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Analysis Techniques for Multivariable Root Loci*

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ABSTRACT

Analysis techniques are developed for the multivariable root locus and the multivariable optimal root locus. The generalized eigenvalue problem is used to compute angles and sensitivities for both types of loci. and an algorithm is presented that determines the asymptotic properties of the optimal root locus.

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I. Introduction

The classical root locus has proven to be a valuable analysis and design tool for single input single output linear control systems. Research is currently underway to extend these methods to multi-input multi-output linear control systems and linear optimal control systems. In this paper we present analysis techniques for both of these multi-variable root loci. We show how to compute angles and sensitivities for both types of loci, and how to determine the asymptotic behavior (as control weights get small) of the optimal root locus.

Previous work on angles and sensitivities is contained in [1,2]. The former uses time domain techniques (the eigenvalue problem) and the latter uses frequency domain techniques. We extend the time domain techniques through the use of generalized eigenvalue problems, and we show this approach to be significantly better for computing angles of approach.

Perhaps the most significant development in the understanding of multivariable root loci was the concept of multivariable transmission zeroes [3]. These form the endpoints of all asymptotically finite branches. Determining the behavior of the asymptotically infinite branches, however, has proven to be a difficult problem and all of the details are not yet known [4]. Frequency domain interpetations of multivariable root loci using Riemann surfaces have been given [5], and the behavior of the closed loop eigenvectors has also attracted some attention.

The root loci of linear quadratic optimal control systems were first described for single-input single-output systems in [6,7]. These methods have been extended to the multi-input case in [8,9,10]. Asymptotic properties (which include the asymptotic behavior of eigenvectors) are used to select quadratic weights in [11]. Optimal root loci can be considered a special case of ordinary linear feedback loci, and it turns out that the asymptotically infinite behavior of this special case is better behaved. Consequently more progress has been made in analyzing this behavior [12,13]. We extend the available analysis techniques for determining the asymptotically infinite behavior to include the behavior of the eigenvectors. In doing so we use a new type of subspace decomposition which simplifies the previous analysis technique [12].

In section II we develop the formulas for computing angles and sensitivities of the multivariable root locus. In section III these formulas are applied to the multivariable optimal root locus. Then in section IV we develop analysis techniques for determing the asymptotically infinite behavior of the multivariable optimal root locus.

The Generalized Eigenvalue Problem

The generalized eigenvalue problem is to find all finite λ and their associated eigenvectors v which satisfy

$$Lv = \lambda Mv \quad .$$

L and M are real valued $p \times p$ matrices which are not necessarily full rank. If M is invertible then premultiplication by M^{-1} changes the generalized eigenvalue problem into a standard eigenvalue problem for which there are exactly p solutions. In general there are 0 to p finite solutions, except for the degenerate case when all λ in the complex plane is a solution. Reliable FORTRAN subroutines based on stable numerical algorithms exist in EISPACK [14] to solve the generalized eigenvalue problem. See [15] for the application of this software to a related class of problems.

Notation

Matrices are denoted by capital letters, scalars and vectors by lower case letters. A^T and y^H are the transpose and Hermitian transpose, respectively, of A and y. A^{-t} indicates $(A^{-1})^T$ or, equivalent, $(A^T)^{-1}$. $A \geq 0$ and $A > 0$ indicates that A is positive semidefinite and positive definite. If A is symmetric then $A^{1/2}$ is the (nonunique) decomposition of A into $A^{1/2}A^{1/2T}$. Subspaces are denoted by script letters, with the exception of the real vector space $\mathbb{R}^n$. "Im A" and "ker A" are the image and kernel of the linear map A. The dimension of U is dim U, subspace inclusion is $\subseteq$, subspace intersection is $\cap$, and a linear combination of subspaces in U + V. An open loop linear system is denoted by (A,B,C).

II. Angles and Sensitivities of the Root Locus

We consider the linear time invariant output feedback problem

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \tag{1}$$

$$y = Cx \qquad y \in \mathbb{R}^m \tag{2}$$

$$u = \frac{-1}{k} Ky . \tag{3}$$

The closed loop system matrix and its eigenvalues, right eigenvectors, and left eigenvectors are defined in the usual way by

$$A_{c\ell} = A - \frac{1}{k} BKC \tag{4}$$

$$(A_{c\ell} - s_i I)x_i = 0 \qquad i = 1,\ldots, n \tag{5}$$

$$y_i^H(A_{c\ell} - s_i I) = 0 \qquad i = 1,\ldots, n . \tag{6}$$

We make the assumptions that (A,B) is controllable, (C,A) is observable, and K is invertable. Only the case where the number of inputs and outputs are equal is treated, and we further assume that the closed loop eigenvalues are distinct.

As k is varied from infinity down to zero the closed loop eigenvalues trace out a root locus. At $k = \infty$ (to be more precise let $\ell = 1/k$ and use $\ell = 0$) the n branches of the root locus start at the open loop eigenvalues. As $k \to 0$, some number $p \leq n-m$ of these branches approach transmission zeros, which are defined here to be those values of s which reduce the rank of

$$\begin{bmatrix} A - sI & B \\ -C & 0 \end{bmatrix} .$$

We further rule out degenerate cases, in other words we assume that A,B, and C do not conspire in such a way that any value of s in the complex plane reduces the rank of the matrix. We note that if all of our assumptions are valid then this is an adequate definition of transmission zeros [3]. Also as $k \to 0$, the remaining n-p branches of the root locus approach infinity. The behavior of these branches concern us in Section IV.

At any point on the root locus an angle can be defined. Consider the closed loop eigenvalue s_i which is computed for some value of k. If k is perturbed by an amount Δk then s_i will be perturbed by Δs_i. As $\Delta k \to 0$ then $\Delta s_i/\Delta k$ approaches the constant ds_i/dk (if this limit exists). The angle of the root locus at point s_i is then defined to be

$$\phi = \arg(ds_i),$$

where "arg" is the argument of a complex number. The angles of the root locus at the open loop eigenvalues are the angles of departure, and the angles at the transmissions zeroes are the angles of arrival, Figure 1 illustrates these definitions.

