NASA TECHNICAL MEMORANDUM

NASA TM-75505

ON WINGS OF CIRCULAR DESIGN

Wilhelm Kinner

Translation of "Über Tragflügel mit kreisförmigem
Grundriss," Zeitschrift für angewandte Mathematik und
Mechanik, Vol. 16, No. 6, Dec. 1936, pp. 349-352

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546   OCTOBER 1979

1. Report No. NASA TM-75505	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle ON WINGS OF CIRCULAR DESIGN		5. Report Date October 1979
		6. Performing Organization Code :
7. Author(s) Wilhelm Kinner		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address Leo Kanner Associates Redwood City, California 94063		11. Contract or Grant No. NASw-3199
		13. Type of Report and Period Covered Translation
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

Translation of "Über Tragflügel mit kreisförmigem Grundriss," Zeitschrift für angewandte Mathematik und Mechanik, Vol. 16, No. 6, Dec. 1936, pp. 349-352.

16. Abstract

Prandtl's method of sources and sinks for aerofoils is used to investigate the aerodynamics of circular wings in constant flow. Lift distribution, total lift, and pitching moment are investigated, as well as the influence of changes in the angle of attack.

17. Key Words (Selected by Author(s))		18. Distribution Statement Unclassified-Unlimited	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 17	22. Price

ON WINGS OF CIRCULAR DESIGN[1]

W. Kinner
Göttingen

In this article, it will be shown how Prandtl's method of
sources and sinks for aerofoils[2] can be used to investigate wings
of circular design in constant flow. The equations are made linear
in the usual fashion; in particular, we move the reference point
of the acceleration potential Φ from the wing to the surface of
the circular base. The center of the circle is made to lie at the
origin, its surface in the xy-plane; the radius is c. Let the
dimensionless Cartesian coordinates be

$$\bar{x} = \frac{x}{c}, \quad \bar{y} = \frac{y}{c}, \quad \bar{z} = \frac{z}{c}.$$

The x-axis is directed to the rear, the y-axis to the right, and
the z-axis is directed vertically. The flow velocity in the x
direction is v. The liquid is assumed to be weightless, friction-
less, and of constant volume.

Since the calculation of the potential for the given lift
distribution leads to difficulties in integration, we work back-
wards, i.e. we use the "potential function of the first kind" (2)
from potential theory, determine the appropriate aerodynamics, and
attempt to satisfy the kinematic condition by linear combinations.
For this we use the elliptical coordinates   which are associated
with the Cartesian coordinates by the following orthogonal transfor-
mations:

$$x = c\sqrt{1-\mu^2}\sqrt{1+\eta^2}\cos\varphi, \qquad y = c\sqrt{1-\mu^2}\sqrt{1+\eta^2}\sin\varphi, \qquad z = c\mu\eta. \tag{1}$$

---

1. Excerpt from the author's dissertation. The investigation of
   circular wings is important for the theory of the autogiro.
2. L. Prandtl, this volume, p. 360. The author's oral presentation
   followed immediately after that of Prandtl.
* Numbers in the margin indicate pagination in the foreign text.

The reverse transformation becomes single-valued when one allows the hyperbolic coordinate μ to change from positive to negative values on going from the upper half-space to the lower. In elliptical coordinates, the well-known[3] solutions of $\Delta \Phi = 0$ have the following real, normalized form:

$$\Phi_n{}^m(\mu, \eta, \varphi) = \frac{1}{i^{n-m+1}} \frac{(n-m)!}{(n+m)!} C_n{}^m P_n{}^m(\mu) Q_n{}^m(i\,\eta) \cos m\,\varphi .$$

(2)

$C_n{}^m$ is a constant, and $P_n{}^m$ and are the associated spherical functions of the first and second kinds, respectively. We are only interested in the functions $\Phi_n{}^m$ with odd n + m, since those with n + m even are constant on the base surface. The lift density at a point (μ, φ) on the base surface is proportional to the potential difference $\Phi_n{}^m$

$$p_u - p_{ob} = \varrho\,(\Phi_{ob} - \Phi_u)$$

(3)

The total lift $A = \iint (p_u - p_{ob})\,dF$ is:

$$A_1{}^\circ = \frac{4}{3}\,\pi\,\varrho\,c^3\,C_1{}^\circ, \qquad A_n{}^m \quad = 0 \text{ for all other } (n,m) \quad (4)$$

