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ANALYSIS OF A UNIDIRECTIONAL COMPOSITE CONTAINING
BROKEN FIBERS AND MATRIX DAMAGE

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ABSTRACT

An analytical solution is developed for the determination of the stresses and displacements in a unidirectional fiber-reinforced composite containing an arbitrary number of broken fibers as well as longitudinal yielding and splitting of the matrix.

The solution is developed using a "materials-modeling" approach which is based on a shear-lag stress transfer mechanism. The equilibrium equation in the axial direction gives a pair of integral equations which are solved numerically.

Excellent agreement is shown to exist between the solution and experimental results for notched unidirectional boron/aluminum laminates without splitting. For brittle matrix composites (ie. epoxy) equally good results are indicated for both matrix yielding and splitting.

For yielding without splitting the fracture strength is found to depend on crack length while for large splitting it is crack length independent.

INTRODUCTION

An attempt is made in this study to develop an analytical model capable of predicting the characteristic strength and fracture properties of a unidirectional composite laminate. The investigation considers a two-dimensional region containing an arbitrary number of broken fibers as well as longitudinal matrix damage as shown in Figure 1. The fiber breaks lie along a transverse line and therefore represent a notch. Damage to the matrix originates at the ends of the notch, i.e., in the region between the last broken and first unbroken fiber and consists of both yielding and splitting. Symmetry is assumed as indicated and only the first quadrant of the region is shown in Figure 1.

The matrix is taken to be an elastic-perfectly plastic material and the fibers are linearly elastic. Load is transferred from adjacent fibers through the matrix by a simple shear-lag mechanism with the shear stresses being independent of transverse displacements. The axial fiber stress is also independent of transverse displacements and the equilibrium equation in the fiber direction reduces to an equation in the longitudinal displacement alone, as is typical of shear-lag solutions.

Similar investigations have been presented by Hedgepeth and Van Dyke in [1] and [2] in which only one broken fiber was considered with yielding alone in [1] and splitting alone in [2]. The extension to more than one broken fiber however is not developed conveniently by the influence function technique as suggested in [1] because the broken fiber adjacent to the damaged region is not typical of any of the remaining broken fibers.

For a given number of broken fibers and a known applied axial stress, taken to be uniform at points remote from the damage, the stress concentration in unbroken fibers, the extent of the matrix damage and the fiber displacements are desired.

An interesting study is presented by Peters in [3] concerning the fracture strength of unidirectional composites which exhibit large matrix splitting (Boron/epoxy, graphite/epoxy) and those such as boron/aluminum which have large plastic yielding but little splitting. For the first class the fracture strength is independent of crack length while in the second crack length dependence is found. This behavior is considered in detail using the present model and the extremes of large splitting and large yielding with no splitting are predicted accurately. The model apparently does not have the capability of accounting for those composites which exhibit matrix yielding and small but stable longitudinal splitting. Some consideration is given to the reasons for such difficulties and possible modifications to give a more complete model are discussed.

FORMULATION

The laminate is modeled as a two-dimensional region, shown in Figure 1, having a single row of parallel, identical, equally spaced fibers with the broken fibers being symmetric about the center line and the matrix damage occuring between the last broken and the first unbroken fiber. It is assumed that the fibers have a much higher elastic modulus in the axial direction than the matrix and therefore the fibers are taken as supporting all of the axial stress in the laminate. The matrix supports transverse normal stresses and shear stresses.

Admittedly, most unidirectional composites consist of more than one lamina with all fibers ~~in each lamina~~ surely not perfectly aligned either through the thickness or, with-in each layer. These variations can have a considerable influence on the stress state. For example, in [4] and [5] it is shown that the shear stress becomes large as the fiber spacing decreases, i.e. $O(1/\sqrt{d})$ for rigid fibers where d is the minimum distance between fibers. Local failure may well occur at critical points through the thickness in advance of laminate splitting which could give an apparent shear stiffness considerably different from that for the matrix alone. It is assumed that such variations can be accounted for by an appropriate choice of the shear modulus G_M and the transfer distance, h. It is with this in mind that the following development will be concerned with an equivalent lamina where G_M and h are to be determined experimentally for any particular laminate.

A free-body diagram for a typical element is given in Figure 2, with the special condition for the last broken fiber, denoted by $n = N$, and for $y \leq L$ that

$$\tau|_{N+1} = -\tau_o \langle y - \ell \rangle \ , \tag{1}$$

where

$$\begin{aligned} \langle y - \ell \rangle &= 1 \ , \ y \geq \ell \ , \\ \langle y - \ell \rangle &= 0 \ , \ y < \ell \ , \quad \text{and} \end{aligned} \tag{2}$$

L equals the total damaged length, ℓ split length, and τ_o the matrix yield stress.

The equilibrium equations in the longitudinal and transverse directions respectively for all fibers n, with the exception of N and $N+1$ when $y \leq L$, are

$$\frac{A_F}{t} \frac{d\, \sigma_f|_n}{dy} + \tau|_{n+1} - \tau|_n = 0 \ , \quad \text{and} \tag{3}$$

$$\sigma_m|_{n+1} - \sigma_m|_n + \frac{h}{2} \frac{d}{dy} \{\tau|_{n+1} + \tau|_n\} = 0 \tag{4}$$

For fiber N, $y \leq L$, $\tau|_{N+1} = -\tau_o \langle y - \ell \rangle$, and the equilibrium equations are

$$\frac{A_F}{t}\frac{d\sigma_f|_N}{dy} - \tau_o\langle y - \ell\rangle - \tau|_N = 0\ , \text{ and} \tag{5}$$

$$\sigma_m|_{N+1} - \sigma_m|_N + \frac{h}{2}\frac{d}{dy}\{-\tau_o\langle y - \ell\rangle + \tau|_N\} = 0\ . \tag{6}$$

For fiber $N+1$, $y \leq L$, $\tau|_{N+1} = -\tau_o\langle y - \ell\rangle$, and the equilibrium equations are

$$\frac{A_F}{t}\frac{d\sigma_f|_{N+1}}{dy} + \tau_{N+2} + \tau_o\langle y - \ell\rangle = 0\ , \text{ and} \tag{7}$$

$$\sigma_m|_{N+2} - \sigma_m|_{N+1} + \frac{h}{2}\frac{d}{dy}\{\tau|_{N+2} - \tau_o\langle y - \ell\rangle\} = 0\ . \tag{8}$$

Further simplifying assumptions are now made regarding the stress-displacement relations which reduce the number of unknowns from three stresses to the two displacements, u_n and v_n. Let

$$\sigma_f|_n = E_F\frac{dv_n}{dy}\ , \tag{9}$$

$$\sigma_m|_n = E_M\{u_{n+1} - u_n\}/h, \text{ and} \tag{10}$$

$$\tau|_{n+1} = G_M\{v_{n+1} - v_n\}/h\ . \tag{11}$$

Substituting into the equilibrium equations, the following pairs of equations are obtained:

For all fibers, except N and $N+1$ when $y \leq L$,

$$\frac{hA_FE_F}{G_Mt}\frac{d^2v_n}{dy^2} + \{v_{n+1} - 2v_n + v_{n-1}\} = 0\ , \text{ and} \tag{12}$$

$$\frac{E_M}{h}\{u_{n+1} - 2u_n + u_{n-1}\} + \frac{G_M}{2}\frac{d}{dy}\{v_{n+1} - v_{n-1}\} = 0 \; . \qquad (13)$$

For fiber N, $y \leq L$,

$$\frac{E_F A_F h}{G_M t}\frac{d^2 v_N}{dy^2} + v_{N-1} - v_N - \frac{h}{G_M}\tau_o\langle y - \ell\rangle = 0 \; , \text{ and} \qquad (14)$$

$$\frac{E_M}{h}\{u_{N+1} - 2u_N + u_{N-1}\} + \frac{h}{2}\frac{d}{dy}\{\frac{G_M}{h}[v_N - v_{N-1}]$$

$$- \tau_o\langle y - \ell\rangle\} = 0 \qquad (15)$$

For fiber $N+1$, $y \leq L$.

$$\frac{E_F A_F h}{G_M t}\frac{d^2 v_{N+1}}{dy^2} + v_{N+2} - v_{N+1} + \frac{h\tau_o}{G_M}\langle y - \ell\rangle = 0 \; , \text{ and} \qquad (16)$$

$$\frac{E_M}{h}\{u_{N+2} - 2u_{N+1} + u_N\} + \frac{h}{2}\frac{d}{dy}\{\frac{G_M}{h}[v_{N+2} - v_{N+1}]$$

$$- \tau_o\langle y - \ell\rangle\} = 0 \; . \qquad (17)$$

The shear stress-displacement form assumed in equation (11) is referred to as the shear-lag assumption and, as can be seen above, the equilibrium equation in the axial direction is independent of the transverse displacement u_n. It is then possible to obtain a solution for the axial displacement v_n, and therefore the fiber stress and shear stress, independently of u_n. Once v_n is known, the transverse displacement and matrix stress may be obtained from the remaining equilibrium equation. References [5] and [6] consider similar three- and two-dimensional solutions, without matrix damage however, in which the shear stress is assumed to depend on the transverse as well as the axial displacement and the equilibrium equations do not uncouple as for the shear-lag assumption.