Next we define the sensitivity of a closed loop eigenvalue to a change in k to be

$$S = \left|\frac{ds_i}{dk}\right| .$$

This definition is motivated by the approximation

$$\frac{ds_i}{dk} \approx \frac{\Delta s_i}{\Delta k} ,$$

from which we obtain

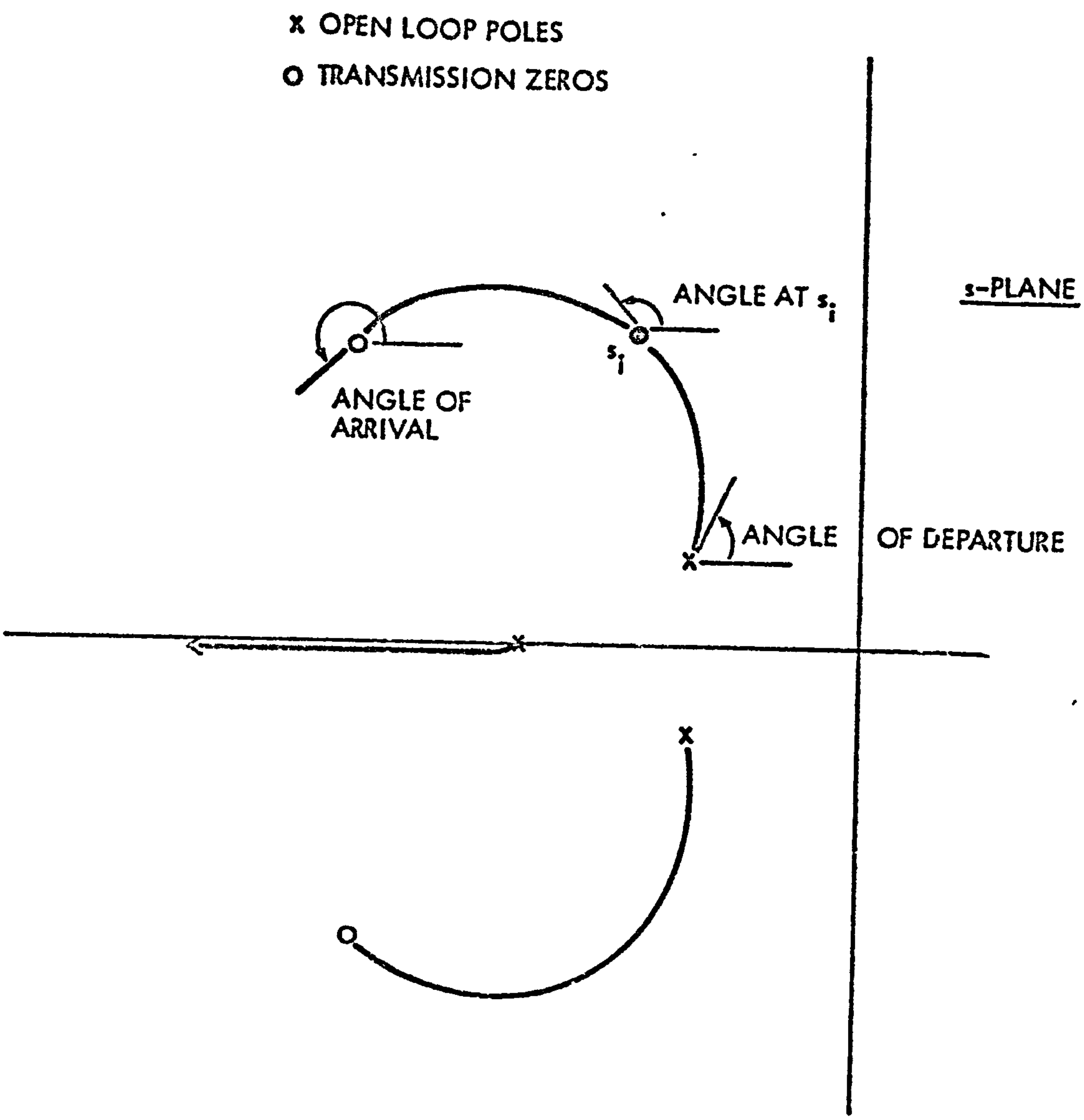

Fig. 1. Definition of Angles

$$|\Delta s_i| \approx \left|\frac{ds_i}{dk}\right| \cdot |\Delta k| \ .$$

So, to first order, a change Δk will move s_i a distance $|\Delta s_i|$ in the direction $\arg(ds_i)$.

Before presenting formulas for these angles and sensitivities, we present the following lemma, which shows how the generalized eigenvalue problem can be used to compute the closed loop eigenstructure.

Lemma 1. The s_i, x_i, and y_i^H are solutions of the generalized eigenvalue problems

$$\begin{bmatrix} A - s_i I & B \\ -C & -kK^{-1} \end{bmatrix} \begin{bmatrix} x_i \\ \nu_i \end{bmatrix} = 0 \qquad i = 1,\ldots, p \tag{7}$$

$$\begin{bmatrix} y_i^H & \eta_i^H \end{bmatrix} \begin{bmatrix} A - s_i I & B \\ -C & -kK^{-1} \end{bmatrix} = 0 \qquad i = 1,\ldots, p \ . \tag{8}$$

Proof. From (7) we see that

$$(A - s_i I)x_i + B\nu_i = 0 \tag{9}$$

$$\nu_i = -\frac{1}{k} KCx_i \ . \tag{10}$$

Substitute (10) into (9) to get

$$(A - s_i I)x_i - \frac{1}{k} BKC\, x_i = 0 \ ,$$

which is the same as (5), the defining equation for the closed loop eigenvalues and right eigenvectors. In a similar way (8) can be reduced to (6) and the proof is complete. Lemma 1 is not a new result but we have been unable to find a reference for it. A precursor of this result (without consideration of closed loop eigenvectors, and without mention of the generalized eigenvalue problem) is the polynomial system matrix representation of Rosenbrock [3].

