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# BIVARIATE NORMAL CONDITIONAL AND RECTANGULAR PROBABILITIES: A COMPUTER PROGRAM WITH APPLICATIONS

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# BIVARIATE NORMAL CONDITIONAL AND RECTANGULAR PROBABILITIES: A COMPUTER PROGRAM WITH APPLICATIONS

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## INTRODUCTION

In applications involving univariate data where estimates and confidence intervals are required, the normal distribution is commonly employed. This distribution is mainly utilized because the probabilities under a normal curve are readily available. In contrast, use of multivariate probabilities in p-variate normal data are less frequent, primarily because probabilities for the multivariate normal case are generally not available. Except for very special cases, the probabilities for sections of p-dimensional space require extensive computations, since the canonical multivariate normal density changes with every change in correlation coefficient parameters. Even the probability computation in the bivariate normal case ($p = 2$) with only one value for the correlation coefficient over arbitrary sections of the $(x, y)$ plane is not easy. Probability computations, therefore, in $p > 2$ dimensions are correspondingly much more difficult. (Ref. 1)

In many applications, problems are posed which not only require the probabilities over a section of p-dimensional space, but also the conditional probabilities of r ($r < p$) variables when the remaining ($p - r$) variables are either fixed, or are within designated intervals. For example, in aircraft target tracking studies, it is of interest to know the probability of X deviations from the target when Y deviations are considered within designated bounds. In aircraft performance studies it is important to know the distribution of the pilot's cardiac R-R intervals either under an assigned difficult aircraft maneuver or under the dynamic flight conditions.

The results on conditional and marginal distributions of r variables when the ($p - r$) remaining variables assume fixed values are well established. (Ref. 1) Similar results, when the remaining ($p - r$) variables assume values within specified ranges involve complexities and are discussed in this report.

In this study, results on bivariate normal distributions ($p = 2$) are reviewed. Various derivations and properties of bivariate normal conditional probabilities are derived. A computer program for conditional probabilities for all assigned values is included. From conditional and marginal probabilities, the rectangle probabilities are then obtained. Examples are presented to illustrate the use of the program. The program listing is appended to this report.

## SYMBOLS

| | |
|---|---|
| $A_y$ | lateral acceleration |
| $A_z$ | vertical acceleration |
| c | a constant with fixed numerical value |
| exp(x) | exponential function at x |

| | |
|---|---|
| $F(s)$ | conditional distribution of $X$ at $X = s$ Given $Y$ is in interval $(a, b)$ |
| $f(u, v)$ | general bivariate normal density |
| $f(x)$, $f(y)$ | standard normal densities |
| $f(x, y)$ | standard bivariate normal density at $X = x$, $Y = y$ |
| $f(x\|a<Y<b)$ | conditional density of $X$ at $X = x$ given $Y$ is in interval $(a, b)$ |
| $f(x\|Y=y)$ | conditional density of $X$ at $X = x$ given $Y = y$ |
| $f(x\|Y<t, \rho<0)$ | conditional density of $X$ at $X = x$ given $Y$ is less than $t$ and correlation is negative |
| $f(x\|Y>-t, \rho>0)$ | conditional density of $X$ at $X = x$ given $Y$ is greater than $-t$ and correlation coefficient $\rho$ is positive |
| $G_\rho(s, t)$ | double integral with two arguments $s$ and $t$ with a fixed value of correlation coefficient $\rho$ |
| $g_t(x)$ | conditional density of $X$ at $X = x$ given $Y$ is in interval $(-t, t)$ |
| $g_t(x\|\rho>0)$ | conditional density of $X$ at $X = x$ when correlation coefficient $\rho$ is positive and $Y$ is in interval $(-t, t)$ |
| $g_t(x\|\rho<0)$ | conditional density of $X$ at $X = x$ when correlation coefficient $\rho$ is negative and $Y$ is in interval $(-t, t)$ |
| $p$, $r$ | dimension of multivariate data or distribution |
| $Pr[a<Y<b]$ | probability that variable $Y$ is in interval $(a, b)$ |
| $Pr[c<X<d, a<Y<b]$ | joint probability that variable $X$ is in interval $(c, d)$ and variable $Y$ is in interval $(a, b)$ |
| $Pr[X<h, Y<k]$ | probability that $X$ is less than $h$ and $Y$ is less than $k$ |
| $U, V, X, Y$ | random variables |
| $u, v, x, y, t$ | specific values of random variables |
| $V_c$ | forward velocity |
| $\alpha$ | fixed positive constant less than 1 |
| $\mu_c$ | mean of forward velocity $V_c$ |

| | |
|---|---|
| $\mu_u$, $\mu_v$ | mean of subscripted random variable |
| $\mu_y$ | mean of lateral acceleration $A_y$ |
| $\mu_z$ | mean of vertical acceleration $A_z$ |
| $\rho$ | correlation coefficient between two random variables |
| $\sigma_c$ | standard deviation of forward velocity $V_c$ |
| $\sigma_u$, $\sigma_v$ | standard deviation of subscripted random variable |
| $\sigma_y$ | standard deviation of lateral acceleration $A_y$ |
| $\sigma_z$ | standard deviation of vertical acceleration $A_z$ |
| $\Phi(t)$ | standard normal distribution at t |

## BIVARIATE NORMAL DISTRIBUTION

A bivariate normal distribution of a random vector (U, V) is characterized by parameters: $\mu_u$, $\mu_v$, $\sigma_u$, $\sigma_v$ and $\rho$. The density function

$$f(u,v) = \left[2\pi\sigma_u\sigma_v\sqrt{(1 - \rho^2)}\right]^{-1} \exp\left( -\left\{\left[(u - \mu_u)/\sigma_u\right]^2 - 2\rho\left[(u - \mu_u)/\sigma_u\sigma_v\right] + \left[(v - \mu_v)/\sigma\right]^2\right\} \Big/ 2(1 - \rho^2)\right)$$

is defined over the entire (u, v) plane. When the variables U and V are standardized, by defining the new variables

$$f(x, y) = \left(2\pi\sqrt{1 - \rho^2}\right)^{-1} \exp\left[ - (x^2 - 2\rho xy + y^2) \Big/ 2(1 - \rho^2)\right]$$

the density function of (X, Y) reduces to the canonical bivariate normal density

$$X = (U - \mu_u)\Big/\sigma_u , \; Y = (V - \mu_v)\Big/\sigma_v$$

defined over the entire (x, y) plane. The parameter $\rho$ is called a correlation coefficient and takes values in the interval (-1, 1). Without any loss of generality, this canonical density f(s, y) is considered in this study.

The density function $f(x, y)$ exhibits certain properties. It is symmetric in opposite quadrants since

$$f(x, y) = f(-x, -y)$$

and

$$f(x, -y) = f(-x, y)$$

Further, $f(x, y)$ is constant over all the ellipses

$$x^2 - 2\rho xy + y^2 = c(1 - \rho^2)$$

for every value of $x$. (Fig. 1) The intercepts made by these ellipses on the $x$ and $y$ axes are equal. If $\rho$ is positive, the major axis of the ellipse is along the 45° line with a length of $2\sqrt{c(1 + \rho)}$; and the minor axis is along the 135° line with a length of $2\sqrt{c(1 - \rho)}$. If $\rho$ is negative, the major axis is along the 135° line with a length of $2\sqrt{c(1 - \rho)}$; the minor axis along the 45° line has a length of $2\sqrt{c(1 + \rho)}$. (Ref. 2) The ellipse

$$x^2 - 2\rho xy + y^2 = (1 - \rho^2) \log 1/(1 - \alpha)^2$$

for all $0 < \alpha < 1$, contains the $\alpha$ proportion of the $(X, Y)$ distribution. (Ref. 3)

The marginal distributions of $X$ and $Y$ are standard normal with the covariance between $x$ and $y$ equal to $\rho$. When $\rho = 0$, then

$$\begin{aligned} f(x, y) &= (\sqrt{2\pi})^{-1} \exp( -x^2/2) \, (\sqrt{2\pi})^{-1} \exp( -y^2/2) \\ &= f(x) \cdot f(y) \end{aligned}$$

which is a product of standard normal densities, implying that $\rho = 0$ if and only if $X$ and $Y$ are independent. When $\rho \neq 0$, bivariate normal probabilities $\Pr(X < h, Y < k)$ for a few selected values of $h$ and $k$ are available from tables and graphs. (Ref. 4, 5) For general values of $h$ and $k$ approximation and interpolation methods are used.