It is the intent of this study to investigate behavior due to broken fibers and matrix damage in which the failure criterion for the matrix is due to shear alone and the matrix is assumed to be elastic-perfectly plastic. In this case, the matrix transverse normal stress plays no role and the remaining discussion will focus on the solution of the axial equilibrium equation and the determination of the fiber stress and shear stress. The inclusion of the matrix normal stresses in a modified failure criterion using the shear-lag model as well as using the coupled equilibrium equations of [6], with damage, is being considered by the first author and will be presented at a later date.

The single equilibrium equation in the longitudinal direction is then:

for all fibers, except N and $N+1$ when $y \leq L$,

$$\frac{E_F A_F h}{G_M t}\frac{d^2 v_n}{dy^2} + v_{n+1} - 2v_n + v_{n-1} = 0 , \qquad (18)$$

for fiber N, $y \leq L$

$$\frac{E_F A_F h}{G_M t}\frac{d^2 v_N}{dy^2} + v_{N-1} - v_N - \frac{h}{G_M}\tau_o \langle y - \ell \rangle = 0 , \qquad (19)$$

and for fiber N+1, $y \leq L$

$$\frac{E_F A_F h}{G_M t} \frac{d^2 v_{N+1}}{dy^2} + v_{N+2} - v_{N+1} + \frac{h}{G_M} \tau_o \langle y - \ell \rangle = 0 . \qquad (20)$$

Noting the coefficient of the second derivative term in the above equations, the following changes in the variables are suggested. Let $y = \sqrt{\frac{E_F A_F h}{G_M t}}\, \eta$ and $\sigma_f \big|_n = \sigma_\infty \bar{\sigma}_n = E_F \frac{dv_n}{dy}$, then the normalized displacement V_n is defined by the equation

$$v_n = \sigma_\infty \sqrt{\frac{A_F h}{E_F G_M t}}\; V_n ,$$

and the normalized shear stress $\bar{\tau}_o$ is given by

$$\tau_o = \sigma_\infty \sqrt{\frac{G_M A_F}{E_F h t}}\; \bar{\tau}_o .$$

Algebraic manipulation then gives

$$\sigma_f \big|_n = \sigma_\infty \frac{dV_n}{d\eta} = \frac{\tau_o}{\bar{\tau}_o} \sqrt{\frac{E_F h t}{G_M A_F}} \frac{dV_n}{d\eta} , \qquad (21)$$

$$\tau \big|_n = \sigma_\infty \sqrt{\frac{G_M A_F}{E_F h t}} \{V_n - V_{n-1}\} = \frac{\tau_o}{\bar{\tau}_o} \{V_n - V_{n-1}\} ,$$

$$L = \sqrt{\frac{E_F A_F h}{G_M t}}\, \alpha , \quad \text{and} \quad \ell = \sqrt{\frac{E_F A_F h}{G_M t}}\, \beta , \quad \text{where}$$

η, $\bar{\sigma}_n$, V_n, $\bar{\tau}_o$, α and β are non-dimensional.

In these equations E_F, A_F, t, L and ℓ are taken as actual fiber modulus, fiber cross-sectional area, lamina thickness and damage dimensions respectively.

The quantities τ_o and G_M/h are equivalent yield stress and lamina stiffness respectively and are to be determined experimentally. The yield stress, τ_o, should be reasonably close to the matrix yield stress obtained from a test using matrix material alone as long as the damage occurs in the matrix rather than along the interface or within the fiber. The quantity G_M/h is felt to be less well defined as discussed above.

The resulting non-dimensional equations are:

For all fibers, except N and $N+1$ when $\eta \leq \alpha$,

$$\frac{d^2V_n}{d\eta^2} + V_{n+1} - 2V_n + V_{n-1} = 0 \ , \tag{22}$$

for fiber N, $\eta \leq \alpha$

$$\frac{d^2V_N}{d\eta^2} - V_N + V_{N-1} - \bar{\tau}\langle\eta - \beta\rangle = 0 \ , \text{ and} \tag{23}$$

and for fiber $N+1$, $\eta \leq \alpha$

$$\frac{d^2V_{N+1}}{d\eta^2} - V_{N+1} + V_{N+2} + \bar{\tau}\langle\eta - \beta\rangle = 0 \ . \tag{24}$$

Defining a new unknown function $f(\eta)$ such that

$$f(\eta) = V_N - V_{N+1} - \bar{\tau}\langle\eta - \beta\rangle \quad \text{if} \quad \eta < \alpha \ , \quad \text{and}$$

$$f(\eta) = 0 \ , \qquad\qquad \eta \geq \alpha$$

with $g(\eta) = V_N - V_{N+1}$ for the same range of η values, the above three equations then become

$$\frac{d^2V_n}{d\eta^2} + V_{n+1} - 2V_n + V_{n-1} = 0 \tag{25}$$

$$\frac{d^2V_N}{d\eta^2} + V_{N+1} - 2V_N + V_{N-1} = -f(\eta), \quad \text{and} \tag{26}$$

$$\frac{d^2V_{N+1}}{d\eta^2} + V_{N+2} - 2V_{N+1} + V_N = f(\eta) \ . \tag{27}$$

These differential-difference equations may be reduced to differential equations by introducing a new function

$$\bar{V}(\eta,\theta) = \frac{2}{\pi} \sum_{n=0}^{\infty} V_n(\eta)\cos(n\theta) \quad \text{from which} \tag{28}$$

$$V_n(\eta) = \frac{2}{\pi} \int_0^{\pi} \bar{V}(\eta,\theta)\cos(n\theta)d\theta \quad \text{and} \tag{29}$$

the three equations become

$$\frac{2}{\pi} \int_0^{\pi} \{\frac{d^2\bar{V}}{d\eta^2} - 2[1 - \cos(\theta)]\bar{V}\}\cos(n\theta)d\theta = 0 \tag{30}$$

$$\frac{2}{\pi} \int_0^{\pi} \{\frac{d^2\bar{V}}{d\eta^2} - 2[1 - \cos(\theta)]\bar{V}\}\cos(n\theta)d\theta = -f(\eta), \text{ fiber } N, \eta \leq \alpha, \text{ and} \tag{31}$$

$$\frac{2}{\pi} \int_0^{\pi} \{\frac{d^2\bar{V}}{d\eta^2} - 2[1 - \cos(\theta)]\bar{V}\}\cos(n\theta)d\theta = f(\eta), \text{ fiber } N+1, \eta \leq \alpha \tag{32}$$

Making use of the orthogonality of the circular functions these three equations may be written as one equation, valid for all values of n and η, as follows:

$$\frac{2}{\pi}\int_0^\pi \{\frac{d^2\bar{V}}{d\eta^2} - 2[1 - \cos(\theta)]\bar{V}\}\cos(n\theta)d\theta = \frac{2}{\pi}<\alpha-\eta> \int_0^\pi f(\eta)\{\cos[(N+1)\theta]$$

$$- \cos(N\theta)\}\cos(n\theta)d\theta \ . \qquad (33)$$

This equation is of the form

$$\frac{2}{\pi}\int_0^\pi F(\eta,\theta)\ \cos(n\theta)d\theta = 0 \quad \text{for all} \quad \eta \quad \text{and} \quad n$$

and noting the definition of $\bar{V}(n,\theta)$ in equations (28) and (29) it is seen that the function $F(\eta,\theta)$ is even valued in θ and therefore, if the integral is to vanish for all n, the function $F(\eta,\theta)$ must be zero. The single equation specifying $\bar{V}(\eta,\theta)$ is then

$$\frac{d^2\bar{V}}{d\eta^2} - \delta^2\bar{V} = -<\alpha - \eta>D^2 f(\eta), \quad \text{where} \qquad (35)$$

$$\delta^2 = 2[1 - \cos(\theta)] = 4 \sin^2(\theta/2), \quad \text{and} \qquad (36)$$

$$D^2 = \cos(N\theta) - \cos[(N+1)\theta] \ . \qquad (37)$$

It is very significant that the irregular boundary condition, equation (1), of specified stress over a finite length, not coincident with either coordinate axis can be accounted for exactly and that the problem reduces to one differential equation which must satisfy boundary conditions along the coordinate axes only. The ability to do so strongly depends on the form of the failure criterion. A condition in which both normal and shear stresses were included generally would couple the axial and transverse equilibrium equations and yield a far more complicated set of differential equations. The apparent need to investigate such modifications is indicated by the results and, as mentioned above, is being considered.