When $k > 0$ then p, the number of finite solutions of s_i in (7) and (8), is equal to n. When $k = 0$ then $0 \leq p \leq n-m$ (under stated assumptions). The ability to use (7) and (8) with $k = 0$ is the major advantage of the generalized eigenvalue problem. The finite solutions s_i when $k = 0$ are the transmission zeroes of the system, and the x_i and y_i^H vectors are the right and left zero directions [15]. From (7) and (8) it is clear that as $k \to 0$ the finite closed loop eigenvalues approach the transmission zeroes and the associated eigenvectors approach the zero directions.[1]

The solutions of the generalized eigenvalue problems contain two vectors ν_i and η_i^H which do not appear in the solutions of the ordinary eigenvalue problems. The importance of the ν_i vectors can be explained as follows. The closed loop right eigenvector x_i is constrained to lie in the m dimensional subspace of $\mathbb{R}^n$ spanned by the columns of $(s_i I - A)^{-1}B$ [17]. Exactly where x_i lies in this subspace is determined by ν_i, via $x_i = (s_i I - A)^{-1} B \nu_i$. This follows from the top part of (7). If the state

[1]In [16], transmission zeroes are computed by solving an eigenvalue problem for equation (5) with k close to zero. This is the high gain feedback method. In [15] this is shown to have the potential to be computationally inferior to solving equation (7) with $k = 0$.

of the closed loop system at time zero is $x_0 = ax_i$, then the state trajectory for time greater than zero is $x(t) = ax_i e^{s_i t}$, and the control action is $u(t) = -(a/k)KCx_i e^{s_i t} = a\nu_i e^{s_i t}$. This follows from the bottom part of (7). The η_i^H vectors play an analogous role in the dual system $S(-A^T, C^T, B^T)$.

For our purposes, however, the ν_i and η_i^H vectors are also significant because they can be used to compute the angles on the root locus. This is shown by the following Theorem:

<u>Theorem 1.</u> The angles of the root locus, for $0 \leq k \leq \infty$ and for distinct s_i, are found by

$$\arg\,(ds_i) = \arg\,\frac{-y_i^H BKCx_i}{y_i^H x_i} \qquad \begin{array}{l} 0 < k \leq \infty \\ i = 1,\ldots,p \end{array} \tag{11}$$

$$\arg\,(ds_i) = \arg\,\frac{\eta_i^H K^{-1}\nu_i}{y_i^H x_i} \qquad \begin{array}{l} 0 \leq k < \infty \\ i = 1,\ldots,p \end{array} \tag{12}$$

<u>Remark.</u> The angles of departure are found using (11) with $k = \infty$, the angles of approach are found using (12) with $k = 0$. For $k > 0$, $p = n$; and for $k = 0$, $0 \leq p \leq n-m$.

<u>Proof.</u> The proof of (11) can be found in [1]. To prove (12), we first show that

$$\frac{ds_i}{dk} = \frac{-\eta_i^H K^{-1}\nu_i}{y_i^H x_i} \,. \tag{13}$$

Rewrite (7) as

$$(L - s_i M)v_i = 0 \qquad\qquad i = 1,\ldots,p \tag{14}$$

where

$$L = \begin{bmatrix} A & B \\ -C & -kK^{-1} \end{bmatrix}, \quad M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad v_i = \begin{bmatrix} x_i \\ \nu_i \end{bmatrix}.$$

Let also,

$$u_i^H = \left[y_i^H \eta_i^H\right]$$

Differentiate (14) to get

$$\left[\frac{d}{dk}(L - s_i M)\right] v_i + (L - s_i M)\frac{dv_i}{dk} = 0 \; .$$

Multiply on the left by u_i^H to get

$$u_i^H \left[\frac{d}{dk}(L - s_i M)\right] v_i = 0 \; .$$

After substituting and rearranging the result is (13). The formula for the angle is shown from (13) to be

$$\arg(ds_i) = \arg(dk) + \arg(-1) + \arg\left(\frac{\eta_i^H K^{-1} \nu_i}{y_i^H x_i}\right) \; .$$

Since k varies negatively from ∞ to 0, arg (dk) = 180°; and since arg(-1) = 180°, the result is (12). This completes the proof.

The angles on the root locus for $0 < k < \infty$ can be found using either (11) or (12). Except for k very close to zero, when numerical problems may be a factor, it is best to use (11) because it involves solving an n dimensional eigenvalue problem rather than an n+m dimensional generalized eigenvalue problem. The following identities, which are obtained from (7) and (8) of Lemma 1, can be used to pass back and forth from (11) to (12):

$$Cx_i = -kK^{-1}\nu_i$$

$$y_i^H B = k\eta_i^H K^{-1} .$$

From these identities we see that when $k = 0$, $Cx_i = 0$ and $y_i^H B = 0$. Therefore, (11) cannot be used when $k = 0$ to compute angles of approach because arg (ds_i) = arg (0), which is not defined.[2]

Equations (11) and (12) are still valid when the controllability and observability assumptions are relaxed. However, angles can only be computed for modes that are both controllable and observable because only these modes move as a function of k, and thus have well defined angles.

The ν_i and η_i^H vectors are also useful for the calculation of eigenvalue sensitivities. This is shown in the next lemma. A separate proof of this lemma is not necessary because the proof follows from intermediate steps in the proof of Theorem 1.

Lemma 2. The sensitivities of distinct closed loop eigenvalues to changes in k, for $0 \leq k \leq \infty$, are found by

$$\left|\frac{ds_i}{dk}\right| = \frac{1}{k^2}\left|\frac{y_i^H BKCx_i}{y_i^H x_i}\right| \qquad \begin{array}{l} 0 < k \leq \infty \\ i = 1,\ldots,p \end{array} \tag{15}$$

$$\left|\frac{ds_i}{dk}\right| = \left|\frac{\eta_i^H K^{-1}\nu_i}{y_i^H x_i}\right| \qquad \begin{array}{l} 0 \leq k < \infty \\ i = 1,\ldots,p \end{array} . \tag{16}$$

Equations (15) and (16) give the same anwers for $0 < k < \infty$. Even though k appears only in (15), we note that both (15) and (16) are dependent on k because the vectors y_i^H, x_i, η_i^H, and ν_i are all dependent on k.

[2] In [1] a limiting argument as $k \to 0$ is used to derive alternate equations for the angles of approach. These results are more complicated than (12) because the rank of CB must be determined. The generalized eigenvalue problem eliminates the need for this rank determination. Furthermore, the equation given in [1] for the Rank (CB) = m case (3.16b [1]) is incorrect due to an error in the derivation after (3.15 [1]). This error leads to the incorrect conclusion that the angles of approach are independent of the output feedback matrix K.

Example 1

To illustrate that above results, we define a system S(A,B,C) and plot root loci for each of 3 output feedback matrices K. The system matrices are

$$A = \begin{bmatrix} -4 & 7 & -1 & 13 \\ 0 & 3 & 0 & 2 \\ 4 & 7 & -4 & 8 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ -2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -5 & 2 & -2 \\ 8 & -14 & 0 & 2 \end{bmatrix}.$$

The output feedback matrices are

Case # 1

$$K = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

Case #2

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Case #3

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 50 \end{bmatrix}$$

Case #2 is the same as used in [1]. The root loci are shown in Figure 2. The angles of departure and approach were computed and are listed in Table 1.

