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EVOLUTION OF WEAK DISTURBANCES IN INERT BINARY MIXTURES

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ABSTRACT

The evolution of weak disturbances in inert binary mixtures is determined for the one-dimensional piston problem. The interaction of the dissipative and nonlinear mechanisms is described by Burgers' equation. The binary-mixture diffusion mechanisms enter as an additive term in an effective diffusivity. Results for the impulsive motion of a piston moving into an ambient medium and the sinusoidally oscillating piston are used to illustrate the results and elucidate the incorrect behavior pertaining to the associated linear theory.

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1. INTRODUCTION

Combined dissipative effects of viscosity, thermal conduction, and mass diffusion play fundamental roles in the propagation of disturbances within mixtures of gases. In many practical applications the behavior of weak disturbances is of primary concern. Whereas the acoustic, or linear, limit leads to great mathematical simplications, there are situations in which nonlinear cumulative effects cannot be ignored. In this paper we wish to describe the interaction of nonlinear and the abovementioned dissipative mechanisms as they pertain to the evolution of weak plane waves in inert binary mixtures. This is done by means of the classical piston problem.

Some of the basics of acoustics for binary mixtures, especially with regards to sound absorption, were set forth by Kohler (1949). The propagation of spherical acoustic disturbances in binary mixtures, with thermal diffusion ignored, was studied by Rasmussen and Frair (1976). For binary mixtures, the study of nonlinear behavior appears to be limited to steady-state shock waves. Dyakov (1954) studied weak shocks and Sherman (1960) studied both weak and strong shocks. On the other hand, for pure gases, when mass diffusion does not play a role, combined nonlinear and dissipative effects on the evolution of finite disturbances have been the subject of investigation for some time. The classic early work is that of Lighthill (1956), and a summary of other related work dealing with Burgers' equation is given by Benton and Platzmann (1972). A more recent work is that of Halabisky and Sirovich (1973). Also, for pure gases, Shidlovsky (1975, 1977) has investigated the evolution of shock waves and other regions of nonuniformity by means of boundary-layer singular-perturbation methods.

General model equations have been proposed by Blythe (1969) and Ockendon and Spence (1969) for the evolution of waves in relaxing gases. The thrust of their work was toward inviscid flows, but their models could also deal with the viscous-type relaxation under consideration in the present work.

In this investigation, we describe how weak, one-dimensional disturbances in binary inert mixtures evolve with time. In particular, we proceed from the basic linear theory and then illustrate how the smallest nonlinear terms modify the evolutionary description. For the piston problem, the wave front is shown to be governed by Burgers' equation. The contributions of the binary-mixture diffusion mechanisms enter as an additive term in an effective diffusivity that is a combination of the Schmidt number and pressure and thermal diffusion coefficients. Besides the mass average velocity perturbation, results are obtained for the species mass-fraction perturbation, which is pertinent to binary mixtures.

2. BASIC EQUATIONS

The equations of motion for mass, species, momentum, and energy for an inert binary mixture are (with body forces neglected)

$$\frac{D\rho}{Dt} + \rho \,\mathrm{div}\, \vec{V} = 0 \tag{1}$$

$$\rho \frac{Dc}{Dt} = - \,\mathrm{div}\, \vec{i} \tag{2}$$

$$\rho \frac{D\vec{V}}{Dt} = - \,\mathrm{grad}\, p + \mathrm{div}\, \overleftrightarrow{\tau} \tag{3}$$

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \overleftrightarrow{\tau} : \overleftrightarrow{\varepsilon} - \mathrm{div}\, \vec{q} \tag{4}$$

where ρ, p, h, and $\vec{V}$ are the density, pressure, specific enthalpy, and velocity of the mixture. The mass fractions of the two inert species of the binary mixture are denoted by $c_1 = c$ and $c_2 = 1 - c$. The rate of strain tensor, $\overleftrightarrow{\varepsilon}$, and viscous stress tensor, $\overleftrightarrow{\tau}$, are given by

$$\overleftrightarrow{\varepsilon} = \frac{1}{2} \left[\mathrm{grad}\, \vec{V} + (\mathrm{grad}\, \vec{V})^t \right] \tag{5}$$

$$\overleftrightarrow{\tau} = 2\mu\overleftrightarrow{\varepsilon} + \lambda(\mathrm{div}\, \vec{V})\, \overleftrightarrow{I} \tag{6}$$

where μ and λ are the first and second coefficients of viscosity. The heat-flux vector, $\vec{q}$, and binary diffusion-flux vector, $\vec{i}$, are determined by means of kinetic theory (Hirschfelder et al., 1954) or principles of continuum mechanics (Landau and Lifshitz, 1959):

$$\vec{q} = - k \,\mathrm{grad}\, T + \left[h_2 - h_1 + M_{12} k_T \right] \vec{i} \tag{7}$$

$$\vec{i} = - \rho D_{12} \left[\mathrm{grad}\, c + k_p \,\mathrm{grad}\, (\ln p) + k_T \,\mathrm{grad}\, (\ln T) \right] \tag{8}$$

Here k is the thermal conductivity, D_{12} is the binary diffusion coefficient, and M_{12} is given by

$$M_{12} \equiv \frac{T}{k_p}\left(R_1 - R_2\right) \tag{9}$$

where T is the temperature, R_1 and R_2 are the specific gas constants for species 1 and 2, and k_p and k_T are the pressure-diffusion and thermal-diffusion coefficients, given by

$$k_p \equiv \frac{\left(R_1 - R_2\right)\left[\left(R_1 - R_2\right)c + R_2\right]c(1 - c)}{R_1 R_2} \tag{10}$$

$$k_T \equiv \alpha c(1 - c) \tag{11}$$

The thermal diffusion factor α is usually positive when $c_1 = c$ refers to the heavier molecular species, but may be slightly negative for exceptional gas pairs. For Maxwellian interaction potentials, α vanishes.

