NASA TECHNICAL MEMORANDUM NASA TM-75758

THEORY OF TWISTED NONUNIFORMLY HEATED BARS

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Translation of "K teorii zakruchennykh neravnomerno nagretykh sterzhney," Izvestiya Akademii nauk SSSR, Otdeleniye tekhnicheskikh nauk, mekhanika i mashinostroyeniye, No. 1, January-February 1960, pp. 141-151.

(NASA-TM-75758) THEORY OF TWISTED NONUNIFORMLY HEATED BARS (National Aeronautics and Space Administration) 21 p HC A02/MF A01 CSCL 20K

N80-19565

Unclas G3/39 47455

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 FEBRUARY 1980

THEORY OF TWISTED NONUNIFORMLY HEATED BARS

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An approximate theory of twisted, nonuniformly heated bars of arbitary cross section is discussed, with the nonlinear distribution of normal stresses taken into account, as applicable, according to experimental data, up to twist parameter $\beta^2 \approx 5$. /141*

The theory of twisted bars, which is of great importance in calculation of airscrews and compressor and turbine blades, has been developed in two directions. In the studies of P.M. Riz [1, 2], A.I. Lur'ye and G.Yu. Dzhanelidze [3] and others, the problem was solved by the methods of elasticity theory. Because of the complexity of the solution, they were reduced to final form, only for bars of the simplest cross sections. V.P. Vetchinkin and N.N. Polyakhov [4] and I.A. Birger and the author (in 1954) proposed approximate methods, which are applicable to uniformly heated bars with specific types of cross sections. A general theory, which imposes no restrictions on cross section shape and is valid for both slightly and moderately twisted bars (a refined classification of bars by degree of twist will be established below), with nonuniform heat- and with account taken of variable elasticity parameters, is presented below.

1. We consider a bar of constant cross section with, in the unstressed state, a uniform twist relative to the rectilinear z_1 axis, which passes through a certain point i normal to the cross section axis (Fig. 1).

We limit ourselves to analysis of bars, for which, before and after deformation, the following condition holds true

$$(\alpha R)^2 \ll 1 \qquad (1.1)$$

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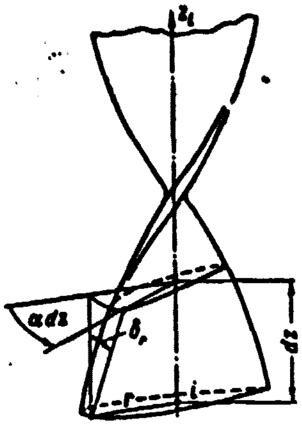

Fig. 1

where α is the relative angle of twist of the bar, R is the distance from the z_1 axis to the most remote point of the section.

Since

$$\alpha r = \operatorname{tg} \delta_r \tag{1.2}$$

where δ_r is the angle between the z_1 axis and the screw line which connects the corresponding points of adjacent sections, condition (1.1) is equivalent to the assumption that this angle is small, which /142 permits it to be considered that

$$\sin \delta_r \approx \delta_r \approx \alpha r, \qquad \cos \delta_r \approx 1 \qquad (0 \leqslant r \leqslant R) \tag{1.3}$$

Together with plane of the cross section Π_0, we introduce "orthogonal section"[1] Π, a surface, the shape of which would be the cross section, if the bar were reduced to the twisted state ($\alpha=\alpha_0$) from the untwisted state ($\alpha=0$) by free twisting (Fig. 2).

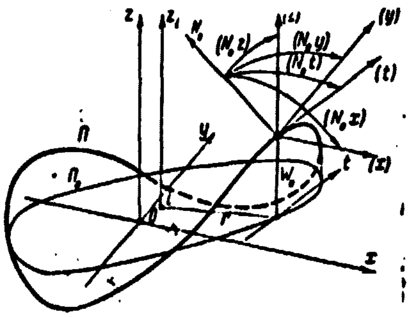

Fig. 2.

By definition, the orthogonal section coincides with the warp surface at $\alpha=\alpha_0$ and, with condition (1.1), it is described by the equation [5, 6]

$$w_0 = \alpha_0 \varphi(x, y) \tag{1.4}$$

in which

$$(\alpha_0 \varphi_x')^2 < 1, \qquad (\alpha_0 \varphi_y')^2 < 1 \tag{1.5}$$

where $\phi(x, y)$ is the twisting function, $\phi_x' = \partial\phi/\partial x, \phi_y' = \partial\phi/\partial y$; x, y, z is the rectangular coordinate system which moves together with the cross section. For bars with variable elasticity parameters E(x, y) and

[1] The term "orthogonal section" was proposed by I.A. Birger, as applied to bars of elongated cross section, for which section Π can be identified approximately with the surface normal to the screw lines.