## DERIVATION OF CONDITIONAL DENSITIES

Conditional Density of $X$ Given $Y = y$. It was stated earlier that if a random vector $(X, Y)$ has a bivariate normal distribution, then the marginal distribution of either $X$ or $Y$ is normal with mean 0 and variance 1. The conditional distribution of $X$ for a fixed value of $Y = y$, however, is normal with mean $\rho y$ and variance $(1 - \rho^2)$. The conditional density $f(x|Y = y)$ is derived below.

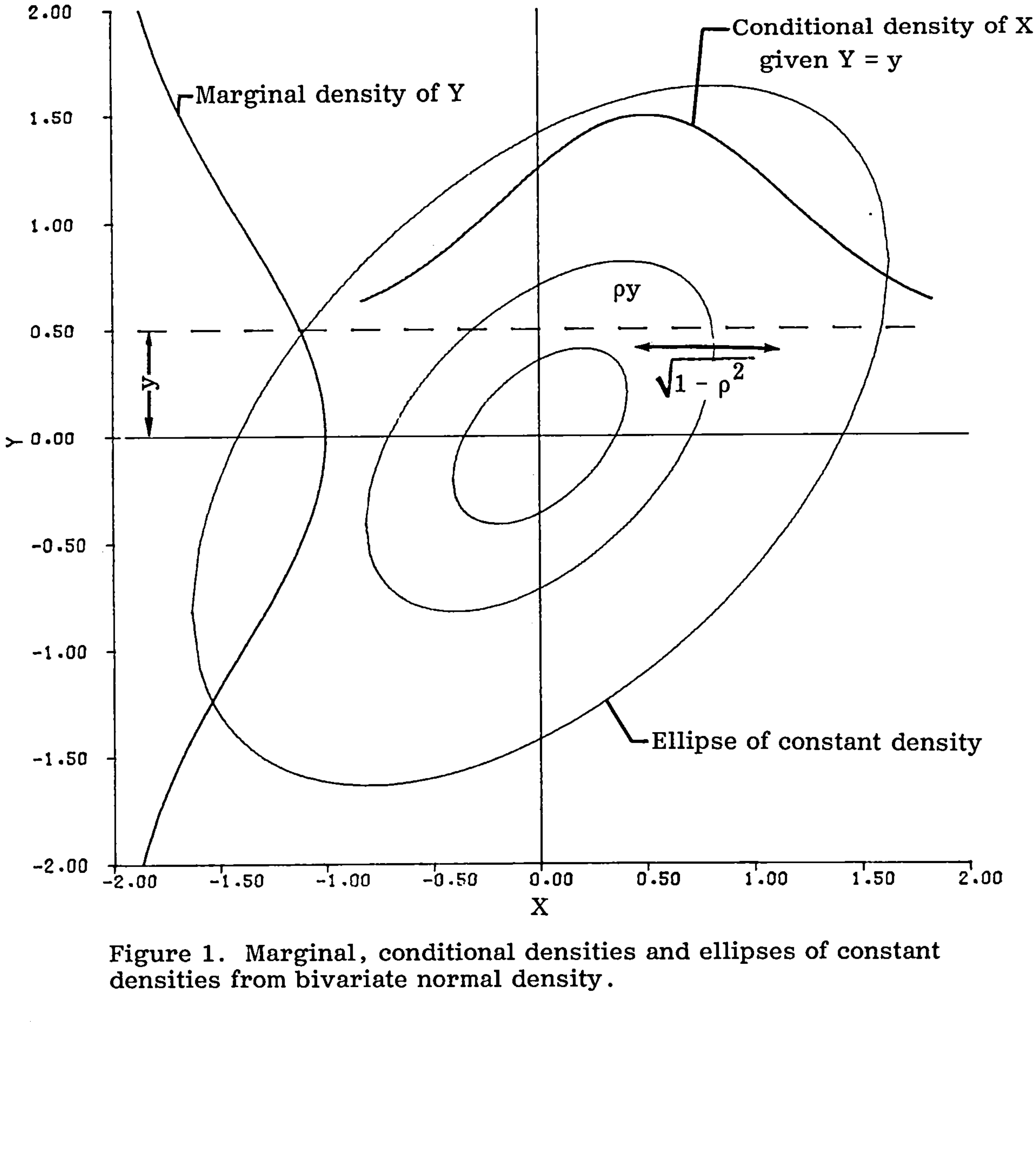


Figure 1. Marginal, conditional densities and ellipses of constant densities from bivariate normal density.

$$f(x|Y = y) = f(x, y)\big/f(y)$$

$$= \frac{\left(2\pi\sqrt{1 - \rho^2}\right)^{-1} \exp\left[-(x^2 - 2\rho xy + y^2) \big/ 2(1 - \rho^2)\right]}{(\sqrt{2\pi})^{-1} \exp(-y^2/2)}$$

$$= \left[\sqrt{2\pi(1 - \rho^2)}\right]^{-1} \exp\left\{-\left[x^2 - 2\rho xy + y^2 - (1 - \rho^2)y^2\right]\right\}$$

$$= \left[\sqrt{2\pi(1 - \rho^2)}\right]^{-1} \exp\left[-(x^2 - 2\rho xy + \rho^2y^2)\big/2(1 - \rho^2)\right]$$

$$= \left[\sqrt{2\pi(1 - \rho^2)}\right]^{-1} \exp\left[-(x - \rho y)^2\big/2(1 - \rho^2)\right]$$

which is the density of a normal distribution with mean $\rho y$ and variance $(1 - \rho^2)$ and is shown in Figure 1.

Conditional Density of $X$ Given $a < Y < b$. The conditional density of $X$ given $a < Y < b$ is not normal and is derived as follows.

$$f(x|a < Y < b) = \frac{\left(2\pi\sqrt{1 - \rho^2}\right)^{-1}\int_a^b \exp\left[-(x^2 - 2\rho xy + y^2)\big/2(1 - \rho^2)\right] dy}{(\sqrt{2\pi})^{-1}\int_a^b \exp(-y^2/2)\, dy}$$

$$= \left[\Phi(b) - \Phi(a)\right]^{-1}\left(2\pi\sqrt{1 - \rho^2}\right)^{-1} \cdot$$

$$\int_a^b \exp\left\{-\left[y^2 - 2\rho xy + \rho^2x^2 + x^2(1 - \rho^2)\right] \big/ 2(1 - \rho^2)\right\} dy$$

$$= \left(\sqrt{2\pi}\right)^{-1} \exp(x^2/2) \left[\Phi(b) - \Phi(a)\right]^{-1} \cdot$$

$$\int_a^b \left[\sqrt{2\pi(1 - \rho^2)}\right]^{-1} \exp\left[-(y - \rho x)^2\big/2(1 - \rho^2)\right] dy$$

$$= f(x)\left[\Phi(b) - \Phi(a)\right]^{-1} \left\{\Phi\left[(b - \rho x)\big/ \sqrt{1 - \rho^2}\right] - \Phi\left[(a - \rho x)\big/\sqrt{1 - \rho^2}\right]\right\}$$

where

$$f(x) = (\sqrt{2\pi})^{-1} \exp(-x^2/2)$$

is a standard normal density and

$$\Phi(t) = \int_{-\infty}^{t} f(x)dx$$

is the standard normal distribution function.

