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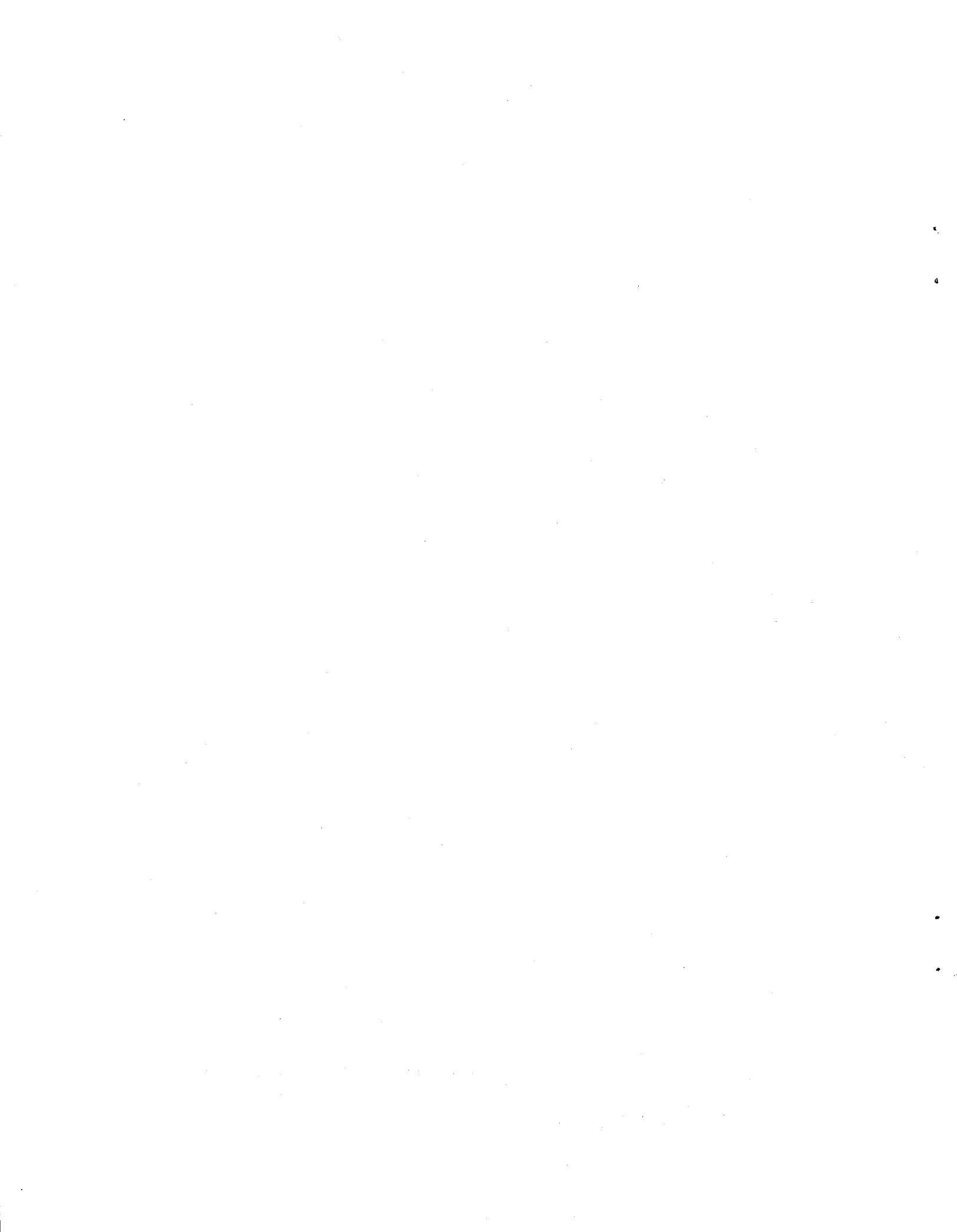LEHIGH UNIVERSITY
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# STRESS INTENSITY FACTORS IN TWO BONDED ELASTIC LAYERS CONTAINING CRACKS PERPENDICULAR TO AND ON THE INTERFACE - PART I. ANALYSIS [*]

by

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## Abstract

In this paper the basic crack problem which is essential for the study of subcritical crack propagation and fracture of layered structural materials is considered. Because of the apparent analytical difficulties, the problem is idealized as one of plane strain or plane stress. An additional simplifying assumption is made by restricting the formulation of the problem to crack geometries and loding conditions which have a plane of symmetry perpendicular to the interface. The general problem is formulated in terms of a coupled system of four integral equations. For each relevant crack configuration of practical interest the singular behavior of the solution near and at the ends and points of intersection of the cracks is investigated and the related characteristic equations are obtained. The edge crack terminating at and crossing the interface, the T-shaped crack consisting of a broken layer and a delamination crack, the cross-shaped crack which consists of a delamination crack intersecting a crack which is perpendicular to the interface, and a delamination crack initiating from a stress-free boundary of the bonded layers are some of the practical crack geometries considered as examples. The formulation of the problem is given in Part I of the paper. Part II deals

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with the solution of the integral equations and presentation of the results.

## 1. Introduction

If one examines the evolution of a typical fracture failure in layered composites, one may invariably trace the initial cause to a localized imperfection which, from the viewpoint of loading, geometry and material strength corresponds to the "weakest link" in the medium. By far the most common forms of such imperfections which may have the potential of growing into a macroscopic crack and of causing eventual failure are the surface flaws, flaws in interlaminar bonding, and the lines of intersection of the interfaces and free boundaries such as holes and other stress-free edges. Under cyclic loading and/or adverse environmental effects a surface flaw, for example, may grow into a part-through surface crack. Upon further application of the loads the surface crack may propagate subcritically through the entire thickness of the first layer. Following the path of least resistance, the crack may either propagate into the adjacent layer or grow along the interface. In analyzing the subcritical growth of these delamination cracks as well as the cracks imbedded into homogeneous layers it is by now generally accepted that the stress intensity factor or some other parameter based on the stress intensity factor (such as the strain energy release rate in the case of delamination cracks) can be used quite effectively as the primary correlation parameter. In studying the fracture of layered materials the basic mechanics problem is then the calculation of the stress intensity factors along the crack front for all physically relevant external loads and crack geometries.

The actual problem is a very complicated three-dimensional problem and at present seems to be analytically intractable. All the existing solutions are therefore

based on the two-dimensional or axisymmetric approximations. Also to keep the analysis within manageable bounds in most of these solutions the medium is generally assumed to be infinite consisting of either semi-infinite spaces with or without a layer in between, or periodically stacked laminates. For example, the plane and axisymmetric problems for a medium which consists of two or three different materials and which contains a crack parallel to or located at a bimaterial interface were considered in [1-4]. The problem of a crack perpendicular to the interfaces may be found in [5-9]. The problem of a T-shaped crack located on and perpendicular to the interface of two bonded half planes was discussed in [10]. The layered composite which consists of periodically arranged two dissimilar bonded layers with cracks perpendicular to the interfaces was considered in [11,12]. The effect of the elastic properties and the thickness of the adhesive in bonded layered materials was studied in [13]. The problem of an infinite medium which consists of periodic dissimilar orthotropic layers having cracks was studied in [14].

In this paper we consider a problem which is also idealized but at the same time is somewhat closer to the actual problem. It is a plane problem of two bonded isotropic infinite layers containing cracks of various orientations and sizes. The particular crack configurations which may be of considerable practical interest and which have been studied in this paper are shown in Figure 1. Unless one is dealing with a composite beam or a plate with through cracks, the idealization made in this paper for solving the problem is also rather severe not only because of the plane stress or plane strain assumption but also because of limiting the number of layers to two and assuming that the materials are isotropic. The composite laminates are, of course, multilayered and orthotropic. However, by a judicious choice of thickness

ratio and material constants, the solution given in this
paper may provide valuable quantitative and qualitative
information which may be useful in studying the real pro-
blem.

2. Solution of Differential Equations

The fundamental problem which forms the basis of all
the crack geometries shown in Figure 1 is described in
Figure 2. Except for the interface cracks, the two iso-
tropic infinite elastic strips are bonded along the y axis,
the strips contain arbitrarily oriented cracks along the
x axis, and the problem is assumed to be symmetric with
respect to the y=0 plane in geometry as well as applied
loads. Furthermore, it is assumed that by a proper super-
position the problem is reduced to a perturbation problem
in which the crack surface tractions are the only external
loads. Let the coordinate systems be selected as in Fig-
ure 2 and let $u_i$, $v_i$, (i=1,2) be the x and y coordinates
of the displacement vector in the strips. The following
differential equations must be solved for each strip under
appropriate boundary and continuity conditions:

$$(\kappa-1)\nabla^2 u+2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}) = 0, \qquad (1a)$$

$$(\kappa-1)\nabla^2 v+2(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}) = 0, \qquad (1b)$$

where $\kappa=3-4\nu$ for plane strain and $\kappa=(3-\nu)/(1+\nu)$ for gener-
alized plane stress, $\nu$ being the Poisson's ratio. Because
of symmetry, the problem will be considered for $0<y<\infty$ only.