The solution to the problem of vanishing stresses and displacements at infinity and uniform compression on the ends of the broken fibers will now be sought. The complete solution is obtained by adding the results corresponding to uniform axial stress and no broken fibers to the following solution.

The boundary conditions are then

$$V_n = 0 \quad \text{as} \quad \eta \to \infty \, , \tag{38}$$

$$\frac{dV_n}{d\eta} = \bar{\sigma}_n = -1, \text{ for } \eta = 0, \text{ broken fibers, and} \tag{39}$$

$$V_n = 0 \, , \text{ for } \eta = 0, \text{ unbroken fibers.} \tag{40}$$

Using a technique such as variation of parameters to determine a particular solution to equation (35), the complete solution satisfying vanishing stresses and displacements at infinity is

$$\bar{V}(\eta,\theta) = A(\theta)e^{-\delta\eta} + \frac{D^2}{\delta} \langle\alpha-\eta\rangle \int_\eta^\alpha \sinh[\delta(\eta-t)]f(t)dt \tag{41}$$

where the unknown functions are $A(\theta)$ and $f(t)$. The remaining two boundary conditions give

$$\frac{dV_n(0)}{d\eta} = \frac{2}{\pi}\int_0^\pi \{-\delta A(\theta) + D^2 \int_0^\alpha \cosh(\delta t)f(t)dt\}\cos(n\theta)d\theta = -1 \tag{42}$$

for all broken fibers

and

$$V_n(0) = \frac{2}{\pi}\int_0^\pi \{A(\theta) - \frac{D^2}{\delta}\int_0^\alpha \sinh(\delta t)f(t)dt\}\cos(n\theta)d\theta = 0 \tag{43}$$

for all unbroken fibers.

Equation (43) is solved exactly by taking

$$A(\theta) - \frac{D^2}{\delta}\int_0^\alpha \sinh(\delta t)f(t)dt = \sum_{m=0}^{N} B_m \cos(m\theta) \tag{44}$$

where the B_m are constants. Equation (42) then gives a system of $N+1$ algebraic equations for the $N+1$ constants B_m in terms of $f(\eta)$ which is, as yet, unknown. For the case of no damage the problem is then solved, i.e. see [7]. For example, consider the special case of no damage and one broken fiber. Then equation (42) gives

$$\frac{2}{\pi}\int_0^\pi - \delta\, B_o \cos(0)d\theta = -1$$

or

$$\frac{2B_o}{\pi}\int_0^\pi 2\sin(\theta/2)d\theta = 1 ,$$

Therefore $B_o = \pi/8 = A(\theta)$ and

$$V_n = \frac{2}{\pi}\int_0^{\pi} A(\theta)e^{-\delta\eta}\cos(n\theta)d\theta \ .$$

The maximum fiber stress is in the first unbroken fiber at $\eta = 0$, and is

$$\frac{\sigma_f|_1(0)}{\sigma_\infty} = \frac{dV_1(0)}{d\eta} = \frac{2}{\pi}\int_0^{\pi} - \delta A(\theta)\cos(\theta)d\theta = \frac{1}{3} ,$$

or, for a unit stress at infinity and an unloaded free end of the broken fiber

$$\bar{\sigma}_1 = 4/3 \ .$$

The normalized crack opening displacement, $2V_o(0)$, is $\pi/2$.

For matrix damage, $\alpha \neq 0$, equation (42) must be supplemented by the condition that

$$f(\eta) = g(\eta) - \bar{\tau}_o\langle\eta-\beta\rangle, \ \eta < \alpha \ , \quad \text{and} \qquad (45)$$

$$g(\alpha) = \bar{\tau}_o .$$

The constants B_m and the function $g(\eta)$ are then specified by requiring that equations (42) and (45) be satisfied. Using equation (41), and after considerable algebraic manipulation, the displacement of any fiber for all values of η is

$$V_n(\eta) = \frac{2}{\pi}\int_0^\pi e^{-\delta\eta} \sum_{m=0}^{N} B_m \cos(m\theta)\cos(n\theta)d\theta$$

$$+ \frac{1}{2}\int_0^\alpha f(t)\{C_n(|t-\eta|) - C_n(t+\eta)\}dt \ , \qquad (46)$$

where

$$C_n(\xi) = \frac{2}{\pi}\int_0^\pi \frac{D^2}{\delta} e^{-\delta\xi} \cos(n\theta)d\theta \ .$$

Equation (42) then becomes

$$\frac{2}{\pi}\int_0^\pi \{-\delta \sum_{m=0}^{N} B_m \cos(m\theta) - D^2 \int_0^\alpha e^{-\delta t} g(t)dt + D^2\bar{\tau}_o\int_\beta^\alpha e^{-\delta t} dt\} \times$$

$$\times \cos(n\theta)d\theta = -1, \ n = 0, 1, \ldots, N \qquad (47)$$

and equation (45) along with (46) gives

$$g(\eta) = \frac{2}{\pi}\int_0^\pi e^{-\delta\eta} \sum_{m=0}^{N} B_m \cos(m\theta)\{\cos(N\theta) - \cos[(N+1)\theta]\}d\theta$$

$$+ \frac{1}{2}\int_0^\alpha g(t)\{C_N(|t - \eta|) - C_N(t + \eta) - C_{N+1}(|t - \eta|) + C_{N+1}(t + \eta)\}dt$$

$$- \frac{\bar{\tau}_o}{2}\int_\beta^\alpha \{C_N(|t - \eta|) - C_N(t + \eta) - C_{N+1}(|t - \eta|) + C_{N+1}(t + \eta)\}dt \ .$$

(48)

The condition that

$$g(\alpha) = \bar{\tau}_o \qquad (49)$$

also must be satisfied.

Physically, it would be more direct to specify the applied stress, σ_∞ and the number of broken fibers, N, and determine the damage zone α and β depending on given yielding and splitting conditions. As α and β appear in the limits of the above integrals this is not convenient mathematically and it is easier to specify the number of broken fibers, N, and the damage zone α and β, and compute the required applied stress σ_∞.

These equations were solved as follows:

I. An initial set of constants B_m was determined for the problem of no damage, $\alpha = \beta = 0$ in equation (47), i.e.,

$$\sum_{m=0}^{N} B_m \frac{2}{\pi} \int_0^\pi \delta \cos(m\theta)\cos(n\theta)d\theta = 1 , \qquad (50)$$

$$n = 0, 1, \ldots, N$$

II. These initial constants were then substituted into the integral equation (48) and, along with equation (49), the function $g(\eta)$ and $\bar{\tau}_o$ were determined using the desired values for α and β.

III. Using $g(\eta)$ and $\bar{\tau}_o$, a new set of constants, B_m, was computed from equation (47) with the desired values of α and β.

IV. This procedure was repeated until the unknowns changed less than a prescribed amount with additional iterations.

In the above solution the unknown function, $g(\eta)$, was assumed to be piece-wise linear over the interval $0 \leq \eta \leq \alpha$ of the form

$$g^i(\eta) = \gamma_o^i + \gamma_1^i \eta, \quad i = 1,2,\ldots k$$

when the interval was divided into k equal subdivisions. The function $g(\eta)$ then contained $2k$ unknowns with one additional unknown being $\bar{\tau}_o$. As $g(\eta)$ is the displacement difference it should be a positive, monotonically decreasing function and its representation as a piece-wise linear function should be sufficiently accurate. The $(2k + 1)$ equations were obtained by requiring that the integral equation, equation (48), be satisfied at the $(k + 1)$ end points; $(k - 1)$ equations resulted from the requirement of continuity of the function $g(\eta)$ between adjacent intervals and the last equation was given by $g(\alpha) = \bar{\tau}_o$.

With the longitudinal displacement v_n now known the transverse displacement u_n is obtained by solving equations (4), (6), and (8). Equation (10) gives the matrix normal stress in terms of u_n. This solution is recorded below for completeness.

$$u_n = - \frac{\sigma_\infty G_M h}{\pi E_M E_F} \int_o^\pi \left\langle \sin (\theta) \frac{d\bar{V}}{d\eta} - \frac{1}{2} \frac{df}{d\eta} \{ \sin (N\theta) + \sin[(N + 1)\theta] \} \right\rangle \frac{\sin (n\theta) \, d\theta}{1 - \cos (\theta)} \tag{51}$$

SOLUTION AND RESULTS

The numerical solution to equations (47), (48) and (49) was developed as described above and was solved using the NASA-Langley Research Center CDC computer. Convergence was fairly rapid with the number of subdivisions, k, required in the representation of the function $g(\eta)$ being on the order of fifteen and the number of iterations necessary to give a change in the normalized shear, $\bar{\tau}$, such that $(\bar{\tau}_{i+1} - \bar{\tau}_i)/\bar{\tau}_i < 10^{-4}$, on the order of ten. Larger values of the damage region, α and β, required an increase in both the number of subdivisions and number of iterations, with $\alpha, \beta \lesssim 2$ being relatively small values and $\alpha, \beta \gtrsim 6$ requiring increased accuracy. Computation time varied both with the above parameters and with the number of broken fibers with typical times being on the order of one minute for $\alpha, \beta \approx 2$ and $N = 1$, to twenty minutes for $\alpha, \beta \approx 4$ and $N = 15$.

