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The Elliptic Anomaly

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SUMMARY

An independent variable different from the time for elliptic orbit integration is used here. Such a time transformation provides an analytical step-size regulation along the orbit. An intermediate anomaly (an anomaly intermediate between the eccentric and the true anomaly) is suggested for optimum performances. A particular case of an intermediate anomaly (the elliptic anomaly) is defined, and its relation with the other anomalies is developed.

INTRODUCTION

The development of integration schemes offering an automatic step-size control (such as the Runge-Kutta-Fehlberg scheme (ref. 1) or the variable-step variable-order multistep method developed by Shampine and Gordon (ref. 2)) may be an invitation to return to the direct integration of the differential equations of the orbital motion in Cartesian coordinates with time as the independent variable. If the automatic step-size control does take care of a step-size reduction near pericenter and a step-size enlargement near apocenter, the step-size distribution along the orbit is not as smooth as through an analytical step-size regulation. This is because the step size is predicted on the basis of a current local truncation error estimation. So that this step size is not rejected too often during the following step, a conservative factor is introduced, prohibiting an excessive predicted step size. As a result, the integration accuracy along the orbit is not homogeneous.

Analytical step-size regulation means that a new independent variable τ is chosen (often called fictitious time) so that equal steps in τ produce the desirable distribution of time steps along the orbit. The time is no longer the independent variable, but it becomes a new dependent variable to be integrated together with the other state variables. The drawback of having one more equation to integrate is compensated for by a more efficient use of the function evaluations. In addition, a highly efficient fixed-step, multistep integration method can be used, which will be free of the overhead involved in a variable-step method.

THE GENERALIZED SUNDMAN TRANSFORMATION

To achieve an analytical step-size regulation for the integration of orbits, a change in the independent variable

$$dt = f d\tau \tag{2.1}$$

is proposed, where t is the physical time, τ the new independent variable and f a function of the state. Equation (2.1) is called a time transformation. By choosing f in the form

$$f = c_\alpha r^\alpha \tag{2.2}$$

where r is the radius vector, α a real number and c_α a coefficient, the new independent variable can be of an angular type describing 2π over one revolution like the classical anomalies used in celestial mechanics. Equation (2.2) is referred to as the generalized Sundman transformation.

Case 1: $\alpha = 0$

By choosing

$$c_0 = \sqrt{|a|^3/\mu}$$

a is the semimajor axis, μ is the universal gravitational constant times the mass of the central body, and τ is equal to the mean anomaly.

Case 2: $\alpha = 1$

Assuming an elliptic orbit, by differentiating Kepler's equation

$$t = \sqrt{a^3/\mu}(u - e \sin u) \tag{2.3}$$

where t is the time since pericenter pass and u the eccentric anomaly, and by using the relation

$$r = a(1 - e \cos u) \tag{2.4}$$

equation (2.1) combined with equation (2.2) becomes

$$\mu^{-1/2} c_\alpha^{-1} a^{3/2-\alpha} (1 - e \cos u)^{1-\alpha} du = d\tau \tag{2.5}$$

where the time is replaced by the eccentric anomaly. With $\alpha = 1$, equation (2.5) becomes

$$\mu^{-1/2} c_1^{-1} a^{1/2} du = d\tau$$

By choosing

$$c_1 = \sqrt{a/\mu}$$

$du = d\tau$ and τ is the eccentric anomaly.

Therefore, equation (2.2) becomes

$$f = \sqrt{a/\mu}\ r$$

which is the original Sundman transformation.

Case 3: $\alpha = 2$

The term function of u in equation (2.5) can be expressed in terms of the true anomaly ϕ as

$$\frac{du}{\sqrt{1 - e^2}(1 - e \cos u)} = d\phi$$

Therefore, by choosing

$$c_2 = \frac{1}{\sqrt{\mu a(1 - e^2)}}$$

$$f = \frac{r^2}{\sqrt{\mu a(1 - e^2)}}$$

defines a time transformation where the new independent variable is the true anomaly.

