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CHARACTERIZATION OF ACOUSTIC DISTURBANCES IN LINEARLY SHEARED FLOWS

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TABLE OF CONTENTS

LIST OF FIGURES

CHARACTERIZATION OF ACOUSTIC DISTURBANCES
IN LINEARLY SHEARED FLOWS

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The equation describing the plane wave propagation, the stability or the rectangular duct mode characteristics in a compressible inviscid linearly sheared parallel, but otherwise homogenous flow, is shown to be reducible to Whittaker's equation. The resulting solutions, which are real, viewed as functions of two variables, depend on a parameter and an argument the values of which have precise physical meanings depending on the problem. The exact solutions in terms of Whittaker functions are used to obtain a number of known results of plane wave propagation and stability in linearly sheared flows as limiting cases in which the speed of sound goes to infinity (incompressible limit) or the shear layer thickness, or wave number, goes to zero (vortex sheet limit). The usefulness of the exact solutions is then discussed in connection with the problems of plane wave propagation and the stability of a finite thickness shear layer with a linear velocity profile. With respect to the plane wave propagation it is shown that, unlike the compressible vortex sheet, the shear layer possesses no resonances and no Brewster angles, whereas with respect to the stability problem it is shown that again unlike the compressible vortex sheet, the thin shear layer is unstable to long wavelength disturbances for all Mach numbers. These results imply that the reflection and stability characteristics of a nonzero thickness but thin shear layer (i.e., the long wavelength characteristics) do not go

over smoothly into the results of the compressible vortex sheet as the wave number approaches zero except for a limited range of generally subsonic relative flow of the two parallel streams bounding the shear layer.

1. Introduction

Although problems pertaining to plane wave propagation, stability and rectangular mode in a compressible inviscid linearly sheared parallel flow have received considerable attention, with the exception of the work of Goldstein and Rice [1], all previous work has been concerned with either asymptotic and/or series solutions of the governing equation which is of course the same for all three above classes of problems. Typical examples of earlier work are that of Küchemann [2] and Pridmore-Brown [3], and of later work, that of Graham and Graham [4] and Goldstein and Rice [1]. Küchemann [2] obtained formal series solutions of the density perturbation equation and also asymptotic solutions purportedly valid for large Mach numbers. His series solution, although it is given in a cumbersome and lengthy form, is correct and agrees with our compact form (see equations (8) and (9) below), but his asymptotic solution is in error (see discussion following equation (17) below). Pridmore-Brown [3] solved the pressure perturbation equation in the short wavelength approximation. His asymptotic solution which is in terms of Airy functions is limited in that the interesting region $\eta \to 0$ is excluded. Graham and Graham [4] obtained a series solution for the density perturbation equation apparently unaware of the earlier work of Küchemann. Their series solution agrees with that of Küchemann and with our series expressions mentioned above. Goldstein and Rice [1] were apparently the first to obtain solutions to the governing equation in terms of special functions. Using an unusual double transformation (the essential part being a differential transformation with a gaussian kernel) they were able to obtain from the original second order equation for the pressure perturbation an exact third order equation from which two independent solutions of the original second order equation were easily extracted in terms of the parabolic cylinder D functions. Unfortunately the solutions obtained by Goldstein and Rice [1] are not in terms of single parabolic cylinder D functions, but combinations of D functions of different orders and in addition the solutions obtained were complex. It is not possible to reduce the solutions given by Goldstein and Rice [1] to our solutions which are in terms of single Whittaker M or W functions, which is not surprising since our solutions are real, whereas those of Goldstein and Rice are complex. But if one does choose either specific linear combinations of the solutions of Goldstein and Rice or the so-called even and odd solutions of Weber's equation instead of the parabolic cylinder functions D used in the above work, then it may be shown that our real solutions, in terms of single Whittaker functions and those of Goldstein and Rice are compatible. Recently in a study Jones [5] considered the stability of an everywhere subsonic ($M < 1$) shear layer with a linear velocity profile with the result that for $0 < M < 1$ there is a characteristic Strouhal number below which the layer is unstable. The problem investigated by Jones [5] is substantially different from the problem treated here. In the first place he considers a source at a finite position from the layer whereas we consider plane waves emanating from $-\infty$ in the propagation problem and the usual formulation of the eigenvalue equation for parallel flow stability (see Betchov and Criminale [6]) without the necessity of invoking causality which Jones does. Moreover, Jones' work pertains to the open-ended region $0 < M < 1$ whereas ours covers the whole range $M \geq 0$ including the incompressible limit $M = 0$ and the supersonic flow regime.

2. Basic Differential Equation and Its Solution

In a homogeneous inviscid compressible parallel shear flow having a linear velocity profile in the z-direction only, i.e., $U = U(z) = bz$, with b constant, it may be easily shown that, starting either from the linearized equations of motion or directly using the appropriate linearized form of the convective wave equation, the z-dependent part $p(z)$ of the pressure perturbation $p(\vec{r})$ is governed by the equation:

$$p_{\eta\eta} - \frac{2}{\eta} p_{\eta} + (4\tau)^2\left(\eta^2 - 1\right) p = 0 \tag{1}$$

where

$$\eta = \frac{1}{K} - M \,, \quad M = M(z) = \frac{bz}{a} \,, \quad 4\tau = \frac{\omega}{b} K \tag{2}$$

M is the local Mach number, a the (uniform) sound speed, and K and ω acquire the following meanings depending on the problem at hand:

For propagation of a plane wave of wave vector $\vec{k}$ and frequency ω impinging on the shear layer from a half-space ($z < 0$) of relative rest and at an angle θ measured from the z-axis $\left(-\frac{\pi}{2} \leq \theta \leq +\frac{\pi}{2}\right)$,

$$K = \sin\theta \quad . \tag{2a}$$

For the stability of a free shear layer to assumed disturbances of the form $p(\vec{r}) = p(z)e^{i\alpha(x-ct)}$ (α and c possibly complex),

$$K = \frac{a}{c} \text{ and } \omega = \alpha c \quad . \tag{2b}$$

For sound propagation in rectangular ducts and for assumed disturbances of the form $p(\vec{r}) = p(z)e^{i\alpha(kx-\omega t)}$ (k and ω real),

$$K = \frac{ak}{\omega} \quad . \tag{2c}$$

The transformation

$$p = \eta^{\frac{1}{2}}W(\xi) \;, \quad \xi = q\eta^2 \tag{3}$$

with q independent of η to be suitably specified later, reduces equation (1) into:

$$W_{\xi\xi} + \left[\frac{4\tau^2}{q^2} - \frac{4\tau^2}{q\xi} - \frac{5/16}{\xi^2}\right] W = 0 \tag{4}$$

If we specify $q = 2\tau$ in equation (4) we obtain:

$$W_{\xi\xi} + \left[1 - \frac{2\tau}{\xi} - \frac{\frac{1}{4}\left(1 + \frac{1}{4}\right)}{\xi^2}\right] W = 0 \tag{5}$$

i.e., the Colomb wave equation with fractional angular momentum $\frac{1}{4}$, whereas if we specify $q = 4i\tau$ in equation (4) we obtain:

$$W_{\xi\xi} + \left[-\frac{1}{4} + \frac{i\tau}{\xi} + \frac{\frac{1}{4} - \left(\frac{3}{4}\right)^2}{\xi^2} \right] W = 0 \tag{6}$$

i.e., Whittaker's equation with independent solutions the M or W functions with parameters $i\tau$, $\pm 3/4$ and argument $4i\tau\eta^2$. It thus follows from equation (6) and (3) that the two independent solutions f and g of equation (1) are:

$$\binom{f}{g} = (4i\tau)^{-\frac{1}{2} \pm m} \eta^{\frac{1}{2}} M_{i\tau, \mp m}\left(4i\tau\eta^2\right) \ldots., \quad m = \frac{3}{4} \tag{7}$$

where M are the Whittaker M-functions and τ and η are defined by equations (2). Using the properties of the Whittaker functions it is easily shown that for τ and η real the functions f and g are also real with f being even and g odd functions of η, whereas both f and g are even functions of τ. The solutions in equation (7) were found earlier by us (see Reference [7]) and used in some preliminary studies on wave propagation through a linearly sheared flow.