The set of equations becomes complete with the addition of thermal and caloric equations of state. For a mixture of thermally perfect gases, we have

$$p = \rho T\left[\left(R_1 - R_2\right)c + R_2\right] \quad , \quad h = \left(h_1 - h_2\right)c + h_2 \tag{12}$$

3. LINEARIZED EQUATIONS

Consider a uniform ambient state denoted by the subscript naught. We consider perturbations about this state and write

$$\left.\begin{aligned} p = p_o\left(1 + p'\right) \quad , \quad \rho = \rho_o\left(1 + \rho'\right) \quad , \quad T = T_o\left(1 + T'\right) \\ c = c_o + c' \quad , \quad \vec{v}' = \vec{V}/a_{f_o} \end{aligned}\right\} \tag{13}$$

where a_{f_o} denotes the frozen speed of sound in the ambient state,

$$a_{f_o}^2 \equiv \gamma p_o/\rho_o \quad , \quad \frac{\gamma - 1}{\gamma} \equiv \frac{\left(R_1 - R_2\right)c_o + R_2}{\left(C_{p_{1_o}} - C_{p_{2_o}}\right)c_o + C_{p_{2_o}}} \quad , \tag{14}$$

and where $C_{p_{1_o}}$ and $C_{p_{2_o}}$ are the ambient-state constant-pressure specific heats of species 1 and 2. Further, we introduce a nondimensional time, τ, and position vector, $\vec{r}$, as

$$\tau \equiv a_{f_o}^2 t/\tilde{\nu}_o \quad , \quad \vec{r} \equiv a_{f_o}\bar{\vec{r}}/\tilde{\nu}_o \tag{15}$$

such that $\nabla \equiv \partial/\partial\vec{r}$ (the barred space variables are dimensional). Characteristic Prandtl and Schmidt numbers are defined as

$$P_r \equiv \rho_o\tilde{\nu}_o C_{p_o}/k_o \quad , \quad S_c \equiv \tilde{\nu}_o/D_{12_o} \tag{16}$$

where

$$\begin{aligned} \tilde{\nu}_o &\equiv \left(2\mu_o + \lambda_o\right)/\rho_o = \left(\frac{4}{3}\mu_o + \kappa_o\right)/\rho_o \quad , \\ C_{p_o} &\equiv \left(C_{p_{10}} - C_{p_{20}}\right)c_o + C_{p_{20}} \quad , \end{aligned} \tag{17}$$

and κ_o is the bulk modulus of viscosity in the ambient medium.

We now consider flows that are irrotational (such as occur with planar, cylindrical, or spherical symmetry) and introduce a velocity potential such that $\vec{v}' = \nabla\phi$. The linearized momentum equation can thus be integrated once, and the governing linearized system of equations becomes

$$\frac{\partial\rho'}{\partial\tau} + \nabla^2\phi = 0 \quad , \quad D_s c' = k_{p_o}\nabla^2 p' + k_{T_o}\nabla^2 T' \tag{18}$$

$$p' = \gamma\left[\nabla^2\phi - \frac{\partial\phi}{\partial\tau}\right] \quad , \quad p' = \rho' + T' + N_o c' \tag{19}$$

$$D_P T' = \frac{\gamma - 1}{\gamma} P_r \frac{\partial p'}{\partial\tau} + \frac{M_o k_{T_o} P_r}{S_c}\left[\nabla^2 c' + k_{p_o}\nabla^2 p' + k_{T_o}\nabla^2 T'\right] \tag{20}$$

where

$$M_o \equiv M_{12_o}/C_{p_o} T_o = \frac{\gamma - 1}{\gamma}\frac{N_o}{k_{p_o}} \tag{21}$$

$$N_o \equiv \frac{R_1 - R_2}{\left(R_1 - R_2\right)c_o + R_2} \tag{22}$$

$$D_s \equiv S_c \frac{\partial}{\partial t} - \nabla^2 \quad , \quad D_P \equiv P_r \frac{\partial}{\partial t} - \nabla^2 \tag{23}$$

The above equations can be manipulated so that the following single seventh-order equation for ϕ is obtained:

$$\gamma\left(1 + N_o k_{p_o}\right)\nabla^6\phi_\tau + \nabla^4\left[\nabla^2\phi - \tilde{\alpha}\phi_{\tau\tau}\right]$$

$$+ \nabla^2\left[\beta_1\phi_{\tau\tau} - \beta_2\nabla^2\phi\right]_\tau + P_r S_c\left[\nabla^2\phi - \phi_{\tau\tau}\right]_{\tau\tau} = 0 \tag{24}$$

where

$$\tilde{\alpha} \equiv P_r\left(1 - N_o k_{T_o}\right) + \gamma\left(1 + S_c\right) + \gamma N_o k_{p_o} + \gamma P_r\left(k_{p_o} + k_{T_o}\right)\left(N_o + M_o k_{T_o}\right)$$

$$\beta_1 \equiv \tilde{\alpha} + S_c P_r - \gamma\left(1 + N_o k_{p_o}\right) \tag{25}$$

$$\beta_2 \equiv S_c + P_r\left(1 + M_o k_{T_o}^2\right) \quad .$$

When $k_{T_o} = 0$, the above linear equations reduce to those of Rasmussen and Frair (1976) who studied spherical waves by means of these equations.

The solution of the linearized equations for a one-dimensional piston moving into an ambient medium requires three boundary conditions at the piston surface: (1) the mass mean velocity, ϕ_x, is equal to the piston velocity, (2) the diffusion-flux normal to the piston vanishes, $\vec{i} \cdot \hat{n} = 0$, and (3) either the temperature or the normal heat flux, $\vec{q} \cdot \hat{n}$, is specified. The solution is obtained analogously to the spherical explosion problem of Rasmussen and Frair (1976). The solution can be written as a part that describes a diffused wave front and two coupled parts that describe heat and mass-diffusion boundary layers adjacent to the piston face. The heat and mass-diffusion boundary layers spread from the piston face as the square root of time whereas the position of the bulk wave front travels like the time itself. When we fix our attention on the wave front for large times, therefore, we can ignore the boundary layers since they are greatly outdistanced by the wave front. The asymptotic long-time wave behavior can be obtained by Laplace-transform techniques (Rasmussen 1975). For the velocity and mass-fraction perturbation we have for the impulsive motion of a piston, where $\phi_x(x = 0, t > 0) = U_p/a_{fo}$,

$$v'(x,\tau) \sim \frac{1}{2}\frac{U_p}{a_{fo}}\left[\operatorname{erfc}\left\{\frac{b(x-\tau)}{\sqrt{\tau}}\right\} + e^{4b^2x}\operatorname{erfc}\left\{\frac{b(x+\tau)}{\sqrt{\tau}}\right\}\right] \tag{26}$$

$$c'(x,\tau) \sim \frac{U_p\left[\gamma k_{po} + (\gamma - 1)k_{To}\right]}{a_{fo}S_c}\ \frac{bx}{\sqrt{\pi\tau^3}}\exp\left\{-\frac{b^2(x-\tau)^2}{\tau}\right\} \tag{27}$$

where

$$b \equiv \left[2\left(\frac{\beta_1 - \beta_2}{P_r S_c}\right)\right]^{-\frac{1}{2}} \tag{28}$$

The above results show that the wave front is centered at $x = \tau$ and spreads out with time. The peak of the mass-fraction perturbation at $x = \tau$ dies out like $\tau^{-\frac{1}{2}}$ and thus ultimately vanishes. If we define the thickness of the wave front as $\delta \equiv \left(\frac{U_p}{a_{fo}}\right) / |\partial v'/\partial \bar{x}|_{max}$, we obtain for large times

$$\frac{a_{fo}\delta}{\tilde{\nu}_o} = \frac{\sqrt{\pi\tau}}{b} \quad . \tag{29}$$

Thus the wave front in the linear theory spreads out like the square-root of time. It is known, however, that a constant piston speed should generate a steady-state shock front. The continual spreading of the wave front and the dying out of the mass-fraction perturbation are incorrect behaviors that arise out of the omission of the convective nonlinearities.

