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A General Time Element Using Cartesian Coordinates

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Eccentric Orbit Integration

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SHUTTLE PROGRAM

A GENERAL TIME ELEMENT USING CARTESIAN COORDINATES ECCENTRIC ORBIT INTEGRATION

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TABLES

1.0 INTRODUCTION

The in-track error resulting in computing elliptic orbits with Cartesian coordinates is reduced by several orders of magnitude with the use of a time element. A general time element, to be used with any arbitrary independent variable, is proposed.

Taking the example of a transfer orbit for a geosynchronous mission, a comparison with the eccentric, true, and elliptic anomaly as the independent variable is presented. The elliptic anomaly (a new anomaly intermediate between the eccentric and the true anomaly) is shown to perform significantly better than the other classical anomalies.

2.0 EFFICIENT INTEGRATION OF ECCENTRIC ORBITS

This document shall denote, by eccentric orbit, an elliptic orbit of eccentricity $e > 0.1$. An orbit is highly eccentric if its eccentricity $e > 0.7$. The transfer orbit linking a near-Earth, near-circular parking orbit to a near-circular geosynchronous orbit is an example of a highly eccentric orbit.

Efficient integration of an orbit is defined as the numerical integration of an orbit with a given accuracy and the minimum number of steps per revolution.

In a preceding document (ref. 1) dedicated to the numerical integration of highly eccentric orbits, the following observations were made:

a. An eccentric orbit should not be integrated with equal stepsizes in time; i.e., time stepsizes should be small near pericenter and large near apocenter.

b. An integration method with automatic stepsize control is adequate, but inefficient, for the integration of eccentric orbits.

c. Better efficiency is obtained by using an independent variable other than time, by providing an analytical stepsize regulation, and by choosing a fixed-step integration method.

d. For large stepsizes, the importance of a proper formulation of the differential equations of motion (DEM) increases.

e. For a certain range of stepsizes, the straightforward formulation of DEM in terms of Cartesian coordinates, together with the use of a time element, compares favorably with more elaborate transformations of DEM.

Automatic stepsize integration methods are based on the consideration of an estimation of the local truncation error per step. Such a method does not provide a stepsize distribution along the orbit as regularly and efficiently as with the help of an analytical stepsize regulation. In addition, by having equal steps in the independent variable, a highly efficient multistep integration method can be used.

Observation (d) is trivial for very large stepsizes. With one step per revolution, for instance, the integration of Cartesian coordinates produces nonvalid results while the integration of a set of orbital elements correctly reproduces the two-body part of the motion.

For reasonably small stepsizes ($\geq$ 50 steps/rev), and for moderate propagation time (<50 revolutions), a formulation of the DEM in terms of Cartesian coordinates is adequate and gives fairly accurate results concerning the orbital shape. However, depending on the physical time, the in-track error can grow outside any limit. This is because of the well-known instability of the DEM direct formulation of the two-body problem (refs. 2 and 3).

If Cartesian coordinates are to be used, a special treatment of the in-track error is required. This is achieved by the introduction of a so-called time element.

A time element, to be used with Cartesian coordinates, has been proposed only in the case of the eccentric anomaly as the independent variable (refs. 2 and 4). As shown in the following sections, the eccentric anomaly does not lead to the most efficient analytical stepsize regulation. The purpose of this document is to propose a general time element, which is valid for any type of anomaly.

From a mathematical standpoint, there is little doubt that a formulation of the DEM in terms of orbital elements is always preferred to Cartesian coordinates. From a practical standpoint, however, it is often observed that the analyst who prepares the software for space missions is hesitant to use variables different from Cartesian coordinates. This is because the observations are physical lengths and velocities directly expressed in Cartesian coordinates; therefore, a valuable contribution is made if the advantage of using orbital elements can be preserved, to a certain extent, in a Cartesian coordinates formulation. This is the primary concern in a time element search.