THE INTERMEDIATE ANOMALY

At this stage, it is interesting to examine the distribution of steps along an elliptic orbit defined by the three transformations of the last section.

Figure 1(a) shows how 12 equal steps in mean anomaly are distributed along an 0.8 eccentricity ellipse. All steps are accumulated around the apocenter, and the pericenter is completely depleted. This is the result of using an independent variable proportional to the time.

Figure 1(b) shows how 12 equal steps in eccentric anomaly are distributed. A significant improvement over the preceding case is noticed, but steps around pericenter are still sparse.

Figure 1(c) shows a distribution of 12 equal steps with respect to the true anomaly. This time, a sparse number of steps around apocenter is noticed.

By simple inspection of figures 1(b) and 1(c), it can be anticipated that a more adequate distribution of steps along the ellipse would be reached by having a time transformation defining an independent variable whose characteristic is intermediate between the eccentric and the true anomaly.

The concept of intermediate anomaly is introduced here by a pure qualitative argument. It will be shown now that one particular intermediate anomaly is the mathematically logical extension to the ellipse of the polar angle for the circle.

THE ELLIPTIC ANOMALY

Recalling that the eccentric anomaly corresponds to $\alpha = 1$ of equation (2.2) and the true anomaly corresponds to $\alpha = 2$, an intermediate anomaly is defined for any value of α such that $1 < \alpha < 2$. As suggested by Nacozy (ref. 3) the first obvious choice is $\alpha = 1.5$. This case will be investigated in more detail as follows:

Case 4: $\alpha = 3/2$

Equation (2.5) becomes

$$\frac{du}{\sqrt{\mu}\ c_{3/2}\sqrt{1 - e\cos u}} = d\tau \tag{4.1}$$

If the auxiliary angle $\theta = \phi/2$ is introduced, equation (4.1) becomes

$$0.5c_{3/2}\sqrt{\mu(1 + e)}\ d\tau = \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} \tag{4.2}$$

with

$$k^2 = 2e/(1 + e)$$

The right-hand side of equation (4.2) is precisely the argument of the normal elliptic integral of the first kind

$$F(\theta, k) = \int_0^\theta d\theta/\sqrt{1 - k^2\sin^2\theta}$$

Therefore, equation (4.2) becomes

$$\tau = \frac{2F(\theta,\ k)}{c_{3/2}\sqrt{\mu(1 + e)}} \tag{4.3}$$

where τ is measured from pericenter. Equation (4.3) gives a direct relationship between the true anomaly $\phi = 2\theta$ and the intermediate anomaly τ. As an elliptic function is involved in its definition, this particular choice of an intermediate anomaly will naturally be called elliptic anomaly and will be denoted from now on by ν.

At apocenter, $\phi = \pi$ and

$$F(\pi/2,\ k) = K(k)$$

where

$$K(k) = \int_0^{\pi/2} d\theta/\sqrt{1 - k^2 \sin^2 \theta}$$

is the complete elliptic integral of the first kind.

The constant $c_{3/2}$ is chosen such that $\nu = \pi$ at apocenter, therefore

$$c_{3/2} = \frac{2K(k)}{\pi\sqrt{\mu(1 + e)}}$$

and equation (4.3) becomes simply

$$\nu = \frac{\pi F(\theta,\ k)}{K(k)} \tag{4.4}$$

Equation (2.2) is now defined as

$$f = \frac{2K(k)}{\pi} \frac{r^{3/2}}{\sqrt{\mu(1 + e)}} \tag{4.5}$$

Figure 1(d) shows the distribution of 12 steps equally spaced with respect to the elliptic anomaly. In contrast to the cases of the eccentric and true anomaly, the distribution of steps given by the elliptic anomaly favors neither the pericenter nor the apocenter and seems to be ideally suited for an analytical step regulation.

