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ON THE OUTPUT OF ACOUSTICAL SOURCES

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by

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§1. Introduction

The performance of an acoustical source, whose existence and nature are revealed by a given inhomogeneous term in a wave equation, depends on its environmental aspects; this means, in particular, that changes in the power output and directivity pattern for any individual source of prescribed type can reflect, in a sensitive manner, those made in the surroundings through the disposition of objects, the admission of a background flow or a variation in configurational parameters. Isolated source characteristics are relatively easy to determine on the basis of linear wave equations and a hypothetical uniformity in the setting, although the interaction or coupling of sources with their surroundings generally poses a formidable problem. Ingard and Lamb (1957) consider the simplest type of interaction problem, with fixed and localized sources of either monopolar, dipolar, or quadrupolar nature lying to one side of a rigid and indefinitely extended plane reflecting surface; and they calculate explicitly the so-called power amplification factors or ratio of total radiated power throughout the half-space containing the source to that generated by the same source in the whole of a uniform free space. Jacques (1971) has subsequently discussed the radiation from fixed mulipolar sources situated above an infinite plane surface at which an invariable complex impedance condition is chosen; his method of specifying power radiation factors, in common with that employed by Ingard and Lamb, rests on the field and energy flux determinations at great distances from the source. Integrals of the flux over an appropriate directional range yield formulae whose evaluation is accomplished by numerical processes.

Acoustical source problems have come to the fore in contemporary developments of aerodynamic sound theory which now encompass both the modelling of the sources themselves and the tracing of effects due to nearby surfaces or structural elements. With their prediction of a substantial rise in power output for suitably oriented quadrupolar sources lying close to the straight edge of an extended plane, Ffowcs Williams and his collaborators (Hall, Crighton, Leppington, and Howe, 1970-) ushered in a more systematic study of interaction effects whose objective is, typically, that of estimating the wave function or sound level far away from the source. Inasmuch as the distant field patterns of virtually all source-surface configurations possess a complicated nature it is difficult to obtain analytical measures of the total power output through direct angular integration. An alternative to the latter procedure is given in what follows, this being arrived at via manipulation (apart from solution) of the pertinent equations for a well posed linear boundary value problem and exemplified by the particular

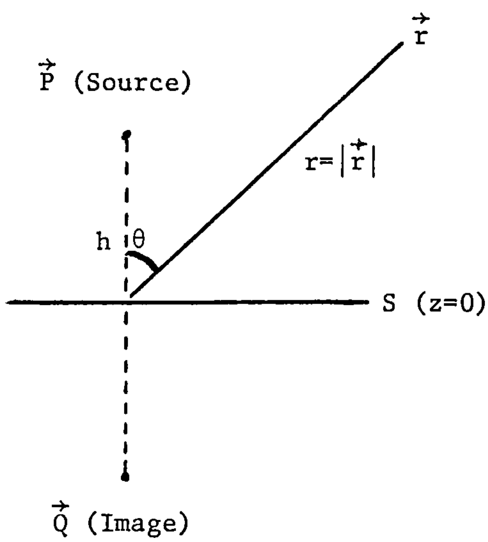
<u>Theorem</u>: Let there be a time-periodic monopole source situated at a fixed point $\vec{P}$ in the homogeneous region exterior to a closed (or infinite) rigid source S ; and suppose that the spatial part of the complete wave function

$$\phi(\vec{r}) = \phi_P(\vec{r}) + \phi_S(\vec{r}) \ , \qquad (1)$$

is realized from a singular component, $\phi_P(\vec{r})$, characteristic of the isolated (free) source and a regular component, $\phi_S(\vec{r})$, attributable to the presence of the rigid surface S . Then the ratio, $\mathscr{P}$, of the time average power output of the source, with and without the reflecting surface, is directly expressible in terms of the imaginary part of the regular component at the location of the source, viz.

$$\mathscr{P} = 1 + \frac{4\pi}{k} \operatorname{Im} \phi_S(\vec{P}) \ . \qquad (2)$$

The advantage of such a representation for calculational purposes, in comparison with a conventional one that involves an integral of the energy flux, or $|\phi|^2$, over the normal directions to a distant control surface, is readily apparent. A first indication of the facility afforded by (2) appears in the problem which envisages a point source at distance h from an infinite rigid plane $(z = 0)$



and the pertinent equations

$$(\nabla^2+k^2)\phi(\vec{r})e^{-i\omega t} = -\delta(\vec{r}-\vec{P})e^{-i\omega t} \ , \quad z > 0$$

$$\frac{\partial\phi}{\partial z} = 0 \quad , \quad z = 0 \ ;$$

it is a simple matter to confirm that

$$\phi(\vec{r}) = \frac{e^{ik|\vec{r}-\vec{P}|}}{4\pi|\vec{r}-\vec{P}|} + \frac{e^{ik|\vec{r}-\vec{Q}|}}{4\pi|\vec{r}-\vec{Q}|} \quad , \quad z \geq 0$$

in these circumstances; and utilization of (2) furnishes the (familiar) result

$$\mathscr{P} = 1 + \frac{4\pi}{k} \operatorname{Im} \frac{e^{ik|\vec{P}-\vec{Q}|}}{4\pi|\vec{P}-\vec{Q}|} = 1 + \frac{\sin 2kh}{2kh} \qquad (3)$$

more expeditiously than does the reduction of an integral

$$\mathscr{P} = 4\pi \lim_{r\to\infty} \{r^2 \int_0^{\pi/2} |\phi(r)|^2 \cdot 2\pi \sin\theta \, d\theta\}$$

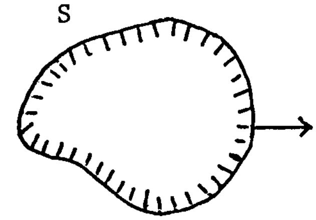
made precise with the asymptotic form

$$\phi(\vec{r}) \sim 2 \cos(kh \cos\theta) \frac{e^{ikr}}{4\pi r} \quad , \quad r \to \infty$$

of the outgoing wave function.

After presenting (in §2) theoretical arguments which justify a calculation of the net power output at the source position in particular instances, the remaining sections (§3-6) are given over to various more complicated applications and extensions having to do with different half-plane problems, an impedance boundary condition and, finally, with the effect of a background flow in the medium.

§2. A Theoretical Basis for Local Power Calculation



Consider a fixed object surrounded by a uniform compressible medium extending indefinitely outwards in all directions; and suppose that the velocity potential Re $(\phi(\vec{r})e^{-i\omega t})$ descriptive of acoustical disturbances in the medium, associated with a time-periodic and isotropic point source at $\vec{P}$, can be sought after through the inhomogeneous wave equation

$$(\nabla^2+k^2)\phi(\vec{r}) = -\delta(\vec{r}-\vec{P}) \ . \tag{4}$$

Let $\vec{v}(\vec{r})$ and $p(\vec{r})$ designate the local velocity and variable pressure, ρ the equilibrium density of the medium, with the expressions

$$\vec{v} = -\nabla\phi \quad , \quad p = -i\omega\rho\phi \tag{5}$$

in terms of the scalar function $\phi(\vec{r})$; and note the concomitant representation for the time-average energy flux vector

$$\vec{N} = \langle p\vec{v} \rangle = \frac{\rho\omega}{2} \, \mathrm{Im} \, \phi^* \, \nabla\phi \ , \tag{6}$$

where * symbolizes complex conjugation.

On multiplying (4) with ϕ^* and rearranging it follows that

$$\phi^*(\nabla^2+k^2)\phi = \nabla\cdot(\phi^*\nabla\phi) - |\nabla\phi|^2 + k^2|\phi|^2 = -\delta(\vec{r}-\vec{P})\phi^*(\vec{r})$$

whence

$$\frac{\omega\rho}{2}\,\mathrm{Im}\,\nabla\cdot(\phi^*\nabla\phi) = -\frac{\omega\rho}{2}\,\mathrm{Im}\,\delta(\vec{r}-\vec{P})\phi^*(\vec{r}) \ .$$

Integration of the latter relation throughout the region V outside the surface S of the object yields

$$\mathbb{P} - \frac{\omega\rho}{2}\,\mathrm{Im}\int_S \phi^* \frac{\partial\phi}{\partial n_S}\,dS = -\frac{\omega\rho}{2}\,\mathrm{Im}\int_V \delta(\vec{r}-\vec{P})\,\phi^*(\vec{r})\,dV$$

where $\mathbb{P}$ is the net power delivered at infinity. In case that either

$$\partial\phi/\partial n_S = 0 \quad \text{or} \quad \phi = 0 \quad \text{on } S$$

then

$$\mathbb{P} = -\frac{\omega\rho}{2}\,\mathrm{Im}\int_V \delta(\vec{r}-\vec{P})\left\{\frac{e^{-ik|\vec{r}-\vec{P}|}}{4\pi|\vec{r}-\vec{P}|} + \phi_S^*(\vec{r})\right\} dV \tag{7}$$

according to the resolution (1), and this implies

$$\mathbb{P} = \frac{\omega\rho k}{8\pi} + \frac{\omega\rho}{2}\,\mathrm{Im}\,\phi_S(\vec{P})$$

or

$$\mathscr{P} = \mathbb{P}/(\frac{\omega\rho k}{8\pi}) = 1 + \frac{4\pi}{k}\,\mathrm{Im}\,\phi_S(\vec{P})$$

as stated in (2).