Let the solution of (1) be expressed in terms of the
following Fourier integrals:

$$u(x,y)=\frac{2}{\pi} \int_0^\infty [f(x,t)\cos y\ t+h(y,t)\sin xt]dt, \qquad (2a)$$

$$v(x,y)=\frac{2}{\pi} \int_0^\infty [g(x,t)\sin yt+k(y,t)\cos xt]dt. \qquad (2b)$$

Substituting from (2) into (1) one would obtain a system of ordinary differential equations for the unknown functions f, h, g, and k. Solving these equations and observing that u, v remain bounded as $y \to \infty$ we find

$$f(x,t) = -\frac{1}{t} \left\{ \left[ F - \frac{\kappa-1}{2} H + x\,t\,G \right] \cosh\ xt \right.$$
$$\left. + \left[ E - \frac{\kappa-1}{2} G + x\,t\,H \right] \sinh\ xt \right\}, \tag{3a}$$

$$g(x,t) = \frac{1}{t} \left\{ \left[ F + \frac{\kappa+1}{2} H + x\,t\,G \right] \sinh\ xt \right.$$
$$\left. + \left[ E + \frac{\kappa+1}{2} G + x\,t\,H \right] \cosh\ xt \right\}, \tag{3b}$$

$$h(y,t) = (C + ytD)e^{-yt}, \tag{3c}$$

$$k(y,t) = (C + \kappa D + ytD)e^{-yt}, \tag{3d}$$

where C, D, E, F, G, and H are functions of the transform variable t and are unknown. Using the condition that $\sigma_{xy}$ must vanish for y=0 in both strips, which follows from the assumed symmetry, and defining A=tD, equations (3c) and (3d) may be replaced by

$$h(y,t) = -\frac{A(t)}{t} \left( \frac{\kappa-1}{2} - yt \right) e^{-yt}, \tag{4a}$$

$$k(y,t) = \frac{A(t)}{t} \left( \frac{\kappa+1}{2} + yt \right) e^{-yt}. \tag{4b}$$

From the stress-displacement relations and from (2), (3) and (4) the stress components in each semi-infinite strip may be expressed as follows:

$$\frac{1}{2\mu} \sigma_{xx}(x,y) = -\frac{2}{\pi} \int_{0}^{\infty} \left\{ \left[ (F+xtG)\sinh xt + (E+xtH)\cosh xt \right] \cos yt \right.$$
$$\left. + (1-yt)Ae^{-yt}\cos xt \right\} dt, \tag{5a}$$

$$\frac{1}{2\mu}\sigma_{yy}(x,y)=\frac{2}{\pi}\int_0^\infty \{[(F+2H+xtG)\sinh\,xt\,+\,(E+2G$$

$$+xtH)\cosh\,xt]\cos yt-(1+yt)Ae^{-yt}\cos\,xt\}dt, \qquad (5b)$$

$$\frac{1}{2\mu}\sigma_{xy}(x,y)=\frac{2}{\pi}\int_0^\infty \{[(E+G+xtH)\sinh\,xt\,+\,(F+H$$

$$+xtG)\cosh\,xt]\sin yt-yt\,Ae^{-yt}\sin\,xt\}dt. \qquad (5c)$$

Referring to Figure 2 we note that equations (2-5) are valid with $\mu=\mu_1$, $\kappa=\kappa_1$, $0<y<\infty$, $0<x=x_1<2h_1$ for strip 1 and with $\mu=\mu_2$, $\kappa=\kappa_2$, $0<y<\infty$, $0<x=x_2<2h_2$ for strip 2. Thus, there are all together ten unknown functions, $A_i$, $E_i$, $F_i$, $G_i$, and $H_i$, $(i=1,2)$ which may be determined from the following boundary and continuity conditions:

$$\sigma_{1xx}(2h_1,y)=0, \quad \sigma_{1xy}(2h_1,y)=0, \quad 0<y<\infty, \qquad (6a,b)$$

$$\sigma_{2xx}(0,y)=0, \quad \sigma_{2xy}(0,y)=0, \quad 0<y<\infty, \qquad (7a,b)$$

$$\sigma_{1xx}(0,y)=\sigma_{2xx}(2h_2,y), \quad \sigma_{1xy}(0,y)=\sigma_{2xy}(2h_2,y),$$

$$0<y<\infty \qquad (8a,b)$$

$$u_1(0,y)=u_2(2h_2,y), \quad 0<y<a_3, \quad b_3<y<\infty, \qquad (9a)$$

$$\sigma_{1xx}(0,y)=p_3(y), \quad a_3<y<b_3 \qquad (9b)$$

$$v_1(0,y)=v_2(2h_2,y), \quad 0<y<a_3, \quad b_3<y<\infty, \qquad (10a)$$

$$\sigma_{1xy}(0,y)=p_4(y), \quad a_3<y<b_3, \qquad (10b)$$

$$v_i(x_i,0)=0, \quad 0<x_i<a_i, \quad b_i<x_i<2h_i, \quad (i=1,2), \qquad (11a)$$

$$\sigma_{iyy}(x_i,0)=-p_i(x_i), \quad a_i<x_i<b_i, \quad (i-1,2). \qquad (11b)$$

Substituting from (5) into the six homogeneous conditions (6)-(8), six of the ten unknown functions may be eliminated. The mixed boundary conditions (9)-(11) would

then give a system of four dual integral equations to determine the remaining four unknowns. The problem may also be reduced directly to a system of four singular integral equations by defining the following four new unknown functions:

$$\frac{\partial}{\partial x_i} v_i(x_i,0) = \phi_i(x_i) \, , \, (i=1,2) \tag{12}$$

$$\frac{\partial}{\partial y}[v_1(+0,y)-v_2(2h_2-0,y)]=\phi_3(y), \tag{13}$$

$$\frac{\partial}{\partial y}[u_1(+0,y)-u_2(2h_2-0,y)]=\phi_4(y). \tag{14}$$

If we now replace the mixed conditions (9)-(11) by (12)-(14) and observe that

$$\phi_i(x_i)=0, \, 0<x_i<a_i, \, b_i<x_i<2h_i, \, (i=1,2), \tag{15}$$

$$\phi_3(y)=0, \, \phi_4(y)=0, \, 0<y<a_3, \, b_3<y<\infty \, , \tag{16a,b}$$

from (2), (3), (4), (5)-(8), and (12)-(14) we obtain

$$A_i(t)=-\frac{2}{\kappa_i+1} \int_{a_i}^{b_i} \phi_i(s)\sin ts \, ds, \, (i=1,2), \tag{17}$$

and

$$(F_1+2h_1tG_1)\sinh2h_1t+(E_1+2h_1tH_1)\cosh2h_1t=R_1(t),$$

$$(E_1+G_1+2h_1tH_1)\sinh2h_1t+(F_1+H_1+2h_1tG_1)\cosh2h_1t=R_2(t),$$

$$E_2=R_3(t), \, F_2+H_2=0,$$

$$\mu_1E_1-\mu_2[(F_2+2h_2tG_2)\sinh2h_2t+(E_2+2h_2tH_2)\cosh2h_2t]=R_4(t),$$

$$\mu_1(F_1+H_1)-\mu_2[(E_2+G_2+2h_2tH_2)\sinh2h_2t$$
$$+(F_2+H_2+2h_2tG_2)\cosh2h_2t]=R_5(t),$$

$$F_1-\frac{\kappa_1-1}{2} H_1-[(E_2-\frac{\kappa_2-1}{2}G_2+2h_2tH_2)\sinh2h_2t$$
$$+(F_2-\frac{\kappa_2-1}{2}H_2+2h_2tG_2)\cosh2h_2t]=R_6(t),$$

$$E_1+\frac{\kappa_1+1}{2} G_1-[(F_2+\frac{\kappa_2+1}{2} H_2+2h_2tG_2)\sinh 2h_2t$$

$$+(E_2+\frac{\kappa_2+1}{2} G_2+2h_2t\ H_2)\cosh 2h_2t]=R_7(t), \quad (18a\text{-}h)$$

where $R_1,..,R_7$ are given in Appendix A in terms of $\phi_1,...,$ $\phi_4$, and the subscripts 1 and 2 in the unknown functions refer to the strips 1 and 2, respectively.