and G(x,y), function ϕ should satisfy the equation [7]

$$(G\varphi_x')_x' + (G\varphi_y')_y' = G_x'(y - y_1) - G_y'(x - x_1) \tag{1.6}$$

and the boundary conditions on the profile

$$(\varphi_x' - y + y_1)\,dy - (\varphi_y' + x - x_1)\,dx = 0 \tag{1.7}$$

Because of condition (1.5), no distinction can be made between the areas and equations of the profiles of the orthogonal and plane sections. To be definite, we assume that a given orthogonal section Π corresponds to the plane section Π_0, with respect to which

$$\int_F G\varphi\,dF = 0 \tag{1.8}$$

Eq. (1.4) characterizes the configuration of a twisted bar, to within the position of twist axis z_1. The final results do not depend on the direction which satisfies conditions (1.1), selected as the direction of the axis; however, as for prismatic bars, the theory is simplified if it is assumed that the z_1 axis passes through the center of rigidity of the section.

Since, corresponding to twist α_0, the initial "shifts"

$$\gamma_{zx}^0 = \alpha_0(\varphi_x' - y + y_1), \qquad \gamma_{zy}^0 = \alpha_0(\varphi_y' + x - x_1) \tag{1.9}$$

should not depend on selection of the position of the axis,

$$\varphi_x' = \varphi_{0x}' - y_1, \qquad \varphi_y' = \varphi_{0y}' + x_1 \tag{1.10}$$

Here, ϕ_0 is the twisting function, determined as usual, on condition that the axis of rotation passes through the coordinate origin.

2. With condition (1.1), the elastic twist deformation can be considered as a continuation of the initial "deformation," and the shifts can be determined by the formulas

$$\frac{\partial u}{\partial z} = -\theta(y - y_i), \qquad \frac{\partial v}{\partial z} = \theta(x - x_i), \qquad w = \theta\varphi \tag{2.1}$$

where θ is the relative angle of elastic twisting which, at small initial twist angles, can be commensurable with α_0.

The twisting stress in an orthogonal section is

$$\tau_{zx}^{(1)} = C\theta(\varphi_x' - y + y_i), \qquad \tau_{zy}^{(1)} = C\theta(\varphi_y' + x - x_i) \tag{2.2}$$

For determination of the normal stresses, we consider that, at all points, the profile of the orthogonal section is normal to the open lateral surface of the bar. Therefore, it can be proposed that the tangential stresses on this section, which are not connected with twisting (at θ=0), are of secondary importance (just as for a plane section in the theory of transverse bending of beams). Therefore, as the initial direction of a longitudinal fiber of a twisted rod N_0 (Fig. 2), it is natural to assume the direction normal to the orthogonal surface with the directing cosines /143

$$l_0 = \cos(N_0x) = -\alpha_0\varphi_x', \quad m_0 = \cos(N_0y) = -\alpha_0\varphi_y', \quad n_0 = \cos(N_0z) = 1 \tag{2.3}$$

and, following the general theory of bars, to disregard the pressure of one longitudinal fiber on another, i.e., to consider

$$\sigma = E(\varepsilon - \gamma t) \tag{2.4}$$

where ε is the longitudinal stress and γt is the thermal expansion of the fiber.

It is evident that, in the general case, the directions of the longitudinal and screw fibers do not coincide.

The length of a unit fiber in the initial state is

$$ds_0 = dz\sqrt{1 + l_0^2 + m_0^2} \approx dz\left[1 + \frac{1}{2}\alpha_0^2(\varphi_x'^2 + \varphi_y'^2)\right] \tag{2.5}$$

and, after deformation,

$$ds = dz\sqrt{(1+\varepsilon_z)^2 + l^2 + m^2} \tag{2.6}$$

where ε_z is the deformation of the fiber parallel to the rod axis, $\ell = \ell_0 + \Delta\ell$, $m = m_0 + \Delta m$ and $\Delta\ell$, Δm are the increments of the angles of inclination of the longitudinal fiber, in connection with elastic twisting of the bar.

Based on the "orthogonal sections" hypothesis similar to the hypotheses of plane or irregular sections, we assume that, in stretching and bending, the orthogonal section does not change its shape, moving according to the laws of solids. Then [8],

$$\varepsilon_z = \varepsilon_0 - \varkappa_x x - \varkappa_y y + \dot{\theta}\varphi \tag{2.7}$$

where ε_0, κ_x, κ_y are components of the plane of deformation, $\dot{\theta} = d\theta/dz$, and

$$\Delta l = -\theta(y - y_i), \qquad \Delta m = \theta(x - x_i) \tag{2.8}$$

In conformance with (2.3) and (2.8), Eq. (2.6) takes the form

$$ds \approx dz\left\{1 + \varepsilon_z + \tfrac{1}{2}[\alpha_0\varphi_x' + \theta(y - y_i)]^2 + \tfrac{1}{2}[\alpha_0\varphi_y' - \theta(x - x_i)]^2\right\} \tag{2.9}$$

from which

$$\varepsilon = \varepsilon_z - \alpha_0\theta\varphi_\psi' + \tfrac{1}{2}\theta^2 r^2 \tag{2.10}$$

where polar coordinates r, ψ, with the pole in the center of rigidity, are introduced, and it is taken into consideration that

$$\varphi_\psi' = \frac{\partial\varphi}{\partial\psi} = (x - x_i)\varphi_y' - (y - y_i)\varphi_x' \tag{2.11}$$