If the source at P has a dipolar nature, with the unit vector $\vec{n}$ pointing along its axis, and the inhomogeneous equation (4) is replaced by

$$(\nabla^2+k^2)\phi(\vec{r}) = -\vec{n}\cdot\nabla_P\delta(\vec{r}-\vec{P}) \tag{8}$$

the counterpart of (7) becomes

$$\mathbb{P} = -\frac{\omega\rho}{2}\,\mathrm{Im}\int_V \vec{n}\cdot\nabla_P\delta(\vec{r}-\vec{P})\{\vec{n}\cdot\nabla_P\frac{e^{-ik|\vec{r}-\vec{P}|}}{4\pi|\vec{r}-\vec{P}|}+\phi_S{}^*(\vec{r})\}\,dV$$

$$= -\frac{\omega\rho}{2}\,\mathrm{Im}\int_V \delta(\vec{r}-\vec{P})\{\vec{n}\cdot\nabla\,\vec{n}\cdot\nabla_P\frac{e^{-ik|\vec{r}-\vec{P}|}}{4\pi|\vec{r}-\vec{P}|}+\vec{n}\cdot\nabla\phi_S(\vec{r})\}\,dV\,.$$

Since

$$\lim_{\vec{r}\to\vec{P}} \vec{n}\cdot\nabla\,\vec{n}\cdot\nabla_P\frac{\sin k|r-P|}{|\vec{r}-\vec{P}|} = \frac{1}{3}k^3$$

it follows that

$$\mathbb{P} = \frac{\omega\rho k^3}{24\pi} + \frac{\omega\rho}{2}\,\mathrm{Im}\,\vec{n}\cdot\nabla\phi_S(\vec{P})$$

and

$$\mathscr{P} = 1 + \frac{12\pi}{k^3}\,\mathrm{Im}\,\vec{n}\cdot\nabla\phi_S(\vec{P}) \tag{9}$$

expresses the dipole power amplification factor.

For a vertical dipole at $\vec{P}(z = h)$ in the presence of a uniform rigid plane $(z = 0)$

$$\phi_S(\vec{r}) = -\frac{\partial}{\partial z_Q}\left(\frac{e^{ik|\vec{r}-\vec{Q}|}}{4\pi|\vec{r}-\vec{Q}|}\right)_{z_Q=-h}\quad,\quad z \geq 0$$

where $\vec{Q}(z = -h)$ is the image of $\vec{P}$ relative to the plane; and the application of (9) yields

$$\mathscr{P} = 1 + \frac{3}{k^3}\frac{d^2}{dz^2}\left(\frac{\sin k(z+h)}{z+h}\right)_{z=h}$$

$$= 1 - \frac{3}{2kh}\left[\left(1-\frac{2}{(2kh)^2}\right)\sin 2kh + \frac{2}{2kh}\cos 2kh\right]$$

in conformity with Ingard and Lamb. The corresponding power representations for other multiple sources can be obtained in similar fashion.

An independent derivation of (2), without recourse to the delta function, relies on the detailed versions

$$\phi_P(\vec{r}) = \frac{e^{ik|\vec{r}-\vec{P}|}}{4\pi|\vec{r}-\vec{P}|} = G(\vec{r},\vec{P})$$

and (10)

$$\phi_S(\vec{r}) = -\int_S G(\vec{r},\vec{r}\,')K_P(\vec{r}\,')\, dS'$$

of the singular and regular parts of the wave function, the latter being formed in terms of an as yet unspecified distribution of (secondary) sources on the surface of the object, with the density

$$K_P(\vec{r}) = \frac{\partial}{\partial n_S}\phi(\vec{r}) \ . \qquad (11)$$

At a considerable distance from S, where

$$G(\vec{r},\vec{r}\,') \simeq \frac{e^{ikr}}{4\pi r} e^{-ik\vec{n}\cdot\vec{r}} \ , \quad r = |\vec{r}| \to \infty \ , \ \vec{n} = \frac{\vec{r}}{r} \ , \ \vec{r}\,' \text{ on } S$$

the total wave function acquires the asymptotic form

$$\phi(\vec{r}) = \phi_P(\vec{r}) + \phi_S(\vec{r}) \simeq A(\vec{n}) \frac{e^{ikr}}{4\pi r} \ , \quad r \to \infty \qquad (12)$$

with an angular or directivity factor

$$A(\vec{n}) = e^{-ik\vec{n}\cdot\vec{P}} - \int_S e^{-ik\vec{n}\cdot\vec{r}} K_P(\vec{r})\, dS \ ; \qquad (13)$$

and it follows, having regard for the energy flux vector (4), that

$$\begin{aligned}\mathscr{P} &= \frac{1}{4\pi}\int |A(n)|^2 \, d\Omega_{\vec{n}} \\ &= \frac{1}{4\pi}\int \{1 - 2\,\mathrm{Re}\, e^{-ik\vec{n}\cdot\vec{P}} \int_S e^{-ik\vec{n}\cdot\vec{r}} K_P(\vec{r})\, dS + \\ &\qquad + \left|\int_S e^{-ik\vec{n}\cdot\vec{r}} K_P(\vec{r})\, dS\right|^2\} \, d\Omega_{\vec{n}}\end{aligned}$$

where $d\Omega_{\vec{n}}$ designates an element of solid angle about the direction of $\vec{n}$ and $\int d\Omega_{\vec{n}} = 4\pi$.

To simplify (14), and thereby confirm the central place of the power formula (2), information about the function $K_P(\vec{r})$ is relevant; thus, if a (soft) boundary condition, $\phi = 0$, holds on S the function $K_P(\vec{r})$ satisfies an integral equation of the first kind,

$$\phi_P(\vec{r}) = \int_S G(\vec{r},\vec{r}')K_P(\vec{r}')\, dS' \qquad \vec{r} \text{ on } S$$

$$= \int_S \frac{\cos k|\vec{r}-\vec{r}'|}{4\pi|\vec{r}-\vec{r}'|} K_P(\vec{r}')\, dS' + i \int_S \frac{\sin k|\vec{r}-\vec{r}'|}{4\pi|\vec{r}-\vec{r}'|} K_P(\vec{r}')\, dS' \ . \qquad (15)$$

The consequence of multiplying in (15) by the complex conjugate function $K_P^*(\vec{r})$, integrating over S and extracting the imaginary part therefrom, is that

$$\text{Im} \int_S K_P^*(\vec{r})\, \phi_P(r)\, dS = \int_S K_P^*(\vec{r})\, \frac{\sin k|\vec{r}-\vec{r}'|}{4\pi|\vec{r}-\vec{r}'|} K_P(\vec{r}')\, dS\, dS'$$

$$= \frac{k}{(4\pi)^2} \int \left| \int_S e^{-ik\vec{n}\cdot\vec{r}} K_P(\vec{r})\, dS\right|^2 d\Omega_{\vec{n}} \qquad (16)$$

inasmuch as

$$\frac{\sin kr}{kr} = \frac{1}{4\pi} \int e^{ik\vec{n}\cdot\vec{r}}\, d\Omega_{\vec{n}} \ . \qquad (17)$$

Use of the latter result along with prior expressions for ϕ_P , ϕ_S and the soft boundary condition on S implies that

$$\text{Im} \int_S K_P^*(\vec{r})\phi_P(\vec{r})\, dS = \text{Im}\left(\int_S K_P(\vec{r})\, \frac{e^{-ik|\vec{r}-\vec{P}|}}{4\pi|\vec{r}-\vec{P}|}\, dS\right)^*$$

$$= \text{Im}\left(\int_S K_P(\vec{r})\, \frac{\cos k|\vec{r}-\vec{P}|}{4\pi|\vec{r}-\vec{P}|}\, dS - i \int_S K_P(\vec{r})\, \frac{\sin k|\vec{r}-\vec{P}|}{4\pi|\vec{r}-\vec{P}|}\, dS\right)^*$$

$$= \text{Im}\left(-\phi_S(\vec{P}) - 2i \int_S K_P(\vec{r})\, \frac{\sin k|\vec{r}-\vec{P}|}{4\pi|\vec{r}-\vec{P}|}\, dS\right)^*$$

$$= \text{Im}\, \phi_S(\vec{P}) + \text{Im}\, 2i\, \frac{k}{(4\pi)^2} \int e^{-ik\vec{n}\cdot\vec{P}} \int K_P^*(\vec{r}) e^{ik\vec{n}\cdot\vec{r}}\, dS\, d\Omega_{\vec{n}}$$

$$= \text{Im}\, \phi_S(\vec{P}) + \frac{2k}{(4\pi)^2}\, \text{Re} \int d\Omega_{\vec{n}}\, e^{ik\vec{n}\cdot\vec{P}} \int_S e^{-ik\vec{n}\cdot\vec{r}} K_P(\vec{r})\, dS \qquad (18)$$

and, after (16), (18) are combined with (14), the relation (2) is directly forthcoming.