## 3. The Integral Equations

By solving the system of algebraic equations (18) and by using (17) it is seen that the stresses and displacements in the strips may be expressed in terms of the new unknown functions $\phi_1,...,\phi_4$ only. The functions $\phi_1,...,$ $\phi_4$ may now be determined by using the boundary conditions (9b), (10b), and (11b) which have not yet been satisfied. Thus, by substituting from (5b), (17) and (18) into (11b), after somewhat lengthy manipulations we obtain

$$\int_{a_i}^{b_i}(\frac{1}{s-x_i} + \frac{1}{s+x_i})\phi_i(s)ds+\sum_{j=1}^{4} \int_{a_j}^{b_j}k_{ij}(x_i,s)\phi_j(s)ds$$

$$=-\frac{\pi(1+\kappa_i)}{4\mu_i}p_i(x_i), \quad i=1,2, \quad a_i<x_i<b_i, \quad a_4=a_3, \quad b_4=b_3, (19)$$

where if $a_i\neq 0$ and $b_i\neq 2h_i$ the kernels $k_{ij}$, $(i=1,2; j=1,...,4)$ are bounded functions in their respective closed domains of definition. The expressions for $k_{ij}$ are, of course, dependent on the solution of the system (18) which is quite cumbersome and hence, will not be given in this paper. The complete details may be found in [15].

Similarly, by substituting from (5a), (5c), (17) and (18) into (9b) and (10b), after separating the dominant parts of the kernels we obtain

$$\frac{1}{\pi} \int_{a_3}^{b_3}(\frac{1}{s-y}+\frac{1}{s+y})\phi_4(s)ds+\gamma\ \phi_3(s)$$

$$-\frac{2}{\pi a_{21}}\sum_{j=1}^{4} \int_{a_j}^{b_j}k_{3j}(y,s)\phi_j(s)ds=\frac{1+\kappa_1}{a_{21}\mu_1}p_3(y), \quad a_3<y<b_3,$$

$$a_4 = a_3, \quad b_4 = b_3, \tag{20}$$

$$\frac{1}{\pi} \int_{a_3}^{b_3} \left( \frac{1}{s-y} - \frac{1}{s+y} \right) \phi_3(s) \, ds - \gamma \phi_4(y)$$

$$+ \frac{2}{\pi a_{21}} \sum_{j=1}^{4} \int_{a_j}^{b_j} k_{4j}(y,s) \phi_j(s) \, ds = \frac{1+\kappa_1}{a_{21} \mu_1} p_4(y),$$

$$a_3 < y < b_3, \quad a_4 = a_3, \quad b_4 = b_3, \tag{21}$$

where

$$m = \mu_1/\mu_2, \quad \gamma = \frac{(1+m\kappa_2)-(m+\kappa_1)}{(1+m\kappa_2)+(m+\kappa_1)},$$

$$a_{22} = (\kappa_1+1) \frac{(m+\kappa_1)-(1+m\kappa_2)}{(m+\kappa_1)(1+m\kappa_2)}, \quad a_{21} = (\kappa_1+1) \frac{(m+\kappa_1)+(1+m\kappa_2)}{(m+\kappa_1)(1+m\kappa_2)}$$

$$\tag{22}$$

and the kernels $k_{3j}$ and $k_{4j}$, $(j=1,..,4)$ are given in [15].

From the definition of the functions $\phi_1,...,\phi_4$ given by (12)-(14) and the conditions (15) and (16) it is clear that for the imbedded cracks shown in Figure 2 the solution of the integral equations (19)-(21) must satisfy the following singlevaluedness conditions:

$$\int_{a_i}^{b_i} \phi_i(s) \, ds = 0, \quad (i=1,..,4), \quad a_4 = a_3, \quad b_4 = b_3. \tag{23}$$

## 4. Singular Behavior of the Solution

For each crack configuration shown in Figure 1 the singular behavior of the solution of the integral equations (19)-(21) and that of the stress state around the crack tips or the irregular points $a_i$, $b_i$, $(i=1,2,3)$ may be examined by using the function theoretic method described in [16] (see also [17] and [18] for applications to crack problems). For $a_1 > 0$, $b_2 < 2h_2$, and $a_3 > 0$ the dominant parts of the integral equations are uncoupled. Following [16], if we express the solution of (19) by

$$\phi_j(s) = \frac{g_j(s)}{(s-a_j)^{\alpha_j}(b_j-s)^{\beta_j}} \, , \quad a_j < s < b_j \, , \quad (j=1,2)$$

$$0 < Re(\alpha_j, \beta_j) < 1 \, , \tag{24}$$

substituting from (24) into (19), for the imbedded cracks it may easily be shown that [17,18]

$$\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0.5. \tag{25}$$

In (24) $g_j$ is a Hölder-continuous function and is nonzero at the ends $a_j$ and $b_j$, $(j=1,2)$. Also, defining

$$\psi(s) = \phi_4(s) + i \, \phi_3(s), \tag{26}$$

the integral equations (20) and (21) may be combined as

$$\frac{1}{\pi i} \int_{a_3}^{b_3} \frac{\psi(s)}{s-y} \, ds - \gamma\psi(y) = \frac{\kappa_1 + 1}{a_{21}\mu_1}[p_4(y) - i \, p_3(y)]$$

$$+ \text{ bounded terms}, \quad a_3 < y < b_3 \, . \tag{27}$$

If we now let

$$\psi(s) = \frac{g_3(s)}{(s-a_3)^{\alpha_3}(b_3-s)^{\beta_3}}, \quad a_3 < s < b_3, \quad 0 < R(\alpha_3, \beta_3) < 1, \tag{28}$$

from (27) following the precedure outlined, for example, in [18] we obtain

$$\alpha_3 = \frac{1}{2} - i\omega, \quad \beta_3 = \frac{1}{2} + i\omega, \quad \omega = \frac{1}{2\pi} \, \log(\frac{1+\gamma}{1-\gamma}), \tag{29}$$

where $g_3$ is again a Hölder continuous function and is non-zero at the ends $a_3$ and $b_3$.

For the limiting cases of cracks intersecting each other, free boundaries, or the interface the kernels $k_{ij}$ which appear in (19)-(21) are no longer bounded for all values of their arguments. In such cases to determine the correct singular behavior of the functions $\phi_1, \ldots, \phi_4$, through a proper asymptotic analysis these unbounded parts of the kernels must be separated and must be taken into consideration in the application of the function theoretic methods to the integral equations. The typical crack tip

behaviors which do not conform to the standard singularities expressed by (24), (25) and (28), (29) are discussed below.

(a) Edge crack, $b_1=2h_1$.

Consider, for example, the problem described by Figure 2 in which $b_1=2h_1$, $a_1>0$, $b_2<2h_2$, $a_2>0$, $a_3>0$, $b_3<\infty$. In this case of all the Fredholm kernels appearing in (19)-(21) only $k_{11}(x_1,s)$ exhibits singular behavior. Analyzing the integrand which defines $k_{11}$ we find that

$$k_{11}(x_1,s)=\int_0^\infty [K_{11}(x_1,s,t)-K_{11}^\infty(x_1,s,t)]dt$$

$$+\int_0^\infty K_{11}^\infty(x_1,s,t)dt=k_{11}^b+k_{11}^s, \tag{30}$$

$$K_{11}^\infty(x_1,s,t)=\frac{1}{2}\{-1+(\frac{3}{2}+x_1t-2h_1t)[-2+4t(2h_1-s)]\}e^{-t(4h_1-x_1-s)},$$

$$k_{11}^s(x_1,s)=\int_0^\infty K_{11}^\infty(x_1,s,t)dt=-\frac{1}{x_1+s-4h_1}$$

$$-\frac{6(2h_1-x_1)}{(x_1+s-4h_1)^2}-\frac{4(2h_1-x_1)^2}{(x_1+s-4h_1)^3}, \tag{31}$$

where $K_{11}^\infty$ is the asymptotic value of $K_{11}$ for $t\to\infty$, $x_1\to2h_1$, and $s\to2h_1$. Now substituting from (30), (31) and (24) into (19) and using the technique described in [18] we obtain the following characteristic equations giving the powers of singularity $\alpha_1$ and $\beta_1$:

$$\cot\pi\alpha_1=0, \tag{32a}$$

$$\frac{g_1(2h_1)}{\sin\pi\beta_1}[2(1-\beta_1)^2-1-\cos\pi\beta_1]=0. \tag{32b}$$

The acceptable roots of (32) are

$$\alpha_1=0.5, \quad \beta_1=0, \tag{33}$$

which are the known results [19]. Powers of singularity identical to (33) are obtained for $a_2=0$, $b_2<2h_2$.

(b)  Crack tip terminating at the interface, $a_1=0$.