Series forms for f and g may be easily obtained from the well-known series expressions of the Whittaker M-functions. The following is one such form:

$$\binom{f}{n} = \sum_{n=0}^{\infty} a_n \eta^n \quad , \quad \begin{array}{l} n = 0,2,4,\ldots \text{ for } f \\ n = 1,3,5,\ldots \text{ for } g \end{array} \tag{8}$$

where a_n is determined by the 3-term recurrence relation:

$$a_n = \frac{(4\tau)^2}{n(n-3)} \left(a_{n-2} - a_{n-4}\right) \quad , \quad n > 3 \tag{9}$$

with

$$a_o = 1 \qquad a_1 = 0$$

$$a_2 = -8\tau^2 \qquad a_3 = 1 \quad .$$

The above series forms were obtained by expanding in series the exponential part of the Whittaker M-functions and are in agreement with the series obtained earlier by both Küchemann [2] and Graham and Graham [4] although the expressions given by these authors are unnecessarily cumbersome. But the expressions in equation (7) for the solutions of equation (1) enable one to obtain even better series-type expressions for f and g that may not only be faster converging but also more suitable in certain applications. These may be obtained from equation (7) using the (series) expressions of the M-functions without expanding in series the exponential part and using the fact that $M_{i\tau,m}(iz)$ is a real function for real τ, m and z. One thus after some algegra obtains the following forms:

$$\begin{pmatrix} f \\ g \end{pmatrix} = \eta^{\frac{3}{2} + 2m} \left[\cos\left(2\tau\eta^2\right) \sum_{n=0}^{\infty} A_n \left(4\tau\eta^2\right)^n + \right. \tag{10}$$

$$\left. + \sin\left(2\tau\eta^2\right) \sum_{n=0}^{\infty} B_n \left(4\tau\eta^2\right)^n \right] \quad ,$$

where the coefficients A_n and B_n are given by:

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} \tau & -\left(\frac{2n-1}{2} + m\right) \\ +\left(\frac{2n-1}{2} + m\right) & \tau \end{bmatrix} \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} \tag{11}$$

with $m = \mp \frac{3}{4}$ and with the upper sign for f and the lower for g, and $A_o = 1$, $B_o = 0$.

The asymptotic expansions for f and g (as well as those of the derivatives of f and g) may be obtained using Olver's method [8]. It should be stressed that one is interested in asymptotic expansions as $\tau \to \infty$ holding uniformly in η when η ranges over unbounded regions as well as asymptotic expansions holding for unbounded τ as $\eta \to \infty$ or 0, i.e., expansions describing the asymptotic behavior of f and g as functions of both τ and η. The point is not one of mathematical rigour only, but also of necessity since the cases $\eta \to 0$ or ∞ correspond to incompressible flow or high Mach number limits respectively and $\tau \to 0$ or ∞ correspond to the long or short wavelength limits; moreover, the cases $(\tau\eta)$ - finite characterize the vortex sheet and the incompressible layer limits as we will see later. Such uniform asymptotic expansions of the Whittaker M-functions have been derived by Skovgaard [9] following Olver's method [8]. The results suitable to the present study are in terms of the modified Bessel functions of the first and second kind I_{2m} and I_{2m+1}. Few of the resulting uniform leading term expressions for the asymptotic expansions for f and g for τ and η real we will need later are:

(i) Leading asymptotic expansion terms for $\eta \to \infty$, τ-finite:

$$\begin{pmatrix} f \\ g \end{pmatrix} \sim 2 \frac{(4\tau)^{-\frac{1}{2} \pm m} e^{\frac{\pi}{2}\tau}}{\left|\Gamma\left(\frac{1}{2} \mp m + i\tau\right)\right|} \eta^{\frac{1}{2}} \left\{ \cos\left[- \tau \ln \left(4\tau\eta^2\right) + \right.\right.$$

$$\left.\left. + 2\tau\eta^2 + \delta - \frac{\pi}{4}(1 \mp 2m)\right]\right\} , \tag{12}$$

with $m = \frac{3}{4}$ and $\delta = \arg\left[\Gamma\left(\frac{1}{2} \mp m + i\tau\right)\right]$

(ii) Leading asymptotic expansion terms for $\tau \to \infty$ or $\frac{\chi}{\tau} \to 0$.

$$\binom{f}{g} \sim \eta \left(\frac{1}{|\eta^2 - 1|}\right)^{\frac{1}{4}} \left[\begin{matrix}\sin h\\ \cos h\end{matrix} (4\tau\chi) - \frac{1}{4\tau\chi} \begin{matrix}\cos h\\ \sin h\end{matrix} (4\tau\chi) + \frac{B_o(\chi)}{4\tau} \begin{matrix}\cos h\\ \sin h\end{matrix} (4\tau\chi) \right] \tag{13}$$

where

$$\left.\begin{aligned} \chi &= -\frac{\pi}{4} + i\,\frac{1}{2}\left[|\eta|\sqrt{\eta^2 - 1} - \ell n\left(|\eta| + \sqrt{\eta^2 - 1}\right)\right] \\ \text{and} \quad B_o &= \frac{-i}{32}\left\{8\left[\frac{i}{\chi} + \frac{\sqrt{\eta^2 - 1}}{|\eta|}\right] - 2\,\frac{|\eta|}{\sqrt{\eta^2 - 1}} + \frac{5}{3}\,\frac{|\eta|^3}{\left(\eta^2 - 1\right)^{\frac{3}{2}}}\right\} \end{aligned}\right\} \tag{14}$$

From the above expressions we may also obtain the following limit forms for f and g that we will also need later:

As $\eta \to \infty$, $\tau \to 0$ but $\tau\eta^2$ - finite we obtain from equation (12):

$$\binom{f}{g} \sim \eta^{\frac{1}{2}} \left\{\cos\left[2\tau\eta^2 - \frac{\pi}{4}\left(1 \mp 2m\right)\right]\right\} \tag{15}$$

As $\eta \to 0^+$, $\tau \to \infty$ but $\tau\eta$ - finite we obtain from equations (13) and (14): (Note that $\lim\limits_{\eta\to 0} \chi = -\eta$ and $\lim\limits_{\eta\to 0} B_o(\chi) = 0$ or $(2\chi)^{-1}$ depending on whether $\eta < 1$ or $\eta > 1$.)

$$\binom{f}{g} \sim (4\tau\eta) \begin{matrix}\sin h\\ \cos h\end{matrix} (4\tau\eta) - \begin{matrix}\cos h\\ \sin h\end{matrix} (4\tau\eta) \tag{16}$$

As $\tau \to \infty$, $\eta > 0$ we obtain from equation (13):

$$\begin{pmatrix} f \\ g \end{pmatrix} \sim \eta \left(\frac{1}{|\eta^2 - 1|}\right)^{\frac{1}{4}} \begin{matrix} \sin h \\ \cos h \end{matrix} (4\tau\chi) \tag{17}$$