When line rather than point sources are contemplated, the two-dimensional analogue of (2),

$$\mathscr{P} = 1 + 4 \text{ Im } \phi_S(\vec{P}) \tag{19}$$

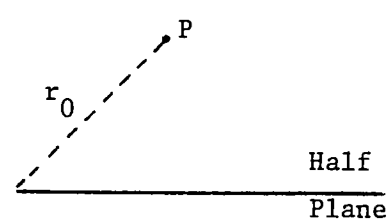
is applicable.

§3. Source Radiation in the Presence of a Half-plane

As the first of several illustrative problems selected to underline the relative ease of power calculations made at the source itself, consider the output from a periodic and fixed monopole which interacts with a rigid half-plane. Suppose that the straight edge of the half-plane extends along the z-axis of a cylindrical coordinate system and that

$$\theta = 0 \, , 2\pi$$

on the respective faces, where $0 < r < \infty$. If the source is located at the point $P(r_0,\theta_0,0)$ the wave or Green's function specified by the differential equation



$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\phi(r,\theta,z;P) = -\frac{\delta(r-r_0)}{r}\,\delta(\theta-\theta_0)\,\delta(z)$$

and the boundary condition

$$\frac{\partial}{\partial\theta}\phi = 0 \quad , \quad \theta = 0 \, , 2\pi$$

admits the representation

$$\phi(r,\theta,z;P) = G(r,\theta,z;r_0,\theta_0,0)$$

$$= \frac{1}{2}\left\{\frac{e^{ikh_-}}{4\pi R_-} + \frac{ik}{4\pi}\int_0^{2\sqrt{rr_0}\cos\frac{\theta-\theta_0}{2}}\left(\frac{H_1^{(1)}(k\sqrt{R_-^2+\xi^2})}{\sqrt{R_-^2+\xi^2}}\right)d\xi\right.$$

$$\left. + \frac{e^{ikh_+}}{4\pi R_+} + \frac{ik}{4\pi}\int_0^{2\sqrt{rr_0}\cos\frac{\theta+\theta_0}{2}}\left(\frac{H_1^{(1)}(k\sqrt{R_+^2+\xi^2})}{\sqrt{R_+^2+\xi^2}}\right)d\xi\right\} \tag{20}$$

where

$$R_{\mp}^2 = r^2 + r_0^2 - 2rr_0\cos(\theta\mp\theta_0) + z^2$$

and $H_1^{(1)}(\zeta)$ designates a first order Hankel function of the first kind.

The power factor $\mathscr{P}$ of this source is, in accordance with (2), found from

$$\mathscr{P} = 1 + \frac{4\pi}{k}\lim_{\substack{r\to r_0\\ \theta\to\theta_0,\, z\to 0}} \operatorname{Im}\{G(r,\theta,z;r_0,\theta_0,0) - \frac{e^{ikh}}{4\pi R_-}\}\,.$$

and the outcome

$$\mathscr{P} = \frac{1}{2} + \frac{1}{2}\,\frac{\sin(2kr_0\sin\theta_0)}{2kr_0\sin\theta_0} + \frac{1}{2}\int_0^{2kr_0}\frac{J_1(\xi)}{\xi}\,d\xi + \frac{1}{2}\int_0^{2kr_0\cos\theta_0}\frac{J_1(\sqrt{(2kr_0\sin\theta_0)^2+\xi^2})}{\sqrt{(2kr_0\sin\theta_0)^2+\xi^2}}\,d\xi \tag{21}$$

involves the first order Bessel function $J_1(\zeta)$. It follows from (21) along with the result

$$\int_0^\infty \frac{J_1(a\sqrt{b^2+\xi^2})}{\sqrt{b^2+\xi^2}}\,d\xi = \frac{\sin ab}{ab}\,, \quad a > 0$$

that

$$\mathscr{P} \simeq 1 + \frac{1}{2}\,\frac{\sin(2kr_0 \sin\theta_0)}{2kr_0 \sin\theta_0} + \frac{1}{2}\int_0^\infty \frac{J_1(\sqrt{(2kr_0 \sin\theta_0)^2+\xi^2})}{\sqrt{(2kr_0 \sin\theta_0)^2+\xi^2}}\,d\xi + 0((kr_0)^{-3/2})$$

$$\simeq 1 + \frac{\sin 2kh}{2kh} \quad , \quad \text{as} \quad kx_0 = kr_0 \cos\theta_0 \to \infty \ , \ 0 < \theta_0 < \pi/2$$

where

$$h = r_0 \sin\theta_0$$

specifies the finite normal distance between the source and the plane; agreement of the latter estimate for a source far removed from the edge of the half-plane with the previous determination (3) for the same source in the presence of a full rigid plane is only to be expected.

An infinitesimally thin rigid screen does not, evidently, exercise any influence on the output of a coplanar monopole point source; and the requisite value $\mathscr{P} = 1$ is assumed by (21) when $\theta_0 = \pi$, inasmuch as

$$\frac{1}{2}\int_0^{-2kr_0} \frac{J_1(\xi)}{\xi}\,d\xi = -\frac{1}{2}\int_0^{2kr_0} \frac{J_1(\xi)}{\xi}\,d\xi$$

and

$$\frac{\sin(2kr_0 \sin\theta_0)}{2kr_0 \sin\theta_0} \to 1 \quad , \quad \theta_0 \to \pi \ .$$

The power factor that replaces (21) in the case of a soft half-plane, at which ϕ itself vanishes, is

$$\mathscr{P} = \frac{1}{2} - \frac{1}{2}\,\frac{\sin(2kr_0 \sin\theta_0)}{2kr_0 \sin\theta_0} + \frac{1}{2}\int_0^{2kr_0} \frac{J_1(\xi)}{\xi}\,d\xi - \frac{1}{2}\int_0^{2kr_0 \cos\theta_0} \frac{J_1(\sqrt{(2kr_0 \sin\theta_0)^2+\xi^2})}{\sqrt{(2kr_0 \sin\theta_0)^2+\xi^2}}\,d\xi \qquad (22)$$

(featuring opposite signs of the second and fourth terms relative to those in (21)) and the particular deductions therefrom, namely

$$\mathscr{P} \simeq 1 - \frac{\sin 2kh}{2kh} \ , \ kr_0 \cos\theta_0 \to \infty \ , \ h = r_0 \sin\theta_0 \ , \ 0 < \theta_0 < \pi/2$$

and

$$\mathscr{P} = \int_0^{2kr_0} \frac{J_1(\xi)}{\xi}\, d\xi \quad , \quad \theta_0 = \pi \quad , \quad 0 < r_0 < \infty \qquad (23)$$

are noteworthy. The latter function of a dimensionless argument $\alpha = kr_0$, where r_0 is the least distance from the source to the edge of a coplanar screen, possesses a regular nature for all α and has the limiting behaviors $\mathscr{P} \to 1$, $\alpha \to \infty$, $\mathscr{P} \to \alpha$, $\alpha \to 0$.

In two dimensions the source or Green's function which satisfy (4) take the respective forms

$$\begin{aligned}\phi(r,\theta;P) &= G(r,\theta;r_0,\theta_0) \\ &= \frac{1}{2}\left\{\frac{i}{4} H_0^{(1)}(kR_-) + \frac{1}{2\pi}\int_0^{2\sqrt{rr_0}\cos\frac{\theta-\theta_0}{2}} \left(\frac{\exp[ik\sqrt{R_-^2+\xi^2}]}{\sqrt{R_-^2+\xi^2}}\right) d\xi\right. \\ &\left.\quad \pm \frac{i}{4} H_0^{(1)}(kR_+) \pm \frac{1}{2\pi}\int_0^{2\sqrt{rr_0}\cos\frac{\theta+\theta_0}{2}} \left(\frac{\exp[ik\sqrt{R_+^2+\xi^2}]}{\sqrt{R_+^2+\xi^2}}\right) d\xi\right\} \qquad (24)\end{aligned}$$

when rigid or soft boundary conditions hold at the trace $\theta = 0(2\pi)$, $0 < r < \infty$ of a half-plane; and the designations

$$R_{\mp}^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta \mp \theta_0)$$

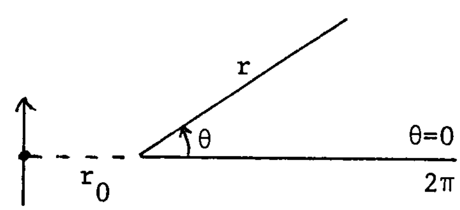
refer to the (squared) distances between the source, and its image in the plane, from an arbitrarily located point r,θ .

The power factor which bears direct correspondence with (23), that is for an isotropic line source located in the extension, and parallel to the edge, of a half-plane proves to be

$$\mathscr{P} = \frac{2}{\pi} \int_0^{2kr_0} \frac{\sin \xi}{\xi}\, d\xi = \frac{2}{\pi}\, \mathrm{Si}(2kr_0) \;, \tag{25}$$

where $\mathrm{Si}(\zeta)$ denotes the sine integral, and this also evidences a regular behavior, as a function of kr_0 , on the range $0 \leq r_0 < \infty$.

