) row of the matrix $[A_n]$

$$\underbrace{0 \;\ldots\; 0}_{\gamma \text{ zeros}} \;\; Q_{\gamma}^{(j)} \;\; Q_{\gamma+1}^{(j)} \;\ldots\; Q_{\gamma+n}^{(j)} \;\; \underbrace{0 \;\ldots\; 0}_{N-n-\gamma \text{ zeros}} \tag{49}$$

We note that, in this notation, the subscript n on the $N+1 \times N+1$ matrix $[A_n]$ denotes that $n+1$ grid points are used to approximate $f(x)$

on each subinterval $[x_j, x_{j+1}]$, not that the approximating polynomial is of degree n. As noted previously, if the degree k of the approximating polynomial does equal n (its maximum possible value), we recover the matrix $[A_n]$ obtained from interpolation.

An example of a matrix $[A_n]$ obtained via least-squares fitting is given in Appendix A. It is interesting to note that when $k = n - 1$, the middle rows of the matrices $[A_n]$ are the same as for the interpolation case when $k = n$. This feature disappears when $k < n - 1$. Additional examples in Appendix B give matrices $[A_n]$ for various sample nonuniform grids.

VERIFICATION OF ACCURACY USING INTEGRATING MATRICES BASED ON GRIDS WITH UNEQUAL SPACING

As a test of accuracy, integrating matrices based on grids with unequal spacing were employed in the analysis of four types of beam problems previously analyzed in reference 4. These include the axial vibrations of a rotating beam, buckling of a rotating beam, vibration and stability of a rotating preconed beam, and the stability of an asymmetric shaft. For the present computations, the length of the beam was divided into 10 subintervals of unequal length (N=10). Eight grid points were used to obtain approximations on each subinterval (n=7). Both interpolation (k=n) and least-squares polynomial fit (k<n) cases were considered.

Numerical results using matrices based on seventh-degree interpolation polynomials (k=n) were found to be in excellent agreement with results in reference 4. These earlier results were obtained using integrating matrices based on equally spaced grid points and seventh-degree polynomials.

Most applications will involve the use of approximating polynomials of maximum degree (k=n). However, if smoothing of experimental data is a factor, a least-squares polynomial fit (k<n) rather than interpolation

may be appropriate. Limited numerical evidence in the present study suggests that if a least-squares fit is used, the degree of the approximating polynomials should be $k=n-1$. Results for $k<n-1$ may be suspect if k is relatively small compared to n.

APPENDIX A

INTEGRATING MATRICES FOR EQUALLY SPACED GRID POINTS

This appendix contains examples of matrices $[A_n]$ for equal grid spacings based on approximating polynomials of degree n. When n is odd, the resulting matrix $[A_n]$ is unique for a given grid. The matrix $[A_7]$ shown in table A1 is an example of this case. When n is even, however, two different matrices $[A_n]$ are possible for a given grid, depending on whether the approximating scheme is left- or right-biased (see fig. 1). This fact has not been previously recognized. Tables A2 to A5 show that the two possible matrices $[A_n]$ are related. However, as can be seen, a different row "marches" across the center of the matrix in the left- and right-biased cases. It should be noted that the derivation in reference 2 corresponds to a left-biased approximation.

The matrix $[A_7]$ obtained using least-squares fit polynomials of degree six rather than interpolating polynomials of degree seven is shown in table A6.

Table A1. The matrix $[A_7]$ for grids with equal spacing $x_{j+1} - x_j = h$, $(j=0, \ldots, 10)$, based on seventh degree interpolation polynomials $(k = n = 7)$.