The above asymptotic expressions differ from some corresponding expansions obtained by Küchemann [2] and Pridmore-Brown [3]. This was expected as mentioned in the introduction since Küchemann [2] essentially seeking a high Mach number (or large η) expansion neglected 1 compared to η^2 in the last term of equation (1) which is tantamount to letting $\tau \to 0$ in Whittaker's equation (6), with the result that the solutions to equation (6) then are the special Whittaker functions $M_{o\pm 3/4}(4i\tau\eta^2)$ which are proportional to the Bessel functions $J_{\pm 3/4}(-2\tau\eta^2)$ as was obtained by Küchemann [2] by solving directly the approximated equation (1) as mentioned above. But this clearly implies that the high Mach number, or large η, solution in terms of $J_{\pm 3/4}(-2\tau\eta^2)$ is only valid for $\tau \to 0$, i.e., for low frequencies ω. In actuality the correct high Mach number asymptotic expansion to lowest order is given by equation (12) for all frequencies whereas the low frequency/high Mach number limit is given by equation (15) and is obtained from equation (12) as $\tau \to 0$, $\eta \to \infty$ but $\tau\eta^2$-finite. It is moreover easily seen that equation (15) constitutes essentially the leading term of the asymptotic expansion of the Bessel functions $J_{\pm 3/4}(-2\tau\eta^2)$ for large η. The reason for Küchemann's [2] incorrect expression lies of course in the fact that it is not sufficient to have $\eta^2 >> 1$ for neglecting 1 in the last term of equation (1) but one should compare the term to be neglected to all the other terms of equation (1) which, after some algebra results into the additional necessary condition, $\tau \to 0$, i.e., low frequencies or large wavelengths. The leading term of our asymptotic expansion for $\tau \to \infty$ differs also from that given by Pridmore-Brown [3]. It should be pointed out that Pridmore-Brown [3] did not obtain a uniform asymptotic expansion but by applying Langer's method [10] he obtained only a nonuniform asymptotic (leading) term which is not valid for $\eta \to 0$. Our expansions are uniform with respect to both τ and η valid in the whole η-plane with the exception of an arbitrary neighborhood around the points $\eta^2 = 1$. These points correspond to points at which the wave normals are parallel to the mean flow there. For the propagation problem, these points define the location of caustics where rays forming ray tubes converge on a line (or a surface). For the stability problem, as pointed out by Betchov and Criminale [6], at these points the pressure fluctuation equation changes from elliptic to hyperbolic but the points $\eta^2 = 1$ are not essential singular points of equation (1). The behavior of the solution around these points may be easily obtained following Skovgaard [9] and Olver [8] who have obtained uniform asymptotic expansions in terms of Airy functions. The leading terms given in equation (13), (16), and (17) are, in particular, uniformly valid in the region $\eta \to 0$, which, for the propagation problem corresponds to the incompressible case and for the stability problem to the solution in the so-called "critical layer".

3. Limiting Cases

In this section we will examine three limiting cases of our solutions to equation (1) which correspond to well-known problems of plane wave propagation in a stratified wind and to stability of vortex sheets which exemplify the significance of the parameter τ and the variable η.

The limiting cases depending on the values of τ and η are shown in Table 1 below:

TABLE 1

	$\eta \to 0$ $(a \to \infty)$	$\lvert\eta\rvert \geq \lvert\eta_1\rvert > 0$	$\lvert\eta\rvert \to \infty$ $(M \to \infty)$
$\tau \to 0$ $\frac{\omega}{b} \to 0$	$a \to \infty$, $\frac{\omega}{b} \to 0$ but $a\frac{\omega}{b} \to 0$ Incompressible Vortex Sheet (Kelvin-Helmholtz)	$\frac{\omega}{b} \to 0$; Compressible Vortex Sheet (Miles, Ribner)	$M \to \infty$ and $\frac{\omega}{b} \to 0$ $(\tau\eta)$-finite Low frequency-High Mach number Finite Shear Layer
$\tau \geq \tau_1 > 0$	Critical Layer	Finite Compressible Shear Layer (Present Case)	$M \to \infty$ and $\frac{\omega}{b}$ = finite
$\tau \to \infty$ $\frac{\omega}{b} \to \infty$ or $a \to \infty$	$a \to \infty$, $\frac{\omega}{b} \neq 0$ but $(\tau\eta)$-finite Incompressible finite Shear Layer (Rayleigh)	$\frac{\omega}{b} \to \infty$ Geometrical Acoustics Limit of a Compressible Finite Shear Layer	$M \to \infty$ and $\frac{\omega}{b} \to \infty$

We will discuss first the case $a \to \infty$ which corresponds to the incompressible shear layer and then we will consider the limiting cases $\frac{\omega}{b} \to 0$ and $\frac{\omega}{b} \to \infty$, i.e., the low and high frequency limits which characterize the compressible vortex sheet and the geometrical acoustics of the compressible finite layer respectively.

3.1. Stability of the Incompressible Shear Layer

For the finite shear layer, letting $a \to \infty$ results in $\eta \to 0$ and $\tau \to \infty$, for finite $\frac{\omega}{b}$, and moreover $4\tau\eta = \frac{\omega}{b} - \alpha z$ is finite. In such a case, since our asymptotic expansion is uniform as $\frac{\eta}{\tau} \to 0$ the solutions of the stability equation are those given by equation (16). Indeed it may be easily verified that the two independent solutions of the stability equation for inviscid incompressible fluctuations in a linearly sheared flow, i.e., of the equation

$$p_{\eta\eta} - \frac{2}{\eta} p_{\eta} - (4\tau)^2 p = 0 \tag{18}$$

are exactly those given by our asymptotic forms in equation (16).

Moreover, the uniformity of our expansions may also be exhibited from the limiting forms of the equation for the eigenvalues. For the geometry shown in Figure 1, it can be shown that the eigenvalue equation is:

$$\begin{vmatrix} -1 & f(\eta_1) & g(\eta_1) & 0 \\ 0 & f(\eta_2) & g(\eta_2) & -1 \\ -4\tau\sqrt{1 - \eta_1^2} & f_\eta(\eta_1) & g_\eta(\eta_1) & 0 \\ 0 & f_\eta(\eta_2) & g_\eta(\eta_2) & 4\tau\sqrt{1 - \eta_2^2} \end{vmatrix} = 0 \tag{19}$$

where f and g are the two independent solutions of the pressure perturbation equation (1) and τ and η have their previous meaning. (See section following equation (2)).

Using in equation (19) the expressions for f and g given by equation (16), we obtain, after some straight forward though lengthy algebra, the following expression for the phase speed c.

$$\frac{c}{b} = \frac{1}{2}\left\{z_1 + z_2 \pm \left[\left(z_1 - z_2\right)^2 - \frac{2}{\alpha}\left(z_1 - z_2\right) + \frac{1}{\alpha^2}\left(1 - e^{-2\alpha(z_1 - z_2)}\right)\right]^{\frac{1}{2}}\right\} \quad (20)$$

The above expression, for $z_1 = 1$, $z_2 = -1$ and $b = 1$ yields the better known special eigenvalue form (Betchov and Criminale [6]).

$$c^2 = 1 - \frac{1}{\alpha} + \frac{1}{4\alpha^2}\left(1 - e^{-4\alpha}\right) \quad (21)$$

The limiting form of the incompressible vortex sheet may be easily obtained from the above equation (20) in the limit $b \to \infty$ and z_1 and $z_2 \to 0$ but $bz_1 \to U_1$, $bz_2 \to U_2$, the two constant velocities respectively on the opposite sides of the vortex sheet; the result is

$$c = \frac{U_1 + U_2}{2} \pm i\,\frac{U_1 - U_2}{2} \quad (22)$$

which for $U_1 = -U_2 = 1$ reduces to the well-known result $c = \pm i$ characterizing the Kelvin-Helmholtz instability of the incompressible vortex sheet. Actually the same result above may be directly obtained not as a limit of equation (20) but directly from the general eigenvalue equation (19) in the limit $\eta \to 0$, $\tau \to 0$ which again exemplifies the uniformity of our asymptotics. In such a case it follows from either equation (8) or equation (16)* that f, g and their derivatives f_η , g_η acquire the forms:

---

*Note that the power series equation (8), and the asymptotic forms, equation (16), of f and g are identical to second order, i.e., as $\eta \to 0$ the uniform asymptotic expansion equation (16) has a second order contact with the power series equation (8).

$$
\left.\begin{aligned}
f &\sim 1 - \frac{1}{2}(4\tau\eta)^2 + o\left(\eta^4\right) &&, \; f_\eta \sim -(4\tau)^2\eta + o\left(\eta^3\right) \\
g &\sim \frac{1}{3}(4\tau\eta)^3 + \frac{1}{30}(4\tau\eta)^5 + o\left(\eta^7\right), && g_\eta \sim (4\tau)^3\eta^2 + o\left(\eta^4\right)
\end{aligned}\right\} \tag{23}
$$

Substituting the above limiting expressions in the eigenvalue equation (19) and keeping lowest order terms by going to the limit $\eta \to 0$, $\tau \to 0$ we obtain directly the eigenvalues:

$$
\left(\frac{U_1 - c}{U_2 - c}\right)^2 = -1 \tag{24}
$$

which is precisely the result in equation (22) derived previously from equation (20).