The time element development in arbitrary independent variable terms was made possible by adopting the Hamiltonian mechanics framework in the extended phase space. Section 4.0 outlines the derivations and results. By combining the time transformation to be used with Cartesian coordinates (sec. 3.0) with the general results of section 4.0, a general time element will be derived in section 5.0.

Finally, a numerical comparison in section 6.0 will show the time element benefit and the outstanding performances of the elliptic anomaly as the independent variable.

3.0 TIME TRANSFORMATIONS

An analytical stepsize regulation is achieved by a change in the independent variable (ref. 5)

$$dt = f \, d\tau \tag{3.1}$$

where t is the physical time, τ is the new independent variable (sometimes called fictitious time), and f is a function of the state. Equation (3.1) is called a time transformation. The function f is chosen to be of the form

$$f = c_\alpha \, r^\alpha \tag{3.2}$$

where r is the radius vector, α is a real number, and c_α is a parameter.

Table I indicates the four classical choices for a new independent variable, which corresponds to the four classical anomalies (mean, eccentric, true, and elliptic). The definition of the parameter c_α involves the semimajor axis a, the universal gravitational constant times the mass of the central body μ, the eccentricity e, and the complete elliptic integral of the first kind $K(e)$. The development of the formula can be found in reference 5 or, following a more general point of view, in reference 6.

TABLE I.- THE FOUR CLASSICAL TIME TRANSFORMATIONS, $f = dt/d\tau = c_\alpha r^\alpha$

α	c_α	Name of angular variable
0	$\sqrt{a^3/\mu}$	Mean anomaly
1	$\sqrt{a/\mu}$	Eccentric anomaly
2	$\left[\mu a(1-e^2)\right]^{-1/2}$	True anomaly
3/2	$\dfrac{2K(e)}{\pi\sqrt{\mu(1+e)}}$	Elliptic anomaly

The case of the elliptic anomaly (a particular choice of an intermediate anomaly, intermediate between the eccentric and the true anomaly) is emphasized here because the elliptic anomaly is, from a mathematical standpoint, the natural way to generalize the circular polar angle to the ellipse.

4.0 CANONICAL ELEMENTS IN TERMS OF AN ARBITRARY INDEPENDENT VARIABLE

In reference 5, a set of canonical orbital elements of the Delaunay type in terms of an arbitrary independent variable τ is proposed. The equations of motion in the extended phase space are

$$\frac{d\underline{b}}{d\tau} = \begin{pmatrix} \mu/(2L)^{3/2} \\ 1 \\ 0 \\ 0 \end{pmatrix} + V\frac{\partial f}{\partial \underline{B}} + f\left[\frac{\partial \underline{x}}{\partial \underline{B}}\right]^T\left(\frac{\partial V}{\partial \underline{x}} - \underline{F}\right) \tag{4.1}$$

$$\frac{d\underline{B}}{d\tau} = -V\frac{\partial f}{\partial \underline{b}} - f\left[\frac{\partial \underline{x}}{\partial \underline{b}}\right]^T\left(\frac{\partial V}{\partial \underline{x}} - \underline{P}\right) \tag{4.2}$$

where

$$\underline{b}^T = (\ell, \psi, g, h)$$

is the quadrivector of canonical elements, and

$$\underline{B}^T = (L, \Psi, G, H)$$

the quadrivector of the conjugate elements:

ℓ = time element	L = total orbital energy
ψ = anomaly	Ψ = conjugate of the anomaly
g = argument of pericenter	G = total angular momentum
h = ascending node	H = component of angular momentum perpendicular to the equator

V is a perturbative gravitational potential, and $\underline{P}$ is a trivector containing all perturbative forces that are not derivable from a potential. The partial derivative of a scalar with respect to a vector is considered to be a column vector.

The Hamiltonian F

$$F \equiv \Psi - \mu/(2L)^{1/2} + fV = 0 \tag{4.3}$$

vanishes in the phase space, where f is the time transformation (eq. (2.2)).