The system has two open loop unstable modes that are attracted to unstable transmission zeroes, so for all values of k the system is unstable. The system has two open loop stable modes that are attracted to $-\infty$ along the negative real axis. One of the branches first goes to the right along the negative real axis and then turns around. The turn around point is called a branch point. The root locus can be thought of as being plotted on a Riemann surface, and the branch points are points at which the root locus moves between different sheets of the Riemann surface [5].

TABLE 1

Angles of Departure and Approach for Example 1

Case	Angles of Departure $-4 \pm 2i$	1	2	Angles of Approach $1 \pm i$
1	$\pm$ 173°	0°	180°	$\mp$ 170°
2	$\pm$ 149	0	180	$\mp$ 121
3	+ 135	0	180	$\mp$ 114

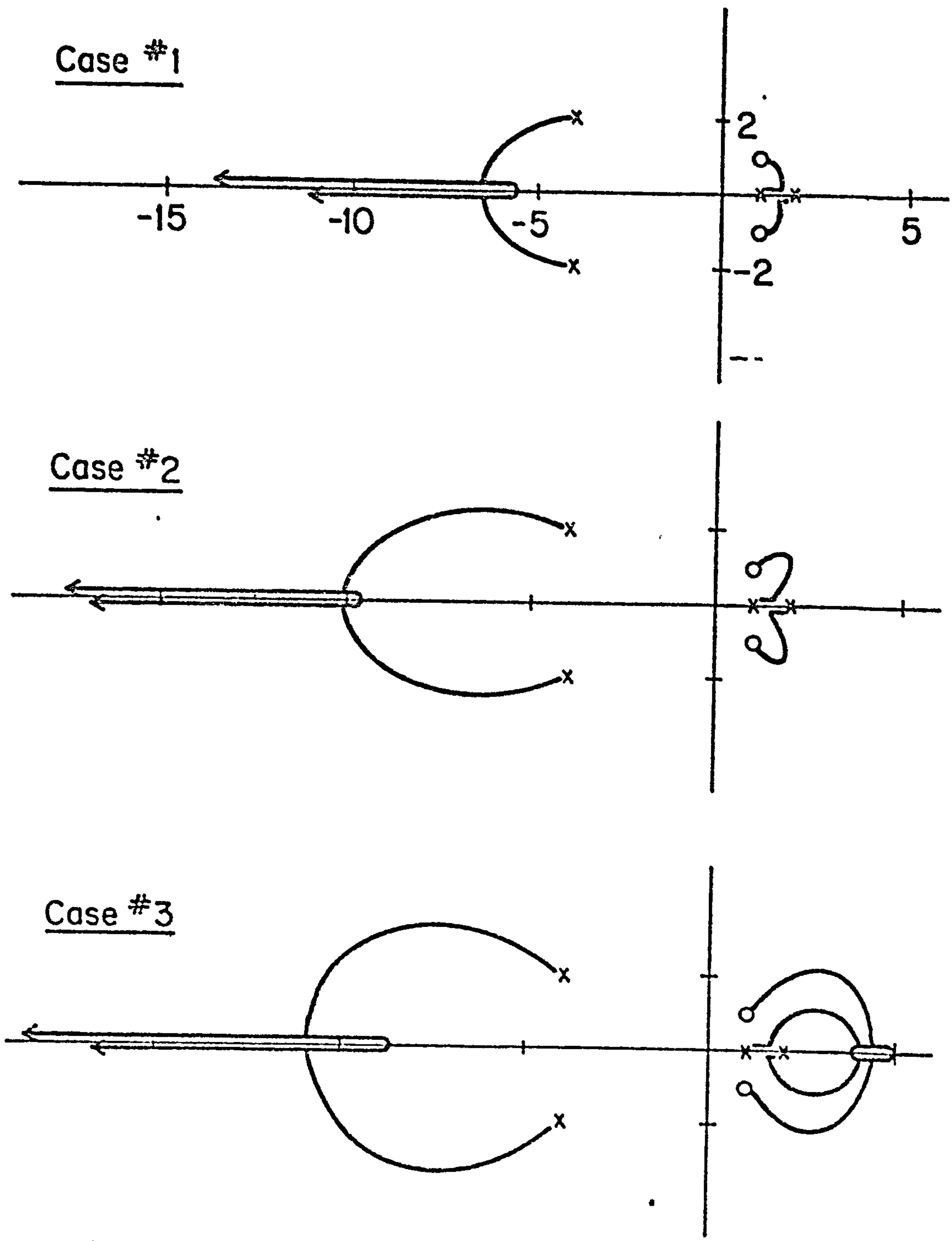

Figure 2. Root Loci of a Linear System with Output Feedback

III. Angles and Sensitivities of the Optimal Root Locus

Our attention now shifts from the linear output feedback problem to the linear optimal state feedback problem with a quadratic cost function. As in [7, 12], we show that the optimal root locus for this problem is a special case of the ordinary output feedback root locus. We show how to compute asymptotically finite properties of the optimal root locus and how to compute angles and sensitivities.

The linear optimal state feedback problem is

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m \tag{17}$$

$$u = F(x) \quad . \tag{18}$$

The optimal control is required to be a function of the state and to minimize the infinite time quadratic cost function

$$J = \int_0^\infty (x^T Qx + \rho u^T Ru)dt \ ,$$

where

$$Q = Q^T \geq 0$$

$$R = R^T > 0$$

$$0 \leq \rho \leq \infty \quad .$$

As usual we assume that (A,B) is controllable and that the state weighting matrix; factored into

$$Q = H^T H \quad ,$$

where Rank (Q) = Rank (H) = r, and $H \in \mathbb{R}^{r\times n}$; produces an observable pair (H,A).

Kalman [18] has shown (for $\rho>0$) that the optimal control is a linear function of the state

$$u = -Fx \quad , \tag{19}$$

where

$$F = \frac{1}{\rho} R^{-1}B^{T}P \quad , \tag{20}$$

and P is the solution of the Riccati equation

$$0 = Q + A^{T}P + PA - \frac{1}{\rho} PBR^{-1}B^{T}P \quad . \tag{21}$$

The closed loop system matrix is

$$A_{c\ell} = A - BF \quad . \tag{22}$$

As ρ is varied from infinity down to zero the closed loop eigenvalues trace out an optimal root locus.