4. NONLINEAR INTERACTION

In order to delineate the balance between the dissipative transport terms and the nonlinear terms, let us first renormalize the variables appearing in (24). We introduce a characteristic length L and define non-dimensional time and distance as

$$\tilde{\tau} \equiv a_{f_o} t/L = \epsilon\tau \quad , \quad \vec{\tilde{r}} \equiv \vec{r}/L = \epsilon\vec{r} \quad , \tag{30}$$

where the parameter $\epsilon \equiv \tilde{\nu}_o/a_{f_o}L$ is to be regarded as small. We note that $\nabla = \epsilon\tilde{\nabla}$. In the new variables, nth-order terms are proportional to ϵ^n, and we can rewrite (24) and display the lowest two orders as

$$\left[\tilde{\nabla}^2\phi - \phi_{\tilde{\tau}\tilde{\tau}}\right]_{\tilde{\tau}\tilde{\tau}} = \frac{\epsilon}{P_r S_c}\,\tilde{\nabla}^2\left[\beta_2\tilde{\nabla}^2\phi - \beta_1\phi_{\tilde{\tau}\tilde{\tau}}\right]_{\tilde{\tau}} + 0\left(\epsilon^2\right) \tag{31}$$

Thus as the viscous (transport) effects go to zero, the classical wave operator prevails, and the first-order correction for small ϵ arises because of the next higher-order derivatives. The terms of order ϵ^2 in (31) correspond to the seventh- and six-order derivatives in (24).

Equation (31) does not account for the nonlinearities in the problem. We account for the lower-order nonlinearities by replacing the perturbation quantities in (13) by $p' = \epsilon\tilde{p}$, and so on for the other primed variables, and utilizing the normalizations (30). The full nonlinear equations then lead to the following equation for the perturbation potential:

$$\left[\tilde{\nabla}^2\phi - \phi_{\tilde{\tau}\tilde{\tau}}\right]_{\tilde{\tau}\tilde{\tau}} = \epsilon\left\{\frac{\tilde{\nabla}^2}{P_r S_c}\left[\beta_2\tilde{\nabla}^2\phi - \beta_1\phi_{\tilde{\tau}\tilde{\tau}}\right]_{\tilde{\tau}} + \left[(\gamma - 1)\phi_{\tilde{\tau}}\tilde{\nabla}^2\phi + \frac{\partial}{\partial\tilde{\tau}}\left(\tilde{\nabla}\phi\right)^2\right]_{\tilde{\tau}\tilde{\tau}}\right\} + 0\left(\epsilon^2\right) \tag{32}$$

In Equation (32) the second-degree nonlinear terms appear to order ε, as shown, and the third and higher degree nonlinearities appear to order ε^2. The nonlinear terms of order ε in (32) are the same as occur in inviscid potential theory. Thus, the first-order corrections to inviscid acoustics are the sum of the first-order linear viscous correction and the first-order nonlinear inviscid correction, which of course might have been anticipated a priori. Our goal now is to examine Equation (32) with the terms of order ε^2 ignored.

5. PISTON PROBLEM

The boundary condition for the one-dimensional impulsive motion of a piston is $v'(\tilde{x} = 0, \tilde{\tau}) = (U_p/a_{fo}) f(\tilde{\tau})$, where U_p is the characteristic speed of the piston and $f(\tilde{\tau})$ is of order unity. The characteristic length is taken to be $L = \tilde{\nu}_o/U_p$, and it follows from (30) that $\tilde{\tau} = U_p a_{f_o} t/\tilde{\nu}_o$, $\tilde{x} = U_p \bar{x}/\tilde{\nu}_o$, and $\varepsilon = U_p/a_{f_o}$. The small parameter ε is thus the frozen Mach number of the piston, and we note that $v' = \varepsilon\phi_{\tilde{x}}$.

A series solution of Equation (32) by a straight-forward expansion in powers of ε will lead to secular behavior such that the first-order correction becomes as large as the zeroth-order term when $\tilde{x} = O\left(\varepsilon^{-1}\right)$. We account for this behavior by introducing new variables defined as

$$\xi = \tilde{x} - \tilde{\tau} \quad , \quad X = \varepsilon\tilde{x} \quad . \tag{33}$$

We further expand ϕ in a series of the form

$$\phi(\tilde{x},\tilde{\tau};\varepsilon) = \phi_o(\xi,X) + \varepsilon\phi_1(\xi,X) + \ldots \tag{34}$$

Expressions (33) and (34) applied to Equation (32) lead to the following equation for $\phi_o(\xi,X)$, which has been integrated twice with respect to ξ and the functions of integration set equal to zero:

$$\phi_{o_{X\xi}} + \frac{\gamma + 1}{2}\phi_{o_\xi}\phi_{o_{\xi\xi}} = \frac{\beta_1 - \beta_2}{2P_r S_c}\phi_{o_{\xi\xi\xi}} \quad . \tag{35}$$

We now note that $U_o \equiv \phi_{o_\xi}$ is the lowest order velocity contribution, that is,

$$v' = \varepsilon U_o + O\left(\varepsilon^2\right) \quad . \tag{36}$$

If we make the further definitions

$$\eta \equiv \frac{\gamma + 1}{2} X = \frac{\gamma + 1}{2} \varepsilon \tilde{x} \quad ,$$

$$\nu^* \equiv \frac{\beta_1 - \beta_2}{(\gamma + 1) P_r S_c} \quad , \tag{37}$$

then Equation (35) can be expressed in the standard form for Burgers' equation:

$$U_{o_\eta} + U_o U_{o_\xi} = \nu^* U_{o_{\xi\xi}} \tag{38}$$

The nonlinear terms of order ε in (32) lead to the convective nonlinear term $U_o U_{o_\xi}$ in (38), and the dissipative terms of order ε in (32) lead to the diffusion term $U_{o_{\xi\xi}}$ in (38). The origin of the diffusivity factor ν^* can be seen by identifying the factors β_1 and β_2 in the original linear Equation (24). To lowest order, the other perturbation variables are found to be

$$\begin{aligned}
p' &= \varepsilon \gamma U_o + O\left(\varepsilon^2\right) \\
\rho' &= \varepsilon U_o + O\left(\varepsilon^2\right) \\
T' &= \varepsilon(\gamma - 1) U_o + O\left(\varepsilon^2\right) \\
c' &= -\frac{\varepsilon^2}{S_c}\left[\gamma k_{P_o} + (\gamma - 1) k_{T_o}\right] U_{o_\xi} + O\left(\varepsilon^3\right)
\end{aligned} \tag{39}$$