The normal stress in the orthogonal section with direction $N(\ell, m, n)$ is

$$\sigma = E(\varepsilon_0 - \varkappa_x x - \varkappa_y y + \dot{\theta}\varphi - \alpha_0\theta\varphi_\psi' + \tfrac{1}{2}\theta^2 r^2 - \gamma t) \tag{2.12}$$

Besides torsional stress $\tau^{(1)}$, tangential shear stresses $\tau^{(2)}$ act in the orthogonal sections. The latter are connected with stress σ by the equation of equilibrium /144

$$\frac{\partial \sigma}{\partial z} + \frac{\partial \tau_{zx}^{(2)}}{\partial x} + \frac{\partial \tau_{zy}^{(2)}}{\partial y} + p = 0 \qquad (2.13)$$

where p is the bulk force component in the direction of the longitudinal fiber.

The total tangential stresses

$$\tau = \tau^{(1)} + \tau^{(2)}$$

The stresses in the plane of the section can be found from the conditions of equilibrium of a longitudinal element bounded by sections Π and Π_0 (Fig. 3), which gives

$$\sigma^0 = \sigma + \alpha(\tau_{zx}\varphi_x' + \tau_{zy}\varphi_y') + \alpha[(\varphi\tau_{zx})_x' + (\varphi\tau_{zy})_y']$$

$$(\alpha = \alpha_0 + \theta) \qquad (2.14)$$

$$\tau_{zx}^0 = \tau_{zx} - \sigma[\alpha_0\varphi_x' + \theta(y - y_1)], \qquad \tau_{zy}^0 = \tau_{zy} - \sigma[\alpha_0\varphi_y' - \theta(x - x_1)]$$

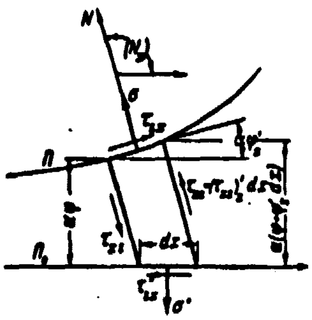

Fig. 3.

In particular, in the case of stretching of the bar, by dropping out all nonlinear terms, we obtain, with $x_1 = y_1 = 0$ and $t = 0$,

$$\sigma^0 = \sigma = \frac{P}{F}, \quad \tau_{zx}^0 = -\alpha_0\sigma\left[y + \frac{I_p}{T}(\varphi_x' - y)\right]$$

$$\tau_{zy}^0 = -\alpha_0\sigma\left[-x + \frac{I_p}{T}(\varphi_y' + x)\right] \qquad (2.15)$$

Here, I_p is the polar moment of inertia, and T is the geometric rigidity in twisting, which coincides with the precise solution of P.M. Riz [1].

At points where $\phi = 0$, Eq. (2.14) coincide with the conventional formulas for the stress in inclined areas.

With the force factors in the plane of the section considered to be known, we write the equilibrium conditions in the form

$$P=\int_F \sigma^\circ dF, \qquad Q_x=\int_F \tau_{zx_t}^\circ dF, \qquad Q_y=\int_F \tau_{zy}^\circ dF$$

$$M_x=\int_F y\sigma^\circ dF, \quad M_y=-\int_F x\sigma^\circ dF, \quad M_{z_t}=\int_F [\tau_{zy}^\circ (x-x_t)-\tau^\circ_{zx}(y-y_t)]\,dF \tag{2.16}$$

We note that, between the force factors in a twisted bar, there are the relationships

$$Q_x=-\frac{dM_y}{dz}+\varkappa (Py_t-M_z)$$
$$Q_y=\frac{dM_x}{dz}-\varkappa (Px_t+M_y) \tag{2.17}$$

which Eq. (2.16) identically satisfy.

By substituting the values of the stresses in the first and last three equations of (2.16), we obtain a system of four equations, which are linear with respect to the components of the plane of deformation ε_0, κ_x, κ_y, and nonlinear relative to angle θ. Without presenting the general expressions, because of their bulk, we consider some applications of theoretical or practical value.

3. For profiles of dimensions x, y of one order, tangential stresses $\tau^{(2)}$ and the constrained twisting effect can be disregarded.