This conditional density is neither normal, nor symmetric. However, in special cases discussed below, symmetry is identifiable.

Symmetry in Conditioning $-t < Y < t$. With $-t < Y < t$, the conditional density of $X$ at specific values of $x$ and $-x$ are

$$g_t(x) = f(x|-t < Y < t)$$

$$= f(x)\left[\Phi(t) - (-t)\right]^{-1} \left\{\Phi\left[(t - \rho x)/\sqrt{1 - \rho^2}\right] - \Phi\left[(-t - \rho x)/\sqrt{1 - \rho^2}\right]\right\}$$

$$g_t(-x) = f(-x|-t < Y < t)$$

$$= f(-x)\left[\Phi(t) - \Phi(-t)\right]^{-1} \left\{\Phi\left[(t + \rho x)\Big/\sqrt{1 - \rho^2}\right] - \Phi\left[(-t + \rho x)\Big/\sqrt{1 - \rho^2}\right]\right\}$$

The symmetry of a standard normal density shows that $f(-x) = f(x)$. With the asymmetry of distribution function $\Phi(t) = 1 - \Phi(-t)$, it is seen that

$$\Phi\left[(t + \rho x)/\sqrt{1 - \rho^2}\right] - \Phi\left[(-t + \rho x)/\sqrt{1 - \rho^2}\right]$$

$$= 1 - \Phi\left[(-t - \rho x)/\sqrt{1 - \rho^2}\right] - \left\{1 - \Phi\left[(t - \rho x)/\sqrt{1 - \rho^2}\right]\right\}$$

$$= \Phi\left[(t - \rho x)/\sqrt{1 - \rho^2}\right] - \Phi\left[(-t - \rho x)/\sqrt{1 - \rho^2}\right]$$

Thus $g_t(-x) = g_t(x)$, showing that for $-t < Y < t$ the conditional density of $X$ is symmetric in $x$, as shown in figure 2.

The conditioning, $-t < Y < t$, with positive and negative values of correlation coefficient $\rho$ also show symmetry of $g_t(x)$. It is to be noted that

$$g_t(x|\rho > 0) = f(x)[\Phi(t) - \Phi(-t)]^{-1} \cdot \left\{\Phi\left[(t - \rho x)/\sqrt{1 - \rho^2}\right] - \Phi\left[(-t - \rho x)/\sqrt{1 - \rho^2}\right]\right\}$$

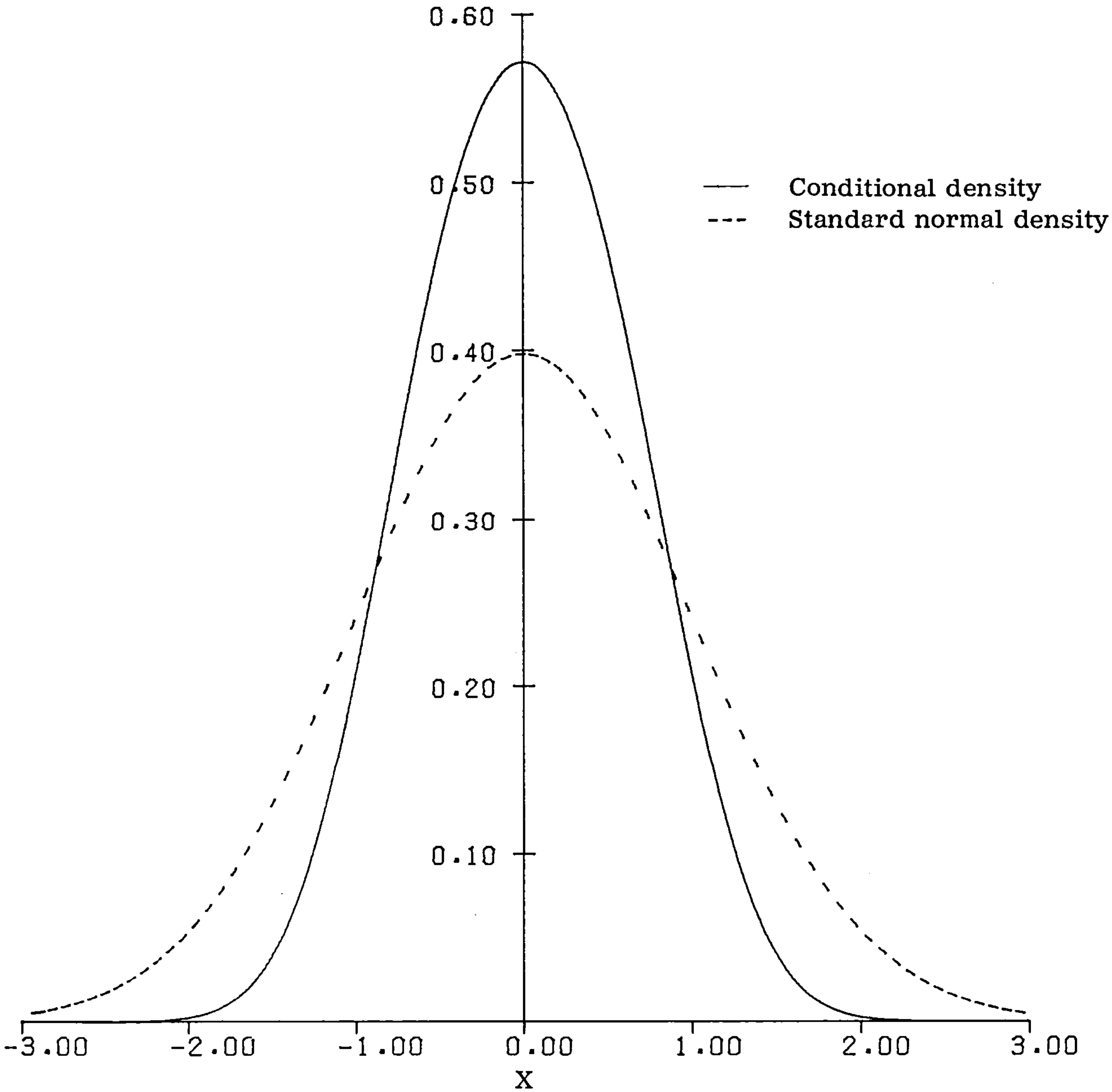


Figure 2. Conditional density of X given $-t < Y < t$ ($t = 1.000$, probability = 0.6826) where (X, Y) is bivariate normal with $\rho = 0.9000$, and standard normal density.

$$g_t(x|\rho < 0) = f(x)[\Phi(t) - \Phi(-t)]^{-1} \cdot$$
$$\left\{\Phi\left[(t + \rho x)/\sqrt{1 - \rho^2}\right] - \Phi\left[(-t + \rho x)/\sqrt{1 - \rho^2}\right]\right\}$$

By the symmetry of $f(x)$, the asymmetry of $\Phi(t)$, and the arguments given earlier, it is seen that $g_t(x|\rho > 0) = g_t(x|\rho < 0)$. The graph of such a density is shown in figure 2.