$$
\frac{h}{120,960}
\begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
36799.0 & 139849.0 & -121797.0 & 123133.0 & -88547.0 & 41499.0 & -11351.0 & 1375.0 & 0.0 & 0.0 & 0.0 \\
-1375.0 & 47799.0 & 101349.0 & -44797.0 & 26883.0 & -11547.0 & 2999.0 & -351.0 & 0.0 & 0.0 & 0.0 \\
351.0 & -4183.0 & 57627.0 & 81693.0 & -20227.0 & 7227.0 & -1719.0 & 191.0 & 0.0 & 0.0 & 0.0 \\
-191.0 & 1879.0 & -9531.0 & 68323.0 & 68323.0 & -9531.0 & 1879.0 & -191.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -191.0 & 1879.0 & -9531.0 & 68323.0 & 68323.0 & -9531.0 & 1879.0 & -191.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -191.0 & 1879.0 & -9531.0 & 68323.0 & 68323.0 & -9531.0 & 1879.0 & -191.0 & 0.0 \\
0.0 & 0.0 & 0.0 & -191.0 & 1879.0 & -9531.0 & 68323.0 & 68323.0 & -9531.0 & 1879.0 & -191.0 \\
0.0 & 0.0 & 0.0 & 191.0 & -1719.0 & 7227.0 & -20227.0 & 81693.0 & 57627.0 & -4183.0 & 351.0 \\
0.0 & 0.0 & 0.0 & -351.0 & 2999.0 & -11547.0 & 26883.0 & -44797.0 & 101349.0 & 47799.0 & -1375.0 \\
0.0 & 0.0 & 0.0 & 1375.0 & -11351.0 & 41499.0 & -88547.0 & 123133.0 & -121797.0 & 139849.0 & 36799.0
\end{bmatrix}
$$

Table A2. The matrix $[\Lambda_4]$ for grids with equal spacing $x_{j+1} - x_j = h$, $(j = 0, \ldots, 10)$, based on left-biased approximations using fourth degree interpolating polynomials $(k = n = 4)$.

$$\frac{h}{720}\begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
251.0 & 646.0 & -261.0 & 106.0 & -19.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-19.0 & 346.0 & 456.0 & -74.0 & 11.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -19.0 & 346.0 & 456.0 & -74.0 & 11.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -19.0 & 346.0 & 456.0 & -74.0 & 11.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & -19.0 & 346.0 & 456.0 & -74.0 & 11.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & -19.0 & 346.0 & 456.0 & -74.0 & 11.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -19.0 & 346.0 & 456.0 & -74.0 & 11.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -19.0 & 346.0 & 456.0 & -74.0 & 11.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 11.0 & -74.0 & 456.0 & 346.0 & -19.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -19.0 & 106.0 & -264.0 & 646.0 & 251.0
\end{bmatrix}$$

Table A3. The matrix $[A_4]$ for grids with equal spacing $x_{j+1} - x_j = h$, $(j=0, \ldots, 10)$ based on right-biased approximations using fourth degree interpolating polynomials $(k = n = 4)$.

$$\frac{h}{720}\begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
251.0 & 646.0 & -264.0 & 106.0 & -19.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-19.0 & 346.0 & 456.0 & -74.0 & 11.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
11.0 & -74.0 & 456.0 & 346.0 & -19.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 11.0 & -74.0 & 456.0 & 346.0 & -19.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 11.0 & -74.0 & 456.0 & 346.0 & -19.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 11.0 & -74.0 & 456.0 & 346.0 & -19.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 11.0 & -74.0 & 456.0 & 346.0 & -19.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 11.0 & -74.0 & 456.0 & 346.0 & -19.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 11.0 & -74.0 & 456.0 & 346.0 & -19.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -19.0 & 106.0 & -264.0 & 646.0 & 251.0
\end{bmatrix}$$

Table A4. The matrix $[A_6]$ for grids with equal spacing $x_{j+1} - x_j = h$, ($j=0, \ldots, 10$) based on left-biased approximations using sixth degree interpolating polynomials ($k = n = 6$).