3.2. Stability of the Compressible Vortex Sheet (Long Wavelength Approximation)

This case is obtained in the limit $\tau \to 0$ (or $\frac{\omega}{b} \to 0$) with η finite. The series expressions given by equation (8) are to lowest order in τ:

$$
f \sim 1 - (4\tau)^2\left(\frac{\eta^2}{2} + \frac{\eta^4}{4}\right) + o(\tau^4) \quad , \quad f_\eta \sim -(4\tau)^2(\eta + \eta^3) + o(\tau^4)
$$

$$
g \sim \eta^3 + \frac{(4\tau)^2}{2}\left(\frac{\eta^5}{5} - \frac{\eta^7}{14}\right) + o(\tau^4), \quad g_\eta \sim 3\eta^2 + \frac{(4\tau)^2}{2}\left(\eta^4 - \frac{\eta^6}{2}\right) + o(\tau^4). \tag{23a}
$$

Using these values in the eigenvalue equation (19) we obtain:

$$
\frac{1}{\eta_1^2} + \frac{1}{\eta_2^2} = 1 \tag{25}
$$

The above equation (25) may be easily reduced to a 4th order equation for c to yield the eigenvalue expression obtained by Miles [11] and others:

$$\frac{c}{a} = \frac{U_1 + U_2}{2a} \pm \frac{1}{2} \left| M^2 + 4 - 4 \sqrt{M^2 + 1} \right|^{\frac{1}{2}} \tag{26}$$

with $M = \dfrac{U_1 - U_2}{a}$.

3.3. Plane Wave Propagation Through A Compressible Shear Layer (Short Wavelength Approximation)

This case is usually handled via the so-called Geometrical Acoustics approximation where for the pressure perturbation $p(\vec{r},t)$, one assumes a solution of the form

$$p(\vec{r},t) = p_o(\vec{r},t)e^{i(\omega t - k\theta)}$$

where the amplitude $p_o(\vec{r},t)$ is assumed to be a slowly varying function (of $\vec{r}$ and t) and by letting k be large. Finally, by expressing all the amplitudes of the perturbations in series of $(ik)^{-1}$ one finds that to the lowest order in $(ik)^{-1}$, θ is determined from the so-called eikonal equation whereas the amplitude p_o may be determined from the energy equation as follows (Blokhintsev [12]):

$$|\nabla\theta|^2 = (1 - \vec{M} \cdot \nabla\theta)^2 \tag{28}$$

and

$$\nabla \cdot \left[\frac{p_o^2}{1 - \vec{M} \cdot \nabla\theta} \left(\frac{\nabla\theta}{|\nabla\theta|} + \vec{M}\right)\right] = 0 \tag{29}$$

with $\vec{M} = \dfrac{\vec{U}}{a}$.

In both equations above the undisturbed and local sound speeds have been assumed to be the same and in the energy equation (29) in addition to neglecting constant multiplicative factors the partial derivative with respect to the time t has also been neglected compared with the divergence term.* For the case of a plane wave incident from a homogenous

---

*This is justified either in the case of a stationary process, such as for a time harmonic field, or on the basis of the original assumption of quasistatic perturbation amplitudes.

half-space ($z < 0$) at an angle ϕ with the z-axis upon a linearly sheared homogeneous medium ($z \geq 0$) with velocity $\vec{U} = bzH(z)\vec{e}_x$ where H is the Heaviside function, the exact solution of the eikonal equation for the phase function θ is known to be (Kornhauser [13]).

$$\theta = \sin \phi \left\{ x \pm \frac{1}{2} \frac{a}{b} \left[\eta \sqrt{\eta^2 - 1} - \ell n \left(\eta + \sqrt{\eta^2 - 1} \right) \right] \right. \tag{30}$$

The amplitude $p_o(\vec{r})$ of the pressure perturbation is not usually given in the standard literature but may be obtained from the reduced energy equation (29) by observing that the quantity $1 - \vec{M} \cdot \nabla\theta$ is, from the eikonal equation (28) equal to $\pm |\nabla\theta| = \pm\eta$.

It follows then that

$$p_o(\vec{r}) \sim \frac{\eta}{\left(\eta^2 - 1\right)^{\frac{1}{4}}} \tag{31}$$

The above results in equation (30) and (31) obtained from the two separate equations (28) and (29) of Geometrical Acoustics actually may be obtained simultaneously from our uniform asymptotic expansion in equation (13). Since the x-dependence of the pressure perturbation $p(\vec{r})$ is $e^{ikx\sin\phi}$, we may write from equation (27) and (13) in the limit $k \to \infty$ or $\tau \to \infty$:

$$\theta \sim \frac{1}{ik} \lim_{k\to\infty} \frac{p(\vec{r})}{p_o(\vec{r})}$$

$$= \frac{1}{ik} \left\{ ik \sin \phi + \ell n \left[\lim_{\tau\to\infty} \left(\frac{f}{g} \right) \right] \right\}$$

$$= x \sin \phi + \frac{1}{ik} \ell n \left[\lim_{\tau\to\infty} \frac{\sin h}{\cos h} (4\tau x) \right]$$

$$= x \sin \phi + \frac{4\tau}{2k} \left[\eta\sqrt{\eta^2 - 1} - \ell n \left(\eta + \sqrt{\eta^2 - 1} \right) \right]$$

or since $4\tau = \frac{\omega}{b} \sin \phi = \frac{ak}{b} \sin \phi$ in this case, the result in the Geometrical Acoustics limit equation (30) follows. Moreover, the amplitude of the pressure fluctuation $\eta(\eta^2 - 1)^{-\frac{1}{4}}$ is obtained immediately by inspection of our asymptotic expansion, equation (13). In fact the leading term of the asymptotic expansion in equation (13) yields the additional information that all the terms of the formal short wavelength expansion have the common term $\eta(\eta^2 - 1)^{-\frac{1}{4}}$, a result not available from the formal Geometrical Acoustics theory (Kornhauser [13]).

Finally it should be pointed out that the Geometrical Acoustics limit in the example of this section has been obtained from the uniform asymptotics of the solutions of equation (1) in the limit $\tau \to \infty$, i.e., $\frac{\omega}{b} \to \infty$ and not in the formal limit $\tau\eta^2 \to \infty$, according to a criterion that has been suggested in the literature (Felsen and Marcuvitz [14]). Clearly the later criterion cannot be true for regions arbitrarily close to the origin $\eta = 0$ whereas our asymptotic solution is valid through the point $\eta = 0$. This observation indicates that care must be exercised in the application of the above mentioned (nonuniform) criterion that has recently been applied to the solution of certain Aeroacoustics problems in the limit of Geometrical Acoustics (Candell [15]).

4. Physical Meaning of η, τ and $4\tau\eta^2$

The physical meaning of η, τ and $4\tau\eta^2$ which characterize the solutions f and g of the pressure perturbation equation (1) follow from the definitions in equations (2) and the meaning of K which depends on the problem at hand. Defining by $\vec{M}_d$ and $\vec{M}_f$ the disturbance and relative mean flow vector Mach numbers and letting $\vec{e}_f$ be the unit vector in the direction of the parallel mean flow we may write

$$\eta = \frac{1}{K} - M = \begin{pmatrix} \frac{1}{\sin\theta} - M & \text{for plane wave propagation} \\ \frac{c}{a} - M & \text{for stability} \\ \frac{\omega/k}{a} - M & \text{for rectangular duct modes} \end{pmatrix}$$

$$= \left(\vec{M}_d - \vec{M}_f\right) \cdot \vec{e}_f \tag{32}$$

Thus η is a relative Mach number measure, i.e., it is the parallel to the mean flow component of the disturbance (vector) Mach number $\vec{M}_d$, relative to the relative mean flow Mach number $\vec{M}_f$. In addition, for propagation of a plane wave at incident angle $-\frac{\pi}{2} \leq \phi_o \leq +\frac{\pi}{2}$ from a homogeneous half-space ($z < 0$) on a parallel flow half-space ($z \geq 0$) it is easily shown that

$$\eta = \frac{1}{\sin\phi} \tag{33}$$

where $\phi = \phi(\vec{r})$ is the (local) angle that the wave normal makes with the z axis.