In this case, we arrive at the following system of equations: /145

$$P+P_t=E_m\left\{\varepsilon_0 F-\varkappa_x S_y-\varkappa_y S_x+\theta\left[\alpha_0 (I_p-T)+\frac{1}{2}\theta I_p\right]\right\}$$
$$M_x+M_{xt}=E_m\left\{\varepsilon_0 S_x-\varkappa_x I_{xy}-\varkappa_y I_x+\theta\left[\alpha_0 (I_{px}-T_x)+\frac{1}{2}\theta I_{px}\right]\right\} \tag{3.1}$$
$$M_y+M_{yt}=-E_m\left\{\varepsilon_0 S_y-\varkappa_x I_y-\varkappa_y I_{xy}+\theta\left[\alpha_0 (I_{py}-T_y)+\frac{1}{2}\theta I_{py}\right]\right\}$$
$$M_z=G_m T\theta+E_m\left\{\varepsilon_0 [\alpha_0 (I_p-T)+\theta I_p]-\varkappa_x [\alpha_0 (I_{py}-T_y)+\theta I_{py}]-\right.$$
$$\left.-\varkappa_y [\alpha_0 (I_{px}-T_x)+\theta I_{px}]+\theta\alpha_0 [\alpha_0 (I_r-T_r)+\frac{3}{2}(I_r-T_{r\varphi})]+\frac{1}{2}\theta^2 I_r\right\}-$$
$$-\alpha_0 (B_t-B_{t\varphi})-\theta B_t$$

Here, we introduce designations for the reduced geometric characteristics

$$
\begin{gathered}
S_x = \int_F E^* y\,dF, \quad S_y = \int_F E^* x\,dF, \\
I_x = \int_F E^* y^2 dF, \quad I_y = \int_F E^* x^2 dF, \quad I_{xy} = \int_F E^* xy\,dF \\
I_\rho = \int_F E^* r^2 dF, \quad I_{\rho x} = \int_F E^* r^2 y\,dF, \quad I_{\rho y} = \int_F E^* r^2 x\,dF, \quad I_r = \int_F E^* r^4 dF \\
T = \int_F E^* (\varphi_\psi' + r^2)\,dF, \quad T_x = \int_F E^*_y (\varphi_\psi' + r^2)\,dF, \quad T_y = \int_F E^* x (\varphi_\psi' + r^2)\,dF \\
T_r = \int_F E^* [r^4 - (\varphi_\psi')^2]\,dF, \quad T_{r\varphi} = \int_F E^* r^2 (\varphi_\psi' + r^2)\,dF \\
\left(E^* = \frac{E}{E_m} = \frac{G}{G_m}, \; E_m = \frac{1}{F}\int_F E\,dF = 2(1+\mu)G_m, \; \mu = \text{const} \right)
\end{gathered} \tag{3.2}
$$

and temperature factors

$$
\begin{gathered}
P_t = \int_F E\gamma t\,dF, \quad M_{xt} = \int_F E\gamma t y\,dF, \quad M_{yt} = -\int_F E\gamma t x\,dF \\
B_t = \int_F E\gamma t r^2 dF, \quad B_{t\varphi} = \int_F E\gamma t (\varphi_\psi' + r^2)\,dF
\end{gathered} \tag{3.3}
$$

In the derivation of system (3.1), it was taken into account that, because of the properties of twisting function ϕ and the assumption $w_0 \ll R$, the role of the terms which reflect the effect of tangential stresses on quantities P, M_x, M_y is insignificant, within the limits of accuracy of the theory. For simplicity in recording, subscript 1 in moment M_x is dropped subsequently.

By proper selection of the coordinate origin and directions of the x, y axes, condition $S_x = S_y = I_{xy} = 0$ can be ensured. By introducing the principal values of the components of the plane of deformation

$$
\begin{gathered}
\varepsilon^* = \varepsilon_0 + \frac{\theta}{F}\left[\alpha_0 (I_\rho - T) + \frac{1}{2}\theta I_\rho\right] \\
\varkappa_x^* = \varkappa_x - \frac{\theta}{I_y}\left[\alpha_0 (I_{\rho y} - T_y) + \frac{1}{2}\theta I_{\rho y}\right] \\
\varkappa_y^* = \varkappa_y - \frac{\theta}{I_x}\left[\alpha_0 (I_{\rho x} - T_x) + \frac{1}{2}\theta I_{\rho x}\right]
\end{gathered} \tag{3.4}
$$

the principal polar coordinate /146

$$r_*^2 = r^2 - \frac{I_p}{F} - \frac{I_{py}}{I_y}x - \frac{I_{px}}{I_x}y \quad (r^2_* \gtrsim 0) \tag{3.5}$$

and the principal value of function ϕ^*

$$(\varphi^*)_\psi' = \varphi_\psi' + \frac{I_p - T}{F} + \frac{I_{py} - T_y}{I_y}x + \frac{I_{px} - T_x}{I_x}y \tag{3.6}$$

which satisfy the conditions

$$I_p{}^* = I_{px}{}^* = I_{py}{}^* = 0, \qquad T^* = T_x{}^* = T_y{}^* = 0$$

where $I_n{}^*$, $T_n{}^*$ (n is an arbitrary subscript), are determined by Eq. (3.2), with substitution of r_*^2 for r^2 and $(\phi^*)_\psi{}'$ for $\phi_\psi{}'$, we represent the first three equations of (3.1) in the form

$$\varepsilon^* = \frac{P + P_t}{E_m F}, \qquad \varkappa_x{}^* = \frac{M_y + M_{yt}}{E_m I_y}, \qquad \varkappa_y{}^* = -\frac{M_x + M_{xt}}{E_m I_x} \tag{3.7}$$

Eq. (3.7) are the same in form as the corresponding relationships for rectangular bars. However, in twisted bars, parameters $\kappa_x{}^*$ and $\kappa_y{}^*$ do not coincide with the components of the elastic curvature of the rod axis or ε^* with its elongation.