In Figure 2 let $a_1=0$.  In this case too $k_{11}$ is the only Fredholm kernel which exhibits singular behavior. Let $k_{11}$ again be expressed by (30).  The asymptotic part of the integrand may be separated as follows:

$$K_{11}^{\infty}(x_1,s,t)=e^{-(x_1+s)t}\{ts[\frac{m\kappa_2}{1+m\kappa_2}-(3-2x_1t)\frac{m}{m+\kappa_1}]$$

$$+\frac{1}{4}(2ts-1-\kappa_1)[\frac{1}{1+m\kappa_2}+(3-2x_1t)\frac{1}{m+\kappa_1}]\},\qquad(34)$$

where $m=\mu_1/\mu_2$.  Noting that for $a_1=0$ in (19) the term $1/(s+x_1)$ also becomes unbounded for $s\to 0$, $x_1\to 0$, the additional singular part of the kernel may now be expressed as

$$k_{11}^{s}(x_1,s)+\frac{1}{s+x_1}=\int_0^{\infty}K_{11}^{\infty}(x_1,s,t)dt+\frac{1}{s+x_1}$$

$$=\frac{c_{11}}{s+x_1}+\frac{c_{12}x_1}{(s+x_1)^2}+\frac{c_{13}x_1^2}{(s+x_1)^3},\qquad(35a)$$

$$c_{11}=\frac{1}{2}-\frac{1+\kappa_1}{2(1+m\kappa_2)}-\frac{3(m-1)}{2(m+\kappa_1)}\ ,\ \ m=\mu_1/\mu_2\ \ ,$$

$$c_{12}=\frac{6(m-1)}{m+\kappa_1}\ ,\ \ c_{13}=-\frac{4(m-1)}{m+\kappa_1}\ .\qquad(35b)$$

The kernel $1/(s-x)$ combined with (35) constitutes a generalized Cauchy kernel.  Following the procedure of [18], from (19), (24), (30), and (35) the characteristic equations which determine $\alpha_1$ and $\beta_1$ for $a_1=0$ may be obtained as

$$\cot\pi\beta_1=0,\qquad(36a)$$

$$\cos\pi\alpha_1+c_{11}+c_{12}\alpha_1+c_{13}\ \frac{\alpha_1(\alpha_1+1)}{2}=0.\qquad(36b)$$

which are the known results [5].

Similarly, for $b_2=2h_2$, $a_2>0$, $a_1>0$, $b_1<2h_1$, and $a_3>0$ we obtain

$$k_{22}^S(x_2,s) = \frac{c_{21}}{4h_2-x_2-s} + \frac{c_{22}(2h_2-x_2)}{(4h_2-x_2-s)^2} + \frac{c_{23}(2h_2-x_2)^2}{(4h_2-x_2-s_2)^3}$$

$$c_{21} = -\frac{1}{2} + \frac{3(1-m)}{2(1+m\kappa_2)} + \frac{m(1+\kappa_2)}{2(m+\kappa_1)} \; , \quad m = \mu_1/\mu_2$$

$$c_{22} = \frac{6(m-1)}{1+m\kappa_2}, \quad c_{23} = \frac{4(1-m)}{1+m\kappa_2}. \tag{37}$$

From (37), (24), (30) and (19) the characteristic equations are found to be

$$\cot\pi\alpha_2 = 0 \; , \tag{38a}$$

$$\cos\pi\beta_2 - c_{21} - \beta_2 c_{22} - \frac{1}{2}\beta_2(1+\beta_2)c_{23} = 0, \tag{38b}$$

which are again the known results [5,6].


(c)  Crack intersecting the interface, $a_1=0$, $b_2=2h_2$.

Referring to Figure 2, if $a_1=0$, $b_2=2h_2$, $a_3>0$, $b_1<2h_1$, and $a_2>0$, in (19) not only $k_{11}$ and $k_{22}$ but also the coupling kernels $k_{12}$ and $k_{21}$ exhibit singular behavior. The singular parts of $k_{11}$ and $k_{22}$ are given by (35) and (37). Using a notation similar to (30), for the singular parts of the coupling kernels we obtain

$$K_{12}^\infty(x_1,s,t) = [\frac{1+\kappa_1}{m+\kappa_1}(x_1 t-\frac{3}{2}) - \frac{1+\kappa_1}{1+m\kappa_2}(\frac{1}{2}+st-2h_2 t)]e^{-t(2h_2-s+x_1)},$$

$$k_{12}^S(x_1,s) = \frac{d_{11}}{x_1-s+2h_2} + \frac{d_{12}x_1}{(x_1-s+2h_2)^2} \; ,$$

$$d_{11} = \frac{1+\kappa_1}{2(1+m\kappa_2)} - \frac{3(1+\kappa_1)}{2(m+\kappa_1)}, \quad d_{12} = \frac{1+\kappa_1}{m+\kappa_1} - \frac{1+\kappa_1}{1+m\kappa_2} ; \tag{39}$$

$$K_{21}^\infty(x_2,s,t) = [\frac{m(1+\kappa_2)}{2(m+\kappa_1)}(1-2st)$$

$$+ \frac{m(1+\kappa_2)}{1+m\kappa_2}(\frac{3}{2}+x_2 t-2h_2 t)]e^{-t(2h_2-x_2+s)} \; ,$$

$$k_{21}^S(x_2,s) = \frac{d_{21}}{2h_2+s-x_2} + \frac{d_{22}(2h_2-x_2)}{(2h_2+s-x_2)^2} \; ,$$

$$d_{21}=\frac{3m(1+\kappa_2)}{2(1+m\kappa_2)}-\frac{m(1+\kappa_2)}{2(m+\kappa_1)}, \quad d_{22}=\frac{m(1+\kappa_2)}{m+\kappa_1}-\frac{m(1+\kappa_2)}{1+m\kappa_2}. \quad (40)$$

Substituting now from (24) into (19), noting that at the irregular point $x_1=0$, $x_2=2h_2$, $\alpha_1=\beta_2$, and using (35), (37), (39), and (40) we find the following characteristic equations to determine $\alpha_1=\beta_2$, $\beta_1$, and $\alpha_2$:

$$\cot\pi\beta_1=0, \quad \cot\pi\alpha_2=0,$$

$$[\cos\pi\alpha_1+c_{11}+\alpha_1c_{12}+\frac{1}{2}\alpha_1(1+\alpha_1)c_{13}][\cos\pi\alpha_1-c_{21}-\alpha_1c_{22}$$

$$-\frac{1}{2}\alpha_1(1+\alpha_1)c_{23}]+(d_{11}+\alpha_1d_{12})(d_{21}+\alpha_1d_{22})=0, \quad (41a\text{-}c)$$

which are the known results [6]. To obtain (41) it is assumed that $g_1(b_1)\neq0$, $g_2(a_2)\neq0$. Equation (41c) is the expression of vanishing determinant of the linear homogeneous algebraic system in $g_1(0)$ and $g_2(2h_2)$. This indicates that the constants $g_1(0)$ and $g_2(2h_2)$ are not independent and are related by

$$[\cos\pi\alpha_1+c_{11}+\alpha_1c_{12}+\frac{1}{2}\alpha_1(1+\alpha_1)c_{13}]g_1(0)$$

$$+ (d_{11}+\alpha_1d_{12})g_2(2h_2)=0. \quad (42)$$

Condition (42) replaces one of the single-valuedness conditions (23) in solving the system of singular integral equations.

For $a_3>0$ the results given by (29), (32), (36), (38), and (41) cover all crack configurations shown in Figures (1a) to (1h).

(d) T-shaped or cross-shaped cracks.