To study the optimal root locus we define a linear output feedback problem with 2n states, m inputs, and m outputs.

$$\tilde{A} = \begin{bmatrix} A & 0 \\ -Q & -A^{T} \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\tilde{C} = [0 \quad B^{T}] \qquad \tilde{K} = R^{-1} \quad .$$

The closed loop system matrix is

$$Z = \tilde{A} - \frac{1}{\rho} \tilde{B}\tilde{K}\tilde{C} = \begin{bmatrix} A & -\frac{1}{\rho} BR^{-1}B^{T} \\ -Q & -A^{T} \end{bmatrix} \quad .$$

The Z matrix is sometimes called the Hamiltonian matrix. Its 2n eigenvalues are known to be symmetric about the imaginary axis, and those eigenvalues in the left half plane (LHP) are the eigenvalues of $A_{c\ell}$[8]. We again assume that the 2n eigenvalues of Z are distinct. Then the right and left eigenvectors of Z can be defined to be

$$z_i, \quad w_i^H \qquad i = 1,\ldots, 2n .$$

The right eigenvectors can be further decomposed into

$$z_i = \begin{bmatrix} x_i \\ \xi_i \end{bmatrix} .$$

Then the x_i vector is a right eigenvector of $A_{c\ell}$ and $\xi_i = Px_i$. There is apparently not a similarly easy way to find the left eigenvector y_i^H of $A_{c\ell}$ from z_i and w_i^H.

The closed loop eigenvalues, right and left eigenvectors of Z can be found by solving ordinary eigenvalues problems. Alternatively, using Lemma 1, they can be found by solving the following generalized eigenvalue problems.

$$\begin{bmatrix} \tilde{A} - s_i I & \tilde{B} \\ - \tilde{C} & -\rho\tilde{K}^{-1} \end{bmatrix} \begin{bmatrix} z_i \\ \nu_i \end{bmatrix} = 0 \qquad i = 1,\ldots,2p \quad (23)$$

$$\begin{bmatrix} w_i^H & \eta_i^H \end{bmatrix} \begin{bmatrix} \lambda - s_i I & \tilde{B} \\ - \tilde{C} & -\rho\tilde{K}^{-1} \end{bmatrix} = 0 \qquad i = 1,\ldots,2p \quad (24)$$

The number of finite generalized eigenvalues is $2p = 2n$ if $\rho > 0$ and is $0 \leq 2p \leq 2(n-m)$ if $\rho = 0$.

We can analyze the optimal root locus by using the LHP portion of the root locus of the Hamiltonian system. At $\rho = \infty$ the n branches of the optimal root locus start at the stable open loop poles (or the mirror image about the imaginary axis of the open loop unstable poles). The branches of the optimal root locus always stay in the LHP. As $\rho \to 0$, p of these branches stay finite and approach transmission zeroes, where $0 \leq p \leq n-m$. These transmission zeroes are the finite LHP solutions of the generalized eigenvalue problem (23) with $\rho = 0$. The right zero directions associated with the transmission zeroes are the x_i portions of the associated z_i vectors.[3]

These asymptotically finite properties will be grouped together in the following way:

$$S^0 = \text{diag } (s_1^0, \ldots, s_p^0)$$

$$X^0 = [x_1^0, \ldots, x_p^0] \quad .$$

Each s_i^0 is a transmission zero and each x_i^0 is a right zero direction. Because each x_i^0 is a direction it is only unique to within a scalar multiple.

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[3] If $Q = H^T H$, where $H \in \mathbb{R}^{m \times n}$, then an n+m dimension generalized eigenvalue problem using S(A,B,H) can be solved to find the transmission zeroes. The p branches that remain finite approach the LHP transmission zeroes, or the mirror image about the imaginary axis of the RHP transmission zeroes. The zero directions are the vectors associated with the LHP transmission zeroes. The zero directions associated with the mirror image of the RHP transmission zeroes cannot be found using this n+m dimension problem.

The angles and sensitivities of the optimal root locus can be found by applying Theorem 1 and Lemma 2 to the Hamiltonian system. The results are the following:

<u>Theorem 2</u>. The angles on the optimal root locus, for $0 \leq \rho \leq \infty$ and for distinct s_i, are found by

$$\arg\ (ds_i) = \arg \left\{ \frac{1}{w_i^H z_i} \ w_i^H \begin{bmatrix} 0 & BR^{-1}B^T \\ 0 & 0 \end{bmatrix} z_i \right\} \quad \begin{matrix} 0 < \rho \leq \infty \\ i = 1,\ldots,p \end{matrix} \tag{25}$$

$$\arg\ (ds_i) = \arg \left\{ \frac{\eta_i^H R \nu_i}{w_i^H z_i} \right\} \quad \begin{matrix} 0 \leq \rho < \infty \\ i = 1,\ldots, p \end{matrix} \tag{26}$$

<u>Remark</u>. The angles of departure are found using (25) with $\rho = \infty$, and the angles of approach by using (26) with $\rho = 0$. For $\rho > 0$, $p = n$; and for $\rho = 0$, $0 \leq p \leq n-m$.

<u>Lemma 3</u>. The sensitivities of distinct closed loop eigenvalues to changes in ρ, for $0 \leq \rho \leq \infty$, are found by

$$\left|\frac{ds_i}{d\rho}\right| = \frac{1}{\rho^2} \left| \frac{1}{w_i^H z_i} w_i^H \begin{bmatrix} 0 & BR^{-1}B^T \\ 0 & 0 \end{bmatrix} z_i \right| \quad \begin{matrix} 0 < \rho \leq \infty \\ i = 1,\ldots, p \end{matrix} \tag{27}$$

$$\left|\frac{ds_i}{d\rho}\right| = \left| \frac{\eta_i^H R \nu_i}{w_i^H z_i} \right| \quad \begin{matrix} 0 \leq \rho < \infty \\ i = 1,\ldots, p \end{matrix} \quad . \tag{28}$$

Remark. The computations for (25-28) can be reduced by using the following identities. First, from (23) and (24), it can be shown that

$$\nu_i = \eta_i \quad .$$

Second, from [8], let $\bar{s}_i$ be the RHP mirror image about the imaginary axis of s_i, and let $\bar{z}_i = (\bar{x}_i^H, \bar{\xi}_i^H)^H$ be the right eigenvector associated with $\bar{s}_i$. Then the left eigenvector associated with s_i is

$$w_i^H = (-\bar{\xi}_i^H \; \bar{x}_i^H) \quad .$$

IV. Asymptotically Infinite Properties of the Optimal Root Locus

In this section we continue to analyze the optimal root locus. We review what is known about the asymptotic behavior and then present an algorithm which can be used to predict the asymptotically infinite behavior.