5.1 Impulsive Compression

For impulsive motion of the piston into the fluid, the function $U_o(\xi,\eta)$ must satisfy the boundary condition $U_o(\xi,0) = H(-\xi)$, where $H(\xi)$ is the Heaviside unit step function. The solution to (38) in this case is derived in Chapter 4 of Whitham (1974) and is

$$U_o(\xi^*,\eta^*) = \frac{\operatorname{erfc}\left\{\dfrac{\xi^* - \eta^*}{\sqrt{4\eta^*}}\right\}}{\operatorname{erfc}\left\{\dfrac{\xi^* - \eta^*}{\sqrt{4\eta^*}}\right\} + \operatorname{erfc}\left\{\dfrac{-\xi^*}{\sqrt{4\eta^*}}\right\} \exp\left\{\frac{1}{2}\left(\xi^* - \frac{1}{2}\eta^*\right)\right\}} \tag{40}$$

where $\xi^* = \xi/\nu^*$ and $\eta^* = \eta/\nu^*$. The binary-mixture contribution appears in the parameter ν^* and hence amounts to a stretching of the coordinates. The derivative $U_{o_{\xi^*}}$ is given by

$$U_{o_{\xi^*}}(\xi^*,\eta^*) = -\frac{\left[\dfrac{1}{\sqrt{\pi\eta^*}}\, e^{-(\xi^*-\eta^*)^2/4\eta^*} + \dfrac{1}{2}\, U_o \operatorname{erfc}\left\{\dfrac{-\xi^*}{\sqrt{4\eta^*}}\right\} e^{\frac{1}{2}\left(\xi^*-\frac{1}{2}\eta^*\right)}\right]}{\operatorname{erfc}\left\{\dfrac{\xi^* - \eta^*}{\sqrt{4\eta^*}}\right\} + \operatorname{erfc}\left\{\dfrac{-\xi^*}{\sqrt{4\eta^*}}\right\} \exp\left\{\frac{1}{2}\left(\xi^* - \frac{1}{2}\eta^*\right)\right\}} \tag{41}$$

The function $U_o(\xi^*,\eta^*) = U_o(x^* - \tau^*, \varepsilon^* x^*)$ is plotted in Figure 1 as a function of $x^* \equiv \tilde{x}/\nu^*$ for various times $\tau^* \equiv \tilde{\tau}/\nu^*$ and for $\varepsilon^* \equiv (\gamma + 1)\varepsilon/2 = 0.1$, where $\eta^* \equiv \varepsilon^* x^*$. In the early stages, evolution of the wave front is dominated by diffusion and hence flattens until a balance is achieved by the steepening effects of nonlinear convection. The ξ-derivative of the velocity, $U_{o_{\xi^*}}$, which is a measure of the slope of the wave front for small ε, is shown in Figure 2. This figure indicates how the maximum slope decreases and the wave front broadens as time increases. The evolution is dramatic for $\tau^* \leq 14$ shown in Figures 1 and 2, but for larger times changes occur more slowly. The maximum value of $-U_{o_{\xi^*}}$ approaches 0.125 as $\tau^* \to \infty$, but at $\tau^* = 140$ the maximum value of

$-U_{o_{\xi^*}}$ has decreased to only 0.141, for $\varepsilon^* = 0.1$. Ultimately, diffusion and nonlinear convection balance and a steady-state shape is attained with $-U_{o_{\xi^*}}$ having a symmetric shape about a maximum value of 1/8.

The function $U_{o_{\xi^*}}$ also describes the mass-fraction perturbation, c', as seen from expressions (39). For the heavier species $\left[\gamma k_{p_o} + (\gamma - 1) k_{T_o}\right]$ is negative, and hence c' is negative in the wave front. Thus, the mass fraction of the heavy species decreases in the wave front, and the mass fraction of the light species increases. The maximum value of these species perturbations is skewed toward the piston side of the wave front. As time increases, the distribution broadens and ultimately becomes symmetric, achieving a steady-state form.

5.2 Thickness of Steady-State Wave Front

As $\tau^* \to \infty$, the wave front evolves into a steady-state form. The thickness δ of the wave front can be defined as

$$\delta = \frac{1}{\left|\frac{\partial U_o}{\partial \bar{x}}\right|_{max}} \tag{42}$$

For $\tau^* \to \infty$ and $\varepsilon \to 0$, we obtain

$$\frac{U_p \delta}{\tilde{\nu}_o} = 8\nu^* = \frac{8\left(\beta_1 - \beta_2\right)}{(\gamma + 1) P_r S_c}$$

$$= \frac{8}{\gamma + 1}\left[\frac{\gamma - 1 + P_r}{P_r} + \frac{\gamma N_o}{S_c k_{p_o}}\left(k_{p_o} + \frac{\gamma - 1}{\gamma} k_{T_o}\right)^2\right] \tag{43}$$

Note here that P_r and S_c are defined in terms of the reduced viscosity $\tilde{\nu}_o \equiv (2\mu_o + \lambda_o)/\rho_o$. Formula (43) is identical to previous results obtained for weak steady shocks, such as by Sherman (1960) when rewritten in our notation, which is a partial check on the correctness of our present analysis. Note that the parameter ν^* is related to the parameter b in the linear theory, Equation (28), by the relation $2(\gamma + 1)b^2\nu^* = 1$. Thus linear theory yields the correct combination of terms arising from viscous, thermal, and mass-diffusion dissipation that contribute to the breadth of the wave front, but raised to the wrong power.

The combination $N_o k_{p_o}$ can be written as

$$N_o k_{p_o} \equiv \frac{m_1}{m_2}\left(1 - \frac{m_2}{m_1}\right)^2 c_o\left(1 - c_o\right) \tag{44}$$

where m_1 and m_2 are the molecular masses of the species. Since this combination is always positive, it can be seen from (43) that pressure and thermal diffusion always tend to broaden the wave front, the more so the greater the disparity in masses of the species. As pointed out by Sherman (1960), k_{p_o} and k_{T_o} are usually of opposite signs and tend to counteract one another.