Symmetry when $-\infty < Y < t$ and $-t < Y < +\infty$. In these cases it is to be noted that $\Phi(-\infty) = 0$, $\Phi(\infty) = 1$. Thus the conditional densities of $X$ are

$$g_t(x|\rho > 0) = f(x|Y < t)$$
$$= f(x)[\Phi(t)]^{-1} \Phi\left[(t - \rho x)/\sqrt{1 - \rho^2}\right]$$

$$g_{-t}(x|\rho > 0) = f(x|-t < Y)$$
$$= f(x)[1 - \Phi(-t)]^{-1}\left\{1 - \Phi\left[(-t - \rho x)/\sqrt{1 - \rho^2}\right]\right\}$$
$$= f(x)[\Phi(t)]^{-1}\left\{\Phi\left[(t + \rho x)/\sqrt{1 - \rho^2}\right]\right\}$$

$$g_{-t}(-x|\rho > 0) = f(-x|-t < Y)$$
$$= f(x)[\Phi(t)]^{-1}\left\{\Phi\left[(t - \rho x)/\sqrt{1 - \rho^2}\right]\right\}$$

Thus $g_t(x) = g_{-t}(-x)$, showing that a one-sided conditioning on $Y$ yields the same density for $x$ as does the conditioning on the other side for the opposite $x$. Further, for negative and positive values of $\rho$, it is to be noted that

$$g_t(x|\rho > 0) = f(x)[\Phi(t)]^{-1}\left\{\Phi\left[(t - \rho x)/\sqrt{1 - \rho^2}\right]\right\}$$

and

$$g_t(x|\rho < 0) = f(x)[\Phi(t)]^{-1}\left\{\Phi\left[(t + \rho x)/\sqrt{1 - \rho^2}\right]\right\}$$
$$= g_{-t}(x|\rho > 0)$$

Therefore, if the conditioning on $Y$ and the sign of the correlation coefficient are reversed, the density remains invariant. An example of these densities is shown in figure 3.

## DERIVATION OF CONDITIONAL DISTRIBUTIONS

Conditional Distribution Function of $X$ Given $Y = y$. The distribution function from the conditional density

$$f(x|Y = y) = \left[\sqrt{2 (1 - \rho^2}\right]^{-1} \exp\left[ -(x - \rho y)^2/2(1 - \rho^2)\right]$$

derived earlier, is easily obtainable via the normal distribution function with mean $\rho y$ and variance $(1 - \rho^2)$. It is to be observed from figure 1, that mean $\rho y$ is a function of the correlation $\rho$ and the specific conditioned value of $y$, but the variance depends only on $\rho$ and is invariant for all values of $y$. Thus the width of any $\alpha$ level confidence interval remains the same irrespective of the conditioned values of $y$.

In applications, the conditioning of variable $Y$ is seldom a fixed value. The conditioning is usually in a range $a < Y < b$, and the formulae for this case are different from the results for $Y = y$.

Conditional Distribution of $X$ Given $a \leq Y \leq b$. The conditional density

$$f(x|a < Y < b) = f(x)[\Phi(b) - \Phi(a)]^{-1} \left\{\Phi\left[(b) - \rho x)/ \sqrt{1 - \rho^2}\right] - \Phi\left[(a - \rho x)/\sqrt{1 - \rho^2}\right]\right\}$$

where

$$f(x) = (\sqrt{2\pi})^{-1} \exp( -x^2/2)$$

and

$$\Phi(t) = \int_{-\infty}^{t} f(x)dx$$

was derived earlier. A general expression for the distribution function

$$F(s) = \int_{-\infty}^{s} f(x|a < Y < b)dx$$

$$= [\Phi(b) - \Phi(a)]^{-1}\int_{-\infty}^{s} f(x) \left\{ \Phi\left[(b - \rho x)/\sqrt{1 - \rho^2}\right] - \Phi\left[(a - \rho x)/\sqrt{1 - \rho^2}\right]\right\}dx$$

for all the values of $s$ involves integration of the expression which is the product of the normal density and distribution function in the appropriate range of the $x$ values. Specifically, for the computation of $F(s)$, the

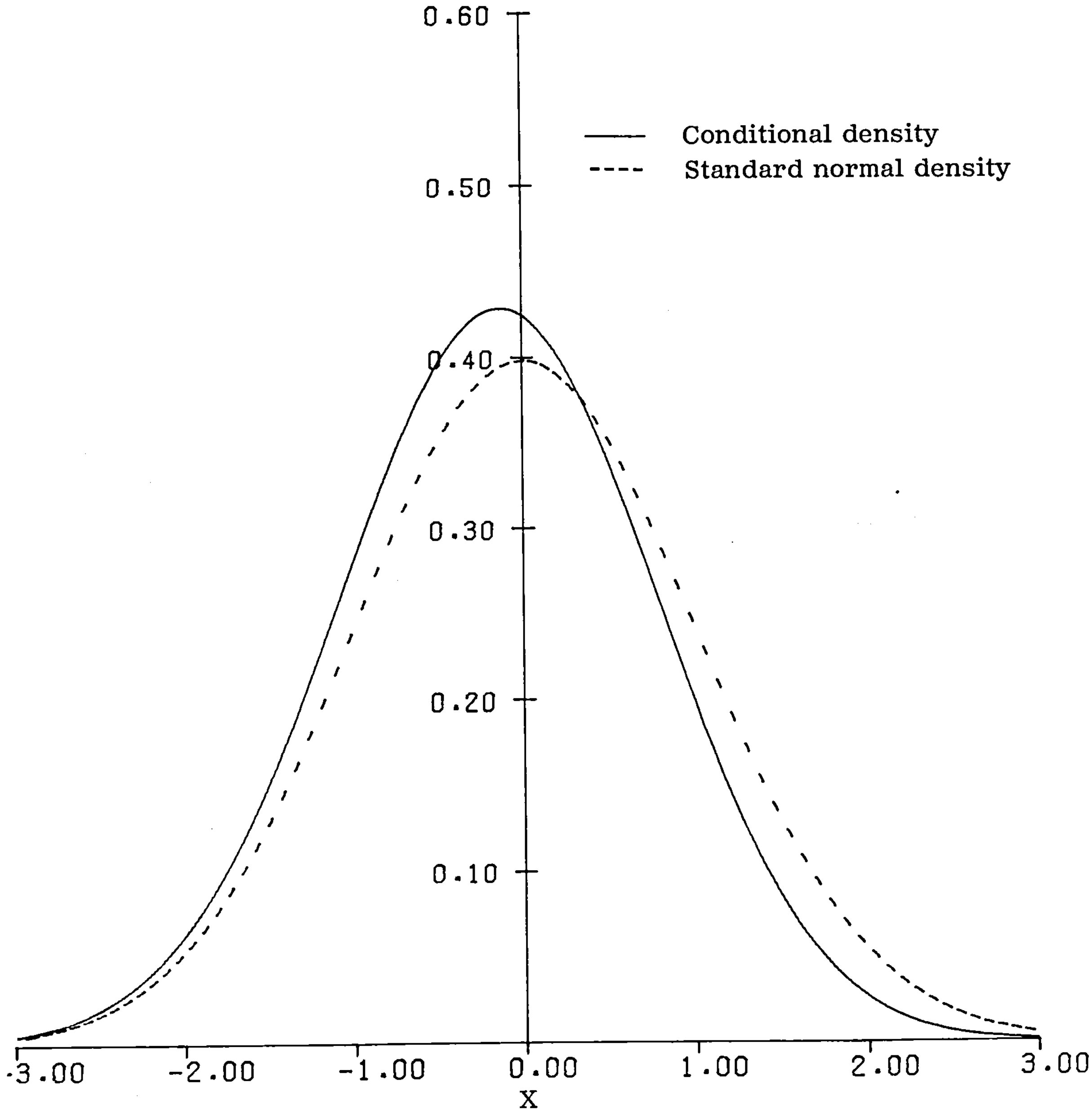


Figure 3. Conditional density of X given $-\infty < Y < t$ ($t = 1.00$, probability = 0.8413) where (X, Y) is bivariate normal with $\rho = 0.6000$, and standard normal density.