A rather different behavior of the output generally obtains if higher order primary sources (with more intense near fields) are chosen; and, specifically, this means an indefinite rise in power amplification as the source moves closer to the edge of the plane. Consider, for example, a dipole source which is oriented normally to the line that issues from the trace of a rigid half-plane and passes through its center. The appropriate (two-dimensional) power formula in this circumstance,



$$\mathscr{P} = 1 + \frac{8}{k^2 r_0}\, \mathrm{Im}\, \frac{\partial}{\partial\theta}\, \phi_S(r_0,\pi) \;,$$

where ϕ_S denotes the regular (secondary) part of a source function defined by the equations

$$(\nabla^2 + k^2) G = - \frac{\delta(r-r_0)}{r_0^2}\, \frac{\partial}{\partial\theta}\, \delta(\theta+\pi)$$

and

$$\frac{\partial G}{\partial\theta} = 0 \quad , \quad \theta = 0, 2\pi \quad , \quad r > 0$$

yields, on employing the (monopole) source representation (24),

$$\mathscr{P} = 1 + \frac{8}{(kr_0)^2}\, \mathrm{Im}\, \frac{\partial}{\partial\theta} \frac{\partial}{\partial\theta_0} \left\{ -\frac{i}{8} H_0^{(1)}(kR_-) + \frac{i}{8} H_0^{(1)}(kR_+) + \frac{1}{4\pi} \int_0^{2r_0 \cos\frac{\theta-\theta_0}{2}} \left(\frac{\exp(ik\sqrt{R_-^2+\xi^2})}{\sqrt{R_-^2+\xi^2}} \right) d\xi + \frac{1}{4\pi} \int_0^{2r_0 \cos\frac{\theta+\theta_0}{2}} \left(\frac{\exp(ik\sqrt{R_+^2+\xi^2})}{\sqrt{R_+^2+\xi^2}} \right) d\xi \right\}_{\theta=\theta_0=\pi}$$

$$= 1 + \frac{8}{(kr_0)^2} \frac{\partial^2}{\partial\theta^2} \left\{ \frac{1}{4} J_0(2kr_0 \cos\frac{\theta}{2}) - \frac{1}{2\pi} \int_0^{2r_0 \sin\frac{\theta}{2}} \frac{\sin(k\sqrt{(2r_0 \cos\frac{\theta}{2})^2+\xi^2})}{\sqrt{(2r_0 \cos\frac{\theta}{2})^2+\xi^2}} \right\}_{\theta=\pi}$$

$$= 1 + \frac{8}{(kr_0)^2} \frac{\partial}{\partial\theta} \left\{ \frac{kr_0}{4} \sin\frac{\theta}{2} J_1(2kr_0 \sin\frac{\theta}{2}) - \frac{1}{4\pi} \cos\frac{\theta}{2} \sin 2kr_0 \right.$$

$$+ \frac{1}{2\pi} r_0{}^2 \sin\theta \int_0^{2r_0 \sin\frac{\theta}{2}} \left(k \frac{\cos(k\sqrt{(2r_0 \cos\frac{\theta}{2})^2+\xi^2})}{(2r_0 \cos\frac{\theta}{2})^2 + \xi^2} \right.$$

$$\left.\left. - \frac{\sin(k\sqrt{(2r_0 \cos\frac{\theta}{2})^2+\xi^2})}{[(2r_0 \cos\frac{\theta}{2})^2+\xi^2]^{3/2}} \right) d\xi \right\}_{\theta=\pi}$$

$$= \frac{\sin 2kr_0}{\pi(kr_0)^2} + \frac{4}{\pi} \int_0^{2kr_0} \left(\frac{\sin\tau}{\tau^3} - \frac{\cos\tau}{\tau^2}\right) d\tau \; .$$

Since

$$\int_\varepsilon^{2kr_0} \frac{\sin\tau}{\tau^3} d\tau = \int_\varepsilon^{2kr_0} \sin\tau \, d(-\frac{1}{2\tau^2})$$

$$= - \frac{\sin 2kr_0}{2(2kr_0)^2} + \frac{\sin\varepsilon}{2\varepsilon^2} + \frac{1}{2} \int_\varepsilon^{2kr_0} \frac{\cos\tau}{\tau} d\tau$$

it turns out, finally, that

$$\mathscr{P} = \frac{\sin 2kr_0}{2\pi(kr_0)^2} + \frac{2}{\pi} \lim_{\varepsilon\to 0} \left\{ \frac{\sin\varepsilon}{\varepsilon^2} - \int_\varepsilon^{2kr_0} \frac{\cos\tau}{\tau^2} dt \right\}$$

$$= \frac{2}{\pi} \mathrm{Si}(2kr_0) + \frac{\sin 2kr_0}{2\pi(kr_0)^2} + \frac{\cos 2kr_0}{\pi kr_0} \; . \qquad (26)$$

Thus the output factor approaches unity when the dipole source recedes far away from the edge $(kr_0 \to \infty)$ and, as the estimate

$$\mathscr{P} \doteq \frac{2}{\pi kr_0} \; , \quad kr_0 \to 0$$

makes plain, becomes arbitrarily large when the source nears the edge.

Before concluding this section an idea may be given of the intricate analysis which is in prospect for total power calculations along conventional lines. As regards, in particular, the relatively simple problem of an isotropic line source symmetrically placed with respect to a half-plane, whose power factor involves the sine integral [cf. (25)], let it be observed that the multiple integral characterization

$$\mathscr{P} = \frac{2}{\pi^2} \int_0^{2\pi} \left| \int_0^{\sqrt{2kr_0}\,\sin\frac{\theta}{2}} e^{i\mu^2}\, d\mu \right|^2 d\theta \tag{27}$$

is consequent to joint use of the far field estimate

$$\phi \simeq \frac{1}{\sqrt{2kr}}\, e^{ik(r+r_0\cos\theta)} \int_0^{\sqrt{2kr_0}\,\sin\frac{\theta}{2}} e^{i\mu^2}\, d\mu \quad , \quad kr \gg 1$$

and the energy flux vector (4). Taking account of the Fourier series development

$$e^{ikr_0\cos\theta} \int_0^{\sqrt{2kr_0}\,\sin\frac{\theta}{2}} e^{i\mu^2}\, d\mu = \sqrt{\pi}\, e^{i\pi/4} \sum_{n=0}^{\infty} e^{-\pi i(n+\frac{1}{2})/2} (-1)^n \sin(n+\tfrac{1}{2})\theta\, J_{n+\frac{1}{2}}(kr_0)$$

the power integral (27) goes over to a single sum formula

$$\mathscr{P} = 2 \sum_{n=0}^{\infty} J^2_{n+\frac{1}{2}}(kr_0)$$

which, after transformations based on Bessel function properties and a change of integration order, reverts to a single definite integral, namely

$$\mathscr{P} = \frac{4}{\pi} \int_0^{\pi/2} \sum_{n=0}^{\infty} J_{2n+1}(2kr_0\cos\psi)\, d\psi = \frac{4}{\pi} \int_0^{\pi/2} \left\{ \frac{1}{2} \int_0^{2kr_0\cos\psi} J_0(\xi)\, d\xi \right\} d\xi$$

$$= \frac{2}{\pi} \int_0^{2kr_0} d\xi\, J_0(\xi) \int_0^{\cos^{-1}(\xi/2kr_0)} d\psi = \frac{2}{\pi} \int_0^{2kr_0} J_0(\xi) \cos^{-1}\frac{\xi}{2kr_0}\, d\xi \ .$$

Since the function

$$I(\alpha) = \int_0^\alpha J_0(\xi) \cos^{-1} \frac{\xi}{\alpha} \, d\xi$$

has the first derivative

$$\frac{dI}{d\alpha} = \int_0^\alpha J_0(\xi) \frac{\xi}{\sqrt{\alpha^2 - \xi^2}} \, d\xi = \int_0^{\pi/2} \sin\psi \, J_0(\alpha \sin \psi) \, d\psi = \frac{\sin \alpha}{\alpha}$$

and the initial value $I(0) = 0$, the determination

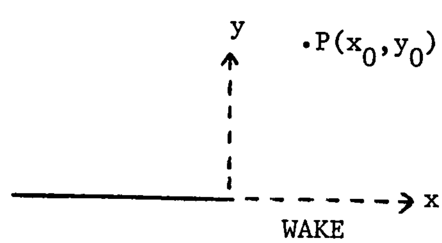
$$I(\alpha) = \int_0^\alpha \frac{\sin \xi}{\xi} \, d\xi = \mathrm{Si}(\alpha)$$

follows, and this secures the anticipated result

$$\mathscr{P} = \frac{2}{\pi} \, \mathrm{Si}(2kr_0) \ .$$

§4. Radiation from a Line Source Near an Edge at which a Kutta Condition Holds

It is natural to speculate on the link between enhanced sound radiation for (multipolar) sources near a sharp edge and the presumption of locally incompressible flow; Jones (1972) undertook, in this regard, to analyze the effect on sound excitation by a time-periodic line source near the edge of a rigid half-plane when a Kutta condition suppresses the unbounded nature of the edge velocity that is characteristic of incompressible and inviscid flow. His model problem, wherein the trace of a rigid plane occupies the half-line $y = 0$, $x < 0$, envisages both a course at $P(x_0, y_0)$ and the existence of a wake (or discontinuity line) elsewhere along the x-axis.