The quantity $\frac{\omega}{b}$ acquires the meaning of a characteristic disturbance Strouhal number, i.e.,

$$S = \frac{\text{(Shear Layer Thickness)} \times \text{Disturbance Frequency}}{\text{Relative Mean Flow Speed}}$$

$$= \frac{z\omega}{bz} = \frac{\omega}{b} \tag{34}$$

The quantity τ is a measure of the disturbance Strouhal number with respect to the disturbance Mach number, i.e.,

$$4\tau = \frac{\omega}{b} K = \frac{\omega/b}{1/K} = \frac{S}{\vec{M}_d \cdot \vec{e}_f} = \frac{\text{Disturbance Strouhal Number}}{\text{Component of the Disturbance Mach Number parallel to the relative Mean Flow}} \tag{35}$$

Finally the argument $4\tau\eta^2$ of the Whittaker M-functions also has in the case of plane wave propagation the elegant meaning of the (local) disturbance wavelength with respect to the relative refractive index change. This may be shown as follows: For propagation of a plane wave incident from a homogeneous half-space ($z < 0$) at an angle $-\frac{\pi}{2} \leq \phi_o < +\frac{\pi}{2}$, with respect to the z-axis, on a homogeneous medium ($z \geq 0$) with speed $\vec{U} = \vec{U}(z)\vec{e}_x$, it is easy to show that the wave normal unit vectors $\vec{e}_n$ are independent of x, i.e, that all wave normals of a given z-stratum are parallel. Thus, defining an index of refraction n by

$$n = \frac{1}{1 + \vec{M}_f \cdot \vec{e}_n} = \frac{1}{1 + M \sin \phi} \tag{36}$$

where ϕ is the local angle of the wave normal and the z-axis and using equation (33) and Snell's Law it follows that

$$n = 1 - M \sin \phi_o = \eta \sin \phi_o = n\eta \sin \phi \tag{37}$$

and one then may write:

$$\frac{\text{Local Disturbance Wavelength}}{\text{Relative Refractive Index Change}} = \frac{k_o n}{\frac{|\nabla n|}{n}}$$

$$= \frac{k_o \frac{\sin \phi_o}{\sin \phi}}{\frac{1}{n}\frac{b}{a} \sin \phi_o} = \frac{ak_o}{b} \frac{n}{\sin \phi} = \frac{\omega}{b} \frac{n^2}{\sin \phi_o} =$$

$$= \frac{\omega}{b} \sin \phi_o \, \eta^2 = 4\tau\eta^2 \tag{38}$$

This interpretation of the argument of the Whittaker functions of the solutions f and g of equation (1) also implies that the necessary condition for the applicability of Geometrical Acoustics suggested in the literature (Felsen and Marcuvitz [14]) and recently applied to aero-acoustics (Candell [15]) is not only not a sufficient one, but also unnecessary and unnecessarily restrictive. This condition is usually given as (Felsen and Marcuvitz [14]; Candell [15]):

$$\frac{|\nabla n|}{k_o n^2} \ll 1 \quad , \tag{39}$$

which, in view of equation (38), implies that $4\tau\eta^2 \gg 1$. But in effect in Section 3 (3.3) we obtained both the exact solution of the eikonal equation <u>and</u> the correct amplitude of the pressure perturbation from our uniform asymptotic expansion in equation (13) in the limit $\tau \to \infty$ independently of the value of η (i.e., of z or upper fluid Mach number). Indeed from equation (13) we see that by comparing the first two terms of the asymptotic expansion we easily conclude that the second term is negligible compared to the first provided that $4\tau|\chi| \gg 1$ which yields the criterion equation (39), i.e., $4\tau\eta^2 \gg 1$, only in the additional limit $|\eta| \to \infty$. But also since our expansion in equation (13) is uniformly valid for $\frac{\chi}{\tau} \to 0$ we conclude the following two conditions for our case (nonzero incidence angle θ and finite velocity profile slope b):

$$4\tau\eta^2 \gg 1 \text{ or } \frac{\lambda}{z^2} \ll 1 \tag{40}$$

$$\frac{\eta}{\tau} \to 0 \text{ or } \lambda z \to 0 \tag{41}$$

Condition (40) above has been interpreted and used as a relaxed criterion for the applicability of Geometrical Acoustics with the implication that far enough from the x-axis (large z) the characteristic wavelength λ need not be unduly small. This interpretation is false in view of the second condition in equation (41) which implies that the further one is in the far field the shorter the wavelength should be to satisfy that condition. These observations indicate that care should be exercised in applying the necessary criterion in equation (39) of Geometrical Acoustics without examining in detail the behaviour of the solutions of the perturbation equations for the particular velocity profile of the problem at hand.

5. Applications

In this section we indicate* how the general solutions of equation (1) may be effectively used for certain problems arising in plane wave propagation and in the stability of a finite thickness shear layer with a linear velocity profile.

5.1. Plane Wave Propagation

One of the problems relating to plane wave propagation through a finite thickness shear layer with a linear velocity profile is that of the existence of "resonances" and the existence of what, in analogy to optical wave propagation, may be called Brewster angles. Resonances and Brewster angles correspond to infinite and zero values of the

*Here only the general outline of the methodology and certain relevant results are given. The details and complete results will be presented in another place.

reflection coefficient respectively. It is known for instance (Ribner [16], Miles [11]) that the vortex sheet, which may be thought as the exact limit of long wavelength wave propagation, exhibits both resonances and Brewster angles. In contrast short wavelength wave propagation through a finite thickness shear layer, i.e., the geometrical acoustics limit, does not exhibit any resonance but only a continuum of Brewster angles. This is shown in Figure 2. The problem of resonances was touched briefly by Graham and Graham [4], who were only able to show that a sufficiently thin, but nonzero thickness, shear layer has no resonances; they were though unable to draw any conclusions for a finite thickness shear layer. The reason for the latter was that Graham and Graham essentially obtained and used the series representation, equation (8), of our solution of the pressure perturbation equation (1). Using our solution in terms of Whittaker functions given by equation (7) it may be shown that the reflection coefficient R (for the amplitude of the pressure perturbation) is given by:

$$R^2 = \frac{(A \pm B)^2 + (C \pm D)^2}{(A \mp B)^2 + (C \mp D)^2} \tag{42}$$

where

$$\begin{aligned}
A &= \frac{1}{4\tau}\left[f_\eta(0)g_\eta(1) - f_\eta(1)g_\eta(0)\right] \\
B &= 4\tau\sqrt{\eta_o^2 - 1}\sqrt{\eta_1^2 - 1}\left[f(1)g(0) - f(0)g(1)\right] \\
C &= \sqrt{\eta_1^2 - 1}\left[f(1)g_\eta(0) - f_\eta(0)g(1)\right] \\
D &= \sqrt{\eta_o^2 - 1}\left[f_\eta(1)g(0) - f(0)g_\eta(1)\right]
\end{aligned} \tag{43}$$

The upper signs in equation (42) hold for $\eta_1 > 1$ and the lower signs for $\eta_1 < -1$, in both cases $|\eta| > 1$. In equations (43) we used the notation 0 and 1 in the arguments of f and g and their derivatives with the understanding that 0 designates evaluation at $\eta = \eta_o = \eta|_{z=0} = \frac{1}{\sin\theta}$ and 1 designates evaluation at $\eta = \eta_1 = \eta|_{z=z_1} = \frac{1}{\sin\theta} - M$, i.e., at the two edges of the shear layer. (See Figure 1 with $z_2 = 0$.)