value of double integrals such as

$$G_\rho(s,t) = \int_{-\infty}^{s} \exp(-x^2/2) \left[ \int_{-\infty}^{(t - \rho x)/\sqrt{1 - \rho^2}} \exp(-u^2/2) du \right] dx$$

for all values of $s$, $t$ and $\rho$ are required. In terms of these functions, it is easily seen that

$$F(s) = \left\{ 2\pi [\Phi(b) - \Phi(a)] \right\}^{-1} \left[ G_\rho(s,b) - G_\rho(s,a) \right]$$

A closed analytical expression for $G_\rho(s, t)$ is not available and for specific values, numerical methods may be employed. However, in cases where symmetry occurs, the numerical computations for a smaller range of values are needed. In order to calculate $F(s)$ for all values of $s$, $a$, $b$ and $\rho$, a computer program using quadratures was developed at DFRC and is given in the Appendix.

Rectangle Probabilities. The region $(c < x < d, a < Y < b)$ is a rectangle in the $(x, y)$ plane. Thus the joint probability $Pr(c < X < d, a < Y < b)$ for real values of $a$, $b$, $c$ and $d$ corresponds to a rectangle probability. The appended computer program can be used to calculate all such rectangle probabilities. The procedure is to identify first that

$$\begin{aligned} Pr[c < X < d, a < Y < b] &= Pr[c < X < d | a < Y < b] \, Pr[a < Y < b] \\ &= [F(d) - F(c)] \, Pr[a < Y < b] \\ &= [F(d) - F(c)] \, [\Phi(b) - \Phi(a)] \end{aligned}$$

for all values of $c < d$ and $a < b$, and then use the computer program with the proper inputs.

## COMPUTER PROGRAM INPUTS AND OUTPUTS

The computer program developed at DFRC computes the conditional density and distribution function as outputs for specified values of $x$ given the end points of the interval of the conditioning variable $Y$, and the correlation coefficient $\rho$. Thus the inputs to the program are specific $x$ values, end points of the $Y$ interval and the $\rho$ value. The output has two options. Either the density or distribution function, or both may be obtained by stating the options in the program.

The rectangle probabilities are to be obtained by finding the conditional probabilities. The computer program with its options is explained in the Appendix.

## EXAMPLES

The following examples illustrate the use of the program and tables shown in the Appendix to calculate various probabilities.

The data for the examples are taken from a Closed Circuit Television (CCTV) experiment. In this experiment, two pilots, A and B, landed an aircraft with the help of an airborne television camera and video monitor. Each pilot made ten (10) touchdowns under visual flight regulations, and eighteen (18) touchdowns utilizing the closed circuit television monitor. The summary of data from the twenty-eight (28) touchdowns is given in Table 1. For this illustration the data parameters are vertical acceleration, $A_z$, forward velocity, $V_c$ and lateral acceleration $A_y$.

TABLE I. SUMMARY OF 28 TOUCHDOWN DATA OF CCTV EXPERIMENT

| Pilot | Parameter (Units) | Mean $\mu$ | S.D. $\sigma$ | Correlation Between |
|---|---|---|---|---|
| A | $A_z$(G) | 1.313 | .2021 | $(A_z,V_c)$ = .2481 |
| | $V_c$(MPH) | 60.25 | 1.3089 | $(A_z,A_y)$ = -.0715 |
| | $A_y$(G) | .023 | .1227 | $(V_c,A_y)$ = .2807 |
| B | $A_z$(G) | 1.294 | .1044 | $(A_z,V_c)$ = -.2569 |
| | $V_c$(MPH) | 62.04 | 1.8747 | $(A_z,A_y)$ = -.2199 |
| | $A_y$(G) | -.007 | .0801 | $(V_c,A_y)$ = -.1993 |

The variables $(A_z, V_c, A_y)$ are assumed to follow a multivariate normal distribution. Thus any two variables follow a bivariate normal distribution and any single variable, a univariate normal distribution, as shown in figure 1. Further, all the values in these date are considered to be parameter values.

Example 1. Computation of a 95% confidence interval of forward velocity ($V_c$) given vertical acceleration ($A_z$) mean is within ± one standard deviation ($\sigma$). It is desired in this example to determine a 95% confidence interval for aircraft forward velocity ($V_c$), in miles per hour, at the point of touchdown, given the pilot's average vertical acceleration ($A_z$), in G's, within ± one standard deviation. The 95% confidence interval end points for $V_c$ given $A_z$ mean is within ± $\sigma$ are obtained by solving for $t$ from the equation

$$.95 = Pr[-t < (V_c - \mu_c)/\sigma_c < t | -1 < (A_z - \mu_z)/\sigma_z < 1]$$

$$= Pr[-t < X < t | -1 < Y < 1]$$

and identifying the interval as $(-t\sigma_c + \mu_c, t\sigma_c + \mu_c)$.

The solution of the equation for pilot A data of $\mu_c = 60.25$, $\sigma_c = 1.3089$, $\mu_z = 1.313$, $\sigma_z = 0.2021$ and correlation $(A_z, V_c) = -.2481$, yields the value of $t = 1.91666$. The 95% confidence interval, therefore, becomes

$$(57.7413, 62.7587)$$

This shows that if in pilot A data, the aircraft's vertical acceleration at touchdown is within +1.3 ±0.2 G's, he has a 95% chance of landing the aircraft between 58 and 63 MPH.

For pilot B data, from table 1, the $t$ value computes to be 1.9136. Thus the 95% confidence interval is

$$(57.7453, 62.7547)$$

indicating if pilot B's vertical acceleration data at touchdown is within +1.3 ±0.1 G's, he also has a 95% chance of landing the aircraft between 58 and 63 MPH.

Example 2. Computation of the probability that the forward velocity $(V_c)$ and $(A_y)$ are both within $\pm\sigma$ of each variable. The probability of $V_c$ and $A_y$ being within $\pm\sigma$ of each respective mean is an example of rectangle probability. In this example, the probability that simultaneously, $V_c$ and $A_y$, will be within one standard deviation of each variable's respective mean is to be computed.

This rectangle probability can be obtained by finding

$$Pr[-1 < (V_c - \mu_c)/\sigma_c < 1 \ , \ -1 < (A_y - \mu_y)/\sigma_y < 1]$$

$$= Pr[-1 < X < 1 \mid -1 < Y < 1] \ Pr[-1 < Y < 1]$$

From univariate tables, $Pr[-1 < Y < 1] = \Phi(1) - \Phi(-1) = .6826$ and is not affected by the correlation coefficients. In order to obtain $Pr[-1 < X < 1 \mid -1 < Y < 1]$, the values of the correlation coefficients are needed.

The correlation coefficient $(V_c, A_y)$ for pilot A data is equal to -0.2807. The computer program output, therefore, for this correlation yields

$$Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c \ , \ \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .47554$$

Thus, for pilot A there is a 48% chance that simultaneously at touchdown, the aircraft's forward velocity will be within 60 ±1.3 MPH and the lateral acceleration is within 0 ±0.1 G's. Conversely, the probability is 0.52 that both variables will not simultaneously be within one standard deviation of their respective means. Similarly, for pilot B with the correlation ($V_c$, $A_y$) equal to .1993, the program yields

$$Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c, \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .47078$$

which represents a 0.47 probability that the forward velocity will be within 62 ±1.9 MPH and lateral acceleration is within 0 ±0.08 G's.