One problem of particular significance is the behavior of a laminate after damage develops and the investigation of the potential for continued longitudinal yielding or splitting or for notch extension due to progressive fiber breaks. Some typical results are given in Figure 3 for one and seven broken fibers using a two/one split strain-to-yield strain condition as shown. The maximum fiber stress normalized by the laminate constant $T_o = \tau_o \sqrt{\frac{E_F ht}{G_M A_F}}$ is plotted against the normalized applied stress where the maximum fiber stress is always found to occur in the first unbroken fiber at the end of the split, $(\eta = \beta)$.

These results are representative of many different cases worked, i.e. increasing the strain at which splitting occurs simply increases both σ_f and σ_∞ at the point of splitting but the nature of the behavior is unchanged. That is, in all cases once the split forms the fiber unloads and the split length becomes unbounded under a five to ten percent further increase in applied stress, at which time the behavior reduces to that of an unnotched laminate with $\sigma_f = \sigma_\infty$. The net section fracture stress is then independent of initial crack length as [3].

This predicted behavior indeed happens in unidirectional graphite/ epoxy or boron/epoxy composites. However, for a less brittle matrix material such as aluminum some small splitting has been observed [3] and [7], but it is stable and the stress in the fiber continues to increase with increased remote stress until the ultimate fiber stress is reached. The model then appears to give reasonable results for large splitting but does not account for small, stable longitudinal splitting. It is of considerable interest to determine if the inability of the model to predict the behavior of stable splitting is due to the assumed failure criterion or the shear-lag model. As mentioned above, a more complex failure criterion including the matrix normal stress as determined from the shear-lag solution as well as more complete shear stress-displacement assumptions are being studied.

Surprisingly, in view of the above difficulties, the present model approximates the behavior of boron/alumimum amazingly well for strains such that splitting does not occur, and gives an accurate estimate of the laminate strength as a function of number of broken fibers (crack length) as well as crack opening displacement. The following results are for no splitting ($\beta = 0$) and conclusions for a specific boron/aluminum laminate will be drawn from the general results.

Figure 4 gives a plot of the maximum fiber stress as a function of the applied stress for different numbers of broken fibers. Defining a function S equal to the normalized ultimate fiber stress as,

$$S = \frac{\sigma_{ult}}{\tau_o}\sqrt{\frac{G_M A_F}{E_F ht}} \ ,$$

the required applied stress for a specified ultimate fiber stress is then given by the intersection of the horizontal lines corresponding to specific values of S as shown.

As the applied stress is increased the matrix undergoes plastic deformation as indicated by Figure 5. For all ranges of α indicated, the maximum shear strain which is at $\eta = 0$ between the N and $N+1$ fibers, is no greater than three times the yield strain and therefore for a boron/aluminum composite splitting probably would not occur. The lines of constant S values on Figure 5 are obtained by locating the corresponding (σ_∞, N) points from Figure 4. Figure 6 is also developed from Figure 4 and gives the strength of the laminate as a function of number of broken fibers (or crack length). That is, σ_∞ is the remote stress required to give a particular ultimate stress in the first unbroken fiber.

In addition to these results the displacement of the center broken fiber, or therefore one-half the crack opening displacement, is obtained and is depicted in Figure 7 as a function of applied stress for different numbers of broken fibers. As the number of broken fibers (crack length) increases it is seen that the matrix yeilding contributes a constant proportion of the total crack opening displacement. This is consistent with Figure 5 where, if the damage length, α, is plotted against number of broken fibers for constant remote stress, Figure 8 is obtained and the relationship is linear as shown.

Using these general results the values of τ_o and the stiffness constant G_M/h will now be determined by comparing with the experimental study of Awerbuch in [8] for unidirectional boron/aluminum. The laminate used in [8] had the following material and geometric properties:

$E_F = 475 \times 10^9$ Pa,

$A_F = 1.59 \times 10^{-8}$ m^2, ($D = 0.1427$ mm),

$t = 0.165$ mm/ply, eight plies,

$\sigma_{ult} = 3.98 \times 10^9$ Pa,

w = width = 25.4 mm, and

fiber centerline spacing = 0.178 mm

For a laminate having seven broken fibers, which corresponds to a crack length of about 1.27 mm, the load vs. COD curve is given in Figure 9. An approximate "best fit" of this curve with [8] requires that the yield stress and stiffness be

$$\tau_o = 0.109 \times 10^9 \text{ Pa, and}$$

$$G_M/h = 65.4 \times 10^{12} \text{ N/m}^3,$$

from which the normalized ultimate stress, S, is

$$S = \frac{\sigma_{ult}}{\tau_o}\sqrt{\frac{G_M A_F}{E_F h t}} = 4.1$$

For reference, the damage length is then $L = 4.71\ \alpha$ fiber spacings and the laminate constant $T_o = 8.65\ \tau_o$.

Now, using these values, the corresponding curve for 29 broken fibers, or a crack length of 5 mm is plotted in Figure 9 and the comparison is seen to be very good.

Further, referring to Figure 6, the failure stress of the laminate as measured in [8] is seen to be of the same form as the predicted value, although the model gives a much larger decrease in strength for small crack length than the experimental result. The value of S determined from the COD measurements does, however, predict the failure stress reasonably well for longer crack lengths. The significant point is that the simple shear-lag model does relate crack opening displacement and laminate strength. Changes in the shear-lag assumption and/or the failure criterion may well improve the agreement with experimental results, and an investigation into which modifications are important should lead to a better understanding of the fracture process.

It should be noted that the value of τ_o given above is close to the measured value for a homogeneous aluminum specimen and is also approximately equal to the value determined experimentally by Peters [3] for a boron/aluminum composite. The equivalent stiffness, G_M/h, for $h = 1.78 \times 10^{-4}$ m, which is the center-line distance between fibers, gives a shear modulus $G_M = 11.6 \times 10^9$ Pa. This is on the order of one half the value for aluminum. Smaller values of h, of course, give smaller values of G_M.

CONCLUSIONS

It has been shown that a model based on the shear-lag assumptions and using a shear stress failure criterion of an elastic-perfectly plastic matrix material is capable of predicting the behavior of a unidirectional boron/aluminum composite quite accurately for strain levels below those sufficient to cause splitting. For brittle matrix composites the model gives equally good results for both yielding and large splitting.

The fracture stress is then predicted for the two limit cases of large yielding with no splitting and large splitting. For yielding alone the fibers nearest the notch continue to increase in stress as the applied load is increased and the fracture stress is crack length dependent. In a brittle matrix longitudinal cracks develop under low loads and become unbounded for a small increase in load, thereby negating the effects of the notch and the fracture stress is crack length independent.