Equation (42) is valid for $-\frac{\pi}{2} \leq \theta \leq +\frac{\pi}{2}$ with the upper signs holding for the regime of ordinary reflection ($\eta > 1$ and $R^2 < 1$) and the lower signs for the regime of the so-called amplified reflection ($\eta_1 < -1$, $R^2 > 1$). (For the total reflection regime $|R|^2 = 1$, $-1 < \eta_1 < +1$, $-\frac{\pi}{2} \leq \theta \leq +\frac{\pi}{2}$.) The conditions for the existence of resonances and Brewster angles become:

$$\begin{aligned} & A + B = C + D = 0 \\ \text{with} \quad & A - B \neq 0 \\ \text{or} \quad & C - D \neq 0 \end{aligned} \tag{44}$$

or both.

Using the Wronskian expression $W(\eta)$ for our solutions f and g, it may be shown that

$$\begin{aligned} AB + CD &= -W(0)W(1)\sqrt{\eta_o^2 - 1}\sqrt{\eta_1^2 - 1} \\ &= -9\eta_o^2\eta_1^2\sqrt{\eta_o^2 - 1}\sqrt{\eta_1^2 - 1} \end{aligned} \tag{45}$$

It follows from equation (45) that in general $A \neq B$ and $C \neq D$.

In order to assess whether the conditions for resonance in equation (44) are possible we consider the thin shear layer case, i.e., τ small but not zero. Using our expressions in equation (23a) we find to lowest order in τ:

$$A + B = 4\tau M\left[3\eta_o\eta_1(1 - \eta_o\eta_1) + \sqrt{\eta_o^2 - 1}\,\sqrt{\eta_o^2 - 1}\,\left(\eta_o^2 + \eta_1^2 + \eta_o\eta_1\right)\right] \qquad (46)$$

To lowest order in τ the right-hand side should be evaluated at the vortex sheet values of η_o and η_1 for which to lowest order in τ the second condition for the existence of resonances, i.e., $C + D = 0$ is fulfilled. Using the known properties of the vortex sheet solution, i.e.,

$$\eta_o^2 + \eta_1^2 = \eta_o^2\eta_1^2 \quad \text{and} \quad \eta_o\eta_1 = 1 - \sqrt{M^2 + 1}$$

we obtain from (46):

$$A + B = -(4\tau)(2M^3) \qquad (47)$$

It follows from (47) that $A + B = 0$ for a nonzero Mach number only if $\tau = 0$, i.e., only for the vortex sheet case. The conclusion is that the conditions for the existence of resonances equation (44) are incompatible for small but finite τ, i.e., for the sufficiently thin shear layer and that layer, in contrast to the compressible vortex sheet exhibits no resonances and no Brewster angles. This is evident also from Figure 2 where the reflection coefficient has been evaluated numerically as a function of Mach number for various values of τ and for a fixed incident angle of 30^0 for which the corresponding compressible vortex sheet has two resonances and one Brewster angle. It is also seen in Figure 2 that for finite nonzero τ the resonances and Brewster angles of the corresponding vortex sheet disappear even for very small values of τ.

5.2. Stability to Long Wavelength Disturbances

It is known that the incompressible vortex sheet is unstable for all wave numbers α whereas the introduction of either finiteness (finite incompressible shear layer) or compressibility (compressible vortex sheet) lead to ranges of α or M for which the motion is stable. This is shown in Figure 4. The question that naturally arises is then whether the introduction of both finiteness and compressibility will yield increased or any ranges of α for which the motion is stable. For small α, i.e., long wavelengths, this question may be readily answered by the use of our series solutions for f and g of the pressure perturbation equation given by equations (23a). For small wave numbers α or small τ using these expressions to lowest order in τ the eigenvalue equation (19) for the normalized disturbance phase speed c/a may be shown to be

$$x^4 - 2Mx^3 + \left(M^2 - 2\right) x^2 + 2Mx - \left(\varepsilon + M^2\right) = 0 \tag{48}$$

where $x = \frac{c}{a}$, M is the upper fluid Mach number (see Figure 1 with $z_2 = 0$) and ε is the small parameter

$$\varepsilon = +\frac{16\tau}{3} \cdot \frac{\eta_o^2\sqrt{1 - \eta_1^2}}{\eta_o + \eta_1} \eta_o\eta_1\left(2 - \eta_o\eta_1\right)\Bigg|_{\left(\frac{c}{a}\right)_v} \quad , M \neq 2\sqrt{2} \quad , \text{ and}$$

ε is to be evaluated at the vortex sheet value of $\left(\frac{c}{a}\right)_v$ as determined in section 3.2. The allowable solutions of the above equation are given exactly by:

$$x = \frac{c}{a} = \frac{1}{2}\left[M \pm \sqrt{M^2 + 4 - 4\sqrt{M^2 + 1 + \varepsilon}}\right] \tag{49}$$

And either by examining the solutions of the 4th order equation (48) by use of Rouché's Theorem (Copson [17]) or the algebraic expression for the solution, equation (49) above, one easily concludes that for all non-zero Mach numbers M (excluding $M = 2\sqrt{2}$) there is always a value of $\frac{c}{a}$ with positive imaginary part. A slight modification of the eigenvalue equation yields for $M = 2\sqrt{2}$ a 5th order equation for $x = \frac{c}{a}$ and again one may show that c acquires always a complex value with positive imaginary part.

Thus the long wavelength properties of the shear layer are drastically different from those of the compressible vortex sheet and in fact contrary to the compressible vortex sheet the thin shear layer is unstable to all Mach numbers for small wave number disturbances.

6. Conclusions

In this study we have considered the characterization of inviscid fluctuations in a compressible linearly sheared, but otherwise homogeneous parallel two-dimensional flow. The behavior of the cross-flow part of the fluctuations were found to be governed by essentially Whittaker's equation with the pressure fluctuations being characterized by the functions.

$$\eta^{1/2} M_{i\tau, \mp \frac{3}{4}} \left(4i\tau\eta^2\right),$$

where M is the Whittaker M-functions and η, τ and $4\tau\eta^2$ admit the following simple interpretations

η = Disturbance Mach number component in the mean flow direction relative to the mean flow Mach number

$$\tau = \frac{\text{Disturbance Strouhal number}}{\text{Disturbance Mach number component in the mean flow direction}}$$

$$4\tau\eta^2 = \frac{\text{Local Disturbance Wavelength}}{\text{Relative Refractive Index Change}}$$

The known solutions to a number of other parallel flow problems may be obtained as limiting cases of our exact solutions. Such cases include the compressible vortex sheet ($\tau \to 0$), the incompressible vortex sheet ($\tau \to 0$, $\eta \to 0$), the incompressible linearly sheared layer ($\tau \to \infty$, $\eta \to 0$, $(\tau\eta)$ - finite) and the short wavelength or Geometrical Acoustics limit of the compressible linearly sheared layer ($\tau \to \infty$ or $\frac{\eta}{\tau} \to 0$).

Finally the exact solutions we have obtained for equation (1) enable us to study exactly the compressible shear layer and to answer some as yet unanswered questions such as those pertaining to the existence of resonances and Brewster angles and to the stability of such a layer. For the shear layer with a linear velocity profile resonances and Brewster angles do not exist except in the limits $\tau \to 0$ (vortex sheet) and $\tau \to \infty$ (geometrical acoustics). Moreover for small wave number disturbances the linear shear layer is unstable for all Mach numbers M. Thus contrary to the compressible vortex sheet, which is stable for $M > 2\sqrt{2}$, one may not regard the compressible vortex sheet as an adequate model of a thin shear layer for all relative Mach numbers of the uniform flows bounding the shear layer.

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LIST OF CAPTIONS

Figure 1.