$$\frac{h}{60,480}\begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
19087.0 & 65112.0 & -46461.0 & 37504.0 & -20211.0 & 6312.0 & -863.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-863.0 & 25128.0 & 46989.0 & -16256.0 & 7299.0 & -2088.0 & 271.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
271.0 & -2760.0 & 30819.0 & 37504.0 & -6771.0 & 1608.0 & -191.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 271.0 & -2760.0 & 30819.0 & 37504.0 & -6771.0 & 1608.0 & -191.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 271.0 & -2760.0 & 30819.0 & 37504.0 & -6771.0 & 1608.0 & -191.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 271.0 & -2760.0 & 30819.0 & 37504.0 & -6771.0 & 1608.0 & -191.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 271.0 & -2760.0 & 30819.0 & 37504.0 & -6771.0 & 1608.0 & -191.0 \\
0.0 & 0.0 & 0.0 & 0.0 & -191.0 & 1608.0 & -6771.0 & 37504.0 & 30819.0 & -2760.0 & 271.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 271.0 & -2088.0 & 7299.0 & -16256.0 & 46989.0 & 25128.0 & -863.0 \\
0.0 & 0.0 & 0.0 & 0.0 & -863.0 & 6312.0 & -20211.0 & 37504.0 & -46461.0 & 65112.0 & 19087.0
\end{bmatrix}$$

Table A5. The matrix $[A_6]$ for grids with equal spacing $x_{j+1} - x_j = h$, $(j=0, \ldots, 10)$ based on right-biased approximations using sixth degree interpolating polynomials $(k = n = 6)$.

$$
\frac{h}{60{,}480}
\begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
19087.0 & 65112.0 & -46461.0 & 37504.0 & -20211.0 & 6312.0 & -863.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-863.0 & 25128.0 & 46989.0 & -16256.0 & 7299.0 & -2088.0 & 271.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
271.0 & -2760.0 & 30819.0 & 37504.0 & -6771.0 & 1688.0 & -191.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-191.0 & 1688.0 & -6771.0 & 37504.0 & 30819.0 & -2760.0 & 271.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -191.0 & 1688.0 & -6771.0 & 37504.0 & 30819.0 & -2760.0 & 271.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -191.0 & 1688.0 & -6771.0 & 37504.0 & 30819.0 & -2760.0 & 271.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & -191.0 & 1688.0 & -6771.0 & 37504.0 & 30819.0 & -2760.0 & 271.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & -191.0 & 1688.0 & -6771.0 & 37504.0 & 30819.0 & -2760.0 & 271.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 271.0 & -2088.0 & 7299.0 & -16256.0 & 46989.0 & 25128.0 & -863.0 \\
0.0 & 0.0 & 0.0 & 0.0 & -863.0 & 6312.0 & -20211.0 & 37504.0 & -46461.0 & 65112.0 & 19087.0
\end{bmatrix}
$$

Table A6. The matrix $[A_7]$ for grids with equal spacing $x_{j+1} - x_j = h$, $(j=0, \ldots, 10)$ based on least-squares fit polynomials of degree $k = n - 1 = 6$.

$$\frac{h}{120,960}\begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
40255.0 & 115657.0 & -49221.0 & 2173.0 & 32413.0 & -31077.0 & 12841.0 & -2081.0 & 0.0 & 0.0 & 0.0 \\
-2765.1 & 57189.9 & 73416.4 & 1757.4 & -19671.4 & 16385.6 & -6311.9 & 979.1 & 0.0 & 0.0 & 0.0 \\
1076.9 & -9264.5 & 72871.5 & 56285.4 & 5188.6 & -8017.5 & 3362.5 & -534.9 & 0.0 & 0.0 & 0.0 \\
-191.0 & 1879.0 & -9531.0 & 68323.0 & 68323.0 & -9531.0 & 1879.0 & -191.0 & 0.0 & 0.0 & 0.0 \\
0.0 & -191.0 & 1879.0 & -9531.0 & 68323.0 & 68323.0 & -9531.0 & 1879.0 & -191.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -191.0 & 1879.0 & -9531.0 & 68323.0 & 68323.0 & -9531.0 & 1879.0 & -191.0 & 0.0 \\
0.0 & 0.0 & 0.0 & -191.0 & 1879.0 & -9531.0 & 68323.0 & 68323.0 & -9531.0 & 1879.0 & -191.0 \\
0.0 & 0.0 & 0.0 & -534.9 & 3362.5 & -8017.5 & 5188.6 & 56285.4 & 72871.5 & -9264.5 & 1076.9 \\
0.0 & 0.0 & 0.0 & 979.1 & -6311.9 & 16385.6 & -19671.4 & 1757.4 & 73416.4 & 57189.9 & -2765.1 \\
0.0 & 0.0 & 0.0 & -2081.0 & 12841.0 & -31077.0 & 32413.0 & 2173.0 & -49221.0 & 115657.0 & 40255.0
\end{bmatrix}$$