The physical time t is obtained via the time element ℓ and other canonical variables through the time element equation

$$t = \ell + \frac{\Psi}{2L}(u - e \sin u - \psi) \tag{4.4}$$

where u is the eccentric anomaly.

Close attention should now be given to the equations of the pair (ℓ, L) of conjugate variables

$$\ell' = \mu/(2L)^{3/2}V \frac{\partial f}{\partial L} + f\frac{\partial \underline{r}}{\partial L}\left(\frac{\partial V}{\partial \underline{r}} - \underline{P}\right) - \frac{\partial t}{\partial L}L' \tag{4.5}$$

$$L' = -f\left(\frac{\partial V}{\partial t} + \underline{P}.\underline{v}\right) \tag{4.6}$$

where ' is an abreviation for the derivative with respect to τ. The partial derivatives of f, $\underline{r}$, and t relative to L are

$$\frac{\partial f}{\partial L} = cf/(2L)$$

where

$c = -3$, true and intermediate anomaly,

$c = -2$, eccentric anomaly;

$$\frac{\partial \underline{r}}{\partial L} = -\underline{r}/(2L)$$

$$\frac{\partial t}{\partial L} = -\frac{\Psi}{2L^2}(u - e \sin u - \psi).$$

5.0 A GENERAL TIME ELEMENT

The sum of two terms is recognized in equation (4.4) as

$$t = \frac{\Psi}{2L}(u - e \sin u) + \left(\ell - \frac{\Psi}{2L}\psi\right)$$

The first term is proportional to the mean anomaly and the second term is the time of pericenter passage. This last term is chosen as a new dependent variable q, where

$$q = \ell - \frac{\Psi}{2L}\psi \tag{5.1}$$

The equation for t now reads

$$t = \frac{\Psi}{2L}(u - e \sin u) + q \tag{5.2}$$

By differentiation of equation (5.1), the differential equation for q is obtained by

$$q' = \ell' - \frac{\psi}{2L}\Psi' + \frac{\Psi}{2L^2}\psi L' - \frac{\Psi}{2L}\psi' \tag{5.3}$$

By assuming that all perturbations, whether conservative or not, are included into the vector $\underline{P}$, $V = 0$, and by using equation (4.3) the expression for Ψ is

$$\Psi = \mu/(2L)^{1/2} \tag{5.4}$$

and Ψ'

$$\Psi' = -\mu L'/(2L)^{3/2}$$

Equation (5.3) now becomes

$$q' = \ell' + 3\mu\psi L'/(2L)^{5/2} - \mu\psi'/(2L)^{3/2}$$

The equation for ψ' is needed. According to equation (4.1) and because $V = 0$,

$$\psi' = 1 - f\frac{\partial \underline{r}}{\partial \Psi}\underline{P} - \frac{\partial t}{\partial \Psi}L'$$

By using two-body relations, after some algebra

$$\frac{\partial \underline{r}}{\partial \Psi} = -\frac{\cos u}{\mu e}(2L)^{1/2}\underline{r}$$

and

$$\frac{\partial t}{\partial \Psi} = (u - e \sin u)/(2L)$$

through (5.2).

By taking equation (4.5) into account, q' becomes

$$q' = f(1 - \cos u/e)\underline{P}\cdot\underline{r}/(2L) + \mu/(2L)^{5/2} \left\{3u + \sin u \left[\cos u - 2(e + 1/e)\right]\right\} L' \qquad (5.5)$$

The elimination of the variable ψ, which was the purpose of defining the new time element q, is apparent.

All quantities in equation (5.5) can be estimated in terms of the Cartesian state ($\underline{r}$, $\underline{v}$) with the help of the two-body relations

$$\underline{r}\cdot\underline{v} = \mu e \sin u/(2L)^{1/2}$$

and

$$r = \mu(1 - e \cos u)/(2L)$$

Therefore, equation (5.5), together with equation (4.6), are ideally suited for supplementing the Cartesian equations of motion with a general time equation.