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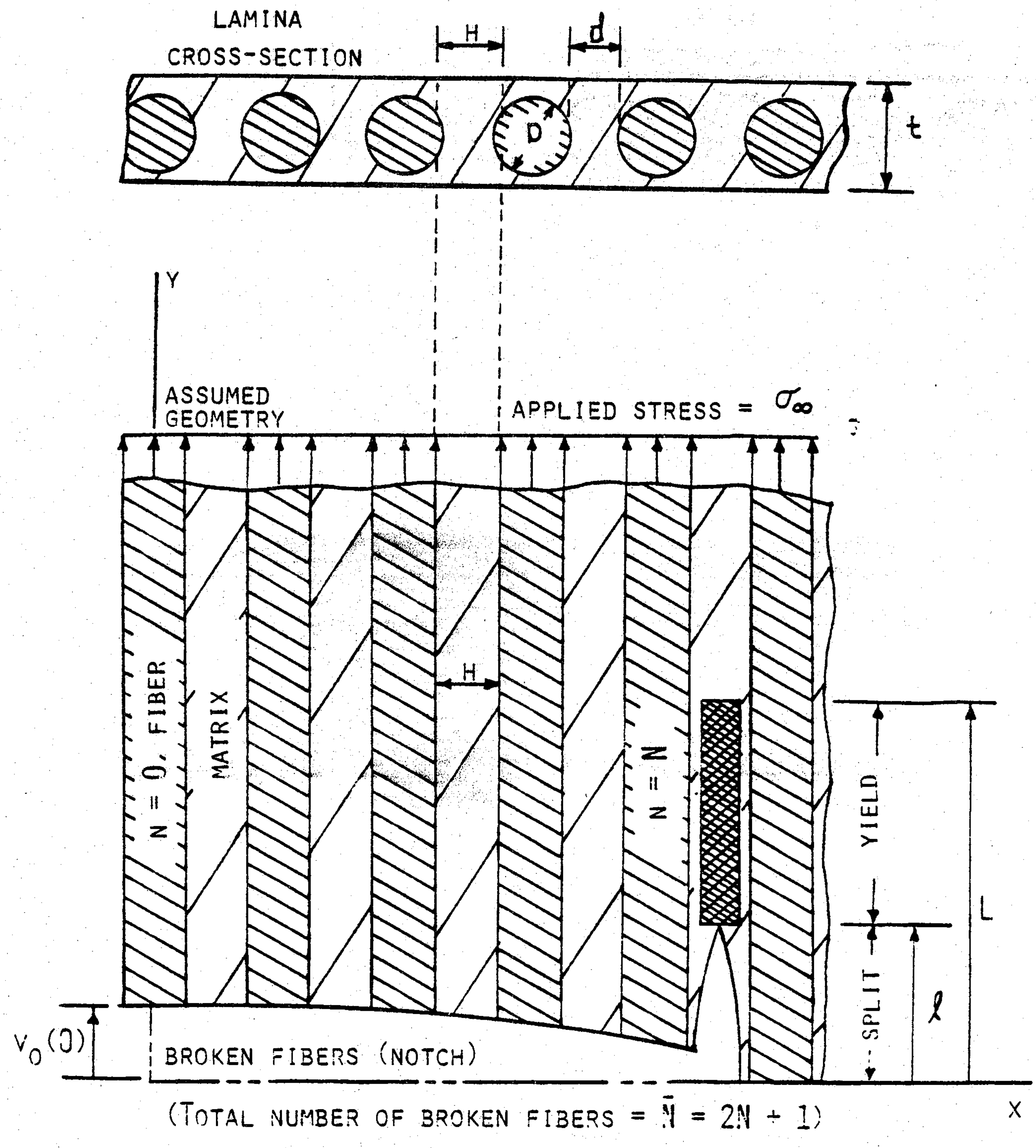
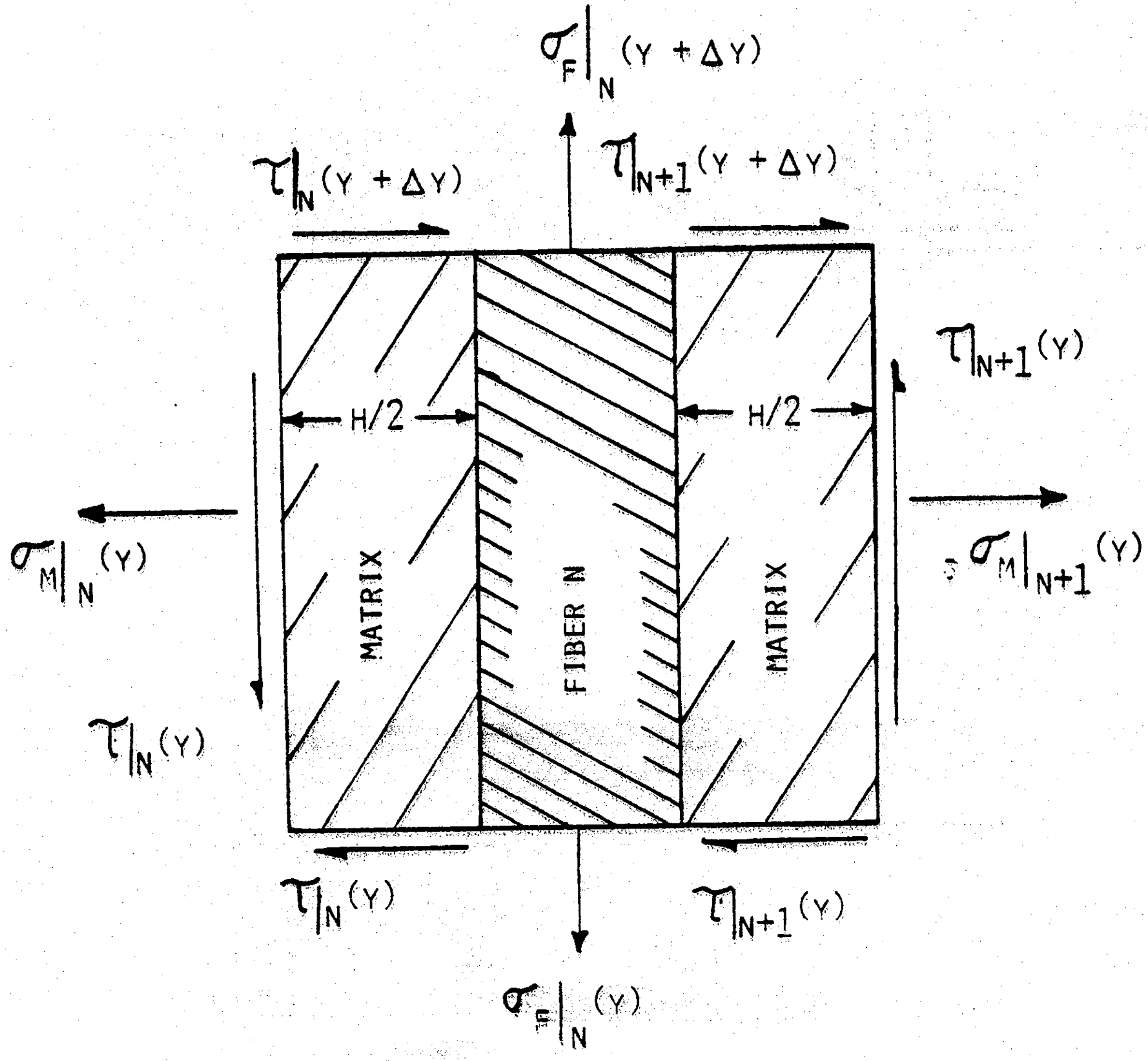


FIGURE 1. TWO-DIMENSIONAL ARRAY OF PARALLEL FIBERS WITH DAMAGE.



NOTE: FOR FIBERS N AND N+1 WITH $Y \leq L$

$$\tau_{N+1} = -\tau_0 \langle Y - L \rangle$$

FIBER CROSS-SECTIONAL AREA = A_F

ELEMENT THICKNESS = t

EQUIVALENT MATRIX YIELD STRESS = τ_0

FIGURE 2. FREE-BODY DIAGRAM OF A TYPICAL ELEMENT.