The problem becomes somewhat more complicated if the interface cracks intersect the cracks which are perpendicular to the boundaries. Consider, for example, the cross-shaped crack shown in Figure 1(j) for which $a_1=0$, $b_1<2h_1$, $a_2>0$, $b_2=2h_2$, $a_3=0$, and $b_3\leq\infty$ (see Figure 2). In this case the point of intersection ($x_1=0$, $y=0$) is an

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irregular point common to all four integral equations given by (19)-(21). Examining the asymptotic behavior of the kernels it is found that $k_{1j}$, $k_{2j}$, $(j=1,2,3,4)$, and $k_{3i}$, $k_{4i}$, $(i=1,2)$ become unbounded as the variables $x_1$, $x_2$, and $y$, approach the common irregular point together with s in pairs. Singular part of each one of these kernels may again be separated in a straightforward manner by analyzing the asymptotic behavior of the related integrands. After separating the singular parts $k_{ij}^s$ of the kernels, the integral equations (19)-(21) may be expressed as

$$\int_o^{b_1} [\frac{1}{s-x_1} + \frac{1}{s+x_1} + k_{11}^s(x_1,s)] \phi_1(s)ds + \int_{a_2}^{2h_2} k_{12}^s(x_1,s) \phi_2(s)ds$$

$$+\sum_{j=3}^4 \int_o^{b_3} k_{1j}^s(x_1,s) \phi_j(s)ds + P_1(x_1) = -\frac{\pi(1+\kappa_1)}{4\mu_1} p_1(x_1),$$

$$0 < x_1 < b_1,$$

$$\int_o^{b_1} k_{21}^s(x_2,s) \phi_1(s)ds + \int_{a_2}^{2h_2} [\frac{1}{s-x_2} + k_{22}^s(x_2,s)] \phi_2(s)ds$$

$$+\sum_{j=3}^4 \int_o^{b_3} k_{2j}^s(x_2,s) \phi_j(s)ds + P_2(x_2) = -\frac{\pi(1+\kappa_2)}{4\mu_2} p_2(x_2),$$

$$a_2 < x_2 < 2h_2$$

$$-\frac{2}{\pi a_{21}} \int_o^{b_1} k_{31}^s(y,s) \phi_1(s)ds - \frac{2}{\pi a_{21}} \int_{a_2}^{2h_2} k_{32}^s(y,s) \phi_2(s)ds$$

$$+\frac{1}{\pi} \int_o^{b_3} (\frac{1}{s-y} + \frac{1}{s+y}) \phi_4(s)ds + \gamma\phi_3(y) + P_3(y)$$

$$= \frac{1+\kappa_1}{a_{21}\mu_1} p_3(y), \quad 0 < y < b_3,$$

$$\frac{2}{\pi a_{21}} \int_o^{b_1} k_{41}^s(y,s) \phi_1(s)ds + \frac{2}{\pi a_{21}} \int_{a_2}^{2h_2} k_{42}^s(y,s) \phi_2(s)ds$$

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$$+ \frac{1}{\pi} \int_0^{b_3} \left(\frac{1}{s-y} - \frac{1}{s+y}\right)\phi_3(s)ds - \gamma\phi_4(y) + P_4(y)$$

$$= \frac{1+\kappa_1}{a_{21}\mu_1}p_4(y), \quad 0 < y < b_3, \tag{45a-d}$$

where the functions $P_k$, $(k=1,..,4)$ represent all the remaining terms in the integral equations corresponding to the bounded kernels, $k_{11}^S$, $k_{12}^S$, $k_{21}^S$, and $k_{22}^S$ are given by equations (35), (39), (40), and (37), respectively, and

$$k_{13}^S(x_1,s) = \frac{A_1 x_1}{s^2+x_1^2} + \frac{A_2 x_1(x_1^2 + 5s^2)}{(s^2+x_1^2)^2} \quad ,$$

$$k_{14}^S(x_1,s) = -\frac{A_1 s}{s^2+x_1^2} + \frac{A_2 s(3s^2-x_1^2)}{(s^2+x_1^2)^2} \quad ,$$

$$k_{23}^S(x_2,s) = -\frac{A_3(2h_2-x_2)}{s^2+(2h_2-x_2)^2} - \frac{A_4(2h_2-x_2)[(2h_2-x_2)^2+5s^2]}{[s^2+(2h_2-x_2)^2]^2} \quad ,$$

$$k_{24}^S(x_2,s) = -\frac{A_3 s}{s^2+(2h_2-x_2)^2} + \frac{A_4 s[3s^2-(2h_2-x_2)^2]}{[s^2+(2h_2-x_2)^2]^2} \quad ,$$

$$k_{31}^S(y,s) = -\frac{2A_1 s}{s^2+y^2} + \frac{2A_2 s(3y^2-s^2)}{(s^2+y^2)^2} \quad ,$$

$$k_{32}^S(y,s) = \frac{2A_2(2h_2-s)}{y^2+(2h_2-s)^2} - \frac{2A_1(2h_2-s)[3y^2-(2h_2-s)^2]}{[(2h_2-s)^2+y^2]^2} \quad ,$$

$$k_{41}^S(y,s) = \frac{2A_1 y}{s^2+y^2} + \frac{2A_2 y(y^2-3s^2)}{(s^2+y^2)^2} \quad ,$$

$$k_{42}^S(y,s) = \frac{2A_2 y}{y^2+(2h_2-s)^2} + \frac{2A_1 y[y^2-3(2h_2-s)^2]}{[(2h_2-s)^2+y^2]^2} \quad , \tag{46a-h}$$

$$A_1 = \frac{1+\kappa_1}{2(1+m\kappa_2)}, \quad A_2 = \frac{1+\kappa_1}{2(m+\kappa_1)}, \quad A_3 = \frac{m(1+\kappa_2)}{2(m+\kappa_1)} \quad ,$$

$$A_4 = \frac{m(1+\kappa_2)}{2(1+mx_2)} \quad, \quad m = \mu_1/\mu_2 . \tag{47}$$

Note that the T-shaped crack shown in Figure 1(i) is a special case of the cross-shaped crack for which $b_1=0$.

In order to simplify the manipulations in the asymptotic analysis we first define

$$x_2^*=2h_2-x_2, \quad s^*=2h_2-s, \quad b_2^*=2h_2-a_2, \quad \phi_2^*(s^*)=\phi_2(s), \tag{48}$$

in the interval $a_2 \le (x_2,s) \le 2h_2$ and then, for convenience, drop the superscript (*). Thus the origin of the coordinate systems becomes the common irregular point of the integral equations (45). Noting that at $s=0$ the unknown functions $\phi_1,\ldots,\phi_4$ must all have the same singular behavior, we let

$$\phi_j(s) = \frac{g_j(s)}{s^\alpha(b_j-s)^{\beta_j}} \quad, \quad 0<Re(\alpha,\beta_j)<1,$$

$$g_j(0)\neq0, \quad g_j(b_j)\neq0, \quad b_4=b_3, \quad (j=1,\ldots,4). \tag{49}$$

Using (49) and the procedure outlined in [16,17,18] one can establish the following asymptotic relations for the relevant singular integrals in the close neighborhood of the end point $x=0, y=0$:

$$\frac{1}{\pi}\int_0^b \frac{\phi(s)}{s-x}ds \cong \frac{g(0)\cot\pi\alpha}{b^\beta x^\alpha} \quad, \quad \frac{1}{\pi}\int_0^b \frac{\phi(s)}{s+x}ds \cong \frac{g(0)}{b^\beta x^\alpha \sin\pi\alpha} ,$$

$$\frac{1}{\pi}\int_0^b \frac{x\phi(s)}{(s+x)^2}ds \cong \frac{\alpha g(0)}{b^\beta x^\alpha \sin\pi\alpha}, \quad \frac{1}{\pi}\int_0^b \frac{x^2\phi(s)}{(s+x)^3}ds \cong \frac{\alpha(\alpha+1)g(0)}{2b^\beta x^\alpha \sin\pi\alpha} ,$$

$$\frac{1}{\pi}\int_0^b \frac{s\phi(s)}{s^2+x^2}ds \cong \frac{g(0)}{2b^\beta x^\alpha \sin\frac{\pi\alpha}{2}}, \quad \frac{1}{\pi}\int_0^b \frac{y\phi(s)}{s^2+y^2}ds \cong \frac{g(0)}{2b^\beta y^\alpha \cos\frac{\pi\alpha}{2}} ,$$

$$\frac{1}{\pi}\int_0^b \frac{sx^2\phi(s)}{(s^2+x^2)^2}ds \cong \frac{\alpha g(0)}{4b^\beta x^\alpha \sin\frac{\pi\alpha}{2}}, \quad \frac{1}{\pi}\int_0^b \frac{ys^2\phi(s)}{(s^2+y^2)^2}ds \cong \frac{(1-\alpha)g(0)}{4b^\beta y^\alpha \cos\frac{\pi\alpha}{2}}.$$

$$\tag{50a-h}$$

Substituting now from (50) into (45) and using (46) and (48), the leading terms of the integral equations around the end point $x=0$, $y=0$ may be expressed as

$$\frac{g_1(0)}{b_1{}^{\beta_1}x_1^{\alpha}\sin\pi\alpha}[\cos\pi\alpha+c_{11}+\alpha c_{12}+c_{13}\alpha(1+\alpha)/2]$$

$$+\frac{g_2(0)}{b_2{}^{\beta_2}x_1^{\alpha}\sin\pi\alpha}(d_{11}+\alpha d_{12})+\frac{g_3(0)}{2b_3{}^{\beta_3}x_1^{\alpha}\cos\frac{\pi\alpha}{2}}[e_{11}+e_{12}(1-\alpha)/2]$$

$$+\frac{g_4(0)}{2b_3{}^{\beta_4}x_1^{\alpha}\sin\frac{\pi\alpha}{2}}(f_{11}+f_{12}\alpha/2)=F_1(x_1),$$