THE ELLIPTIC ANOMALY AS ARGUMENT OF THE JACOBIAN ELLIPTIC FUNCTIONS

In order to express the elliptic anomaly in terms of the traditional anomalies, it is necessary to introduce the inverse of the elliptical integral of the first kind: the Jacobian elliptic functions (ref. 4)

$$\mathrm{sn}(v, k) = \sin\theta$$

$$\mathrm{cn}(v, k) = \cos\theta \qquad (5.1)$$

such that

$$v = F(\theta, k)$$

The functions sn and cn are periodic of period $4K(k)$ and are a generalization of the circular functions sine and cosine for the ellipse. For zero eccentricity $(k = 0)$

$$\mathrm{sn}(v, 0) = \sin v$$

$$\mathrm{cn}(v, 0) = \cos v$$

By introducing equation (4.4) into equation (5.1)

$$\sin\theta = \mathrm{sn}(K\,\nu/\pi, k)$$

$$\cos\theta = \mathrm{cn}(K\,\nu/\pi, k)$$

the true anomaly $\phi = \theta/2$ is expressed in terms of the elliptic anomaly, as follows:

$$\sin\phi = 2\mathrm{sn}(n, k)\,\mathrm{cn}(n, k)$$

$$\cos\phi = 1 - 2\mathrm{sn}^2(n, k)$$

where

$$n = K\,\nu/\pi$$

By using the usual expressions relating the true anomaly to the eccentric anomaly, the eccentric anomaly u can be expressed directly in terms of the elliptic anomaly as follows:

$$\sin u = 2\sqrt{\frac{1-e}{1+e}}\;\frac{\mathrm{sn}(n,\ k)\ \mathrm{cn}(n,\ k)}{1 - k^2\mathrm{sn}^2(n,\ k)}$$

$$\cos u = \frac{1 + e - 2\mathrm{sn}^2(n,\ k)}{1 + e - 2e\,\mathrm{sn}^2(n,\ k)}$$

Equation (2.4) expressing the radius r becomes (in terms of the elliptic anomaly)

$$r = \frac{a(1 - e)}{1 - k^2\mathrm{sn}^2(n,\ k)}$$

By using the focal unit vectors $\underline{i}$ directed toward pericenter and $\underline{j}$ normal to $\underline{i}$, position $\underline{r}$ and velocity $\underline{v}$ are expressed as

$$\underline{r} = r(1 - 2\mathrm{sn}^2(n,\ k))\underline{i} + 2r\ \mathrm{sn}(n,\ k)\ \mathrm{cn}(n,\ k)\underline{j}$$

$$\underline{v} = \sqrt{\mu(1+e)}/r\left[\left(\frac{e}{a(1 - e^2)}\,\underline{r} - \underline{i}\right)2\mathrm{sn}(n,\ k)\mathrm{cn}(n,\ k) + (1 - 2\mathrm{sn}^2(n,\ k))\underline{j}\right]\mathrm{dn}(n,\ k)$$

Formulas involving the elliptic anomaly were first developed in reference 3, where an intermediate variable proportional to the elliptic anomaly is defined.

The three time transformations corresponding to the eccentric, true, and elliptic anomalies were already outlined in reference 5 and are developed along an alternate way in reference 6.

THE NUMERICAL EVALUATION OF ELLIPTIC FUNCTIONS

An apparent drawback to using the elliptic anomaly as an independent variable is the need to estimate elliptical functions at each integration step.

However, fast and accurate algorithms for computing elliptic integrals and functions are given in reference 7.

The algorithm for estimating the Jacobian elliptic functions sn and cn is based on the Gauss transformation and iteratively applied. A total of only two trigonometric functions is required. For an accuracy of at least 15 decimal places, no more than 100 elementary operations are needed.