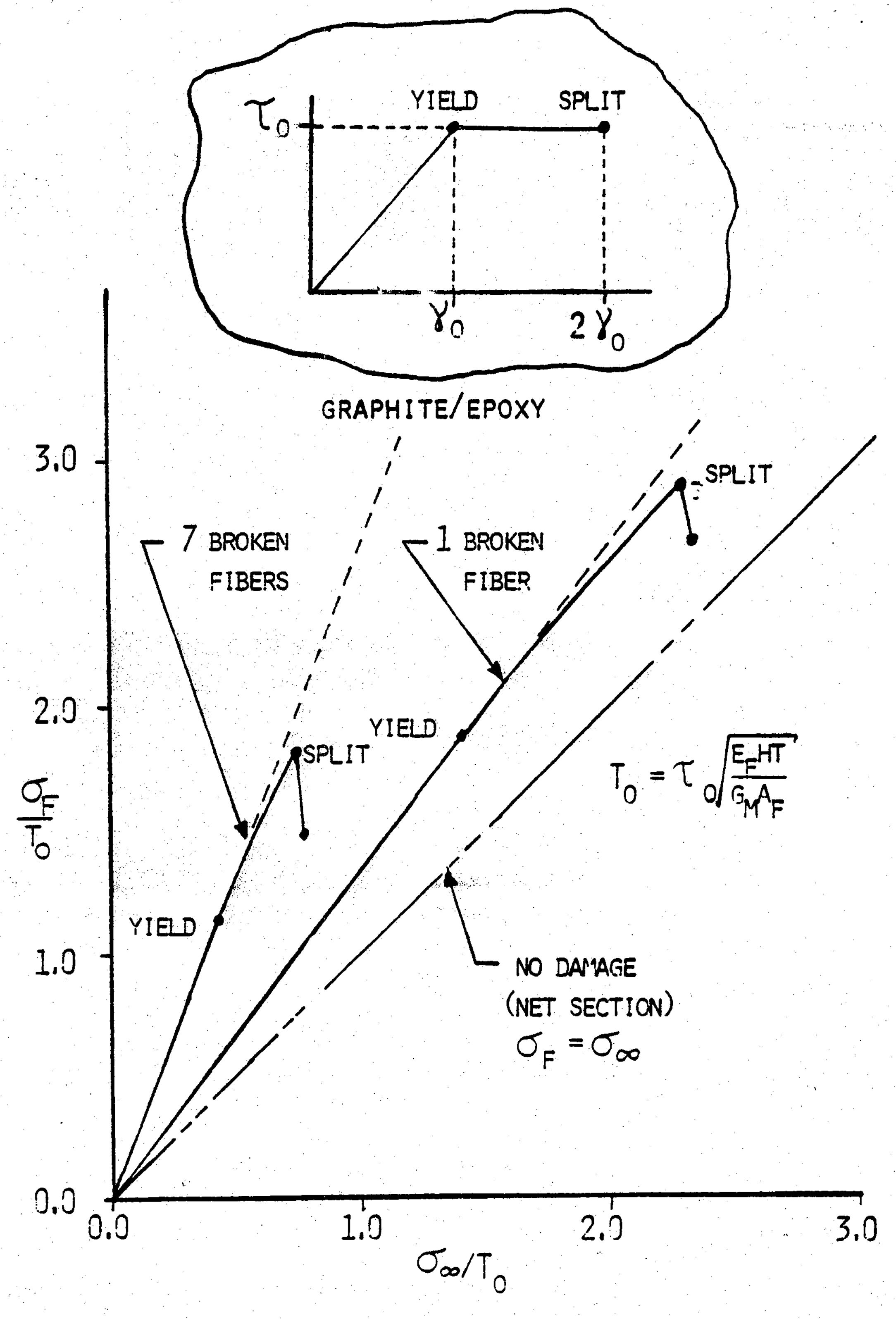


FIGURE 3. MAXIMUM FIBER STRESS FOR YIELDING AND SPLITTING ELASTIC-PERFECTLY PLASTIC.

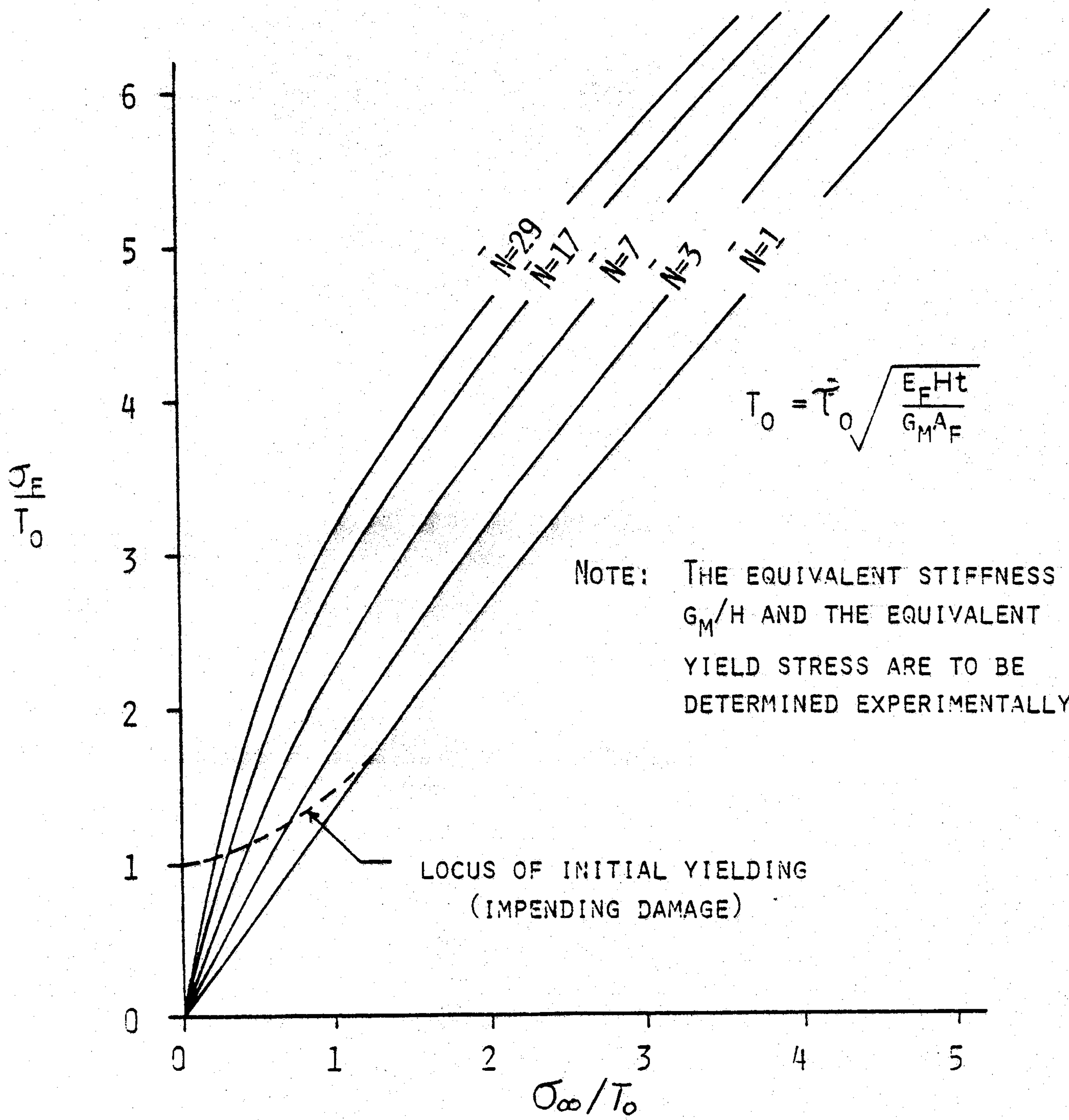


FIGURE 4. MAXIMUM FIBER STRESS AS A FUNCTION OF APPLIED STRESS FOR NO SPLITTING.