The moment about the y-axis $M = \iint (p_u - p_{ob})\,x\,dF$ is:

$$M_2{}^1 = \frac{4}{15}\,\pi\,\varrho\,c^3\,C_2{}^1, \qquad M_n{}^m \quad = 0 \text{ for all other } (n,m) \quad (5)$$

The downdraught on the wing follows from $w(x,y) = \frac{1}{v}\int_{-\infty}^{x} \frac{\partial \Phi}{\partial z}\,dx$ by

integrating in the plane of the wing parallel to the direction of flow. One obtains:

$$w_1{}^\circ(\mu, \varphi) = \frac{C_1{}^\circ}{v}\left[Q_1(\bar{y}) - \frac{\pi}{2} P_1{}^1(\mu) \cos\varphi \right], \qquad w_2{}^1(\mu, \varphi) = \frac{\pi}{4}\,\frac{C_2{}^1}{v} P_2(\mu) \quad (6)$$

In general, $w_n{}^m(\mu, \varphi)$ may be represented by tesseral spherical functions, to which for odd n the factor $(-1)^{\frac{n-1}{2}} \frac{C_n{}^m}{v} Q_n(\bar{y})$ is attached.

---

3. M. J. O. Strutt, Lamé, Mathieu, and Related Functions in Physics and Technology, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 1, 1932, p. 59.

2

In this manner, the downdraught for the functions concerned is made to change logarithmically.

The shape of the wing is given by $z(x,y) = \frac{1}{v} \int\limits^{x} w\, dx.$

The drag $W_n{}^m$, according to the separation law of Munk, corresponds to its one dimensional analog. For odd m, the lift density $a(y) = \int\limits_{-x_0}^{+x_0} (p_u - p_{ob})\, dx$ (with $x_0 = \sqrt{c^2 - y^2}$) vanishes identically; for even m one obtains:

$$a_{2\,i+1}^{2\,i}(\bar{y}) = (-1)^{i+1} 2\,\pi\,\varrho\,c\,C_{2\,i+1}^{2\,i} \int\limits_{-1}^{\bar{y}} P_{2\,i+1}(\bar{y})\, d\bar{y}, \qquad w_{2\,i+1}^{2\,i}(\bar{y}) = (-1)^{i} \frac{C_{2\,i+1}^{2\,i}}{v} Q_{2\,i+1}(\bar{y}) \tag{7}$$

$$W_1{}^0 = \pi\,\frac{\varrho\,c^2\,C_1{}^{0\,2}}{v^2}, \qquad W_3{}^0 = \frac{\pi}{6}\,\frac{\varrho\,c^2\,C_3{}^{0\,2}}{v^2} \quad \cdots \cdots \cdots \tag{8}$$

For the function $\Phi_n{}^m$, there is no suction pressure at the leading edge since the induced speeds (except at the wing ends) are finite. It follows that these functions alone are insufficient to permit investigation of the influence of changes in the angle of attack.

The "potential function of the second kind," Φ_n necessary for this is obtained from the $\overline{\Phi_{n+1}^n}$ (n = 0, 1, 2,....) by the limiting process

$$\Phi_n(\mu, \eta, \varphi) = \lim_{\Delta c \to 0} \frac{\left[1 + (n+1)\frac{\Delta c}{c}\right]\Phi_{n+1}^n(\mu + \Delta\mu, \eta + \Delta\eta, \varphi) - \Phi_{n+1}^n(\mu, \eta, \varphi)}{\Delta c} \tag{9}$$

from which

$$\Phi_n = C_n \frac{\mu(1-\mu^2)^{n/2}}{(\mu^2 + \eta^2)(1+\eta^2)^{n/2}} \cos n\,\varphi \cdots \cdots \cdots \tag{10}$$

The Φ_n become infinite at the edges of the circle; they have no /351 physical meaning in and of themselves since the outflow condition at the rear edge (that the lift density approach zero) is not fulfilled.

The lift is

$$A_0 = 4\,\pi\,\varrho\,c^2\,C_0, \qquad A_n = 0 \text{ for } n = 1, 2, 3,\ldots \tag{11}$$

3

The moment about the y-axis is

$$M_1 = \frac{4}{3}\pi \varrho c^3 C_1, \qquad M_n = 0 \text{ for } n = 0, 2, 3, \dots \qquad (12)$$

To calculate the downdraught on the wing, one integrates to
a line parallel to the x-axis outside the plane of the wing base,
and, after the integration has been carried out, shifts the terminus
to the base surface. The downdraught is independent of   inside
the base circle:

$$w_{2\varkappa}(\bar{y}) = (-1)^{\varkappa+1}\frac{C_{2\varkappa}}{v}\frac{d\,Q_{2\varkappa}(\bar{y})}{d\,\bar{y}}, \qquad w_{2\lambda+1}(\bar{y}) = (-1)^{\lambda}\frac{\pi}{2}\frac{C_{2\lambda+1}}{v}\frac{d\,P_{2\lambda+1}(\bar{y})}{d\,\bar{y}} \qquad (13)$$

Thus the wing surfaces are ruled surfaces.