The postulated equations for the coordinate factor $\phi(\vec{r},\vec{P})$ of a velocity potential comprise

$$(\nabla^2+k^2)\phi(\vec{r},\vec{P}) = -\delta(x-x_0)\delta(y-y_0)$$

$$\frac{\partial}{\partial y}\phi = 0 \quad , \quad y = 0 \quad , \quad x < 0 \tag{28}$$

and

$$\phi(x,0+;P) - \phi(x,0-;P) = Ae^{i\kappa x} \quad , \ x > 0$$

wherein the latter (or wake) condition involves two constant parameters A and κ ; only one of these constants, namely A , can be fixed by means of the foregoing equations along with the further hypothesis that $\partial\phi/\partial y$ (i.e., the normal velocity) is continuous at the wake and the Kutta requirement. The other parameter, κ , is assigned a positive (though indeterminate) value, as befits an outgoing wave or radiation condition.

Let the superscripts $\pm$ differentiate the wave factors in $y \gtrless 0$, respectively, and define the function

$$V(x) = \left.\frac{\partial\phi}{\partial y}\right|_{y=0} \quad , \quad x > 0 \ ; \tag{29}$$

then, if $y_0 > 0$, say, the representations

$$\phi^{(+)}(x,y;P) = \frac{i}{4} H_0^{(1)}(k\sqrt{(x-x_0)^2+(y-y_0)^2}) + \frac{i}{4} H_0^{(1)}(k\sqrt{(x-x_0)^2+(y+y_0)^2}) - $$
$$- \frac{i}{2} \int_0^\infty H_0^{(1)}(k\sqrt{(x-x')^2+y^2})V(x')\,dx'$$

and

$$\phi^{(-)}(x,y;P) = \frac{i}{2} \int_0^\infty H_0^{(1)}(k\sqrt{(x-x')^2+y^2})V(x')\,dx' \tag{30}$$

comply with the first two equations of the set (28) and jointly provide, on imposing the wake condition, a linear integral equation for V , viz.

$$\int_0^\infty H_0^{(1)}(k|x-x'|)V(x')\,dx' = \frac{1}{2} H_0^{(1)}(k\sqrt{(x-x_0)^2+y_0^2}) + iAe^{i\kappa x} \quad , \; x > 0 \; . \qquad (31)$$

A formal resolution of the boundary value problem is thus achieved by solving the integral equation and employing the representations (30). There is no need, in fact, to investigate that part of the solution which obtains in the absence of a wake discontinuity $(A = 0)$, since the corresponding source or Green's function of a rigid half-plane is known. Application of the Wiener-Hopf method to the integral equation that contains the wake term alone,

$$\int_0^\infty H_0^{(1)}(k|x-x'|)\hat{V}(x')\,dx' = iAe^{i\kappa x} \quad , \; x > 0 \qquad (32)$$

yields, in sequential order, the complex Fourier transform of $\hat{V}(x)$, namely

$$\hat{V}(\zeta) = \int_0^\infty e^{i\zeta x}\,\hat{V}(x)\,dx = -\frac{A}{2}\sqrt{\kappa+k}\,\frac{\sqrt{k+\zeta}}{\zeta+\kappa+i\varepsilon} \; , \quad \varepsilon > 0 \qquad (33)$$

and thence the function itself,

$$\hat{V}(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-i\zeta x}\,\hat{V}(\zeta)\,d\zeta = -\frac{A}{4\pi}\sqrt{\kappa+k}\int_{-\infty}^\infty \frac{e^{-i\zeta x}\sqrt{\zeta+k}}{\zeta+\kappa+i\varepsilon}\,d\zeta \; . \qquad (34)$$

On referring to an alternative form of the latter,

$$\hat{V}(x) = -\frac{A}{4\pi}\sqrt{\kappa+k}\left\{\int_{-\infty}^\infty \frac{e^{-i\zeta x}}{\sqrt{k+\zeta}}\,d\zeta - (\kappa-k)\int_{-\infty}^\infty \frac{e^{-i\zeta x}\,d\zeta}{\sqrt{k+\zeta}\,(\zeta+\kappa+i\varepsilon)}\right\}$$

and the result

$$\int_{-\infty}^\infty \frac{e^{-i\zeta x}\,d\zeta}{\sqrt{k+\zeta}} = 2e^{-i\pi/2}\int_k^\infty \frac{e^{i\tau x}}{\sqrt{\tau-k}}\,d\tau = 2\sqrt{\frac{\kappa}{x}}\,e^{i(kx-\pi/4)} \quad , \; x > 0$$

the singular part of $\hat{V}(x)$ at $x = 0$ is isolated, viz.

$$\hat{V}(x) \doteq -\frac{A}{2\pi}\sqrt{\frac{\pi}{x}\,(\kappa+k)}\;e^{i(kx-\pi/4)} \quad , \; x \to 0+ \; . \qquad (35)$$

A calculation made with the Green's function of a rigid half-plane lying on the negative x-axis reveals that

$$\overline{V}(x) = \frac{1}{2\pi x}\frac{\partial}{\partial\theta}\left\{\int_0^{2\sqrt{xr_0}} \frac{\exp(ik\sqrt{x^2+r_0^2-2xr_0\cos(\theta-\theta_0)+\xi^2})}{\sqrt{x^2+r_0^2-2xr_0\cos(\theta-\theta_0)+\xi^2}}\,d\xi\right\}_{\theta=0}$$

$$= \frac{1}{2\pi}\sqrt{\frac{r_0}{x}}\sin\frac{\theta_0}{2}\,\frac{e^{ik(x+r_0)}}{x+r_0}$$

$$-\frac{1}{2\pi}ikr_0\sin\theta_0\int_0^{2\sqrt{xr_0}\cos\frac{\theta_0}{2}} \frac{\exp(ik\sqrt{x^2+r_0^2-2xr_0\cos\theta_0+\xi^2})}{x^2+r_0^2-2xr_0\cos\theta_0+\xi^2}\,d\xi \qquad (36)$$

$$+\frac{1}{2\pi}r_0\sin\theta_0\int_0^{2\sqrt{xr_0}\cos\frac{\theta_0}{2}} \frac{\exp(ik\sqrt{x^2+r_0^2-2xr_0\cos\theta_0+\xi^2})}{(x^2+r_0^2-2xr_0\cos\theta_0+\xi^2)^{3/2}}\,d\xi$$

where the first term alone exhibits a singularity at $x = 0$. Hence the full solution of (31), $V(x) = \overline{V}(x) + \hat{V}(x)$, remains finite in the limit $x \to 0+$ if

$$\frac{1}{2\pi}\frac{\sin\frac{\theta_0}{2}}{\sqrt{r_0}}e^{ikr_0} - \frac{A}{2\pi}\sqrt{\pi(\kappa+k)}\,e^{-i\pi/4} = 0$$

or

$$A = \frac{1}{\sqrt{\pi r_0}}\,\frac{\sin\frac{\theta_0}{2}}{\sqrt{\kappa+k}}\,e^{i(kr_0+\pi/4)} . \qquad (37)$$

This agrees with a determination by Jones utilized, in turn, for estimating the distant sound field amplitude along directions close to that of the wake and elsewhere.

A different measure of the effect brought about by imposing a Kutta condition, namely the change in total energy radiated, can be given; in order to arrive at this, consider the integrals of an identity obtained from the first equation in (28),

$$\frac{\omega\rho}{2} \operatorname{Im} \nabla \cdot (\phi^* \nabla \phi) = - \frac{\omega\rho}{2} \operatorname{Im}\{\delta(x-x_0)\delta(y-y_0)\phi^*\} \quad ,$$

which are taken, separately, over the adjacent domains $y \gtrless 0$ with a common boundary at the half-plane and the wake. After replacing the (two-dimensional) divergence integrals by curvilinear ones along great semi-circular arcs in $y \gtrless 0$ and line integrals along $y = 0\pm$, adding the results and employing the last two equations of (28), it turns out that

$$\mathscr{P} = 1 + 4 \operatorname{Im} \phi_S(x_0,y_0) + 4 \operatorname{Im} \int_0^\infty A^* e^{-i\kappa x} V(x)\, dx \qquad (38)$$

where the (familiar) term involving the value of the secondary wave function

$$\phi_S(x,y) = \phi(x,y;P) - \frac{i}{4} H_0^{(1)}\left(k\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$$

at the fixed source point $P(x_0,y_0)$ is supplemented by a non-local term, on account of the postulated wake or discontinuity line for the velocity potential. There is no contribution to the latter term stemming from the function $\hat{V}(x)$ and, specifically, its Fourier transform (cf. (33))

$$\hat{V}(-\kappa) = \int_0^\infty e^{-i\kappa x}\, \hat{V}(x)\, dx = - \frac{A}{2} \sqrt{\kappa+k}\, \frac{\sqrt{k-\kappa}}{i\varepsilon}$$

if $\kappa > k$ and $\sqrt{k-\kappa}$ has a wholly imaginary value, for then

$$\operatorname{Im} A^* \hat{V}(-\kappa) = 0 \ .$$

This means that the Kutta condition affects terms in (38) which depend linearly on the complex magnitude A and appear in the combined form

$$\Delta\mathscr{P} = 4 \text{ Im} \left\{ -\frac{i}{2} \int_0^\infty H_0^{(1)}(k\sqrt{(x-x_0)^2+y_0^2})\hat{V}(x)\, dx + A^* \int_0^\infty e^{-i\kappa x}\overline{V}(x)\, dx \right\}$$

$$= -\frac{2}{\pi} \text{ Im} \left\{ i \int_{-\infty}^\infty \frac{e^{-i\zeta x_0 + i\sqrt{k^2-\zeta^2}\, y_0}}{\sqrt{k^2-\zeta^2}} \hat{V}(\zeta)\, d\zeta \right\} + 4 \text{ Im } A^* \overline{V}(-\kappa) \qquad (39)$$

whose details are shaped by the representation (30) and the Hankel function integral

$$H_0^{(1)}(k\sqrt{(x-x_0)^2+y_0^2}) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{e^{i\zeta(x-x_0)+i\sqrt{k^2-\zeta^2}\,|y_0|}}{\sqrt{k^2-\zeta^2}}\, d\zeta \, .$$