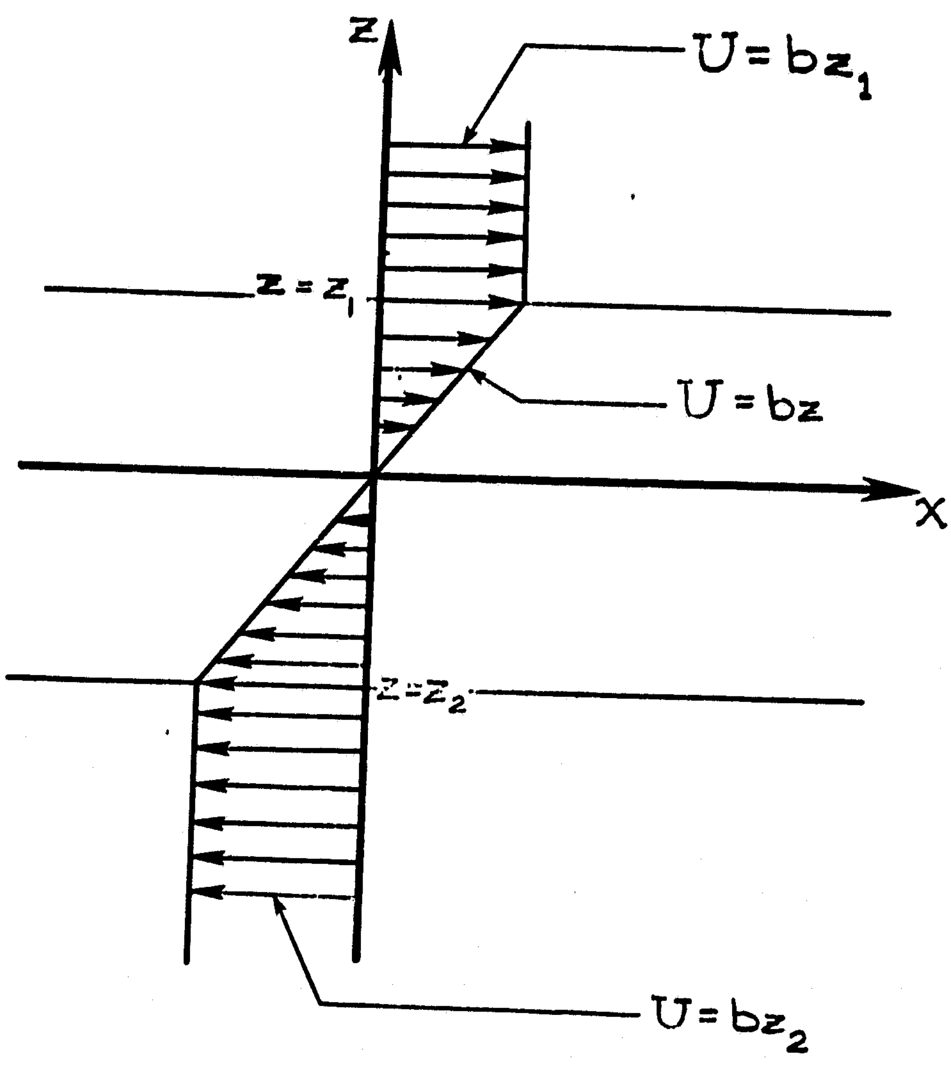
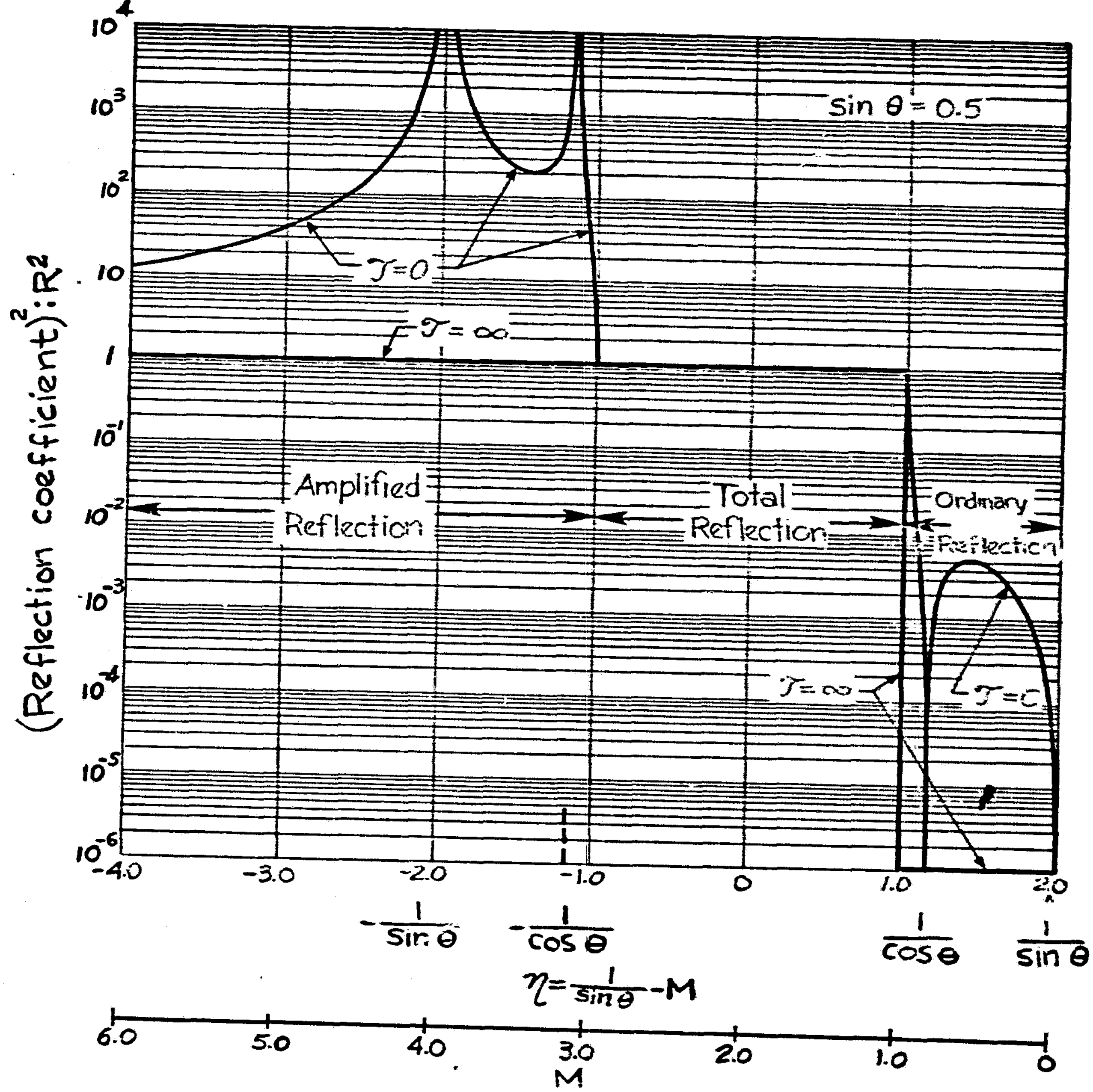


Figure 2.



10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
(Reflection coefficient)^2: R^2
Sin θ = 0.5
T=0
T=∞
Amplified Reflection
Total Reflection
Ordinary Reflection
T=∞
T=0
-4.0
-3.0
-2.0
-1.0
0
1.0
2.0
$-\frac{1}{\sin\theta}$
$-\frac{1}{\cos\theta}$
$\frac{1}{\cos\theta}$
$\frac{1}{\sin\theta}$
$\eta = \frac{1}{\sin\theta} - M$
6.0
5.0
4.0
3.0
2.0
1.0
0
M

Figure 3.