5.3 Oscillating Piston

For the oscillating piston problem, the boundary condition is $v'(0,\tilde{\tau}) = \varepsilon \sin \tilde{\omega}\tilde{\tau}$, where $\varepsilon \equiv U_p/a_{f_o}$ as before, and $\tilde{\omega}$ is the nondimensional frequency defined in terms of the physical frequency, ω, by $\tilde{\omega} \equiv \tilde{\nu}_o\omega/a_{f_o}U_p$. The initial condition on the Burgers' function is thus $U_o(\xi,\eta = 0) = -\sin \tilde{\omega}\xi$. By means of Fourier series expansions, we find the solution of (38) to be (Benton and Platzmann, 1972)

$$U_o(\xi,\eta) = \frac{4\tilde{\omega}\nu^*\sum_{n=1}^{\infty} nA_n \exp\left\{-n^2\tilde{\omega}^2\nu^*\eta\right\}\sin(n\tilde{\omega}\xi)}{1 + 2\sum_{n=1}^{\infty} A_n \exp\left\{-n^2\tilde{\omega}^2\nu^*\eta\right\}\cos(n\tilde{\omega}\xi)}, \tag{45}$$

where

$$A_n(\tilde{\omega}\nu^*) \equiv (-1)^n I_n\left(\frac{1}{2\tilde{\omega}\nu^*}\right) \Big/ I_o\left(\frac{1}{2\tilde{\omega}\nu^*}\right) \quad , \tag{46}$$

and I_n is the modified Bessel function of the first kind of order n. The appearance of the factor ν^* delineates the binary-mixture, or mass diffusion, effects.

When $\tilde{\omega}^2\nu^*\eta >> 1$, the term n = 1 dominates the series, and thus in the limit of ultimate decay expression (45) becomes

$$U_o(\xi,\eta) \sim 4\tilde{\omega}\nu^* A_1(\tilde{\omega}\nu^*)\exp(-\tilde{\omega}^2\nu^*\eta)\sin\tilde{\omega}\xi \tag{47}$$

Aside from the exponential damping, the amplitude of the disturbance in the far field is a function of the frequency of oscillation. This is a nonlinear effect and is akin somewhat to the dependency of the amplitude on frequency in the nonlinear oscillations of spring-mass systems. The amplitude function $4\tilde{\omega}\nu^* A_1(\tilde{\omega}\nu^*)$ is plotted as a function of $\tilde{\omega}\nu^*$ in Figure 3. The function is a maximum of unity when $\tilde{\omega}\nu^* \to \infty$, and decreases as $\tilde{\omega}\nu^*$ becomes smaller. When $\tilde{\omega}\nu^* >> 1$, expression (47) approaches the form

$$U_o(\xi,\eta) \sim -\exp(-\tilde{\omega}^2\nu^*\eta)\sin\tilde{\omega}\xi \tag{48}$$

This is the solution that is obtained from the linear theory, described by Equation (27), in the limit $\varepsilon\tilde{\omega}\nu^* << 1$. Thus, the linear theory according to Equation (27) is valid in the final stage of decay when $\tilde{\omega}\nu^* >> 1$ and $\varepsilon << (\tilde{\omega}\nu^*)^{-1}$. Except for this special limit, nonlinear effects always make a contribution, even in the final stages of decay. Even though small, the nonlinear effects are cumulative. At high frequencies the steepening of the compression part of the wave tends to cancel the flattening of the expansion part, whereas for low frequencies the compressions and expansions are less closely spaced and hence lead to a nonvanishing cumulative effect in the far field.

The evolution of the wave disturbances just before final decay can be examined by keeping the term n = 1 in the denominator as well as the numerator of (45) and ignoring all terms for $n \geq 2$. Then we have, for $\tilde{\omega}^2\nu^*\eta \gtrsim 1$,

$$U_o(\xi,\eta) \sim \frac{-\left(\frac{4\tilde{\omega}\nu^* I_1}{I_o}\right) \exp(-\tilde{\omega}^2\nu^*\eta) \sin \tilde{\omega}\xi}{1 - \left(\frac{2I_1}{I_o}\right) \exp(-\tilde{\omega}^2\nu^*\eta) \cos \tilde{\omega}\xi} \tag{49}$$

The wave forms produced by expression (49) are shown in Figure 4. The curves are shown for $\tilde{\omega}^2\nu^*\eta = 1$ and $\tilde{\omega}\nu^* = \infty$, 0.5, 0.25. The linear sine wave is represented by $\tilde{\omega}\nu^* = \infty$. The values $\tilde{\omega}\nu^* = 0.5$ and 0.25 represent the effects of the nonlinearities, which yield wave forms that are flatter in the middle and steeper on the ends. As $\eta \to \infty$, for $\tilde{\omega}\nu^*$ fixed, all the curves evolve toward the sine-wave shape.

The other perturbation variables are also given by expressions (39), the pressure, density, and temperature being proportional to the velocity. The mass-fraction perturbation, c', is proportional to the ξ-derivative of U_o. For the behavior approaching ultimate decay, $\tilde{\omega}^2\nu^*\eta \gtrsim 1$, the derivative of (49) yields a valid description:

$$U_{o_{\xi^*}}(\xi,\eta) \sim -\frac{\left(\frac{2\tilde{\omega}\nu^* I_1}{I_o}\right) \exp\left(-\tilde{\omega}^2\nu^*\eta\right) \left[2\tilde{\omega}\nu^* \cos \tilde{\omega}\xi + U_o \sin \tilde{\omega}\xi\right]}{1 - \frac{2I_1}{I_o} \exp\left(-\tilde{\omega}^2\nu^*\eta\right) \cos \tilde{\omega}\xi} \tag{50}$$

The behavior of expression (50), and thus of c', is shown in Figure 5 for $\tilde{\omega}^2 \nu^* \eta = 1$ and $\tilde{\omega}\nu^* = \infty$, 0.5, and 0.25. The limiting value $\tilde{\omega}\nu^* = \infty$ yields a damped cosine wave that corresponds to the associated linear theory. The perturbation c' is thus 90 degrees out of phase with U_o. When $\tilde{\omega}\nu^*$ is finite, the magnitude of $U_{o_{\xi^*}}$, and thus c', is larger in the compression part of the wave than it is in the expansion part. This is a nonlinear effect. If c' is identified with the heavier species, such that $\{\gamma k_{p_o} + (\gamma - 1) k_{T_o}\}$ is negative, then we find that c' is negative in the compression part of the wave and positive with a smaller magnitude in the expansion part of the wave. It also can be seen that more time is spent in the expansion part of the wave than in the compression part. As $\eta \to \infty$, for $\tilde{\omega}\nu^*$ fixed, all the curves tend toward the cosine-wave shape.

6. CONCLUDING REMARKS

The interaction of nonlinear and dissipative mechanisms for weak disturbances in inert binary mixtures has been investigated. As for pure gases, the basic description of one-dimensional problems is by means of Burgers' equation. The effects of the binary mixture are delineated by the effective diffusion coefficient $\tilde{\nu}^*$ which contains the effects of Schmidt number and pressure and thermal diffusion coefficients associated with binary mixtures as well as the Prandtl number and ratio of specific heats associated also with pure gases. The binary-mixture mechanisms tend to broaden the wave fronts and to modify the amplitude-frequency relation and damping rate of oscillatory disturbances. The mass-fraction perturbation is skewed toward the piston side of evolving shock fronts, but becomes symmetrical as the steady-state is approached. For oscillatory motions, the magnitude of the mass-fraction perturbation is larger in the compression part of the wave and of smaller magnitude and opposite sign in the expansion part of the wave.