Example 3. Computation of the probability of forward velocity ($V_c$) and lateral acceleration ($A_y$) being within $\pm\sigma$ of each variable, given vertical acceleration is equal to its mean ($A_z = \mu_z$). This rectangle probability can be obtained as in Example 2, except in this case the vertical acceleration ($A_z$) is set equal to the variable's mean value ($\mu_z$). The probability in other words, is a function of a conditional correlation coefficient which is different from the coefficient given in the table.

For the pilot A data, this conditional coefficient is equal to .3809 and the program output yields

$$Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c , \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .48391$$

which represents a 0.48 probability that the forward velocity will be within 60.25 ±1.309 MPH, and lateral acceleration within .023 ±0.1227 G's given that vertical acceleration is 1.313 G's.

For pilot B, the conditional correlation coefficient is equal to -.2807 and the corresponding rectangle probability is

$$Pr[\mu_c - \sigma_c < V_c < \mu_c + \sigma_c , \mu_y - \sigma_y < A_y < \mu_y + \sigma_y] = .47554$$

This represents a 0.48 probability that the forward velocity will be within 62.0 ±1.9 MPH and lateral acceleration is within 0 ±0.08 G's given the vertical acceleration is +1.294 G's.

*Dryden Flight Research Center*
*National Aeronautics and Space Administration*
*Edwards, California, March 17, 1980*