It follows from (2.12), (3.4)-(3.7) that, with $\dot{\theta} = 0$,

$$\sigma = E\left[\frac{P + P_t}{E_m F} - \frac{M_y + M_{yt}}{E_m I_y}x + \frac{M_x + M_{xt}}{E_m I_x}y - \alpha_0\theta(\varphi^*)_\psi' + \frac{1}{2}\theta^2 r_*^2 - \gamma t\right] \tag{3.8}$$

The last equation of (3.1) can be presented in the form

$$\theta_0 - \alpha_0 v^* = \theta(1 + \nu + \beta^2) + 3(1 + \mu)\alpha_0\theta^2\frac{I_{r^*} - T_{r^*}{}^*}{T} + (1 + \mu)\theta^3\frac{I_r{}^*}{T} \tag{3.9}$$

where

$$\theta_0 = \frac{M_t}{C_m T} \tag{3.10}$$

$$v^* = \frac{1}{C_m T}\left[\frac{P}{F}(I_p - T) - \frac{M_y}{T_y}(I_{py} - T_y) + \frac{M_x}{T_x}(I_{px} - T_x) - (B_t^* - B_{t\varphi}^*)\right] \tag{3.11}$$

$$v = \frac{1}{C_m T}\left(\frac{P}{F} I_p - \frac{M_y}{T_y} I_{py} + \frac{M_x}{T_x} I_{px} - B_t^*\right) \tag{3.12}$$

where B_t^* and $B_{t\phi}^*$ are determined by Eq. (3.3), with the substitution of r_*^2 for r^2 and $(\phi^*)_\psi{}'$ for $\phi_\psi{}'$, and

$$\beta^2 = 2(1+\mu)\,\alpha_0^2 \frac{I_r^* - T_r^*}{T} \tag{3.13}$$

With $\alpha_o=0$ (initially untwisted bar), Eq. (3.9) changes to

$$(1+\mu)\frac{I_r^*}{T}\theta^3 + (1+v)\theta = \theta_0 \tag{3.14}$$

For cases of stretching and twisting bars of circular and elongated rectangular cross sections, with t=0, formulas are obtained from (3.14), which coincide with the results of S.P. Timoshenko [9]. With $\theta_o=0$, Eq. (3.14), besides the trivial solution $\theta=0$, has a second solution

$$\theta = \pm\sqrt{-\frac{T(1+v)}{I_r^*(1+\mu)}} \tag{3.15}$$

which takes on a real value at $v \leq -1$. /147

By assuming in (3.12), the value $v=-1$, we find the condition for loss of stability of a rectangular bar due to twisting. At $M_x=M_y=0$, t=0, this gives the known formula for the value of the critical force [10]

$$P_* = -\frac{GTF}{I_p}$$

4. By disregarding the θ^2 and θ^3 terms in Eq. (3.9), we solve it for θ,

$$\theta = k_\theta(\theta_0 - \alpha_0 \nu^*) \qquad \left(k_\theta = \frac{1}{1+\nu+\beta^2}\right) \tag{4.1}$$

The formula for the effective geometric torsional rigidity of the twisted bar results from relationship (4.1)

$$T_\alpha = T(1+\nu+\beta^2)$$

which, at $\nu \ll 1+\beta^2$, coincides with the formula obtained for this case by Chen Chu [11], as applied to hollow tubes with an elongated bisymmetrical section.

Dimensionless coefficient β^2 plays an important part in the theory of twisted bars, and it can be called the twist parameter. Depending on the value of β^2, it is expedient to provisionally divide twisted bars into three groups.

a. Slightly twisted bars, $\beta^2 \ll 1$. In this case, with $\theta_0 = 0$, elastic twisting angle θ is proportional to angle α_0, strains ϵ and ϵ_z practically coincide, and normal stresses σ can be calculated by the conventional formula for a rectangular bar.