The nonzero mass-fraction perturbations suggest that pressure and thermal diffusion may have significant effects in chemically reacting mixtures where nonequilibrium effects play essential roles. Research towards these ends may yield significant results for the behavior of detonation waves and combustion noise.

Perturbation methods applied to higher-order forms of the nonlinear Equation (32) may yield more general model equations than Burgers' equation. Also spherical and cylindrical disturbances might be studied fruitfully by this approach. These are areas for future research.

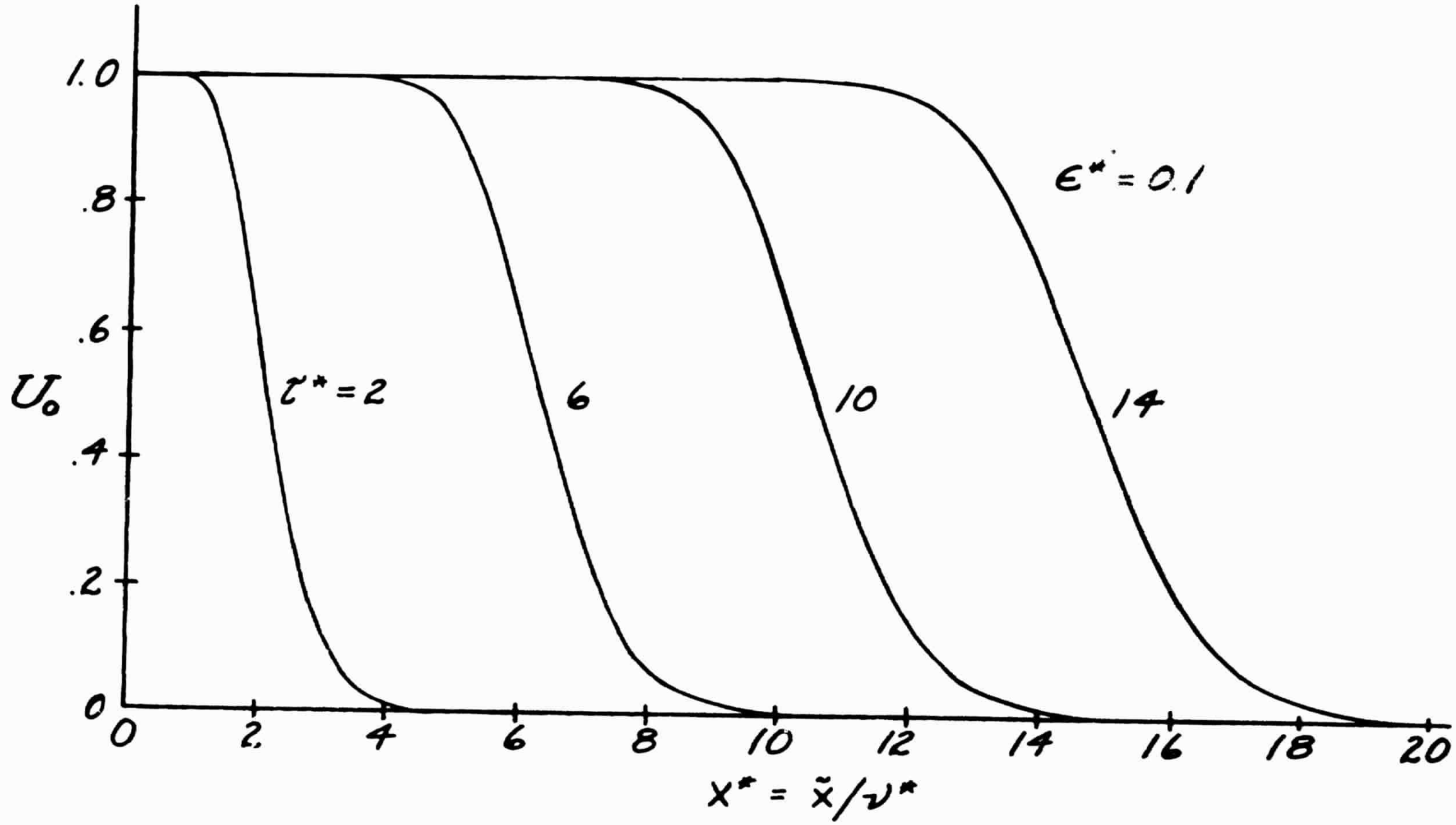

Figure 1. Evolution of Velocity Profile

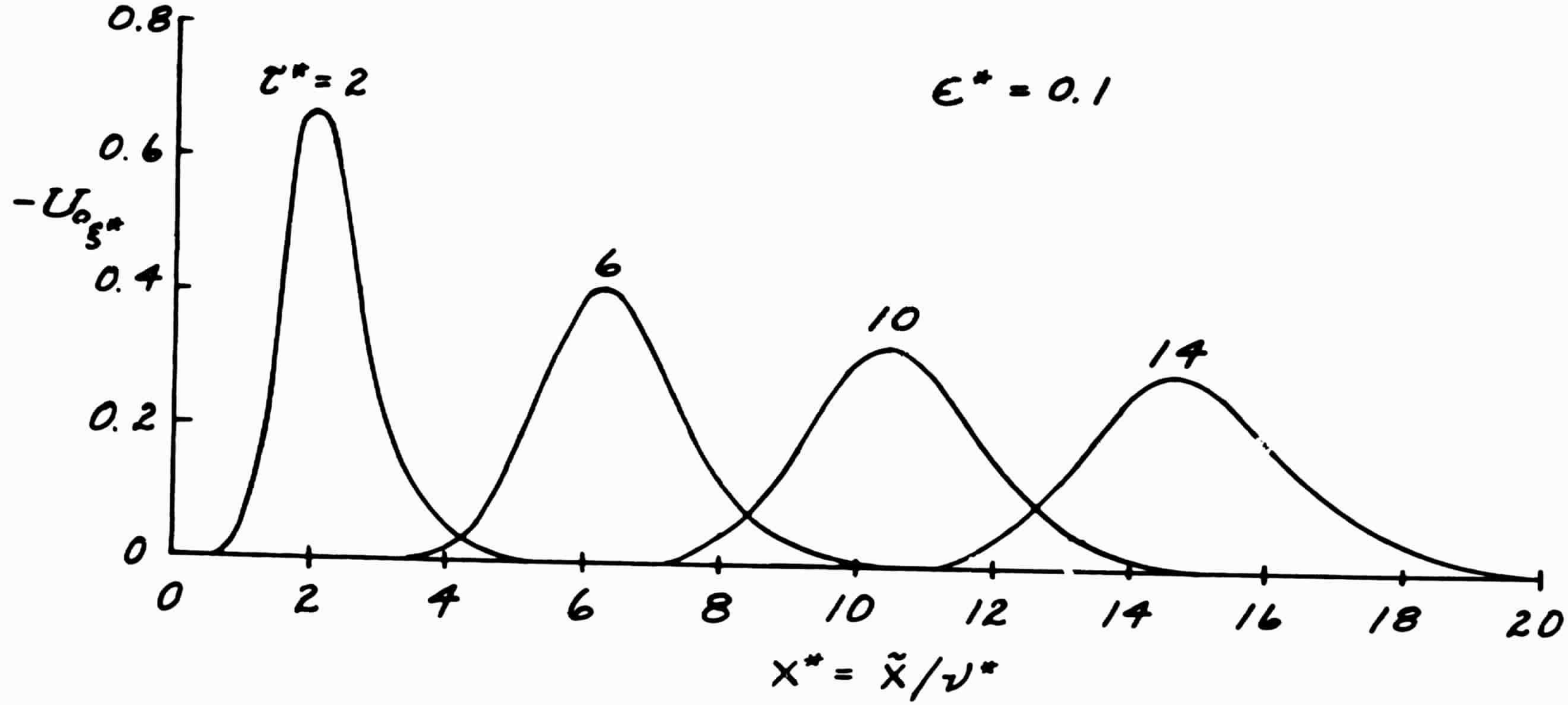

Figure 2. Evolution of Mass-Fraction Perturbation