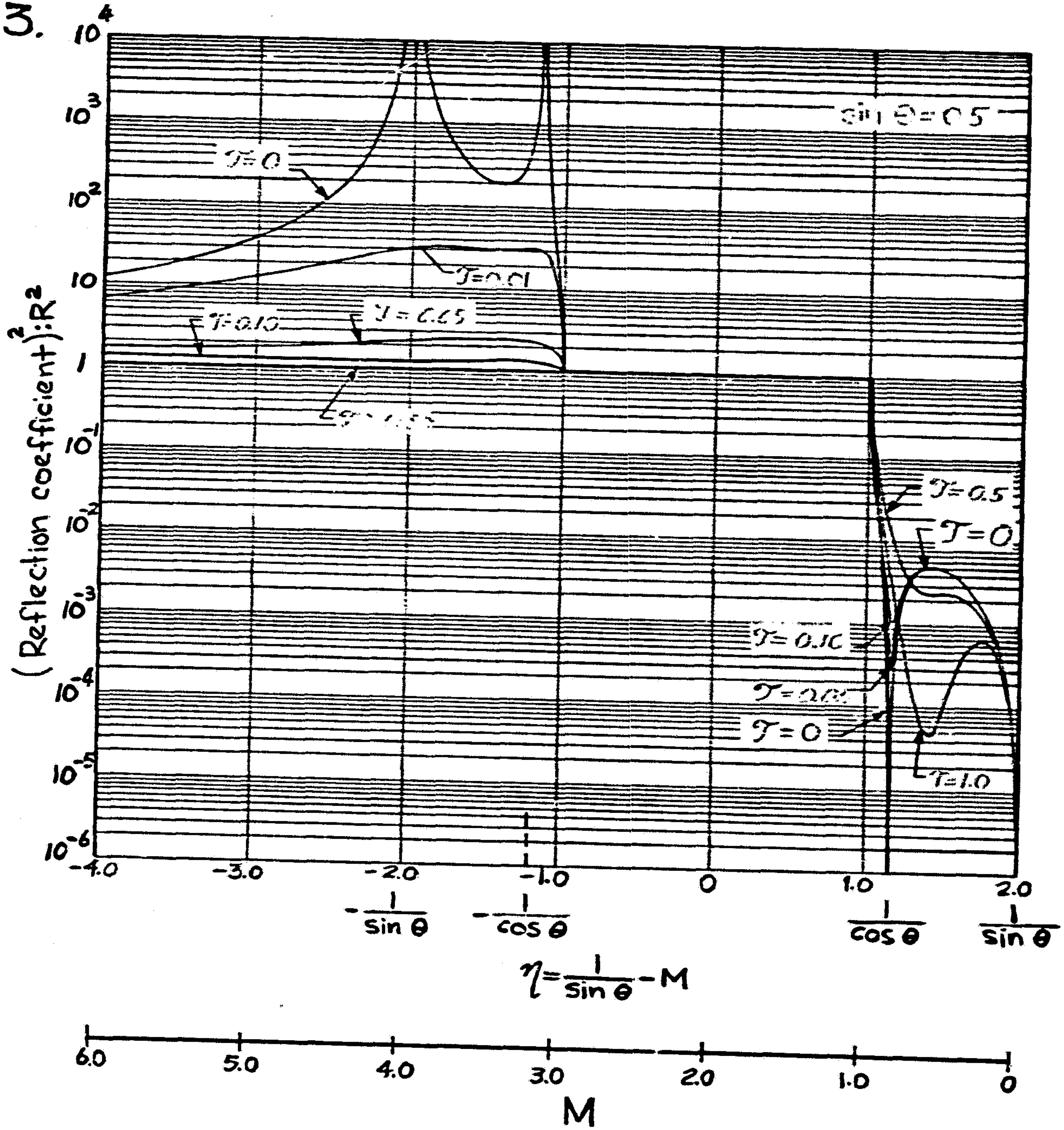
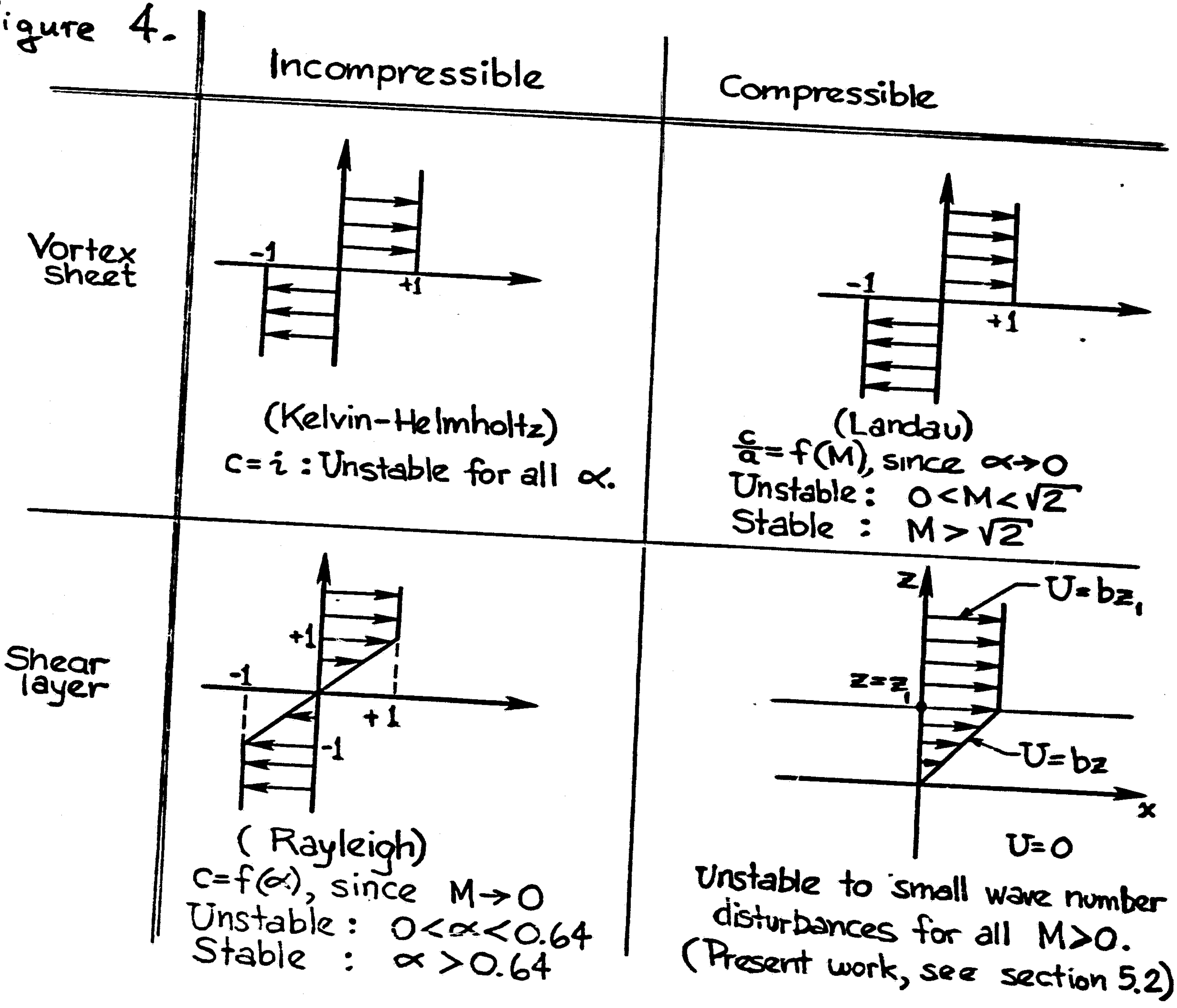


Figure 4.



Incompressible
Compressible
Vortex sheet
-1
+1
(Kelvin-Helmholtz)
c = i : Unstable for all α.
(Landau)
c/a = f(M), since α→0
Unstable : 0 < M < √2
Stable : M > √2
Shear layer
+1
-1
+1
-1
(Rayleigh)
c=f(α), since M→0
Unstable : 0<α<0.64
Stable : α>0.64
z
U = bz1
z = z1
U=bz
x
U=0
Unstable to small wave number disturbances for all M>0.
(Present work, see section 5.2)