For a section with two axes of symmetry, with $t=0$ and $\nu \ll 1 (k_\theta \approx 1)$, we have

$$\begin{aligned} \varepsilon_0 &= \frac{P}{EF} - \alpha_0 \frac{M_t}{CF}\left(\frac{I_p}{T} - 1\right), & \varkappa_x &= \frac{M_y}{EI_y} \\ \theta &= \frac{M_t}{CT} - \alpha_0 \frac{P}{CF}\left(\frac{I_p}{T} - 1\right), & \varkappa_y &= -\frac{M_x}{EI_x} \end{aligned} \tag{4.2}$$

which coincides with the results of G.Yu. Dzhanelidze [12].

b. Moderately twisted bars, the value of β^2 is commensurable with unity. With increase of β^2, the torsional rigidity of the bar increases considerably, which is confirmed by experiments [11], the tensile and bending rigidities decrease (for asymmetrical profiles), and the normal stresses in the orthogonal section (and the stresses practically equal to them in the plane of the cross section) are redistributed according to relationship (3.8).

As was shown by the tensile tests of twisted bars by the author [13], the approximate theory reported, based on assumptions (1.1), gives good accuracy up to $\beta^2 \leqslant 5$.

For blades and air screws, the value of β^2 can reach 2-3 or more.

Together with the concise notation of (3.7) and (4.1), the basic system of equations of a twisted bar of arbitary cross section, with $\beta^2 \leqslant 5$ and $\nu \ll 1+\beta^2$, can be represented in the following developed form

$$\varepsilon_j = \sum_{k=1}^{4} a_{jk} L_k \qquad (4.3)$$

where ε_j should be understood to be strain components ε_o, κ_x, κ_y, θ and L_k, the force and temperature factors $P+P_t$, M_y+M_{yt}, M_x+M_{xt}, $M_z+\alpha_o(B_t-B_{t\phi})$, and coefficients a_{jk}, which satisfy the theorem of reciprocity, have the following values /148

$$(4.4)$$

$$a_{11} = \frac{1}{E_m F}\left[1 + 2(1+\mu)k_\psi \alpha_0^2 \frac{(I_p - T)^2}{TF}\right], \qquad a_{14} = a_{41} = -k_\psi \alpha_0 \frac{I_p - T}{G_m T F}$$

$$a_{22} = \frac{1}{E_m I_y}\left[1 + 2(1+\mu)k_\psi \alpha_0^2 \frac{(I_{py} - T_y)^2}{T I_y}\right], \qquad a_{24} = a_{42} = k_\psi \alpha_0 \frac{I_{py} - T_y}{G_m T I_y}.$$

$$a_{33} = -\frac{1}{E I_x}\left[1 + 2(1+\mu)k_\psi \alpha_0^2 \frac{(I_{px} - T_x)^2}{T I_x}\right], \qquad a_{34} = -a_{43} = k_\psi \alpha_0 \frac{I_{px} - T_x}{G_m T I_x}$$

$$a_{44} = \frac{k_\psi}{G_m T}, \quad a_{23} = -a_{32} = -k_\psi \alpha_0^2 \frac{(I_{px} - T_x)(I_{py} - T_y)}{G_m T I_x I_y}$$

$$a_{12} = a_{21} = -k_\psi \alpha_0^2 \frac{(I_p - T)(I_{py} - T_y)}{G_m T F I_y}, \quad a_{13} = -a_{31} = k_\psi \alpha_0^2 \frac{(I_p - T)(I_{px} - T_x)}{G_m T F I_x},$$

c. Strongly twisted bars, $\beta^2 > 5$. A general theory of such bars has not been developed.

It follows from Eq. (3.2) that, in the case of slightly warped profiles, for which $\phi \approx 0$, the equality $T_n \approx I_n$ occurs, and all the terms which contain angle α_0 revert to zero, i.e., in this case, the initial twist does not play a part. An example of such a bar is a round cylinder.

5. Twisting has the greatest effect on bars with elongated, strongly warped profiles, for which the value of integrals (3.2) depends basically on the values of the subintegral functions at points remote from the rod axis, where r>>h (h is the greatest thickness of the profile), and the initial "shifts" in the plane normal to the radius

$$\gamma_{z\psi}{}^{\circ} = \alpha_0\left(r + \frac{1}{r}\varphi_\psi{}'\right) \tag{5.1}$$

become small, compared with the initial "bends" of the fiber

$$\omega_{z\psi}{}^{\circ} = \frac{1}{2}\alpha_0\left(r - \frac{1}{r}\varphi_\psi{}'\right) \tag{5.2}$$

With the assumption that $\gamma_{z\psi}{}^{\circ} \approx 0$, we find (Fig. 2)

$$\varphi_\psi{}' \approx -r^2, \qquad \cos(N_0 t) = -\frac{\alpha_0}{r}\varphi_\psi{}' \approx \alpha_0 r = \delta_r \tag{5.3}$$

i.e., at points sufficiently remote from the axis, the longitudinal fiber coincides with the screw line, and the orthogonal section, with a section of the corresponding screw surface, as was adopted by I.A. Birger.

Because of equality (5.3), we have

$$T_n \approx 0,\ (\varphi^*)_\psi{}' \approx -r_*^2, \quad T_n{}^* \approx 0,\ B_{t\varphi} \approx 0,\ B_{t\psi}{}^* \approx 0,\ \nu^* \approx \nu$$

and all formulas of Sections 3 and 4 are significantly simplified.