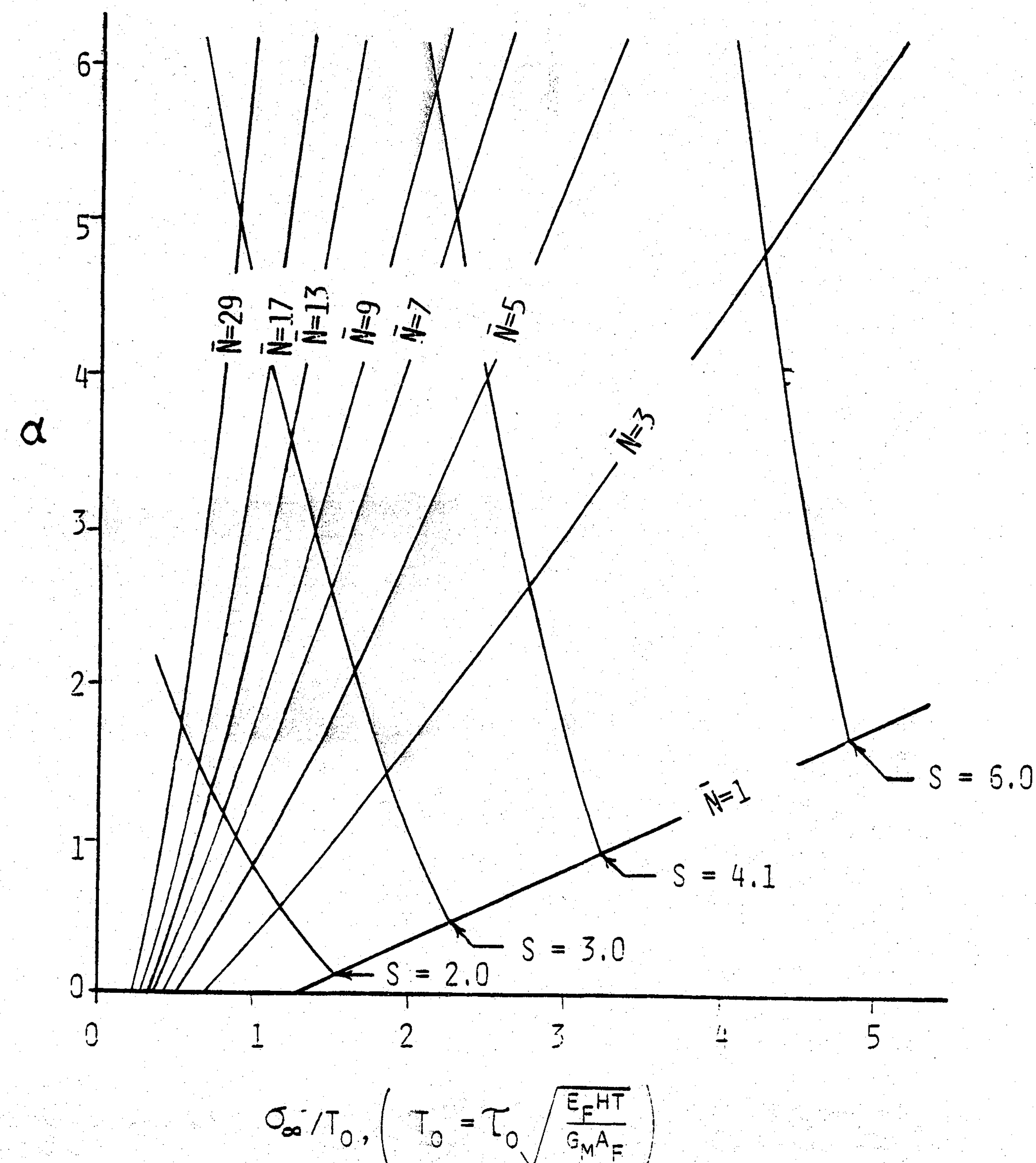


FIGURE 5. YIELD LENGTH AS A FUNCTION OF APPLIED STRESS, NO SPLITTING.

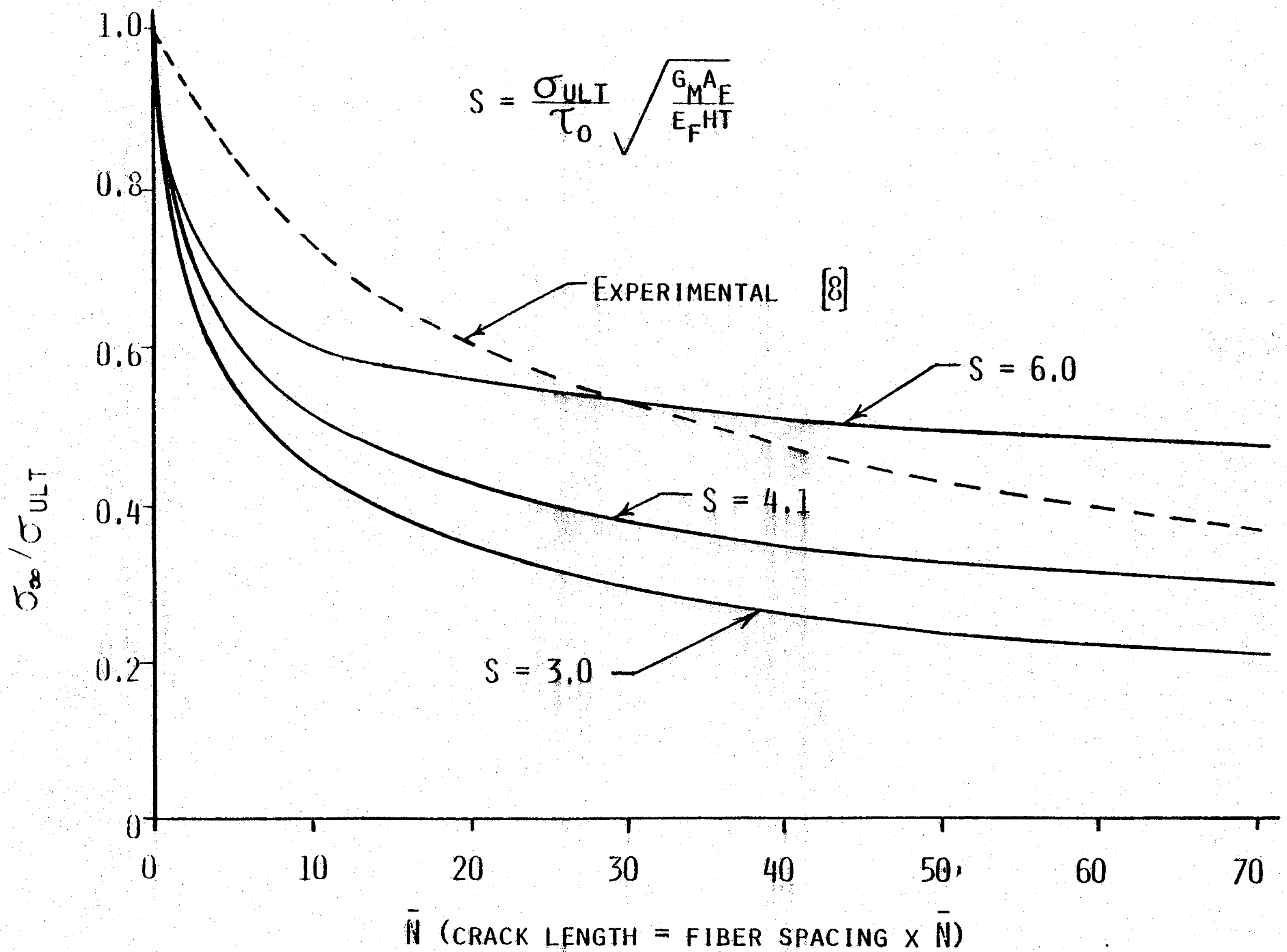


FIGURE 6. APPLIED STRESS AS A FUNCTION OF NUMBER OF BROKEN FIBERS FOR DIFFERENT ULTIMATE FIBER STRESS.

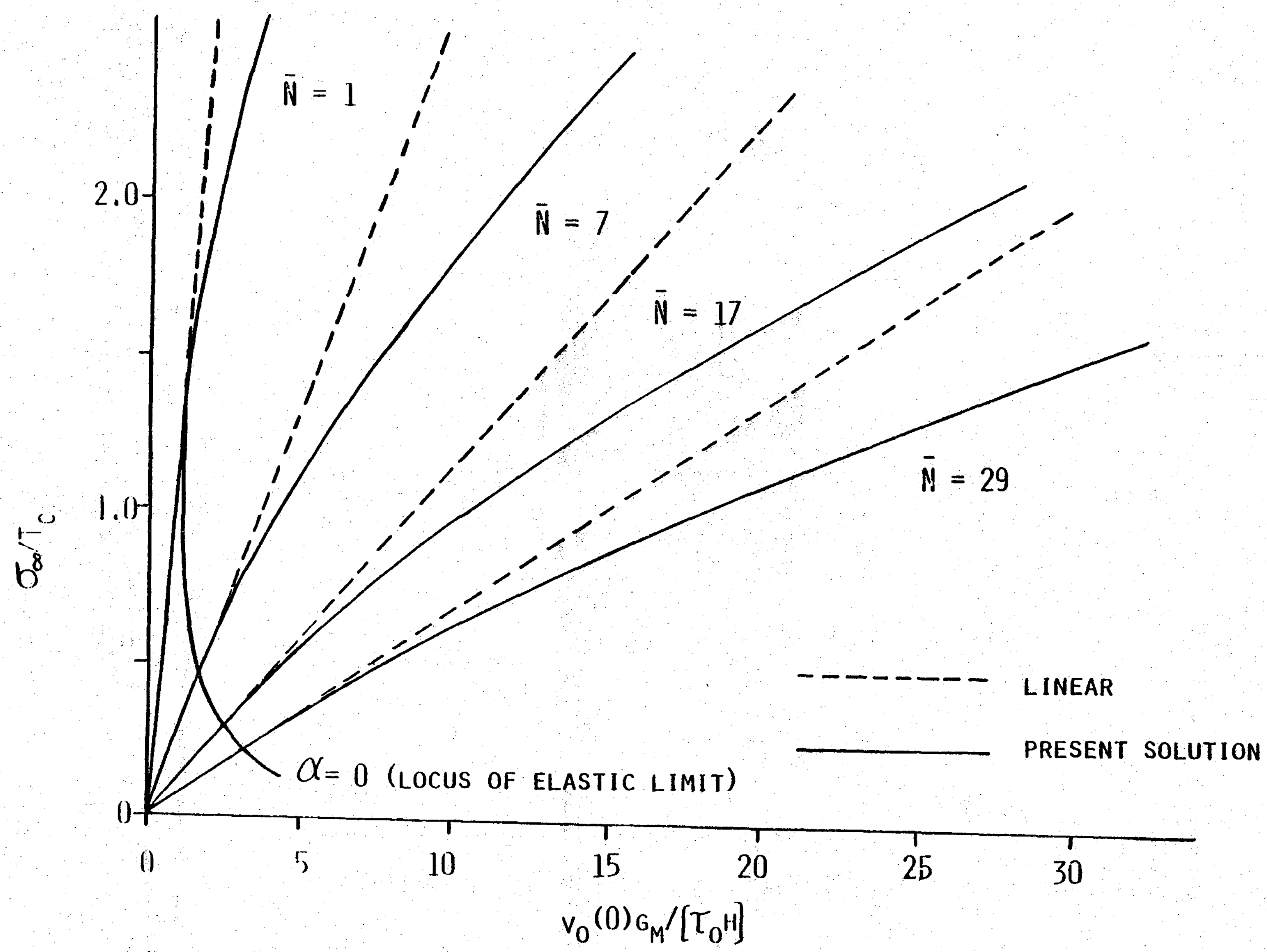


FIGURE 7. MAXIMUM FIBER DISPLACEMENT AS A FUNCTION OF APPLIED STRESS.