If just the time transformation equation (eq. (4.5)) is used, only the complete elliptic integral of the first kind $K(k)$ is required, and the corresponding algorithm does not involve more than 20 elementary operations.

CONCLUDING REMARKS

The use of an independent variable different from the time leading to an analytical step-size regulation is the most efficient way to compute elliptic orbits.

Instead of the commonly used eccentric or true anomaly as a new independent variable, it is suggested to use an intermediate anomaly (intermediate between the eccentric and the true anomaly). One choice among the possible intermediate anomalies is the elliptic anomaly, so called because it involves elliptic functions. The elliptic anomaly appears to be the most natural way for extending the polar angle of the circle to the ellipse.

The anomalies discussed in this paper were used as the independent variable during a numerical comparison. An orbit of eccentricity, 0.73, was chosen, and a time element was included. Results of the comparison showed that the use of the elliptic anomaly as the independent variable led to an accuracy about one order of magnitude higher than any other classical anomaly. These results will be presented in a subsequent paper.

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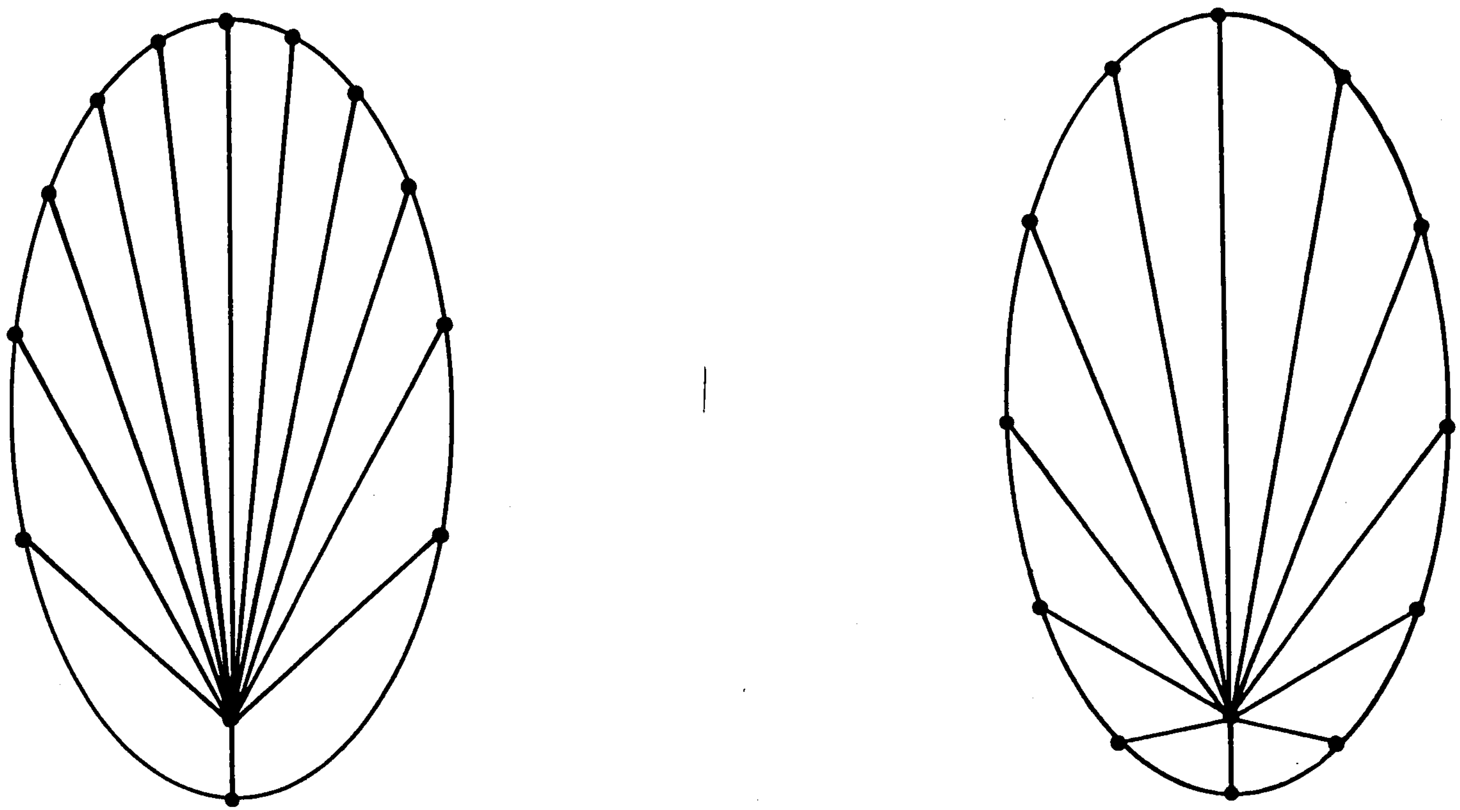