The drag vanishes for odd n and becomes infinite for even n;
for a curved section of the front edge the suction pressure becomes
infinite for even n if the integration is extended to the ends
of the wing.

For the corresponding one dimensional equation, one has

$$a_{2\varkappa}(\bar{y}) = (-1)^{\varkappa} 2\pi \varrho c\, C_{2\varkappa} P_{2\varkappa}(\bar{y}), \qquad a_{2\lambda+1}(\bar{y}) = 0 \quad \dots \dots \qquad (14)$$

$$w_{2\varkappa}(\bar{y}) = (-1)^{\varkappa+1}\frac{C_{2\varkappa}}{v}\frac{d\,Q_{2\varkappa}(\bar{y})}{d\,\bar{y}}, \qquad w_{2\lambda+1}(\bar{y}) = 0 \quad \dots \dots \qquad (15)$$

With the mathematical apparatus we have developed the second
basic problem can be solved in the following manner. From the
given wing form $z(\bar{x},\bar{y})$ one calculates the downdraught $w(\bar{x},\bar{y}) = v\frac{\partial z(\bar{x},\bar{y})}{\partial \bar{x}}$

and separates from the result the downdraught function of the first
kind that is contained in it. The remaining function $w(\bar{y})$ is a
function of $\bar{y}$ alone; one manipulates this as a downdraught function
of the second kind so that the outflow condition at the rear edge
is fulfilled. The potential coefficients $c_{2\varkappa}$ are determined by an
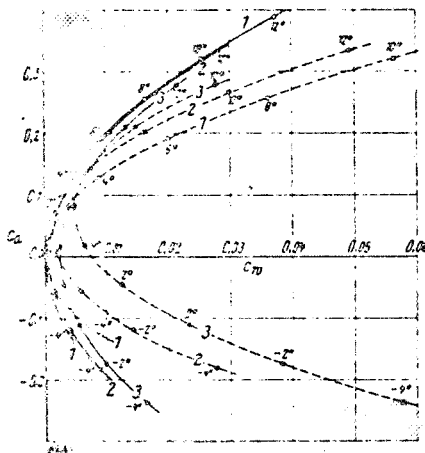infinite system of linear equations; from these follow the $c_{2\lambda+1}$ by
means of the outflow condition.

4

Fig. 1. Polar Curve of the Theoretical Drag.

1. Planar circular disc.
2. Surface with fixed aerodynamic center $c_1^0 + c_2^0 + c_3^0$ with $c_1^0 = c_3^0$, $c_2^1 = -2.58 c_1^0$ and $\frac{c_1^0}{c_2} = 0.025$
3. As in 2. with $\frac{c_1^0}{c_2} = 0.050$.

solid curves: line drag
stippled curves: surface drag

The distribution of lift, the total lift, and the pitching moment are thus known. The influences of changes in the angle of attack can be investigated by the linear superposition of a planar circular disc. The "surface drag" $W_F = -\frac{1}{v} \iint (p_u - p_{ob}) w(x,y) \, dF$ is greater

than the actual drag by the amount of the suction pressure. The actual drag is identical with the "line drag" $W_L = -\frac{1}{v} \int_{-c}^{+c} q_L(y) w_L(y) \, dy$,

since the outflowing vortex sheet, and thus also its kinetic energy, /352
the equivalent of the drag work, is not changed by displacement of the lift elements. According to their analytical form, the lift and its moment appear as linear expressions, the two drags and the suction pressure as quadratic expressions in the two parameters $\frac{H}{c}$ (camber parameter) and tg a.

According to the procedure outlined, the planar circular disc, the calotte, a wing with fixed aerodynamic center, and S-flap (?) were investigated (see illustration); also, the rolling moment of a warped wing was calculated, using the appropriate antisymmetric potential function (trigonometric factor sin ϕ).

Experiments to check the theory are in preparation at the Institute for Aerodynamic Research in Göttingen. A theoretical investigation of elliptically shaped wings in a constant stream, using Lamé functions, is underway.