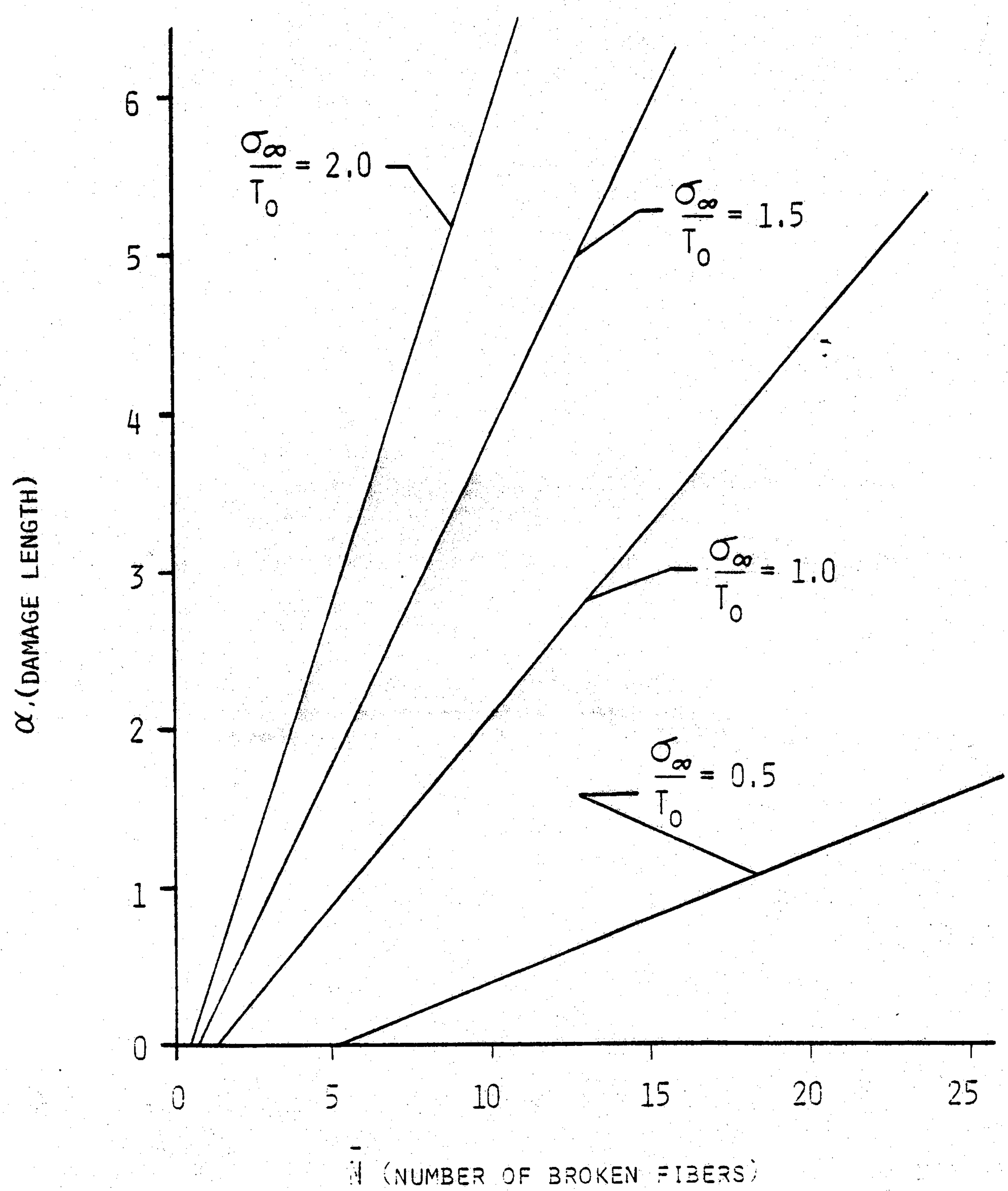


FIGURE 8. DAMAGE LENGTH AS A FUNCTION OF NUMBER OF BROKEN FIBERS FOR DIFFERENT VALUES OF APPLIED STRESS.

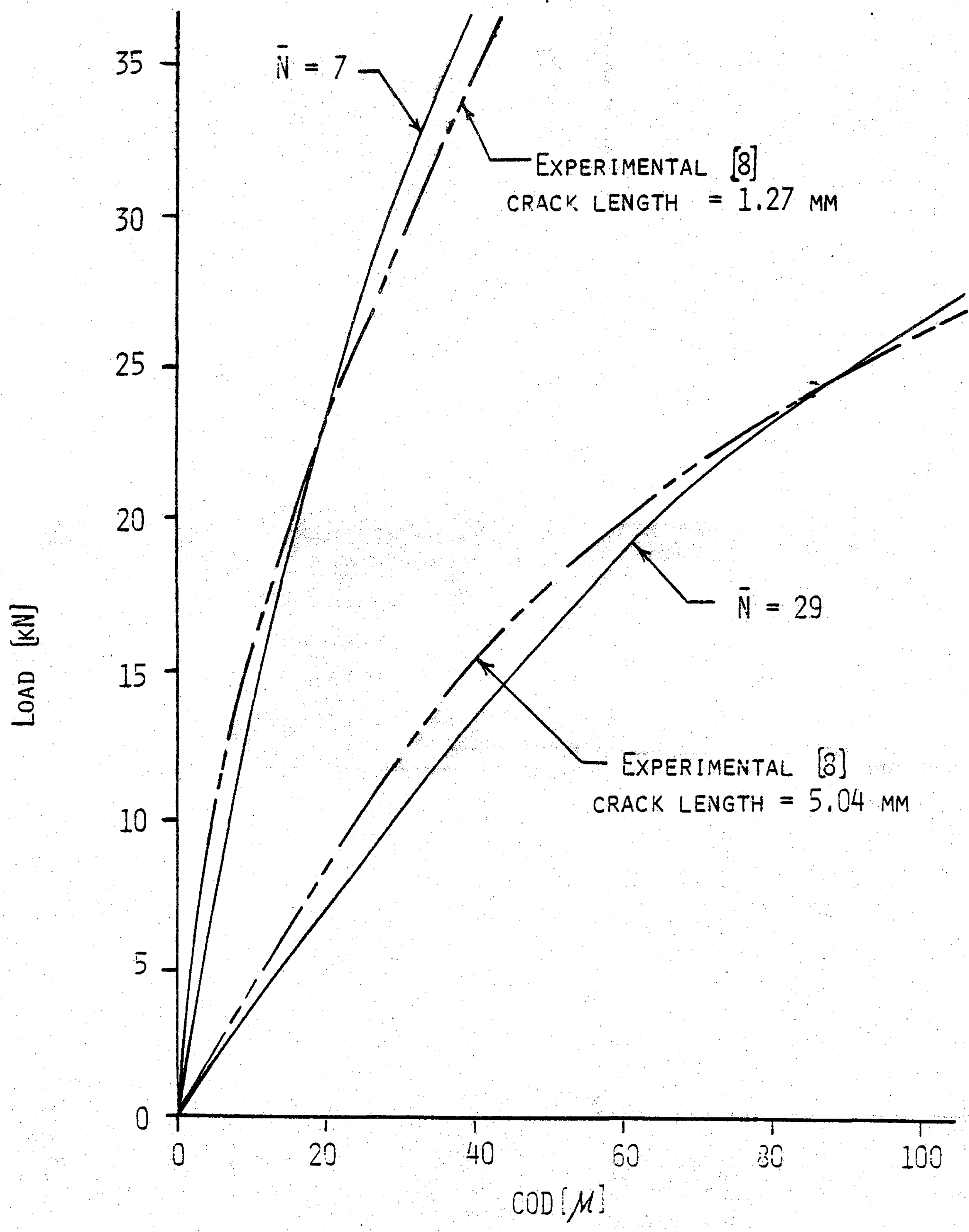


FIGURE 9. COMPARISON OF PRESENT RESULTS WITH EXPERIMENTAL STUDY [8] FOR A 25.4 MM BY 8 PLY LAMINATE.