It suffices here, in view of the crude model, to merely estimate when kr_0 and κr_0 are very small; thus, relying on the explicit determination (33) for $\hat{V}(\zeta)$ the first term of (39) becomes

$$\frac{2}{\pi} \text{ Im} \left\{ i\, \frac{A}{2} \sqrt{\kappa+k} \int_{-\infty}^\infty \frac{e^{-i\zeta x_0 + i\sqrt{k^2-\zeta^2}\, y_0}}{\zeta + \kappa + i\varepsilon} \frac{d\zeta}{\sqrt{k-\zeta}} \right\}$$

and, since the integral remains finite in the limit $x_0, y_0 \to 0$, this has an approximate value

$$\frac{1}{\pi} \sqrt{\kappa+k} \text{ Im} \left\{ A \int_{-\infty}^\infty \frac{1}{\zeta + \kappa + i\varepsilon} \frac{d\zeta}{\sqrt{k-\zeta}} \right\}$$

$$= 2 \text{ Im } A \quad , \quad \kappa r_0 \to 0 \, .$$

On retaining just the initial and predominant term in the representation (36) of $\overline{V}(x)$ and noting the relation

$$\int_0^\infty e^{i\zeta x} \left(\frac{1}{2\pi} \sqrt{\frac{r_0}{x}} \frac{e^{ik(x+r_0)}}{x + r_0} \right) dx = \frac{1}{2\pi} e^{ikr_0} \int_0^\infty \frac{e^{i(k+\zeta) r_0 \tau}}{\sqrt{\tau}\,(1+\tau)}\, d\tau$$

it follows that

$$4 \text{ Im } A^* \overline{V}(-\kappa) = \frac{2}{\pi} \text{ Im} \left\{ A^* \sin\frac{\theta_0}{2} \int_0^\infty \frac{d\tau}{\sqrt{\tau}\,(1+\tau)} \right\}$$

$$= 2 \sin\frac{\theta_0}{2} \text{ Im } A^* \quad , \quad \kappa r_0 \to 0 \, .$$

Accordingly,

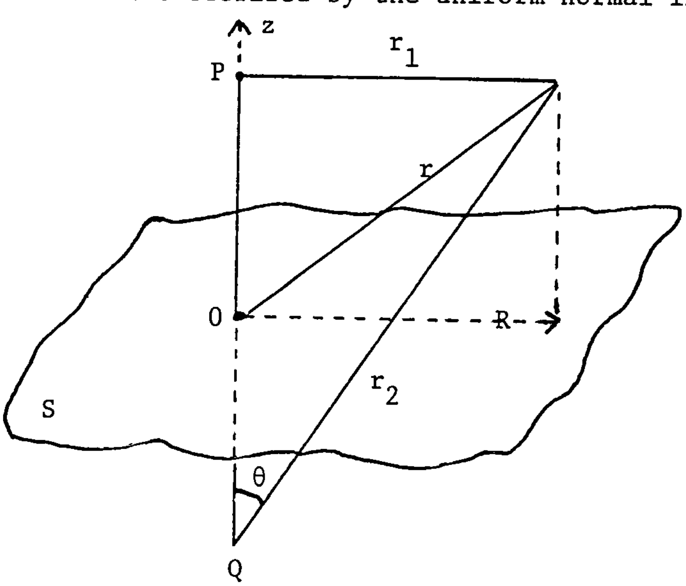
$$\Delta \mathscr{P} = 2 \text{ Im } (A - A \sin \frac{\theta_0}{2})$$

$$= \frac{\sin \frac{\theta_0}{2} (1 - \sin \frac{\theta_0}{2})}{\sqrt{\frac{\pi}{2} (\kappa + k) r_0}} , \quad (\kappa + k) r_0 << 1 \qquad (40)$$

which result indicates that enforcement of a Kutta (or finite velocity) condition at the edge of a half-plane significantly elevates the power output from a fixed line source situated nearby, if θ_0 is not too small.

§5. Radiation by a Point Source Above a Plane Impedance Boundary

Consider an infinite plane surface S which is locally reflecting and characterized by the uniform normal impedance $\rho c \zeta$; then, if $z = 0$ on the plane, the boundary condition



$$\frac{\partial \phi}{\partial z} + ik\eta\phi = 0 , \quad z = 0 \qquad (41)$$

applies, with $\eta = \frac{1}{\zeta}$ termed the specific admittance. It follows directly from (41) and the normal (z-) component of the flux vector (6) at the surface that

$$N_z = \frac{\omega\rho}{2} \text{ Im}(\phi^* \frac{\partial \phi}{\partial z})_{z=0} = - \frac{\omega\rho}{2} \text{ Im}(ik\eta|\phi|^2)$$

and thus energy leaves, rather than enters the half-space $z > 0$ if

$$\text{Re } \eta > 0 . \qquad (42)$$

Let the equation (4) describe a source at $\vec{P}(0,0,h > 0)$ and introduce the split

$$\phi(\vec{r}) = \phi_P(\vec{r}) + \phi_S(\vec{r})$$

$$= \frac{e^{ik|\vec{r}-\vec{P}|}}{4\pi|\vec{r}-\vec{P}|} + \phi_S(\vec{r}) \ , \ z > 0 \tag{43}$$

where $\phi_S(\vec{r})$ is regular everywhere above the surface S . Multiplication of (4) by ϕ^* and subsequent integration over the half-space $z > 0$ leads to the relationship

$$\mathscr{P} = 1 + \frac{4\pi}{k} \text{Im } \phi_S(\vec{P}) = \mathscr{E} + \mathscr{D} \tag{44}$$

in which

$$\mathscr{E} = 4\pi \lim_{r\to\infty} \{r^2 \int |\phi|^2 \, d\Omega\} \tag{45}$$

represents the energy outflow rate in all directions above the impedance surface and

$$\mathscr{D} = 4\pi \text{ Re } \{\eta \int_S |\phi|^2 \, dx \, dy\} \tag{46}$$

represents the rate of energy loss or absorption at the complex impedance surface, both of these measures relative to the power output from an identical source in a completely homogeneous environment.

If the component parts of the wave function (43) are given explicit and individual expression through the integral

$$\phi(\vec{r}) = \phi(R,z) = \frac{1}{4\pi} \int_0^\infty \frac{J_0(\xi R)}{\sqrt{\xi^2-k^2}} \left\{ e^{-\sqrt{\xi^2-k^2}|z-h|} + \frac{\sqrt{\xi^2-k^2} + ik\eta}{\sqrt{\xi^2-k^2} - ik\eta} e^{-\sqrt{\xi^2-k^2}(z+h)} \right\} d\xi$$

$$\arg \sqrt{\xi^2-k^2} = \begin{matrix} 0 \ , \ \xi > k \\ -\pi/2 \ , \ \xi < k \end{matrix} \tag{47}$$

that takes account of both the cylindrical symmetry round the line OP normal to the surface and the boundary condition (41), it appears that

$$\phi_S(\vec{P}) = \phi_S(0,h) = \frac{1}{4\pi}\int_0^\infty \frac{\xi}{\sqrt{\xi^2-k^2}}\,\frac{\sqrt{\xi^2-k^2}+ik\eta}{\sqrt{\xi^2-k^2}-ik\eta}\,e^{-2\sqrt{\xi^2-k^2}h}\,d\xi$$

and hence

$$\text{Im}\,\phi_S(0,h) = \frac{k\alpha}{2\pi}\int_k^\infty \frac{\xi\,\exp(-2\sqrt{\xi^2-k^2}h)}{(\sqrt{\xi^2-k^2}+k\beta)^2+(k\alpha)^2}\,d\xi + \frac{k\beta}{2\pi}\int_0^k \frac{\xi\,\sin(2\sqrt{k^2-\xi^2}h)}{(\sqrt{k^2-\xi^2}+k\alpha)^2+(k\beta)^2}\,d\xi$$

$$-\frac{1}{4\pi}\int_0^k \frac{\xi}{\sqrt{k^2-\xi^2}}\,\frac{k^2\alpha^2+k^2\beta^2+\xi^2-k^2}{(\sqrt{k^2-\xi^2}+k\alpha)^2+(k\beta)^2}\cos\,(2\sqrt{k^2-\xi^2}h)\,d\xi \qquad (48)$$