Review of Known Asymptotic Behavior

As $\rho \to 0$ the number of asymptotically finite branches is p, where $0 \le p \le n-m$. These branches approach the LHP transmission zeroes of the Hamiltonian system. The associated eigenvectors approach zero directions, which are part of the zero directions of the Hamiltonian system, as explained in section III. The remaining n-p branches group in m Butterworth patterns and approach infinity. Let the order of the ith pattern by n_i. Each of the n_i eigenvalues in this pattern lies on one of n_i asymptotes [4] with a distance from the origin approximately equal to

$$\left(\frac{s_i}{\rho^{1/2}}\right)^{\frac{1}{n_i}} .$$

There are n_i right eigenvectors associated with the pattern. These span the same subspace of $\mathbb{R}^n$ spanned by

[4] A first order Butterworth pattern has one asymptote which coincides with the negative real axis. A second order Butterworth pattern has two asymptotes which have angles of $\pm 45°$ with the negative real axis. In general, an ith order Butterworth pattern has i asymptotes, each of which starts at the origin and goes through the LHP solutions s of $s^{2i} + (-1)^i = 0$.

$$Bv_i^\infty, \ ABv_i^\infty, \ldots, A^{n_i-1} Bv_i^\infty .$$

The ordering of these vectors can be conveniently summarized in terms of a multi-index defined in the following way:

$$\gamma = (01,\ 11,\ 21, \ldots, [n_1-1]1,\ 02,\ 12,\ 22, \ldots, [n_2-1]2, \ldots, [n_m-1]m) \ .$$

Each component ij of γ describes the vector $A^i B v_j^\infty$ associated with the m Butterworth patterns.

One special case of the above asymptotically infinite structure deserves special notice. When Rank $(B^T QB) = m$ then there are (n-m) finite modes and the remaining m infinite modes all form first order Butterworth patterns. This is called the "generic" case. For an explanation of the word "generic," see [19]. The m asymptotically infinite eigenvalues lie on the negative real axis an approximate distance $s_i^\infty/\rho^{1/2}$ from the origin. Their associated eigenvectors approach Bv_i^∞ and, hence, the multi-index γ is

$$\gamma = (01,\ 02,\ \ldots,\ 0m) \ .$$

The asymptotically infinite eigenstructure $\{s_i^\infty, v_i^\infty, i=1,\ldots, m\}$ of a generic problem can be readily computed by solving the following m-dimensional eigenvalue problem:

$$[(s_i^\infty)^2 I - R^{-1}B^T QB]v_i^\infty = 0 \qquad i = 1,\ldots, m.$$

The resulting solutions will be grouped together in the following way:

$$S^\infty = \mathrm{diag}(s_1^\infty, \ldots, s_n^\infty)$$

$$N^\infty = [v_1^\infty, \ldots, v_m^\infty]$$

Each of the ν_i^∞ vectors is a direction and is therefore only unique to within a scalar multiple.

In contrast to the generic problem, the nongeneric case does not yield to a similarly simple calculation of its infinite asymptotic eigenstructure. For this case, it is necessary to evaluate vectors ν_i^∞, scalars s_i^∞, and also the Butterworth dimensions n_i, $i = 1, \ldots, m$. An algorithm for this purpose is provided below.

<u>An Algorithm for the Non-Generic Case</u>

Under our earlier assumptions, the ν_i^∞ vectors form a basis for $\mathbb{R}^m$. The algorithm presented here decomposes $\mathbb{R}^m$ into these basis vectors. This is done in two steps. The first is to compute basis vectors for a sequence of $\mathcal{U}_i$ subspaces of $\mathbb{R}^m$ (defined below). The second step is to use a series of eigenvalue problems to further break down the $\mathcal{U}_i$ subspaces into the ν_i^∞ basis vectors. These same eigenvalue problems compute the s_i^∞'s.

Let $k \le n-m+1$ be the highest order Butterworth pattern. Define the matrices

$$J_i = HA^{i-1}B \qquad i = 1, \ldots, k,$$

and define the subspaces of $\mathbb{R}^m$

$$\mathcal{U}^0 = \mathbb{R}^m$$

$$\mathcal{U}_1 = \mathcal{U}^0 \cap \ker J_1$$

$$\vdots$$

$$\mathcal{U}_k = \mathcal{U}_{k-1} \cap \ker J_k \quad .$$

These subspaces are shown pictorially in Figure 3. They are nested such that

$$0 = \mathcal{U}_k \subseteq \cdots \subseteq \mathcal{U}_1 \subseteq \mathcal{U}_0 = R^m ,$$

and their dimensions satisfy

$$m_i = \dim \mathcal{U}_{i-1} - \dim \mathcal{U}_i \qquad i = 1,\ldots, k$$

$$\sum_{i=1}^{k} m_i = m .$$

A basis for each of the $\mathcal{U}_i$ subspaces can be recursively computed.[5] The recursion stops at the k^{th} step when $\mathcal{U}_k = 0$. Define

$$U_i \qquad i = 0,1,\ldots, k-1$$

to be matrices whose columns form a basis for the $\mathcal{U}_i$ subspaces. The basis vectors are not unique, and without loss of generality let

$$U_0 = I .$$

Though it is not obvious at this point, we note that the number of ith order Butterworth patterns is m_i. If there are no ith order Butterworth patterns then $m_i = 0$ and $\mathcal{U}_{i-1} = \mathcal{U}_i$. The dimensions of U_i are $m \times \ell_i$, where

$$\ell_i = m_{i+1} + \cdots + m_k ,$$

and ℓ_i is the number of Butterworth patterns of order greater than i. When the U_i matrices are computed we have enough information to form

[5] Numerically this may be difficult to do. The decomposition requires determination of the kernal of a matrix and the intersection of subspaces. For both of the singular value decomposition is a good tool to use [14].