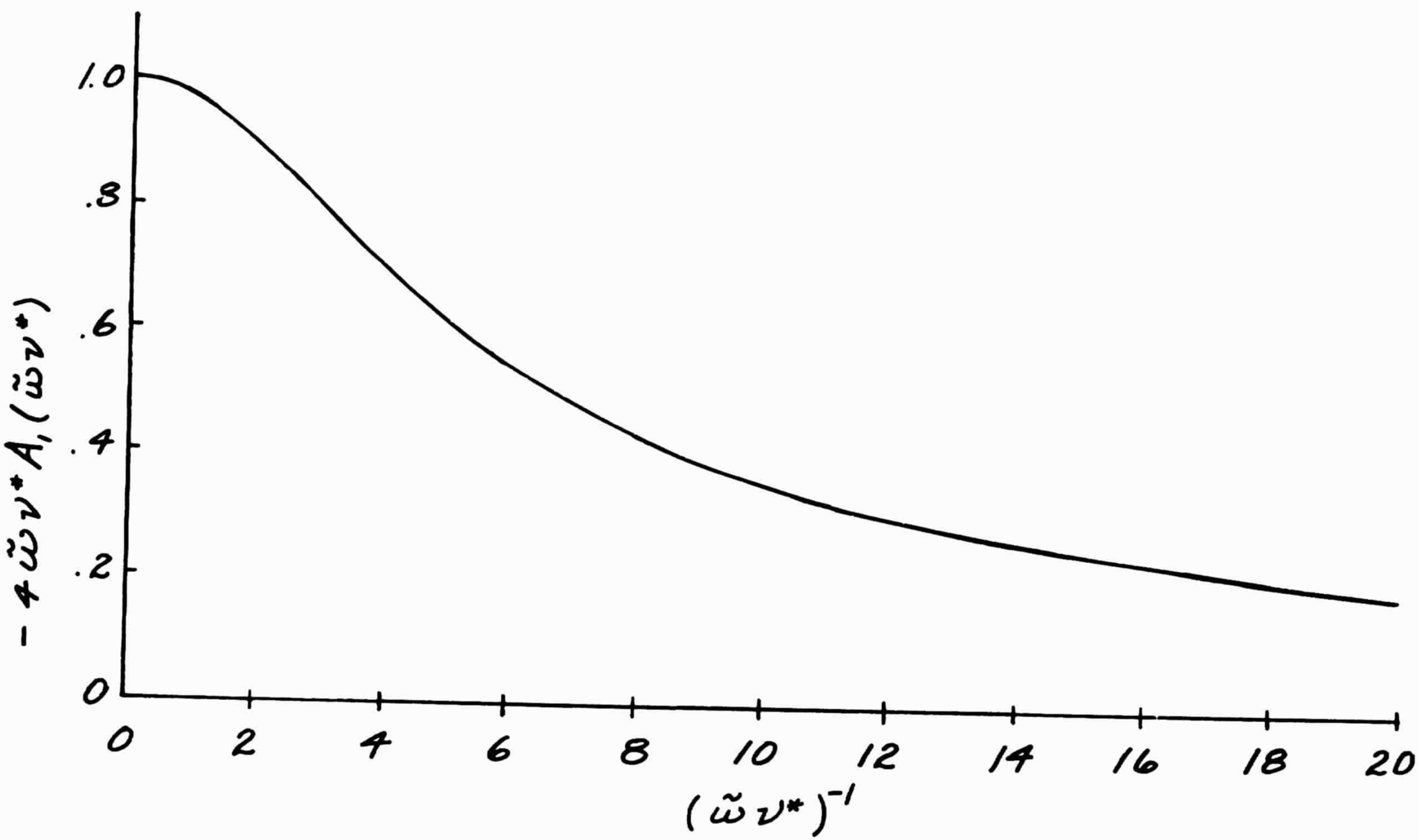

Figure 3. Variation of Amplitude Function with Frequency

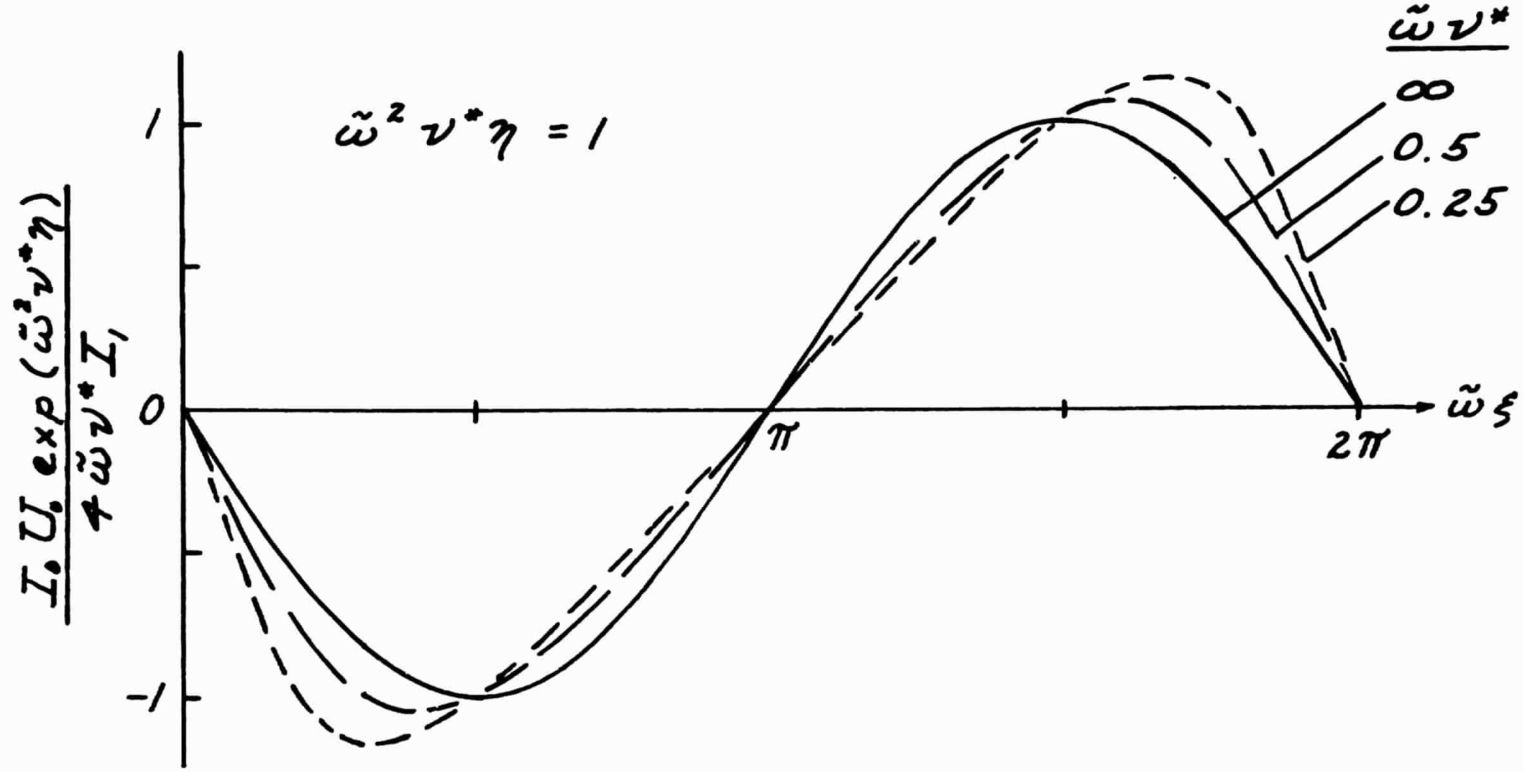

Figure 4. Velocity Wave Forms for Final Decay

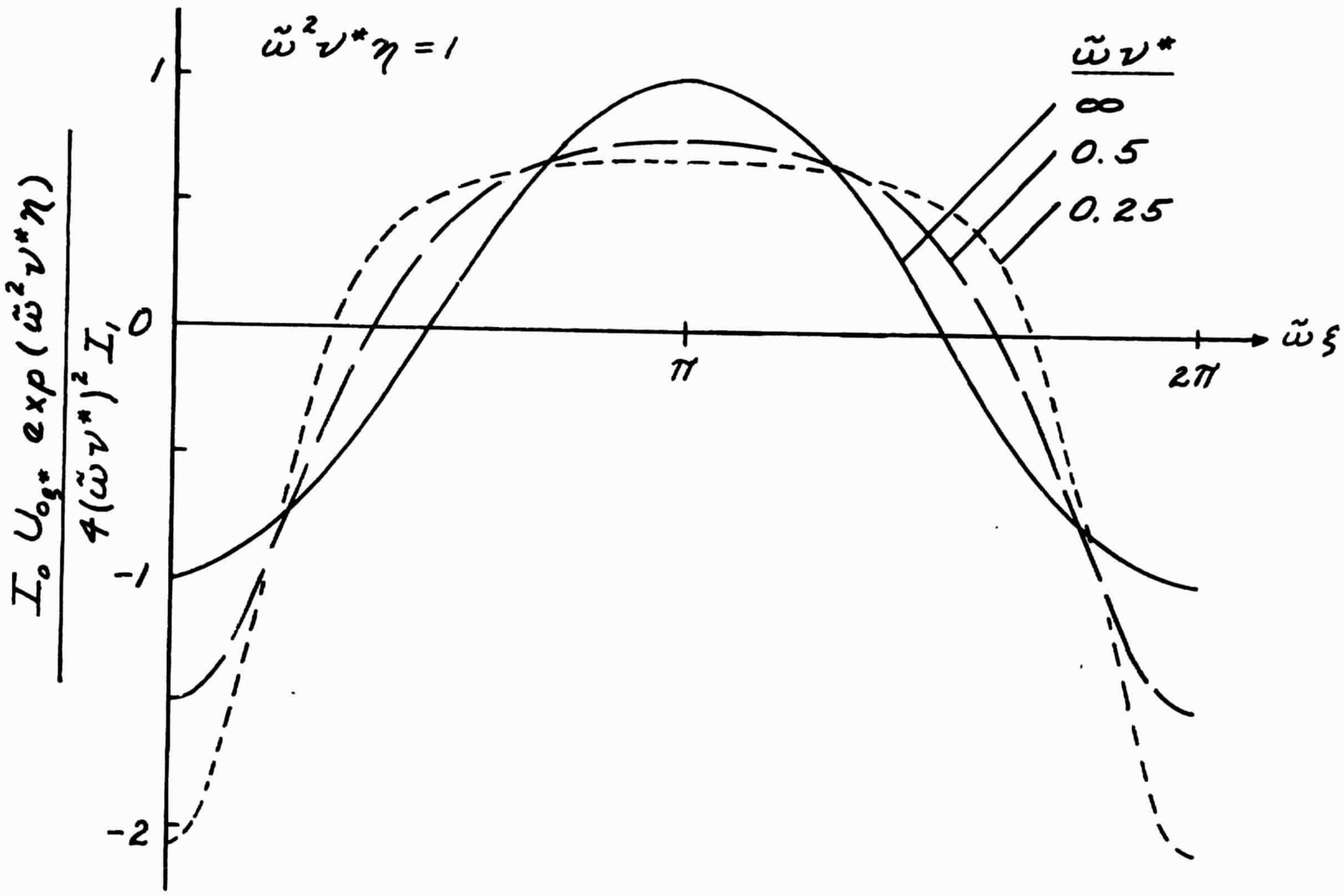

Figure 5. Mass-Fraction Perturbation Wave Forms for Final Decay

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