where α,β denote the real and imaginary parts of the admittance, i.e.,

$$\eta = \alpha + i\beta \quad . \qquad (49)$$

On referring to the pair of Fourier-Bessel integrals which stem from (47), namely

$$\phi(R,0) = \frac{1}{2\pi}\int_0^\infty F(\xi)\xi J_0(\xi R)\,d\xi$$

and

$$F(\xi) = \int_0^\infty \phi(R,0)RJ_0(\xi R)\,dR = \frac{e^{-\sqrt{\xi^2-k^2}h}}{\sqrt{\xi^2-k^2}-ik(\alpha+i\beta)} \qquad (50)$$

it is next found that

$$\int_{-\infty}^\infty |\phi(R,0)|^2\,dx\,dy = 2\pi\int_0^\infty |\phi(R,0)|^2\,R\,dR$$

$$= \frac{1}{2\pi}\int_0^\infty R\,dR\int_0^\infty \xi\xi' J_0(\xi R)J_0(\xi' R)F(\xi)F^*(\xi')\,d\xi\,d\xi'$$

$$= \frac{1}{2\pi}\int_0^\infty \delta(\xi-\xi')\xi F(\xi)F^*(\xi') = \frac{1}{2\pi}\int_0^\infty \xi|F(\xi)|^2\,d\xi$$

$$= \frac{1}{2\pi}\int_k^\infty \frac{\xi\,\exp\,(-2\sqrt{\xi^2-k^2}h)}{(\sqrt{\xi^2-k^2}+k\beta)^2+(k\alpha)^2}\,d\xi + \frac{1}{2\pi}\int_0^k \frac{\xi d\xi}{(\sqrt{k^2-\xi^2}+k\alpha)^2+(k\beta)^2} \quad . \qquad (51)$$

All the integrals encountered in (48) and (51), save for

$$I = \int_0^k \frac{\xi d\xi}{(\sqrt{k^2-\xi^2}+k\alpha)^2+(k\beta)^2} = \int_0^1 \frac{\mu d\mu}{|\eta+\sqrt{1-\mu^2}|^2}$$

$$= \frac{1}{2} \log \frac{(1+\alpha)^2 + \beta^2}{\alpha^2 + \beta^2} - \frac{\alpha}{|\beta|} \left(\tan^{-1} \frac{1 + \alpha}{|\beta|} - \tan^{-1} \frac{\alpha}{|\beta|}\right) \quad ,$$

are reducible to the exponential type, i.e.,

$$E_1(z) = \int_z^\infty \frac{e^{-\xi}\, d\xi}{\xi} \quad , \quad |\arg z| < \pi \quad ; \tag{52}$$

and the consequent determinations of the total power factor $\mathscr{P}$, radiation factor $\mathscr{E}$, and absorption factor $\mathscr{D}$ prove to be

$$\mathscr{P} = 1 + \frac{\sin 2kh}{2kh} + 2\mathrm{Re}\{\eta e^{-2ikh\eta} E_1(-2ikh(1+\eta))\} \quad , \tag{53}$$

$$\mathscr{E} = 1 + \frac{\sin 2kh}{2kh} - 2\alpha \int_0^1 \frac{\mu d\mu}{|\eta + \sqrt{1-\mu^2}|^2}$$

$$+ 2\ \mathrm{Re}\{\eta e^{i2ikh\eta}(E_1(-2ikh(1+\eta))-E_1(-2ikh\eta))\} \quad , \tag{54}$$

and

$$\mathscr{D} = 2\alpha \int_0^1 \frac{\mu d\mu}{|\eta + \sqrt{1-\mu^2}|} + 2\mathrm{Re}\{\eta e^{-2ikh\eta} E_1(-2ikh\eta)\} \tag{55}$$

with proper limiting values if $\alpha , \beta \to 0$ or $\alpha , \beta \to \infty$ (the hard or soft surface conditions, respectively).

Ingard and Lamb state that "a rigorous solution is not so easily obtained (for absorbing surfaces) since the evaluation of the reflected field cannot be obtained directly by simple image procedure but would have to draw upon the analysis of reflection of a spherical wave by a boundary." Ingard's own study of the latter problem (1951), which expresses the complete wave function as a sum of primary and "image" source contribution (see figure)

$$\phi(\vec{r}) = \frac{e^{ikr_1}}{4\pi r_1} + Q\,\frac{e^{ikr_2}}{4\pi r_2} \quad , \tag{56}$$

with a complex and variable amplitude factor

$$Q(r_2,\theta) = 1 - 2kr_2\eta \int_0^\infty \frac{e^{-kr_2\xi}\,d\xi}{\sqrt{(1+\eta\cos\theta + i\xi)^2 + \sin^2\theta(1-\eta^2)}} \quad , \tag{57}$$

affords, it may be noted, a ready determination of $\mathscr{P}$; thus, from (44),

$$\begin{aligned}\mathscr{P} &= 1 + \frac{4\pi}{k}\,\mathrm{Im}\left\{Q(2h,0)\,\frac{e^{2ikh}}{4\pi\cdot 2h}\right\} \\ &= 1 + \frac{\sin 2kh}{2kh} - 2\,\mathrm{Im}\left\{\eta e^{2ikh}\int_0^\infty \frac{e^{-2kh\xi}}{1+\eta+i\xi}\,d\xi\right\} \\ &= 1 + \frac{\sin 2kh}{2kh} + 2\mathrm{Re}\left\{\eta e^{-2ikh\eta}E_1(-2ikh(1+\eta))\right\}\end{aligned}$$

in agreement with (53). The wave function (56) is, on the other hand, less suited than the version (47) in terms of cylindrical coordinates (R,z) for determining the absorption factor $\mathscr{D}$.

§6. Power Output of a Point Source in a Uniform Flow

Acoustical excitations and their power indices are strikingly affected by the existence of a background flow or a relative motion between the source and its environs. Assume that the convected wave equation

$$\nabla^2\Phi - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \vec{U}\cdot\nabla\right)^2\Phi = -\delta(\vec{r}-\vec{P})f(t) \tag{58}$$

characterizes the velocity potential for a monopole source at the fixed point $\vec{P}$, with temporally variable amplitude rendered by the function $f(t)$, in a fluid which streams uniformly and steadily at velocity $\vec{U}$. Multiplication of (58) by $-\rho\partial\Phi/\partial t$, where ρ designates the constant density of the fluid in the absence of acoustical perturbation therein, and the subsequent recasting of individual terms leads to the relation

$$\frac{\partial W}{\partial t} + \nabla \cdot \vec{N} = \rho\delta(\vec{r}-\vec{P})f(t)\,\frac{\partial\Phi}{\partial t} \tag{59}$$

where

$$W = \frac{1}{2}\,\rho(\nabla\Phi)^2 + \frac{\rho}{2c^2}\,(\frac{\partial\Phi}{\partial t})^2 - \frac{\rho}{2c^2}\,(\vec{U}\cdot\nabla\Phi)^2 \tag{60}$$

and

$$\vec{N} = -\rho\,\frac{\partial\Phi}{\partial t}\,\nabla\Phi + \frac{\rho}{c^2}\,\vec{U}\,\frac{\partial\Phi}{\partial t}\,(\frac{\partial\Phi}{\partial t} + \vec{U}\cdot\nabla\Phi)\ . \tag{61}$$

The scalar and vector quantities W, $\vec{N}$ may be identified as measures of the acoustical energy density and flux, respectively, the latter being directly applicable to calculations of energy transfer across <u>fixed</u> surfaces in the fluid.

When $\vec{P}$ lies at the origin, $f(t) = e^{-i\omega t}$ and the streaming velocity $\vec{U} : (U,0,0)$ is along the positive x-direction, the time-periodic (complex) source function deduced from (58),

$$\Phi(\vec{r},0,t) = \frac{e^{ikR}}{4\pi R_1}\,e^{-i\omega t}\quad ,\quad k = \omega/c \tag{62}$$

has both non-symmetric amplitude and phase factors with the specifications

$$R_1{}^2 = x^2 + (1-M^2)(y^2+z^2)\quad ,\quad R = \frac{-Mx + R_1}{1 - M^2} \tag{63}$$

at Mach numbers $M = U/c$ less than unity.