APPENDIX B

INTEGRATING MATRICES FOR GRIDS WITH ARBITRARY SPACING

This appendix contains three examples of the matrix $[A_7]$ derived using approximation by interpolating polynomials (k=n=7) on grids with unequal spacing.

Table B1. The matrix $[A_7]$ based on seventh degree interpolation polynomials $(k = n = 7)$ for the grid $\{x_0, x_1, \ldots, x_{10}\} = \{0, 1, 3, 6, 18, 30, 42, 54, 57, 59, 60\}$.

$$\frac{1}{20,160}\begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
8162.6 & 13249.4 & -1463.1 & 215.6 & -5.3 & 0.9 & -0.2 & 0.0 & 0.0 & 0.0 & 0.0 \\
-4583.7 & 25501.0 & 20488.5 & -1104.1 & 21.3 & -3.6 & 0.6 & -0.1 & 0.0 & 0.0 & 0.0 \\
9054.4 & -24056.6 & 51871.5 & 23718.5 & -119.6 & 18.4 & -2.9 & 0.3 & 0.0 & 0.0 & 0.0 \\
-556373.3 & 1248657.5 & -1131904.6 & 597792.0 & 87968.0 & -4814.3 & 647.1 & -52.4 & 0.0 & 0.0 & 0.0 \\
0.0 & -62839.3 & 158494.0 & -127315.8 & 160548.1 & 126028.3 & -16602.6 & 7229.7 & -3622.4 & 0.0 & 0.0 \\
0.0 & 0.0 & -3622.4 & 7229.7 & -16602.6 & 126028.3 & 160548.1 & -127315.8 & 158494.0 & -62839.3 & 0.0 \\
0.0 & 0.0 & 0.0 & -52.4 & 647.1 & -4814.3 & 87968.0 & 597792.0 & -1131904.6 & 1248657.5 & -556373.3 \\
0.0 & 0.0 & 0.0 & 0.3 & -2.9 & 18.4 & -119.6 & 23718.5 & 51871.5 & -24056.6 & 9054.4 \\
0.0 & 0.0 & 0.0 & -0.1 & 0.6 & -3.6 & 21.3 & -1104.1 & 20488.5 & 25501.0 & -4583.7 \\
0.0 & 0.0 & 0.0 & 0.0 & -0.2 & 0.9 & -5.3 & 215.6 & -1463.1 & 13249.4 & 8162.6
\end{bmatrix}$$

Table B2. The matrix $[A_7]$ based on seventh degree interpolation polynomials ($k = n = 7$) for the grid $\{x_0, x_1, \ldots, x_{10}\} = \{0, 9, 18, 27, 36, 45, 48, 51, 54, 57, 60\}$.