# APPENDIX

The program to compute the conditional density and distribution function for specified values of $x$ given the conditioning on variable $Y$.

```
      PROGRAM MAIN(OUTPUT)
C****
C****              ILLUSTRATIVE USE OF THE ENCLOSED COMPUTER
C****              PROGRAMS TO COMPUTE VARIOUS
C****              PROBABILITIES ASSOCIATED WITH THE EXAMPLES
C****              GIVEN IN THE TEXT OF THIS PAPER...
C****
C****              BY BROWNLOW, SDC/ISI
C****
      PRINT 1,  RPROB(-1.,1.,-1.,1.,-.2807)
      PRINT 1, RPROB(-1.,1.,-1.,1.,-.1993)
      PRINT 1, RPROB(-1.,1.,-1.,1.,.3809)
C****
      PRINT1,  TINV(.95,-1.,1.,-.2481)
      PRINT 1,  TINV(.95,-1.,1.,-.2569)
    1 FORMAT(* *F10.5)
      END
```

```
      FUNCTION CD(X)
C****
C****             CONDITIONAL DENSITY FUNCTION
C****             CD(X\A<Y<B) = 1./SQRT(2*PI*EXP(X*X))*
C****              ( PHI( (B-R*X)/(SQRT(1-R*R)) - PHI( (A-R*X)/(SQRT(1-R*R)))
C****              /( PHI(B) - PHI(A) )
C****
C****             WHERE R = COEFFICIENT OF CORRELATION BETWEEN
C****             X AND Y
C****
C****                          T
C****             PHI(T) = INTEGRAL F(X) DX
C****                         -INF
C****
C****             AND F(X) = 1./SQRT(2*PI*EXP(X*X))
C****             BY BROWNLOW, SDC/ISI
C****
C****
      COMMON/PARAM/A,B,R,SQR
      CD = .39894228/SQRT(EXP(X*X)) * ( PHI( (B-R*X)/SQR)-
     . PHI((A-R*X)/SQR))/(PHI(B)-PHI(A))
C****
      RETURN
      END
```

```
      FUNCTION FINT2(F,A,B)
C****
C****               INTEGRAL OF THE FUNCTION F FROM A TO B
C****               BY GAUSSIAN-LEGENDRE QUADRITURE,  96 POINT FORM
C****               REQUIRES 96 EVALUATIONS OF F(X).
C****
C****               F MUST BE DECLARED EXTERNAL IN
C****               THE CALLING PROGRAM.
C****               BY BROWNLOW, SDC/ISI
C****
      DOUBLE PRECISION ROOT(48), WEIGHT(48), ANSWER,DA,DB
      DOUBLE PRECISION ARY(48,2)
      EQUIVALENCE (ARY(1,1),ROOT(1)),  (ARY(1,2),WEIGHT(1))
C****
C****
C****               SET UP ROOTS AND WEIGHTS...
      DATA ((ARY(I,J),J=1,2),I=1,18)   /
     .  0.01627 67448 49602 96957900,  0.03255 06144 92363 16624200,
     .  0.04881 29851 36049 73111200,  0.03251 61187 13868 83598700,
     .  0.08129 74954 64425 55899400,  0.03244 71637 14064 26936400,
     .  0.11369 58501 10665 92091100,  0.03234 38225 68575 92842900,
     .  0.14597 37146 54896 94198900,  0.03220 62047 94030 25066900,
     .  0.17809 68823 67618 60275900,  0.03203 44562 31992 66321800,
     .  0.21003 13104 60567 20360300,  0.03182 87588 94411 00653500,
     .  0.24174 31561 63840 01232800,  0.03158 93307 70727 16855800,
     .  0.27319 88125 91049 14148700,  0.03131 64255 96861 35581300,
     .  0.30436 49443 54496 35302400,  0.03101 03325 86313 83742300,
     .  0.33520 85228 92625 42261600,  0.03067 13761 23669 14901400,
     .  0.36569 68614 72313 63503100,  0.03029 99154 20827 59379400,
     .  0.39579 76498 28908 60328500,  0.02989 63441 36328 38598400,
     .  0.42547 89884 07300 54536500,  0.02946 10899 58167 90597000,
     .  0.45470 94221 67743 00863600,  0.02899 46141 50555 23654300,
     .  0.48345 79739 20596 35976800,  0.02849 74110 65085 38564600,
     .  0.51169 41771 54667 67358600,  0.02797 00076 16848 33444000,
     .  0.53938 81083 24357 43622700,  0.02741 29627 26029 24282300/
C****
      DATA  ((ARY(I,J),J=1,2),I=19,37)/
     .  0.56651 04185 61397 16840400,  0.02682 68667 25591 76219800,
     .  0.59303 23647 77572 08368400,  0.02621 23407 35672 41391300,
     .  0.61892 58401 25468 57038600,  0.02557 00360 05349 36149900,
     .  0.64416 34037 84967 10679800,  0.02490 06332 22483 61028800,
     .  0.66871 83100 43916 15395300,  0.02420 48417 92364 69128200,
     .  0.69256 45366 42171 56134400,  0.02348 33990 85926 21984200,
     .  0.71567 68123 48967 62622500,  0.02273 70696 58329 37400100,
     .  0.73803 06437 44400 13285100,  0.02196 66444 38744 34919500,
     .  0.75960 23411 76647 49870300,  0.02117 29398 92191 29898800,
     .  0.78036 90438 67433 21760400,  0.02035 67971 54333 32459500,
     .  0.80030 37441 39140 81722900,  0.01951 90811 40145 02241000,
     .  0.81940 03107 37931 67553900,  0.01866 06796 27411 46738500,
     .  0.83752 35112 28127 12149400,  0.01778 25023 16045 26083800,
     .  0.85495 90334 34601 45546300,  0.01688 54798 64245 17245000,
     .  0.87138 85059 09296 50287400,  0.01597 05629 02562 29138100,
     .  0.88689 45174 02420 41605700,  0.01503 87210 26994 93800600,
     .  0.90146 06353 15852 34131900,  0.01409 09417 72314 86091600,
     .  0.91507 14231 20393 07420600,  0.01312 82295 66961 57263700,
     .  0.92771 24567 22308 69096500,  0.01215 16046 71088 31963500/
C****
```

```
            DATA((ARY(I,J),J=1,2),I=38,48)/
           .  0.93937 03397 52755 216932D0,  0.01116 21020 99838 493591D0,
           .  0.95003 27177 84437 635756D0,  0.01016 07705 35008 415758D0,
           .  0.95968 82914 48742 539300D0,  0.00914 86712 30783 386633D0,
           .  0.96832 68284 63264 212174D0,  0.00812 68769 25698 759217D0,
           .  0.97593 91745 35135 466453D0,  0.00709 64707 91153 865269D0,
           .  0.98251 72635 63014 677447D0,  0.00605 85455 04235 961683D0,
           .  0.98805 41263 29623 799481D0,  0.00501 42027 42927 517693D0,
           .  0.99254 39003 23762 624572D0,  0.00396 45543 38444 686674D0,
           .  0.99598 18429 87209 290650D0,  0.00291 07318 17934 946408D0,
           .  0.99836 43758 63181 677724D0,  0.00185 39607 38946 921732D0,
           .  0.99968 95038 83230 766828D0,  0.00079 67920 65552 012429D0/
      C****
      C****
      C****               INTEGRATION DONE BY TRANSLATING F TO THE
      C****               INTERVAL -1 TO 1
      C****
            ANSWER = 0.D0
            DB = B
            DA = A
      C****
            DO 1 I=1,48
            T = ((DB-DA)*ROOT(I) +(DB+DA))/2.D0
            ANSWER = ANSWER + WEIGHT(I) * F(T)
            T = ((DB-DA)*(-ROOT(I)) +(DB+DA))/2.D0
          1 ANSWER = ANSWER + WEIGHT(I) * F(T)
      C****
            FINT2 = (DB-DA)*ANSWER/2.D0
      C****
            RETURN
            END
```

```
      FUNCTION RECT(A,B,R)
C****
C****              RECTANGLE PROBABILITY...
C****              VOLUME UNDER THE NORMAL BIVARIATE DENSITY,
C****              -INF<X<A,   -INF<Y<B.
C****
C****              BY BROWNLOW, SDC/ISI
C****
      COMMON/GPARM/ AA,BB,RR,SQR
      EXTERNAL G
C****
      AA = A
      BB = B
      RR = R
      SQR = SQRT(1.-R*R)
C****
      RECT = FINT2(G,-15.,A)
C****
      RETURN
      END
```

```
      FUNCTION G(X)
C****
C****                   CONDITIONAL DISTRIBUTION FUNCTION...
C****
      COMMON/GPARM/A,B,R,SQR
C****
      T = (B-R*X)/SQR
      G = EXP(-X*X/2.)*PHI(T)*2.506628275
      RETURN
      END
```

```
      FUNCTION TINV(P,A,B,R)
C****
C****                 GIVEN A<Y<B FIND T SO THAT -T<X<T AND
C****                 P( -T<X<T \ A<Y<B) = P
C****
C****                 WITH COEFFICIENT OF CORRELATION BETWEEN X AND
C****                 Y EQUAL TO R.
C****
C****                 P( -T<X<T \ A<Y<B) = P( -T<X<T, A<Y<B)/P(A<Y<B)
C****
C****                 T IS FOUND BY INTERVAL HALVING.
C****                 BY BROWNLOW, SDC/ISI
C****
C****
      DENOM = PHI(B)-PHI(A)
      TMAX = 10.
      TMIN = 0.
C****
      DO 1 I=1,50
      T = (TMAX + TMIN)/2.
C****
      PCOMPT = RPROB(A,B,-T,T,R)/DENOM
C****
      IF(PCOMPT .GT. P) TMAX = T
      IF(PCOMPT .LT. P) TMIN = T
      IF( ABS(PCOMPT-P) .LE. 1.E-5) GO TO 2
    1 CONTINUE
C****
      PRINT 100, P.A,B,R
  100 FORMAT(*  COULDN'T FIND T IN 50 ITERATIONS: P=*,F7.4,*  A=*,F7.4,
     . *  B=*,F7.4,*  R=*,F7.4)
      TINV = (TMIN+TMAX)/2.
      RETURN
C****
    2 TINV = T
      RETURN
      END
```

```
      FUNCTION RPROB(A,B,C,D,R)
C****
C****                 RECTANGLE PROBABILITY FOR BIVARIATE
C****                 NORMAL DISTRIBUTION...
C****
C****                   C<X<D
C****                   A<Y<B
C****
C****                 AND THE COEFFICIENT OF CORRELATION BETWEEN