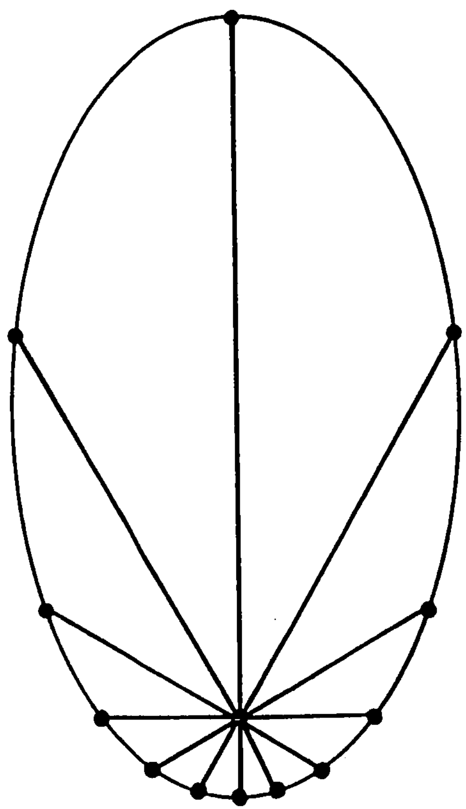
(a) The mean anomaly

$\alpha = 0$

(b) The eccentric anomaly

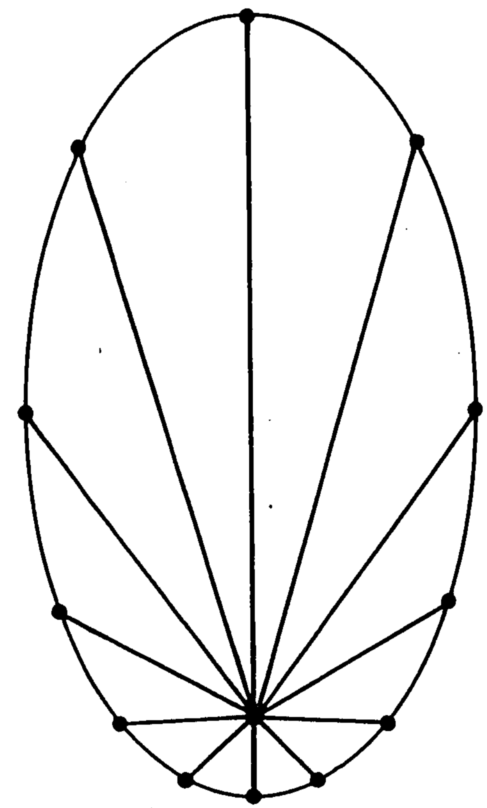
$\alpha = 1$

Figure 1.- Distribution of 12 equidistant steps with respect to the four classical anomalies.



(c) The true anomaly

$\alpha = 2$



(d) The elliptic anomaly

$\alpha = 3/2$

Figure 1.- Concluded.

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16. Abstract

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