$$\frac{1}{20,160}\begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
54413.8 & 216583.6 & -209551.9 & 249881.5 & -251655.8 & 393214.5 & -367014.9 & 95569.3 & 0.0 & 0.0 & 0.0 \\
-1863.3 & 69970.1 & 158837.7 & -83726.0 & 70431.8 & -100894.5 & 92318.1 & -23633.9 & 0.0 & 0.0 & 0.0 \\
419.1 & -5342.6 & 82760.1 & 131441.5 & -46527.8 & 55534.5 & -49058.8 & 12206.0 & 0.0 & 0.0 & 0.0 \\
-180.8 & 1902.4 & -10689.5 & 91754.0 & 118311.8 & -57550.5 & 46895.8 & -11003.1 & 0.0 & 0.0 & 0.0 \\
0.0 & -67.3 & 788.3 & -5329.6 & 74623.5 & 241249.5 & -199317.4 & 83141.5 & -13648.5 & 0.0 & 0.0 \\
0.0 & 0.0 & -1.1 & 16.2 & -182.2 & 24830.6 & 44019.2 & -10739.7 & 2906.9 & -369.9 & 0.0 \\
0.0 & 0.0 & 0.0 & -0.9 & 24.1 & -2079.5 & 30083.5 & 37856.3 & -6781.0 & 1553.8 & -176.2 \\
0.0 & 0.0 & 0.0 & 0.8 & -19.3 & 1194.2 & -6464.0 & 37551.0 & 30606.6 & -2637.2 & 247.1 \\
0.0 & 0.0 & 0.0 & -1.3 & 30.8 & -1594.2 & 7163.5 & -16743.8 & 47601.2 & 24833.8 & -810.0 \\
0.0 & 0.0 & 0.0 & 4.7 & -108.7 & 5039.5 & -20744.0 & 44407.8 & -49411.0 & 66434.8 & 18856.9
\end{bmatrix}$$

Table B3. The matrix $[A_7]$ based on seventh degree interpolation polynomials ($k = n = 7$) for the grid $\{x_0, x_1, \ldots, x_{10}\} = \{0, 7, 14, 21, 28, 30, 32, 39, 46, 53, 60\}$.

$$\frac{1}{20{,}160}\begin{bmatrix}
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
41570.5 & 175861.5 & -198941.0 & 325645.6 & -1186430.9 & 1557432.3 & -584757.9 & 10739.8 & 0.0 & 0.0 & 0.0 \\
-1264.0 & 52596.1 & 132392.5 & -97389.9 & 296804.7 & -378037.4 & 138404.7 & -2386.7 & 0.0 & 0.0 & 0.0 \\
231.1 & -3221.2 & 59850.8 & 118673.2 & -158054.5 & 187955.0 & -65319.7 & 1005.3 & 0.0 & 0.0 & 0.0 \\
-57.7 & 664.6 & -4382.8 & 58759.0 & 193548.3 & -150822.6 & 43929.0 & -517.8 & 0.0 & 0.0 & 0.0 \\
0.0 & -1.1 & 15.2 & -148.2 & 17830.1 & 25674.7 & -3099.4 & \text{51.[illegible]} & -2.4 & 0.0 & 0.0 \\
0.0 & 0.0 & -2.4 & 51.2 & -3099.4 & 25674.7 & 17830.1 & -148.2 & 15.2 & -1.1 & 0.0 \\
0.0 & 0.0 & 0.0 & -517.8 & 43929.0 & -150822.6 & 193548.3 & 58759.0 & -4382.8 & 664.6 & -57.7 \\
0.0 & 0.0 & 0.0 & 1005.3 & -65319.7 & 187955.0 & -158054.5 & 118673.2 & 59850.8 & -3221.2 & 231.1 \\
0.0 & 0.0 & 0.0 & -2386.7 & 138404.7 & -378037.4 & 296804.7 & -97389.9 & 132392.5 & 52596.1 & -1264.0 \\
0.0 & 0.0 & 0.0 & 10739.8 & -584757.9 & 1557432.3 & -1186430.9 & 325645.6 & -198941.0 & 175861.5 & 41570.5
\end{bmatrix}$$

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16. Abstract

Integrating matrices form the basis of an efficient numerical procedure for solving differential equations associated with rotating beam configurations. By expressing the equations of motion in matrix notation, utilizing the integrating matrix as an operator, and applying the boundary conditions, the spatial dependence is removed from the governing partial differential equations and the resulting ordinary differential equations can be cast into standard eigenvalue form. Previous derivations of integrating matrices are limited to the case of equally spaced grid points and approximation by interpolating polynomials. The restriction to equally spaced grid points may not be appropriate for some beam configurations. This report derives integrating matrices for arbitrarily spaced grid points using either interpolating or least-squares fit orthogonal polynomials. Several previously unnoticed features of the equally spaced grid case are also discussed.

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