A few remarks about equation (5.5) can be formulated as follows:

a. Equation (5.5) is valid for any type of independent variable.

b. Equation (5.5) is singular for vanishing eccentricity. This is a result of the development from the Delaunay elements, which are singular for zero eccentricity. However, this restriction is irrelevant because near-circular orbits do not require a time transformation.

c. A secular term is present in the coefficient of L'. This term may cause a reduction of accuracy for long-term orbit propagation; however, in this case, using a formulation with Cartesian coordinates is not recommended. Instead, a set of elements should be used.

d. When the independent variable is the elliptic anomaly, the only elliptical function to be computed for estimating the right-hand side of the differential equations is $K(e)$, which is a complete elliptic integral of the first kind appearing in the time transformation evaluation f. As discussed in reference 6, a fast algorithm is available for this integral so that the corresponding overhead is of negligible effect on the computing time.

To be complete, the Cartesian equations of motion, to be integrated with equations (4.6) and (5.5), are recalled as

$$\underline{r}' = f\underline{v}$$

$$\underline{v}' = f(\underline{P} - \mu\underline{r}/r^3)$$

6.0 NUMERICAL COMPARISON

In order to have a preliminary assessment of the efficiency of the time element proposed in the preceding section, a numerical comparison is made. As a sample problem, a standard transfer orbit between a near-Earth parking orbit and a geosynchronous orbit is chosen. Orbital elements of the transfer orbit are as follows:

semimajor axis	a	=	24 371 km
eccentricity	e	=	0.73
inclination	i	=	30°
longitude of ascending node	Ω	=	0°
argument of perigee	ω	=	270°
initial true anomaly	ψ	=	0°

As an integration method, the standard fourth-order Runge-Kutta scheme is chosen. By no means should this imply that such a simple method is adequate for orbit computation; a higher-order Runge-Kutta or a multistep method would be far more efficient. However, for comparison, the primitive character of the fourth-order Runge-Kutta scheme will emphasize the stabilizing role of a time element.

The force models, as defined in reference 7, are chosen. The following four cases are investigated.

a. No perturbations (table II)

b. Earth oblateness perturbation (table III)

c. Earth oblateness and Moon perturbation (table IV)

d. Earth oblateness and atmospheric drag, and Moon perturbation (table V)

One revolution is integrated with four different stepsizes (25, 50, 100, and 200 steps/revolution) and three anomalies: eccentric ($\alpha = 1$), true ($\alpha = 2$), and elliptic ($\alpha = 1.5$), with and without time elements.

The reference orbits were estimated by a more refined integration. The result of the experiments, displayed in tables II through V, show the accuracy reached at the end of an integer number of revolutions by comparison with the reference trajectory at a given time corresponding to perigee pass. The error measured is mostly in-track.

It should first be noticed that the use of the mean anomaly for such an orbit would give totally meaningless results.

With a simple time transformation without the use of a time element, tables II through IV show that meaningful results are obtained. An accuracy of less than 1 kilometer after one revolution can be obtained with the true and elliptic anomaly and 200 steps/rev.

TABLE II.- ACCURACY (KM) AFTER ONE REVOLUTION ALONG A TRANSFER ORBIT OF ECCENTRICITY 0.73 WITH (WITHOUT) USE OF A TIME ELEMENT - NO PERTURBATIONS

	Step/revolution			
α	25	50	100	200
1	4.4 (12 000)	1.2 (840)	0.094 (50)	0.0062 (3.0)
2	15.0 (1 000)	1.2 (72)	.079 (4.7)	.0050 (.30)
1.5	2.5 (2 700)	.1 (170)	.0058 (11)	.00034 (.66)

TABLE III.- ACCURACY (KM) AFTER ONE REVOLUTION ALONG A TRANSFER ORBIT OF ECCENTRICITY 0.73 WITH (WITHOUT) USE OF A TIME ELEMENT - OBLATENESS PERTURBATION