Instead of relationship (2.3), there will be

$$l_0 \approx -\alpha_0(y - y_1), \qquad m_0 \approx \alpha_0(x - x_1) \tag{5.4}$$

and, instead of (2.12), with $\theta \ll \alpha_0$, /149

$$\sigma = E(\varepsilon_0 - \varkappa_x x - \varkappa_y y + \dot{\theta}\varphi + \alpha_0 \theta r^2 - \gamma t) \tag{5.5}$$

If, to condition (1.8) adopted above

$$\int_F E\varphi y dF = 0 \qquad \int_F E\varphi x dF = 0 \tag{5.6}$$

is added, formulas (3.7) remain in force, and conditions (5.6) are used to determine the coordinates of the center of rigidity.

By the substitution of (5.5) in torque Eq. (2.16), without consideration of the θ^2 and θ^3 terms, we obtain

$$\theta_0 = \alpha_0 \nu = \theta(1 + \nu + \beta^2) + \frac{M_z^{(\tau)}}{G_m T} + \frac{2(1+\mu)}{T}\alpha_0 \dot{\theta} \int_F \bar{E}\varphi r_*^2 dF \tag{5.7}$$

where $M_z^{(\tau)}$ is the torque of tangential stresses $\tau^{(2)}$, which can be found for elongated sections without determining the stresses themselves, if it is assumed that $\phi_r' \ll r^{-1}\phi_\psi'$, i.e., that warping is reduced primarily to rotation of individual elements of the cross section relative to the radius. Then,

$$\varphi_x' \approx y - y_i, \qquad \varphi_y' \approx -(x - x_i) \tag{5.8}$$

$$M_z^{(\tau)} = -\int_F [(\tau_{zx}^{(2)}\varphi)_x' + (\tau_{zy}^{(2)}\varphi)_y'] dF + \int_F \varphi[(\tau_{zx}^{(2)})_x' + (\tau_{zy}^{(2)})_y'] dF =$$
$$= -\oint \varphi[\tau_{zx}^{(2)} dy - \tau_{zy}^{(2)} dx] - \int_F \varphi \frac{\partial\sigma}{\partial z} dF - \int_F \varphi p dF$$

Because of the boundary conditions on the profile, the first integral equals zero, so that, with $\partial t/\partial z = 0$, we have

$$M_z^{(\tau)} = -E_m I_\varphi \ddot{\theta} - E_m \alpha_0 \dot{\theta} \int_F E^\circ \varphi r_*^2 dF - \int_F \varphi p dF \tag{5.9}$$

where

$$I_\varphi = \int_F E^\circ \varphi^2 dF \tag{5.10}$$

By substitution of (5.9) in (5.7), we arrive at the differential equation of constrained twisting

$$\ddot{\theta} - \lambda^2 \theta = -k_0 \lambda^2 (\dot{\theta}_0 - \alpha_0 \nu + \zeta) \tag{5.11}$$

where

$$\lambda^2 = \frac{T}{2(1+\mu) k_0 I_\varphi}, \qquad \zeta = \frac{1}{G_m T} \int_F \varphi p \, dF \tag{5.12}$$

with the boundary conditions $\theta=0$ in the end connection and $\dot{\theta}=0$ at the free end.

With $\alpha_0=0$, p=0, t=0, Eq. (5.12) coincides with the equation of constrained twisting of the theory of thin bars [8].

By designating

$$B_\varphi = -E_m I_\varphi \dot{\theta}, \quad M_\varphi = \frac{dB_\varphi}{dz}, \qquad M_z^* = M_z - M_\varphi + \int_F \varphi p \, dF - G_m T \alpha_0 \nu \tag{5.13}$$

we obtain

$$\theta = k_0 \frac{M_z^*}{G_m T} \tag{5.14}$$

and, consequently, /150

$$\sigma = E^\circ \left[\frac{P + P_t}{F} - \frac{M_y + M_{yt}}{I_y} x + \frac{M_x + M_{xt}}{I_x} y - \frac{B_\varphi}{I_\varphi} \varphi + \right. \tag{5.15}$$

$$\left. + 2(1+\mu) k_0 \alpha_0 \frac{M_z^*}{T} r_s^2 - E_m \gamma t \right]$$

$$\tau_{\max}^{(1)} \approx (G^\circ h)_{\max} k_0 \frac{M_z^*}{T} \tag{5.16}$$

For thin wall profiles, on the basis of (5.8),

$$\varphi_s' = (y - y_t) x_s' - (x - x_t) y_s' \tag{5.17}$$

where s is a coordinate read along the midline of the profile.

After integrating (5.17), with changes of ϕ through the profile disregarded, we obtain, to within a constant,

$$\varphi \approx -\omega \tag{5.18}$$

where ω is the doubled sector area.