$$\frac{g_1(0)}{b_1{}^{\beta_1}x_2^{\alpha}\sin\pi\alpha}(d_{21}+\alpha d_{22})+\frac{g_2(0)}{b_2{}^{\beta_2}x_2^{\alpha}\sin\pi\alpha}[-\cos\pi\alpha+c_{21}+\alpha c_{22}$$

$$+c_{23}\alpha(1+\alpha)/2]+\frac{g_3(0)}{2b_3{}^{\beta_3}x_2^{\alpha}\cos\frac{\pi\alpha}{2}}[e_{21}+e_{22}(1-\alpha)/2]$$

$$+\frac{g_4(0)}{2b_3{}^{\beta_4}x_2^{\alpha}\sin\frac{\pi\alpha}{2}}[f_{21}+f_{22}\alpha/2]=F_2(x_2),$$

$$\frac{2g_1(0)}{a_{21}b_1{}^{\beta_1}y^{\alpha}\sin\frac{\pi\alpha}{2}}[A_1+(1-2\alpha)A_2]-\frac{2g_2(0)}{a_{21}b_2{}^{\beta_2}y^{\alpha}\sin\frac{\pi\alpha}{2}}[A_2+(1-2\alpha)A_1]$$

$$+\frac{\gamma g_3(0)}{b_3{}^{\beta_3}y^{\alpha}}+\frac{g_4(0)}{b_3{}^{\beta_4}y^{\alpha}\sin\pi\alpha}(1+\cos\pi\alpha)=F_3(y),$$

$$\frac{2g_1(0)}{a_{21}b_1{}^{\beta_1}y^{\alpha}\cos\frac{\pi\alpha}{2}}[A_1-(1-2\alpha)A_2]+\frac{2g_2(0)}{a_{21}b_2{}^{\beta_2}y^{\alpha}\cos\frac{\pi\alpha}{2}}[A_2-(1-2\alpha)A_1]$$

$$+\frac{g_3(0)}{b_3{}^{\beta_3}y^{\alpha}\sin\pi\alpha}(\cos\pi\alpha-1)-\frac{\gamma g_4(0)}{b_3{}^{\beta_4}y^{\alpha}}=F_4(y), \qquad (51a-d)$$

where the constants $c_{ij}$ and $d_{ij}$ are given by (35b), (37),

(39), and (40),

$$e_{11}=(A_1+A_2), \quad e_{12}=4A_2, \quad e_{21}=-(A_3+A_4), \quad e_{22}=-4A_4,$$

$$f_{11}=3A_2-A_1, \quad f_{12}=-4A_1, \quad f_{21}=-(A_3+A_4), \quad f_{22}=4A_4, \qquad (52)$$

and the functions $F_1,..,F_4$ contain all remaining terms which are bounded at the end point $x=0$, $y=0$. If we now multiply both sides of equations (51a)-(51d) by $x_1^\alpha$, $x_2^\alpha$, $y^\alpha$, and $y^\alpha$, respectively, and let $x_1 \to 0$, $x_2 \to 0$, and $y \to 0$ we obtain a system of algebraic equations for $g_1(0),...,g_4(0)$ of the following form:

$$\sum_{j=1}^{4} B_{kj} g_j(0) = 0, \quad k=1,..,4 \qquad (53)$$

where the coefficients $B_{kj}$ are given by (51). Since $g_1(0),...,g_4(0)$ are assumed to be generally nonzero, from (53) the characteristic equation which accounts for $\alpha$ is obtained to be

$$\det|B_{kj}| = 0, \quad (k,j)=(1,2,3,4). \qquad (54)$$

After some simple manipulations from (54) it is found that

$$\frac{1}{\sin\pi\alpha}[\cos\pi\alpha+1-2(\alpha-1)^2]=0, \qquad (55)$$

which is identical to the result found for the edge crack (see equation 32b).

For the T-shaped crack shown in Figure 1(i) $\phi_1=0$, the problem is formulated by the integral equations (45b)-(45d), and the characteristic equation for $\alpha$ may be obtained from (51b)-(51d) as follows:

$$\det|B_{ij}|=0, \quad (i,j)=(2,3,4). \qquad (56)$$

It can be shown that (56) too reduces to (55).

The foregoing asymptotic analysis is restricted to the singular behavior of the solution at the common

irregular point $x=0$, $y=0$. The analysis given in the previous sections indicate that $\beta_3 = \beta_4$ and the characteristic equations for $\beta_1, \beta_2, \beta_3$, and $\alpha$ are uncoupled. Hence $\beta_1, \beta_2$, and $\beta_3$ are given by (25) and (29).

One should note that theoretically the solution of the system of singular integral equations (19)-(21) contains four arbitrary real constants [16]. In the case of nonintersecting cracks these constants are determined from four singlevaluedness conditions of displacements given by (24). However, in the case of intersecting cracks, such as T or cross-shaped cracks considered in this section, kinematically it is clear that there is only one single-valuedness condition, namely (see the definitions (12) and (13))

$$\int_{a_2}^{2h_2} \phi_2(x_2)dx_2 - \int_0^{b_3} \phi_3(y)dy + \int_0^{b_1} \phi_1(x_1)dx_1 = 0. \tag{57}$$

The additional three conditions which are necessary for a unique solution of the system of integral equations are provided by (53). Note that with (54) satisfied, (53) gives three equations relating the end values $g_1(0),\ldots,$ $g_4(0)$. In the case of T-shaped crack $\phi_1 = 0$, (53) consists of three homogeneous equations and gives two independent conditions relating $g_2$, $g_3$ and $g_4$.

## 5. Stress Intensity Factors

A careful examination of the integral equations (19)-(21) would indicate that at a given irregular point if the displacement derivatives have a singularity of power $\alpha$, then the stress state is also singular having the same power $\alpha$. In applications it is important to know not only the power of stress singularity but also the coefficient of the singular term in the plane of projected (or conjectured) crack extension. This coefficient is known as the strength of the stress singularity or the stress intensity

factor. These coefficients can be evaluated in terms of the displacement derivatives $\phi_1,..,\phi_4$ by simply observing that in the integral equations (19)-(21) the expressions on the left hand side represent the related stress component outside as well as inside the cuts $(a_i, b_i)$, $(i=1,..,4)$.

(a) Imbedded Crack.

Consider the crack in material 2 and let $0<a_2<b_2<2h_2$ (Figure 2). Defining the sectionally holomorphic function

$$F(z) = \frac{1}{\pi} \int_{a_2}^{b_2} \frac{\phi_2(s)}{s-z} \, ds, \tag{58}$$

and using (24) and (25) it can be shown that

$$F(z) = \frac{g_2(a_2)e^{\pi i/2}}{(b_2-a_2)^{\frac{1}{2}}} \frac{1}{(z-a_2)^{\frac{1}{2}}} - \frac{g_2(b_2)}{(b_2-a_2)^{\frac{1}{2}}} \frac{1}{(z-b_2)^{\frac{1}{2}}}$$

$$+ O((z-c)^{\omega}), \quad (c=a_2 \text{ or } b_2, \ \text{Re}(\omega)>-\tfrac{1}{2}). \tag{59}$$

In (58) since $F(z)$ is holomorphic outside the cut, from (58) and (59) it follows that

$$\frac{1}{\pi} \int_{a_2}^{b_2} \frac{\phi_2(s)\,ds}{s-x_2} = F(x_2) = \frac{g_2(a_2)}{(b_2-a_2)^{\frac{1}{2}}} \frac{1}{(a_2-x_2)^{\frac{1}{2}}}$$

$$- \frac{g_2(b_2)}{(b_2-a_2)^{\frac{1}{2}}} \frac{1}{(x_2-b_2)^{\frac{1}{2}}} + O(|x_2-c|^{\omega}), \quad (c=a_2 \text{ or } b_2). \tag{60}$$

Referring to (19) and observing that the equation gives the expression for $\sigma_{2yy}(x_2, 0)$ for $0<x_2<2h_2$ and that the terms containing the kernels which are bounded at $x_2=a_2$ and $x_2=b_2$ would have no contribution to the singular behavior of the stress state at $a_2$ and $b_2$, the asymptotic behavior of $\sigma_{2yy}$ may now be obtained from (60). Thus, defining the stress intensity factors by

$$k(a_2) = \lim_{x_2 \to a_2} [2(a_2-x_2)]^{\frac{1}{2}} \, \sigma_{2yy}(x_2,0),$$

$$k(b_2) = \lim_{x_2 \to b_2} [2(x_2 - b_2)]^{\frac{1}{2}} \sigma_{2yy}(x_2, 0), \qquad (61a,b)$$

we find

$$k(a_2) = \frac{4\mu_2}{1+\kappa_2} \frac{g_2(a_2)}{[(b_2-a_2)/2]^{\frac{1}{2}}} = \lim_{x_2 \to a_2} \frac{4\mu_2}{1+\kappa_2} [2(x_2-a_2)]^{\frac{1}{2}} \phi_2(x_2),$$

$$k(b_2) = - \frac{4\mu_2}{1+\kappa_2} \frac{g_2(b_2)}{[(b_2-a_2)/2]^{\frac{1}{2}}}$$

$$= - \lim_{x_2 \to b_2} \frac{4\mu_2}{1+\kappa_2} [2(b_2-x_2)]^{\frac{1}{2}} \phi_2(x_2). \qquad (62a,b)$$

(b)  Crack Terminating at the Interface.