	Step/revolution			
α	25	50	100	200
1	3.6 (12 000)	1.2 (840)	0.087 (50)	0.0054 (3.0)
2	15 (1 000)	1.2 (720)	.079 (4.7)	.0051 (.30)
1.5	2.3 (2 700)	0.097 (1 700)	.0054 (11)	.00044 (.66)

TABLE IV.- ACCURACY (KM) AFTER ONE REVOLUTION ALONG A TRANSFER ORBIT OF ECCENTRICITY 0.73 WITH (WITHOUT) USE OF A TIME ELEMENT - OBLATENESS AND THIRD-BODY PERTURBATIONS

	Step/revolution			
α	25	50	100	200
1	3.4 (12 000)	1.2 (840)	0.090 (50)	0.0061 (3.0)
2	15 (1 000)	1.2 (720)	.078 (4.7)	.0045 (.30)
1.5	2.2 (2 700)	.089 (1 700)	.0045 (11)	.00014 (.66)

TABLE V.- ACCURACY (KM) AFTER 10 REVOLUTIONS ALONG A TRANSFER ORBIT OF ECCENTRICITY 0.73 WITH (WITHOUT) USE OF A TIME ELEMENT - OBLATENESS, DRAG, AND THIRD-BODY PERTURBATIONS

	Step/revolution		
α	50	100	200
1	50 (15 000)	2.8 (850)	0.17 (41)
2	11 (250)	0.74 (32)	.047 (2.5)
1.5	3.2 (2 000)	.19 (1 200)	.011 (7.0)

By including the time element, the accuracy is improved by three to four orders of magnitude, as shown in tables II through IV. For 50 steps/revolution or more, the elliptic anomaly gives one order of magnitude better accuracy than the eccentric or true anomaly.

Without time element, the true anomaly performs slightly better than the elliptic anomaly. With the time element, the elliptic anomaly performs considerably better than the true anomaly, showing that the combination time element and elliptic anomaly as independent variable reduces the in-track error most efficiently.

The eccentric anomaly performs poorly without the time element, while it performs as well as the true anomaly with the time element.

The inclusion of the oblateness (table III) and third-body perturbations (table IV) does not seem to significantly affect the accuracy after one revolution. This illustrates the well-known fact that the direct integration of a Keplerian orbit with Cartesian coordinates is unstable. Therefore, the perturbative effect of the force model does not add significantly more instabilities.

This is no longer true with the atmospheric drag perturbation. This perturbation acts on a highly eccentric orbit like a shock at pericenter. Table V, showing a comparison including oblateness, drag, and third-body perturbations along 10 revolutions, indicates a strong decrease in accuracy in all cases. The integration with 25 steps/revolution had to be dropped.

However, even in this worst-case example, a two- to three-order magnitude improvement in accuracy is shown by including a time element. The elliptic anomaly leaves an error after 10 revolutions 5 times smaller than the true anomaly, which is about 5 times smaller than the eccentric anomaly.

7.0 CONCLUSION

A general time element valid with any choice of the independent variable is proposed together with the use of Cartesian coordinates for the integration of the elliptic motion.

This time element is derived from a set of canonical elements of the Delaunay type developed in the extended phase space, and is valid for an arbitrary independent variable.

By neglecting to separate the conservative from the nonconservative perturbations, a very simple expression for a general time element differential equation is found.

Numerical comparisons show that an improvement of about three orders of magnitude in accuracy can be obtained by introducing a time element for integrating eccentric orbits. The accuracy obtained with the elliptic anomaly (an anomaly intermediate between the eccentric and the true anomaly) can be as much as one order of magnitude better than the eccentric or true anomaly used as independent variable.

The spectacular nature of these results, to be confirmed by more refined numerical experiments, renders the use of the eccentric or the true anomaly obsolete as the independent variable when used with coordinates, while the elliptic anomaly combined with a time element appears to be the ideally suited independent variable for elliptic orbit computation.

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