Tangential stress $\tau^{(2)}$, which acts along the midline of the profile, is

$$\tau_{zs}^{(2)} = -\frac{1}{h(s)}\left[\int_0^s p(s_1)\,h(s_1)\,ds_1 + \frac{\partial}{\partial z}\int_0^s \sigma(s_1)\,h(s_1)\,ds_1\right] \tag{5.19}$$

which, with $\partial t/\partial z=0$ and with (2.17) taken into account, gives

$$\tau_{zs}^{(2)} = \frac{1}{h(s)}\left\{\frac{F(s)}{F}\int_0^{s_0} p(s)\,h(s)\,ds - \int_0^s p(s_1)\,h(s_1)\,ds_1 + \frac{Q_x S_y(s)}{I_y} + \right.$$
$$\left. + \frac{Q_y S_x(s)}{I_x} + \frac{M_\omega S_\omega(s)}{I_\omega} - \alpha_0\left[(Py_t - M_x)\frac{S_y(s)}{I_y} - (Px_t + M_y)\frac{S_x(s)}{I_x} + B_\omega\frac{I_p^\circ(s)}{I_\omega}\right]\right\} \tag{5.20}$$

where

$$F(s) = \int_0^s E^\circ h\,ds_1, \quad S_x(s) = \int_0^s E^\circ hy\,ds_1, \quad S_y(s) = \int_0^s E^\circ hx\,ds_1$$

$$S_\omega(s) = \int_0^s E^\circ h\omega\,ds_1, \qquad I_p^\circ(s) = \int_0^s E^\circ hr_\circ^2\,ds_1 \tag{5.21}$$

and S_o is the length of the midline.

6. In a number of cases, the constraint effect appears only near fastened cross sections, where θ=0 and the initial twisting has no effect but, in sufficiently remote sections from the end connection, the twisting can be considered practically free, which considerably simplifies the calculations.

Without consideration of constraint, Eq. (3.9), for elongated profiles, is reduced to the form

$$\xi^3 + 3\beta\xi^2 + 2(1 + \nu + \beta^2)\xi = 2(\xi_0 - \beta\nu) \quad \left(\xi = \frac{\beta}{\alpha_0}\theta,\ \xi_0 = \frac{\beta}{\alpha_0}\theta_0\right) \tag{6.1}$$

Analysis of Eq. (6.1) which reduces arbitrary cases of deformation of twisted bars to the single relationship $\xi=\xi(\xi_0\nu,\beta)$, confirms that the role of nonlinear terms ξ^2, ξ^3 is small, as a rule.

By using the equations of elasticity theory in a nonorthogonal curvilinear coordinate system, V.M. Marchenko developed, in general form, a theory of free torsion and tension without bending of a uniformly heated twisted bar, with an arbitrary value of angle α_0, in which, for an elongated elliptical section with semiaxes ab($a>b$) and $\mu=1/3$, /151 the solution in approximate form (by the method of variations) was reduced to calculation formulas, which have the form (in our notation)

$$0 = k_1\left(\frac{M_z}{GT} - \frac{P\alpha_0}{4GF\eta}\right) \quad \left(k_1 = \left[1 + \frac{5}{28}\frac{a^2\alpha_0^2}{\eta}\right],\ \eta = \left[\frac{b}{a}\right]^2\right)$$
$$\varepsilon_0 = k_1\left[\frac{P}{EF}\left(1 + \frac{20}{84}\frac{a^2\alpha_0^2}{\eta}\right) - \frac{M_z\alpha_0}{4GF\eta}\right] \tag{6.2}$$

For an elongated ellipse ($\eta<1$), with $\mu=1/3$ and $\nu \ll 1+\beta^2$, by formulas (3.13), (4.1) and (4.4), we have

$$k_0 = \left[1 + \frac{8}{45}\frac{a^2\alpha_0^2}{\eta}\right]^{-1}, \quad a_{11} = \frac{k_0}{EF}\left(1 + \frac{31}{90}\frac{a^2\alpha_0^2}{\eta}\right),$$
$$a_{44} = \frac{k_0}{GT}, \quad a_{14} = a_{41} = -k_0\frac{\alpha_0}{4GF\eta}$$

and comparing solutions (4.3) and (6.2) shows that they practically coincide. The normal stresses in the center of the section are:

according to V.M. Marchenko

$$\sigma(0) = k_1\left[\left(1 + \frac{69}{224}\frac{a^2\alpha_0^2}{\eta}\right)\frac{P}{F} - \frac{29}{56}\alpha_0\frac{M_z}{F\eta}\right] \tag{6.3}$$

according to formula (5.15)

$$\sigma(0) = k_0\left[\left(1 + \frac{31}{90}\frac{a^2\alpha_0^2}{\eta}\right)\frac{P}{F} - \frac{2}{3}\alpha_0\frac{M_z}{F_\eta}\right]$$

The divergence of the results also is small.

Thus, comparison of the approximate theory with experimental data and with the partial solutions obtained with the use of the system of elasticity theory, indicates its sufficient generality and accuracy.
The author thanks I.A. Birger for discussion of the work.

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