If the crack tip touches the interface the stress component of primary interest is the "cleavage" stress in the adjacent medium. For example, let $a_2=0$, $b_2=2h_2$, $a_1>0$, and $a_3>0$ (Figure 2). In this case the first equation of (19) may be written as

$$\frac{1+\kappa_1}{4\mu_1} \sigma_{1yy}(x_1, 0) = \frac{1}{\pi} \int_0^{2h_2} k_{12}^S(x_1, s) \phi_2(s) ds + H_1(x_1),$$

$$0 < x_1 < 2h_1, \qquad (63)$$

where $\phi_2$ is given by (24) with $\beta_2$ defined by (38b), $k_{12}^S$ is the singular part of $k_{12}$ and is given by (39), and $H_1(x_1)$ essentially represents all the remaining bounded terms. Defining again the sectionally holomorphic function $F(z)$ by (58) and substituting from (39) and (24), for the leading terms around $x_1=0$ we find

$$\frac{1}{\pi} \int_0^{2h_2} \frac{\phi_2(s) ds}{x_1 - s + 2h_2} \simeq \frac{g_2(2h_2)}{(2h_2)^{\frac{1}{2}} \sin \pi \beta_2} \frac{1}{x_1^{\beta_2}},$$

$$\frac{1}{\pi}\int_0^{2h_2} \frac{x_1\phi_2(s)ds}{(x_1-s+2h_2)^2} \cong \frac{\beta_2 g_2(2h_2)}{(2h_2)^{\frac{1}{2}}\sin\pi\beta_2} \frac{1}{x_1^{\beta_2}}, \qquad (64a,b)$$

If we define the stress intensity factor by

$$k(2h_2) = \lim_{x_1\to 0} \sqrt{2} \, x_1^{\beta_2} \, \sigma_{1yy}(x_1,0), \qquad (65)$$

from (63), (39) and (64) we obtain

$$k(2h_2) = \frac{4\mu_1}{1+\kappa_1} \frac{d_{11}+\beta_2 d_{12}}{\sin\pi\beta_2} \frac{g_2(2h_2)}{\sqrt{h_2}}. \qquad (66)$$

It should be noted that the stress components in the small neighborhood of the point $x_1=0$, $y=0$ may be expressed as

$$\sigma_{kij}(r,\theta)= \frac{k(2h_2)}{\sqrt{2} \, r^{\beta_2}} f_{kij}(\theta), \quad (k=1,2, \ 0\le\theta<\pi, \ i,j=r,\theta), \qquad (67)$$

where the functions $f_{kij}$ are given in [5].

(c)  Crack Crossing the Interface.

In the point of intersection of the crack and the interface even though one may again define a single stress intensity factor, from the viewpoint of applications it is more convenient to express the asymptotic forms of the normal and shear stresses along the interface separately. For example, let $b_2=2h_2$, $a_1=0$, $a_2=0$, $b_1<2h_1$, and $a_3>0$ (Figure 2).  In this case the interface stresses are given by (20) and (21), which, for small values of $y$, may be expressed as

$$\sigma_{xx}(0,y)= -\frac{2\mu_1}{\pi(1+\kappa_1)}\Big[\int_0^{b_1} k_{31}^s(y,s)\phi_1(s)ds$$

$$+ \int_0^{2h_2} k_{32}^s(y,s)\phi_2(s)ds\Big] + H_2(y),$$

$$\sigma_{xy}(0,y) = \frac{2\mu_1}{\pi(1+\kappa_1)}[\int_0^{b_1} k_{41}^S(y,s)\phi_1(s)ds$$

$$+ \int_0^{2h_2} k_{42}^S(y,s)\phi_2(s)ds] + H_3(y), \qquad (68a,b)$$

where the singular kernels are given by (46) and the functions $H_2$ and $H_3$ represent the remaining bounded terms. Substituting from (24) and (46) into (58) and a similar Cauchy integral with the density $\phi_1$ and using the results given by (50), from (68) one may easily obtain the asymptotic expressions for $\sigma_{xx}$ and $\sigma_{xy}$. The stress intensity factors for $\sigma_{xx}$ and $\sigma_{xy}$ may then be defined and evaluated as follows:

$$k_{xx}(0) = \lim_{y\to 0} y^\alpha \sigma_{xx}(0,y)$$

$$= -\frac{2\mu_1}{1+\kappa_1}\{-\frac{[A_1+(1-2\alpha)A_2]g_1(0)}{\sqrt{b_1}\sin(\pi\alpha/2)} + \frac{[A_2+(1-2\alpha)A_1]g_2(0)}{\sqrt{2h_2}\sin(\pi\alpha/2)}\},$$

$$k_{xy}(0) = \lim_{y\to 0} y^\alpha \sigma_{xy}(0,y)$$

$$= \frac{2\mu_1}{1+\kappa_1}\{\frac{[A_1-(1-2\alpha)A_2]g_1(0)}{\sqrt{b_1}\cos(\pi\alpha/2)} + \frac{[A_2-(1-2\alpha)A_1]g_2(0)}{\sqrt{2h_2}\cos(\pi\alpha/2)}\}$$

$$(69a,b)$$

where $\alpha=\alpha_1=\beta_2$ is obtained from the characteristic equation (41c) and the constants $A_1$ and $A_2$ are given by (47).

(d) Interface Crack.

In the case of an interface crack substituting from (28) into (27) it can be shown that the asymptotic behavior of the contact stresses in the small neighborhood of the crack tips is of the following form [2]:

$$\sigma_{1xy}(0,y)-i\sigma_{1xx}(0,y) \cong \frac{M(y)}{w_0(y)},$$

$$w_0(y) = (y-b_3)^{\beta_3}(a_3-y)^{\alpha_3}, \quad y<a_3, \quad b_3<y, \tag{70}$$

where M is a bounded function. Since $\alpha_3$ and $\beta_3$ are complex, one may define the stress intensity factors as follows:

$$k_1(b_3)+ik_2(b_3)=\lim_{y\to b_3} w_0(y)[\sigma_{1xx}(0,y)+i\sigma_{1xy}(0,y)],$$

$$k_1(a_3)+ik_2(a_3)=\lim_{y\to a_3} w_0(y)[\sigma_{1xx}(0,y)+i\sigma_{1xy}(0,y)]. \tag{71a,b}$$

In terms of the density function $\psi(y)$ defined by (26) and (28) these complex stress intensity factors may be expressed as [2]

$$k_1(b_3)+ik_2(b_3)=-g_3(b_3)\frac{\mu_1 a_{21}}{1+\kappa_1}\sqrt{1-\gamma^2},$$

$$k_1(a_3)+ik_2(a_3)=g_3(a_3)\frac{\mu_1 a_{21}}{1+\kappa_1}\sqrt{1-\gamma^2}, \tag{72a,b}$$

where $g_3(y)$ is obtained from the solution of the system of integral equations (19)-(21). Note that because of the definition (71) the dimension of the stress intensity factors is $\sigma\ell$ rather than the conventional $\sigma\sqrt{\ell}$.