The determination of energy output from this source by integrating the flux vector $\vec{N}$ is facilitated with a suitable choice of the enveloping control surface; and an evident one comprises a cylindrical (curved) portion, aligned with the flow direction and having the source on its symmetry axis, together with plane end sections normal to the flow. If $\sigma = \sqrt{y^2+z^2}$ = constant defines such a cylindrical surface, the normal flux thereat is

$$N_\sigma = -\rho \frac{\partial\Phi}{\partial t}\frac{\partial\Phi}{\partial\sigma}$$

and thus, taking into account the representation

$$\frac{\partial\Phi}{\partial\sigma} = \frac{1}{4\pi}\left\{-\frac{(1-M^2)\sigma}{R_1^3} + ik\frac{\sigma}{R_1^2}\right\}\exp(-i\omega(t-\frac{R}{c})) \quad ,$$

the time-average measure

$$\overline{N}_\sigma = \frac{1}{2}\left(\frac{1}{4\pi}\right)^2 \frac{\rho k^2}{c}\frac{\sigma}{R_1^3}$$

follows. Since a vanishingly small energy flux passes outwards across finite plane areas perpendicular to the cylinder axis at remote sites (where $|x| \to \infty$), the whole power delivered by the source has the magnitude

$$\begin{aligned}\mathscr{E} &= 2\pi\int_{-\infty}^{\infty}\overline{N}_\sigma\sigma\,dx = \pi\rho\frac{k^2}{c}\left(\frac{1}{4\pi}\right)^2\int_{-\infty}^{\infty}\frac{\sigma^2\,dx}{[x^2+(1-M^2)\sigma^2]^{3/2}} \\ &= \frac{\rho\omega^2}{8\pi c}\frac{1}{1-M^2} \; . \qquad (64)\end{aligned}$$

An identical result is secured on integration of the right-hand side, or source term, in the time-averaged version of the energy balance equation (59), namely

$$\mathscr{E} = \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} \rho\delta(x)\delta(y)\delta(z)e^{i\omega t} \frac{\partial\Phi}{\partial t}\, dx\, dy\, dz$$

$$= \frac{1}{2} \rho\omega \operatorname{Im} \int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\Phi(\vec{r},0,t)e^{i\omega t}\, dx\, dy\, dz \tag{65}$$

$$= \frac{\omega\rho}{8\pi} \operatorname{Im} \int_{-\infty}^{\infty} \exp\left(\frac{ik\sigma}{\sqrt{1-M^2}}\right) \frac{\delta(y)\delta(z)\, dy\, dz}{\sqrt{1-M^2}}$$

$$= \frac{\omega\rho}{8\pi} \lim_{\sigma\to 0} \frac{\exp(ik\sigma/\sqrt{1-M^2})}{\sqrt{1-M^2}} = \frac{\rho\omega^2}{8\pi c} \frac{1}{1 - M^2} .$$

Consider next an identical source at distance h above a rigid wall occupying the entire plane $z = 0$; then the relevant wave function is composed of two like terms

$$\Phi = \Phi(\vec{r},\vec{P},t) + \Phi(\vec{r},\vec{Q},t) \quad , \quad z > 0 \tag{66}$$

where $\vec{P}(0,0,h)$ and $\vec{Q}(0,0,-h)$ are mirror image points in the plane, and the radiated energy can be directly found by recourse to a source integral of the form (65), viz.

$$\mathscr{E} = \frac{1}{2} \rho\omega \operatorname{Im} \int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z-h)(\Phi(\vec{r},\vec{P},t) + \Phi(\vec{r},\vec{Q},t))e^{i\omega t}\, dx\, dy\, dz$$

$$= \frac{\rho\omega^2}{8\pi c} \frac{1}{1 - M^2} + \frac{\rho\omega}{8\pi} \frac{\sin(2kh/\sqrt{1-M^2})}{2h\sqrt{1-M^2}} . \tag{67}$$

The relative power factor obtained from the ratio of (67) and (64)

$$\mathscr{P} = 1 + \sqrt{1-M^2}\, \frac{\sin(2kh/\sqrt{1-M^2})}{2kh} \tag{68}$$

which incorporates the effects of both a background flow and a hard reflecting surface, generalizes the earlier expression (3).

It requires a rather more involved procedure to arrive at (67) by calculation in the far field; thus, introducing the spherical polar coordinates r,θ,ψ through the transformation formulas

$$x = r\cos\theta \quad , \quad y = r\sin\theta\cos\psi \quad , \quad z = r\sin\theta\sin\psi \quad ,$$

with the ranges

$$0 \leq \theta \leq \pi \quad , \quad 0 \leq \psi \leq \pi \quad , \quad r > 0$$

appropriate to the half-space $z > 0$, and employing the asymptotic characterizations

$$R_{\mp} = \sqrt{x^2+(1-M^2)[y^2+(z\mp h)^2]} \simeq \sqrt{1-M^2\sin^2\theta}\; r \mp h(1-M^2)\frac{\sin\theta\sin\psi}{\sqrt{1-M^2\sin^2\theta}}$$

$$r \to \infty$$

$$\frac{-Mx + R_{\mp}}{1-M^2} \simeq \frac{1}{1-M^2}\left\{(\sqrt{1-M^2\sin^2\theta} - M\cos\theta)r \mp h(1-M^2)\frac{\sin\theta\sin\psi}{\sqrt{1-M^2\sin^2\theta}}\right\}$$

an estimate for the complete wave function,

$$\Phi \simeq \frac{1}{2\pi r}\,\frac{\cos(kh\sin\theta\sin\psi/\sqrt{1-M^2\sin^2\theta})}{\sqrt{1-M^2\sin^2\theta}}\exp\left\{ik\,\frac{\sqrt{1-M^2\sin^2\theta}-M\cos\theta}{1-M^2}\,r - i\omega t\right\} ,$$

$$r \to \infty \qquad (69)$$

is obtained from (66). On employing the latter along with the radial component of the energy flux

$$N_r = -\rho\frac{\partial\Phi}{\partial t}\left\{(1-M^2\cos^2\theta)\frac{\partial\Phi}{\partial r} + \frac{M^2\sin\theta\cos\theta}{r}\frac{\partial\Phi}{\partial\theta} - \frac{M}{c}\cos\theta\frac{\partial\Phi}{\partial t}\right\}$$

the average radiated power in the half-space $z > 0$ acquires the versions

$$\mathscr{E} = \int_0^{\pi} d\psi \int_0^{\pi} \sin\theta \, d\theta \lim_{r\to\infty} (r^2 N_r)$$

$$= \int_0^{\pi} d\psi \int_0^{\pi} \sin\theta \, d\theta \left\{ \frac{\rho\omega}{2} \left(\frac{1}{2\pi}\right)^2 \cos^2\left(kh \frac{\sin\theta \sin\psi}{\sqrt{1-M^2 \sin^2\theta}} \right) \right.$$

$$\cdot \left[k \frac{\sqrt{1-M^2 \sin^2\theta} - M\cos\theta}{1-M^2} \cdot \frac{1-M^2\cos^2\theta}{1-M^2\sin^2\theta} \right.$$

$$+ k \frac{M^2 \sin\theta\cos\theta}{1-M^2\sin^2\theta} \frac{1}{1-M^2} \frac{d}{d\theta} \left(\sqrt{1-M^2\sin^2\theta} - M\cos\theta\right)$$

$$\left.\left. + kM \frac{\cos\theta}{1-M^2\sin^2\theta} \right] \right\}$$

$$= \frac{\rho}{2c} \left(\frac{\omega}{2\pi}\right)^2 \int_0^{\pi} d\psi \int_0^{\pi} \frac{\sin\theta}{(1-M^2\sin^2\theta)^{3/2}} \cos^2\left(kh \frac{\sin\theta\sin\psi}{\sqrt{1-M^2\sin^2\theta}} \right) d\theta$$

$$= \frac{\pi\rho}{4c} \left(\frac{\omega}{2\pi}\right)^2 \int_0^{\pi} \frac{\sin\theta}{(1-M^2\sin^2\theta)^{3/2}} \left\{ 1 + J_0\left(2kh \frac{\sin\theta}{\sqrt{1-M^2\sin^2\theta}} \right) \right\} d\theta \qquad (70)$$

where J_0 denotes the zero order Bessel function.

That part of (70) which does not involve the Bessel function agrees precisely with the first term in (67); to link the remaining parts a change of the integration variable proves convenient. If θ is replaced by γ in (70), such that

$$\frac{\sin\theta}{\sqrt{1-M^2\sin^2\theta}} = \frac{\sin\gamma}{\sqrt{1-M^2}}, \qquad 0 \le \theta \le \pi/2, \quad 0 \le \gamma \le \pi/2$$

and

$$\sin\theta = \frac{\sin\gamma}{\sqrt{1-M^2\cos^2\theta}}, \quad \frac{\cos\theta \, d\theta}{(1-M^2\sin^2\theta)^{3/2}} = \frac{\cos\gamma \, d\gamma}{\sqrt{1-M^2}}, \quad \tan\theta = \sqrt{1-M^2}\tan\gamma ,$$

then

$$\int_0^{\pi} \frac{\sin\theta}{(1-M^2\sin^2\theta)^{3/2}} J_0\left(2kh\frac{\sin\theta}{\sqrt{1-M^2\sin^2\theta}}\right)d\theta =$$

$$= 2\int_0^{\pi/2} \frac{\sin\theta}{(1-M^2\sin^2\theta)^{3/2}} J_0\left(2kh\frac{\sin\theta}{\sqrt{1-M^2\sin^2\theta}}\right)d\theta$$

$$= \frac{2}{1-M^2}\int_0^{\pi/2} \sin\gamma\, J_0\left(2kh\frac{\sin\gamma}{\sqrt{1-M^2}}\right)d\gamma$$

$$= \frac{1}{\sqrt{1-M^2}}\,\frac{\sin(2kh/\sqrt{1-M^2})}{kh}$$

since

$$\int_0^{\pi/2} \sin\tau\, J_0(\alpha\sin\tau)\,d\tau = \frac{\sin\alpha}{\alpha}\ ;$$

and the full compatibility of (67), (70) is demonstrated.

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