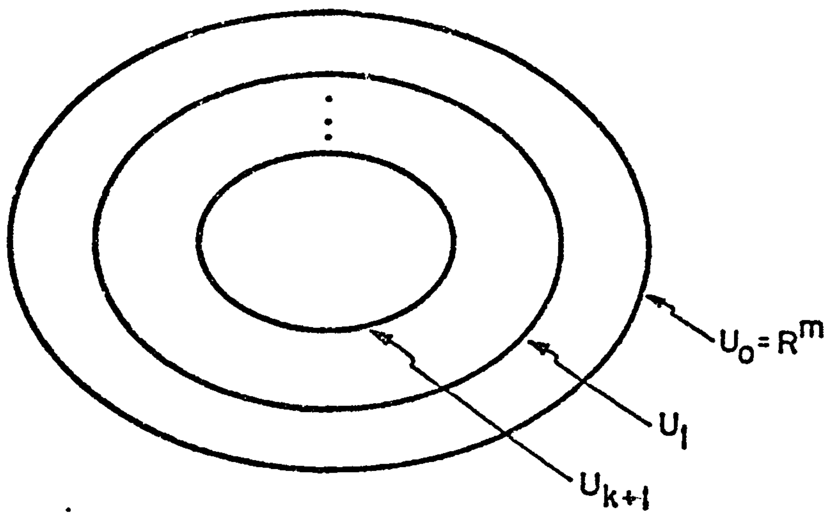

Fig. 3. The U_i Subspaces.

γ, the multi-index which lists the orders of the m Butterworth patterns. In the generic case $k=1$, $m_1 = m$, and $\mathcal{U}_1 = 0$.

The next step in the algorithm is to use the U_i matrices to compute N^∞ and S^∞. We decompose N^∞ and S^∞ into

$$N^\infty = [N_1^\infty, \ldots, N_k^\infty]$$

$$S^\infty = \mathrm{diag}(S_1^\infty, \ldots, S_k^\infty) \quad .$$

N_i^∞ is an $m \times m_i$ matrix whose columns are the ν_j^∞'s associated with ith order Butterworth patterns. S_i^∞ is an $m_i \times m_i$ diagonal matrix whose diagonal elements are the s_j^∞'s associated with ith order Butterworth patterns. We note that

$$\mathcal{U}_i = \mathrm{Im}N_{i+1}^\infty + \ldots + \mathrm{Im}N_k^\infty \qquad i = 0, \ldots, k-1$$

and that in general

$$\mathrm{Im}N_i^\infty \cap \mathrm{Im}N_j^\infty \neq 0 \qquad \text{for } i \neq j \quad .$$

In words, these two equations tell us that one basis for the subspace $\mathcal{U}_i$ consists of the vectors ν_i^∞ associated with Butterworth patterns of order greater than i, and that in general the ν_i^∞'s are not orthogonal.

We define two more sets of matrices:

$$G_i = J_i^T J_i \qquad i = 1, \ldots, k$$

$$T_i = (U_{i-1}^T R U_{i-1})^{-1} (U_{i-1}^T G_i U_{i-1}) \quad i = 1, \ldots, k \quad .$$

When the U_i matrices are known then the T_i matrices can be computed in a straightforward manner. The dimensions of T_i are $\ell_{i-1} \times \ell_{i-1}$. The significance of the T_i matrices is that they can be decomposed by an eigenvalue problem to find the N_i^∞ and S_i^∞ matrices. This connection is made clear in the following Theorem.

Theorem 3. The Jordan canonical form of T_i is

$$T_i = [W_{i1}\ W_{i2}] \begin{bmatrix} \Lambda_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_{i1} & W_{i2} \end{bmatrix}^{-1} \qquad i = 1,\ldots, k. \tag{29}$$

Λ_i is a diagonal matrix with positive real eigenvalues, and

$$N_i^\infty = U_{i-1}W_{i1} \tag{30}$$

$$(S_i^\infty)^2 = \Lambda_i \quad . \tag{31}$$

Proof: Appendix A.

Note that in the decomposition (29), W_{i1} has as many columns as there are ith order Butterworth patterns, and W_{i2} has as many columns as there are Butterworth patterns of greater than ith order. So the dimensions of W_{i1} and W_{i2}, respectively, are $\ell_{i-1} \times m_i$ and $\ell_{i-1} \times \ell_i$. If there are no ith order Butterworth patterns then $T_i = 0$ and W_{i1} is not present.

Note further that there are no restrictions on the multiplicity of the s_i^∞'s (these are positive real numbers and not closed loop eigenvalues). If $s_i^\infty = s_j^\infty$ but they are associated with Butterworth patterns of different orders then they are solutions to different eigenvalue problems (29). Consequently there is no ambiguity in the asscciated ν_i^∞ and ν_j^∞ vectors. However, if $s_i^\infty = s_j^\infty$ and they are associated with Butterworth patterns of the same order then we can only say that ν_i^∞ and ν_j^∞ form a nonunique basis for a two dimensional subspace of $\mathbb{R}^m$. This is known from properties of the Jordan canonical form of T_i (29) and from (30).

Example 2

The algorithm described above is illustrated with the following A, B, $Q = H^TH$, and R matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -5 & -4 & 0.1 & 1 \\ 0.1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} -65 & 0 & 0 & 1 \\ 100 & 10 & 0 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} .01 & -.10 \\ -.01 & .1211 \end{bmatrix}$$

The asymptotically finite properties are found by solving a generalized eigenvalue problem using the system S(A,B,H). The results are

$$S^0 = [10] \qquad X^0 = \begin{bmatrix} 1 \\ -10 \\ 0 \\ 65 \end{bmatrix}$$

The asymptotially infinite properties are found by the algorithm of Theorem 3. First we find the $\mathcal{U}_i$ subspaces and their matrices U_i:

$$J_1 = HB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad J_2 = HAB = \begin{bmatrix} 0 & -5 \\ 1 & 10 \end{bmatrix}$$

Since $\mathcal{U}_1 = \ker J_1$, $\quad U_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since $\ker J_2 = 0$, $\mathcal{U}_2 = \mathcal{U}_1 \cap \ker J_2 = 0$.
The number of first and second order Butterworth patterns are

$$m_1 = \dim \mathcal{U}_0 - \dim \mathcal{U}_1 = 1$$

$$m_2 = \dim \mathcal{U}_1 - \dim \mathcal{U}_2 = 1 \quad .$$

The T_i matrices and their Jordan canonical forms are

$$T_1 = R^{-1}G_1 = \begin{bmatrix} 0 & 9 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

$$N_1^\infty = W_{11} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad S_1^\infty = \Lambda_1^{1/2} = [3]$$

$$T_2 = (U_1^T R U_1)^{-1} U_1^T G_2 U_1 = [100]$$

$$N_2^\infty = U_1 W_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad S_2^\infty = \Lambda_2^{1/2} = [10]$$

Therefore

$$N^\infty = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad S^\infty = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix} \quad \Upsilon = (01,\ 02,\ 12) \ .$$

Each column of N^∞ represents a direction and is therefore only unique to within a scalar multiple.