After calculating the stress intensity factors the strain energy release rate for the crack propagating along the interface may be obtained from [2]

$$\frac{\partial U}{\partial \ell_3} = \frac{\pi(1+\kappa_1)}{2\mu_1 a_{21}\ell_3}(k_1^2 + k_2^2), \quad \ell_3=(b_3-a_3)/2. \tag{73}$$

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## Appendix A

The functions $R_1(t),..,R_7(t)$.

$$R_1(t) = \frac{1}{\kappa_1+1}\int_{a_1}^{b_1} t[(2h_1+s)e^{-t(4h_1+s)}$$

$$-(2h_1-s)e^{-t(4h_1-s)}]e^{2h_1t}\phi_1(s)ds,$$

$$R_2(t) = \frac{1}{\kappa_1+1}\int_{a_1}^{b_1}\{[1-(2h_1+s)t]e^{-t(4h_1+s)}$$

$$-[1-(2h_1-s)t]e^{-t(4h_1-s)}\}e^{2h_1t}\phi_1(s)ds,$$

$$R_3(t) = \frac{1}{\kappa_2+1}\int_{a_2}^{b_2}2tse^{-ts}\phi_2(s)ds,$$

$$R_4(t) = \frac{\mu_1}{\kappa_1+1}\int_{a_1}^{b_1}2tse^{-ts}\phi_1(s)ds - \frac{\mu_2}{\kappa_2+1}\int_{a_2}^{b_2}t[(2h_2+s)e^{-t(4h_2+s)}$$

$$-(2h_2-s)e^{-t(4h_2-s)}]e^{2h_2t}\phi_2(s)ds,$$

$$R_5(t) = -\frac{\mu_2}{\kappa_2+1}\int_{a_2}^{b_2}\{[1-(2h_2+s)t]e^{-t(4h_2+s)}$$

$$-[1-(2h_2-s)t]e^{-t(4h_2-s)}\}e^{2h_2t}\phi_2(s)ds,$$

$$R_6(t) = \frac{1}{\kappa_2+1}\int_{a_2}^{b_2}\{[\frac{\kappa_2-1}{2} + (2h_2+s)t]e^{-t(2h_2+s)}$$

$$-[\frac{\kappa_2-1}{2} + (2h_2-s)t]e^{-t(2h_2-s)}\}\phi_2(s)ds + \int_{a_3}^{b_3}\phi_4(s)\sin ts\, ds,$$

$$R_7(t) = \frac{1}{\kappa_1+1}\int_{a_1}^{b_1}[2ts-(\kappa_1+1)]e^{-ts}\phi_1(s)ds$$

$$+ \frac{1}{\kappa_2+1}\int_{a_2}^{b_2}\{[\frac{\kappa_2+1}{2} - (2h_2+s)t]e^{-t(2h_2+s)}$$

$$-[\frac{\kappa_2+1}{2} - (2h_2-s)t]e^{-t(2h_2-s)}\}\phi_2(s)\, ds$$

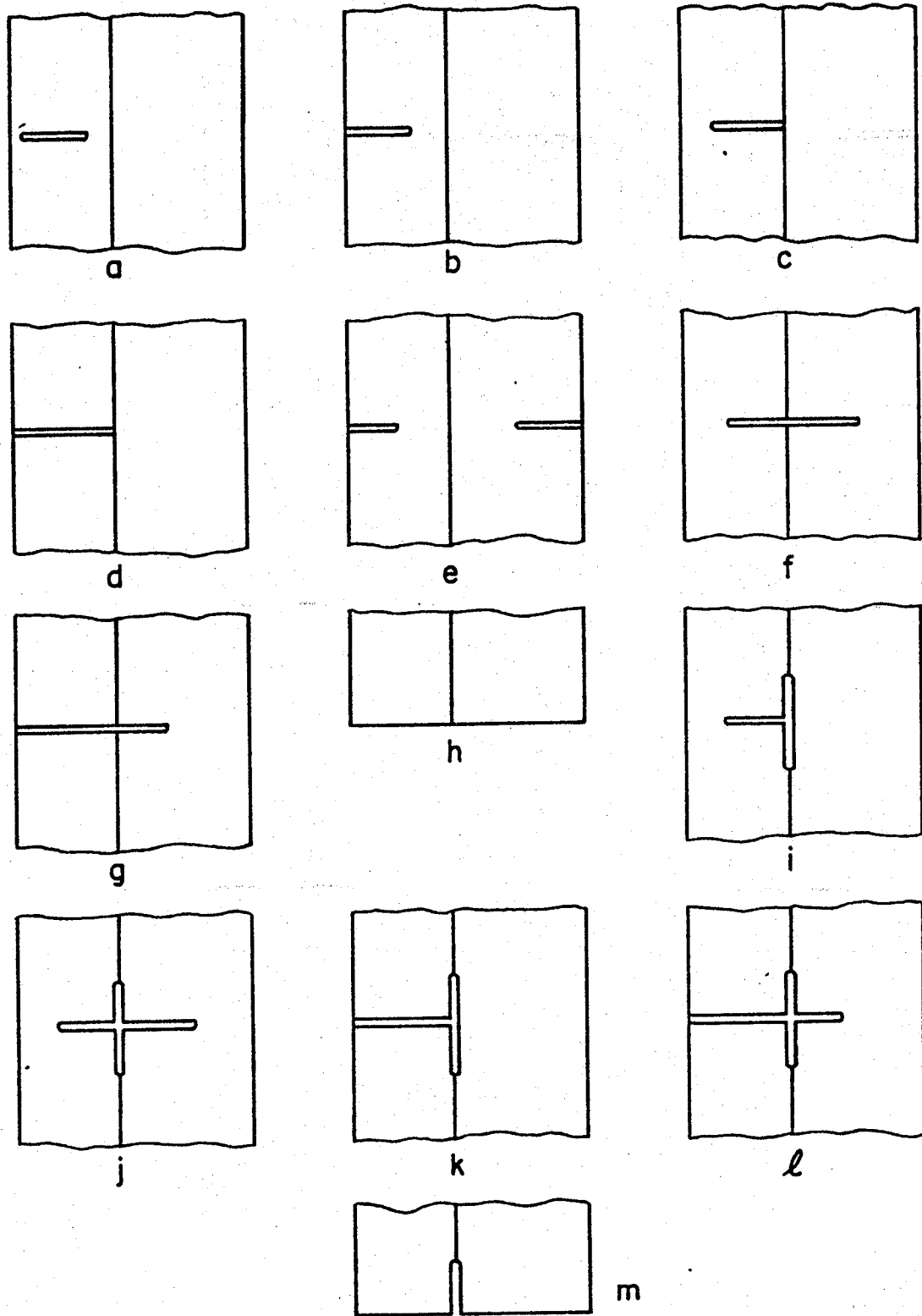$$+ \int_{a_3}^{b_3}\phi_3(s)\cos ts\, ds.$$
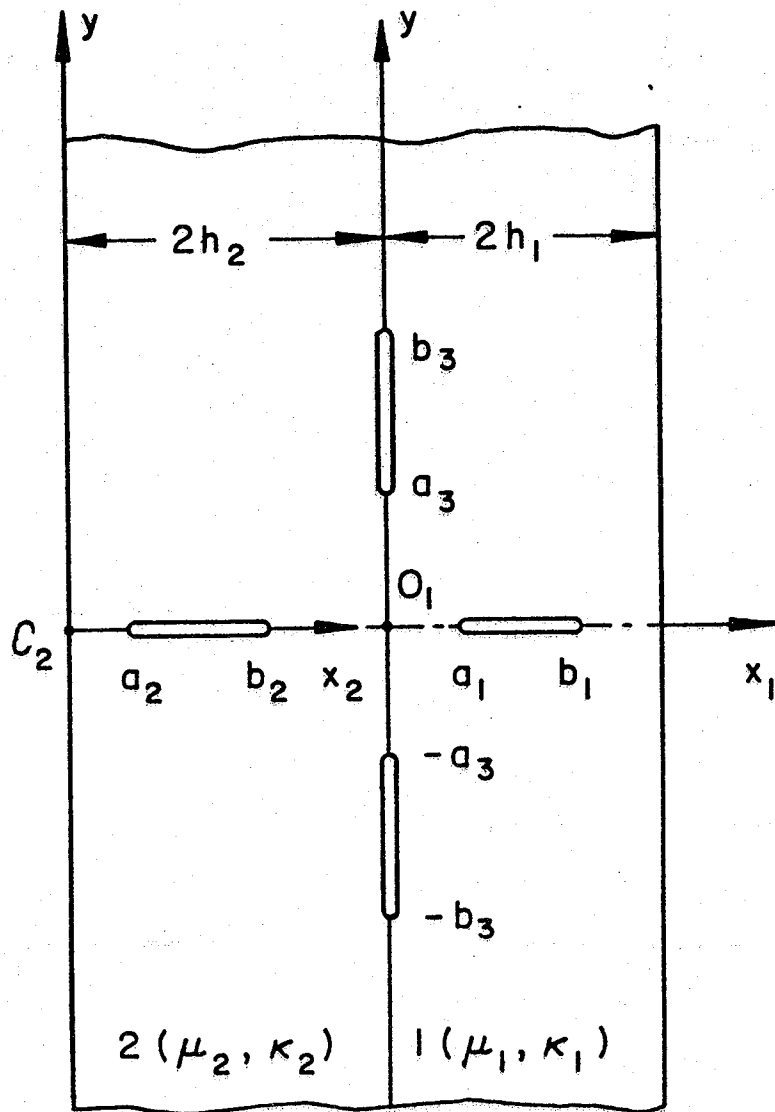
Figure 1. Crack configurations considered in the paper.

Figure 2.    Geometry and notation of the crack problem.