V. Conclusions

Both the eigenvalue and generalized eigenvalue problems can be used to compute angles and sensitivities of multivariable root loci and optimal root loci. The generalized eigenvalue problem is superior to use for computing the angles of approach.

The elementary matrices A, B, Q, and R can be used to determine the asymptotic behavior of the optimal root locus and the associated eigenvectors. A generalized eigenvalue problem can be used to compute S_i^0 and X_1^0, the asymptotically finite properties. A subspace decomposition of the control space R^m and a series of eigenvalue problems can be used to compute S^∞, N^∞, and γ; the asymptotically infinite properties.

We are hopeful that a similar type of subspace decomposition can be used to determine the asympotically infinite behavior of arbitrary multivariable root loci. The extension of the present method is difficult, however, because we do not in general have the symmetry of closed loop eigenvalues about the imaginary axis forced by the optimal Hamiltonian system.

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Appendix A - Proof of Theorm 3

The proof is by induction and uses the fact that all closed loop eigenvalues s_j and closed loop vectors ν_j must satisfy

$$[\rho R + \Phi^T(-s_j)\Phi(s_j)]\nu_j = 0 \qquad j = 1,\ldots, n \tag{A.1}$$

where

$$\Phi(s) = H(sI-A)^{-1}B$$

$$= H(\frac{1}{s}I + \frac{1}{s^2} A + \ldots)B$$

$$= \sum_{i=1}^{\infty} \frac{1}{s^i} J_i \ .$$

Equation (A.1) is derived in [20]. It can also be derived by manipulations of (23). An expanded version of $\Phi^T(-s)\Phi(s)$ is

$$\Phi^T(-s)\Phi(s) = \sum_{i=2}^{\infty} \left[\frac{1}{s^i} \sum_{j=1}^{i-1} (-1)^j J_j^T J_{i-j}\right] .$$

The first step in the induction proof is to show that the theorem is valid for N_1^{∞} and S_1^{∞}. We assume without loss of generality that first order Butterworth patterns exist. Equation (A.1) can be rewritten

$$\left[\rho R + \frac{1}{s_j^2} B^T QB + 0\left(\frac{1}{s_j^2}\right)\right]\nu_j = 0 \qquad j = 1,\ldots, n \ . \tag{A.2}$$

As $\rho \to 0$ the s_j^{-2} term dominates (for the asymptotically infinite eigenvalues s_j) and (A.2) can be rewritten

$$(\lambda_j I - R^{-1}B^TQB)\nu_j^\infty = 0 \qquad j = 1,\ldots, m \tag{A.3}$$

$$\lambda_j = \rho s_j^2 \ . \tag{A.4}$$

The eigenvalues λ_j of $R^{-1}B^TQB$ are real and nonnegative. (This is because the eigenvalues are the same as those of $R^{-1/2}B^TQBR^{-1/2}$, which is a matrix of the form X^TX, which is known to have real and nonnegative eigenvalues). When $\lambda_j > 0$ we can use (A.4) to solve for s_j. The LHP solution is $s_j = -\lambda_j^{1/2}/\rho^{1/2}$, which is a first order Butterworth pattern with $s_j^\infty = \lambda_j^{1/2}$. Therefore S_1^∞ as given in Theorem 3 is valid. N_1^∞ is also valid, because from (A.3) we see that the ν_j^∞ vectors associated with first order Butterworth patterns are eigenvectors of $R^{-1}B^TQB$.

The ν_j^∞ vectors associated with Butterworth patterns of order greater than one form a basis for the kernal of $R^{-1}B^TQB$, which is $\mathcal{U}_1$. Heuristically speaking, these ν_j^∞ vectors are not "trapped" by the s_j^{-2} term of (A.2).

The next step in the induction proof is to assume that S_{i-1}^∞ and N_{i-1}^∞ are valid and then show that S_i^∞ and N_i^∞ are valid. If S_{i-1}^∞ and N_{i-1}^∞ are valid then the ν_j^∞ vectors associated with Butterworth patterns of order $\geq i$ form a basis for $\mathcal{U}_{i-1}$. Therefore for each of these ν_j^∞ vectors there exists an ω_j vector such that $\nu_j^\infty = U_{i-1}\omega_j$. Substitute this into (A.1) to get

$$[\rho R + \Phi^T(-s_j)\Phi(s_j)]U_{i-1}\omega_j = 0 \qquad j = 1,\ldots, \ell_{i-1} \ .$$

Multiply on the left to get

$$U_{i-1}^T[\rho R + \Phi^T(-s_j)\Phi(s_j)]U_{i-1}\omega_j = 0 \qquad j = 1,\ldots, \ell_{i-1}.$$

After some algebra this reduces to

$$[\rho U_{i-1}^T R U_{i-1} + (-1)^i \frac{1}{s_j^{2i}} U_{i-1}^T G_i U_{i-1} + 0\left(\frac{1}{s_j^{2i}}\right)\omega_j = 0 .$$

As $\rho \to 0$ the s_j^{-2i} term dominates and we get

$$(\lambda_j I - T_i)\omega_j = 0 \qquad j = 1,\ldots, \ell_{i-1} \tag{A.5}$$

$$\lambda_j = -(-1)^i \rho s_j^{2i} \quad . \tag{A.6}$$

The eigenvalues λ_j are real and nonnegative, for the same reasons as for the i=1 case. When $\lambda_j > 0$ we can solve for s_j, and the LHP solutions are recognized as an ith order Butterworth pattern with $s_j^\infty = \lambda_j^{1/2}$. Therefore S_i^∞ of Theorem 3 is valid. The eigenvectors ω_j of T_i associated with the nonzero eigenvalues λ_j are the columns of W_{i1} and therefore $N_i^\infty = U_{i-1}W_{i1}$. The ν_j^∞ vectors associated with Butterworth patterns of order greater than i are not "trapped" by the s_j^{-2i} term, and therefore they lie in $\mathcal{U}_i$. This completes the proof.

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