C****                 X AND Y IS R.
C****                 BY BROWNLOW, SDC/ISI
C****
      RPROB = (RECT(D,B,R) - RECT(C,B,R)-RECT(D,A,R) + RECT(C,A,R))
     .         *0.159154943
C****
C****
      RETURN
      END
```

```
      C****
      C****
60          RETURN
            END
```

```
      FUNCTION PHI(X)
C****
C****
C****                   NORMAL(0,1) DISTRIBUTION FUNCTION
C****                   PHI(X) = INTEGRAL OF NORMAL DENSITY
C****                   FROM -INFINITY TO X.
C****                   BY BROWNLOW, SDC/ISI
C****
C****
      LOGICAL FLAG
      IF(X .GT. -10.) GO TO 1
      PHI = 0.
      RETURN
C****
    1 IF(X .LT. 10.) GO TO 2
      PHI = 1.
      RETURN
C****
    2 FLAG = .T.
C****
C****                   DETERMINE IF X>0, SERIES EXPANSION IS FOR
C****                   POSITIVE VALUES OF X..
C****
      IF(X .GT. 0.) GO TO 3
      FLAG = .F.
C****
C****                   INITIALIZE VALUES FOR PARTIAL SUM OF THE SERIES...
C****
    3 Z = ABS(X)
      D = 1.
      SUM = 0.
      TOP = Z
      BOT = 1.
C****
    4 CONTINUE
      SAVE = SUM
      SUM = SUM + TOP/BOT
C****
C****                   CONTINUE TO SUM UNTIL MACHINE UNDERFLOWS...
C****
      IF(SAVE .EQ. SUM) GO TO 5
C****
C****                   UPDATE EXPRESSIONS FOR THE SUM...
C****
      TOP = TOP*Z*Z
      D = D + 2.
      BOT = BOT*D
      GO TO 4
C****
C****
C****                   DEPENDING UPON WHETHER ORIGINAL X>0 OR X<0,
C****                   GET APPROPRIATE INTEGRAL VALUE...
C****
    5 PHI = SUM/SQRT(6.283185308*EXP(X*X)) +.5
      IF(FLAG) RETURN
C****
      PHI = 1.-PHI
```

```
      FUNCTION FRAC(PV,Y1,Y2)
C****
C****                 GIVEN A BIVARIATE NORMAL DISTRIBUTION
C****                 WITH COEFFICIENT OF CORRELATION RHO
C****                 AND Y1 < Y < Y2, FRAC(PV,Y1,Y2) RETURNS
C****                 THAT VALUE T, SUCH THAT:
C****
C****                 PROB( -T < X < T, Y1 < Y < Y2 ) = PV
C****
C****                 BINARY SEARCH, LIMITED TO A MAXIMUM OF 20 ITERATIONS
C****
C****
C****                 BY BROWNLOW, SDC/ISI, 11/79
C****
C****
      KOUNT = 0
      TMIN = 0.
      TMAX = 10.
C****
    1 T = (TMIN+TMAX)/2.
C****
      VAL = RECT(-T,T,Y1,Y2)
      PRINT 100, VAL,T
  100 FORMAT(*  *2F10.6//)
C****
C****
C****                 IF WERE WITHIN 1.E-5 OF THE VALUE, WE
C****                 HAVE FOUND THE SOLUTION...
C****
      IF( ABS(VAL-PV) .LT. 1.E-5)  GO TO 2
C****
      IF(VAL .LT. PV) TMIN=T
      IF(VAL .GT. PV) TMAX = T
C****
C****                 CHECK FOR MAXIMUM NUMBER OF ITERATIONS...
C****
      IF(KOUNT .GE.20)   RETURN
      KOUNT = KOUNT + 1
C****
C****
      GO TO 1
C****
    2 FRAC = T
C****
C****
      RETURN
      END
```

```
      FUNCTION CONDEN(X)
C****
C****                CONDITIONAL DENSITY FUNCTION OF X, GIVEN
C****                A<Y<B  FROM BIVARIATE NORMAL DISTRIBUTION
C****                F(X,Y) WITH COEFFICIENT OF CORRELATION RHO.
C****
C****                BY BROWNLOW, SDC/ISI,  11/79
C****
C****
      COMMON/ARG/RHO,A,B
C****
C****                SET UP THE PARAMETERS, PHI IS THE UNIVARIATE
C****                NORMAL DISTRIBUTION FUNCTION.
C****
      R = SQRT(1.-RHO*RHO)
      D = PHI(B)-PHI(A)
      T = PHI( (B-RHO*X)/R) - PHI( (A-RHO*X)/R )
C****
      CONDEN = EXP(-X*X/2.)*T/(D*2.506628275)
C****
      RETURN
      END
```

```
      FUNCTION G(S,T)
C****
C****              BIVARIATE NORMAL DISTRIBUTION FUNCTION.
C****              G(S,T) = DOUBLE INTEGRAL OF NORMAL
C****              BIVARIATE DENSITY FUNCTION, -INF TO S,
C****              -INF TO T.
C****
C****              NOTICE THAT THE NUMERICAL COMPUTATIONS
C****              USE THE FACT THAT THE CONTRIBUTION TO THE
C****              INTEGRAL VALUE FROM -INF TO -15.
C****              IS  INSIGNIFICANT.
C****
C****              BY BROWNLOW, SDC/ISI,  11/79
C****
      COMMON/PASS/TT
C****
      EXTERNAL FCN
C****
      TT = T
C****
      G = FINT2(FCN,-15.,S)/6.283185308
C****
      RETURN
      END
```

```
      FUNCTION FCN(X)
C****
C****              DENSITY FUNCTION FOR DOUBLE INTEGRAL,
C****              PHI(X)*Z(X), WHERE PHI AND Z ARE THE
C****              NORMAL DISTRIBUTION AND DENSITY FUNCTIONS
C****              RESPECTIVELY.
C****
C****
C****              BY BROWNLOW, SUC/ISI, 11/79
C****
C****
      COMMON/ARG/RHO,A,B
      COMMON/PASS/TT
C****
      Z(ARG) = EXP(-ARG*ARG/2.)
C****
C****
      U = (TT-RHO*X)/SQRT(1.-RHO*RHO)
C****
      FCN = PHI(U)*Z(X)*2.506628275
C****
C****
      RETURN
      END
```

```
      FUNCTION RECT(X1,X2,Y1,Y2)
C****
C****                 RECTANGLE PROBABILITIES FOR BIVARIATE NORMAL
C****                 DISTRIBUTION, X1<X<X2,  Y1<Y<Y2,  AND THE
C****                 COEFFICIENT OF CORRELATION IS RHO.
C****
C****                 BY BROWNLOW, SDC/ISI,  11/79
C****
C****
      RECT= G(X2,Y2)-G(X1,Y2)-G(X2,Y1)+G(X1,Y1)
C****
      RETURN
      END
```

```
      FUNCTION CONDIST(X)
C****
C****              CONDITIONAL DISTRIBUTION FUNCTION OF X GIVEN
C****              A<Y<B FROM BIVARIATE NORMAL DISTRIBUTION
C****              F(X,Y) WITH COEFFICIENT OF CORRELATION RHO.
C****
C****              BY BROWNLOW, SDC/ISI,  11/79
C****
C****
      COMMON/ARG/RHO,A,B
C****
C****              PHI IS THE UNIVARIATE NORMAL DISTRIBUTION
C****              FUNCTION.
C****
      CONDIST=(G(X,B)-G(X,A))/((PHI(B)-PHI(A))*6.283185308)
C****
      RETURN
      END
```

```
      FUNCTION FINT2(F,A,B)
C****
C****               INTEGRAL OF THE FUNCTION F FROM A TO B
C****               BY GAUSSIAN-LEGENDRE QUADRITURE,  96 POINT FORM
C****               REQUIRES 96 EVALUATIONS OF F(X).
C****
C****               F MUST BE DECLARED EXTERNAL IN
C****               THE CALLING PROGRAM.
C****               BY BROWNLOW, SDC/ISI
C****
      DOUBLE PRECISION ROOT(48), WEIGHT(48), ANSWER,DA,DB,ARY(48,2)
      EQUIVALENCE (ARY(1,1),ROOT(1)),  (ARY(1,2),WEIGHT(1))
C****
C****
C****               SET UP ROOTS AND WEIGHTS...
      DATA ((ARY(I,J),J=1,2),I=1,18)  /
     . 0.01627 67448 49602 96957900,  0.03255 06144 92363 16624200,
     . 0.04881 29851 36049 73111200,  0.03251 61187 13868 83598700,
     . 0.08129 74954 64425 55899400,  0.03244 71637 14064 26936400,
     . 0.11369 58501 10665 92091100,  0.03234 38225 68575 92842900,
     . 0.14597 37146 54896 94198900,  0.03220 62047 94030 25066900,
     . 0.17809 68823 67618 60275900,  0.03203 44562 31992 66321300,
     . 0.21003 13104 60567 20360300,  0.03182 87588 94411 00653500,
     . 0.24174 31561 63840 01232800,  0.03158 93307 70727 16855800,
     . 0.27319 88125 91049 14148700,  0.03131 64255 96861 35581300,
     . 0.30436 49443 54496 35302400,  0.03101 03325 86313 83742300,
     . 0.33520 85228 92625 42261600,  0.03067 13761 23669 14901400,
     . 0.36559 68614 72313 63503100,  0.03029 99154 20827 59379400,
     . 0.39579 76498 25903 60328500,  0.02989 63441 36328 38598400,
     . 0.42547 89884 07300 54535500,  0.02946 10899 58167 90597000,
     . 0.45470 94221 67743 00863600,  0.02899 46141 50555 23654300,
     . 0.48345 79739 20596 35976300,  0.02849 74110 65085 38564600,
     . 0.51159 41771 54567 67358600,  0.02797 00076 16848 33444000,
     . 0.53938 81083 24357 43622700,  0.02741 29627 26029 24232300/
C****
      DATA  ((ARY(I,J),J=1,2),I=19,37)/
     . 0.56661 04185 61397 16840400,  0.02682 68667 25591 76219800,
     . 0.59303 23647 77572 08368400,  0.02621 23407 35672 41391300,
     . 0.61892 58401 25468 57038600,  0.02557 00360 05349 36149900,
     . 0.64415 34037 54967 15679600,  0.02490 06332 22483 61028800,
     . 0.66871 33103 43916 15395300,  0.02420 48417 92364 69128200,
     . 0.69255 45366 42171 56134400,  0.02348 33990 85926 21984200,
     . 0.71567 63123 48967 52622500,  0.02273 70696 58329 37400100,
     . 0.73803 05437 44400 13285100,  0.02196 66444 38744 34919500,
     . 0.75960 23411 76647 49870300,  0.02117 29398 92191 29898300,
     . 0.78036 90438 67433 21760400,  0.02035 67971 54333 32459500,
     . 0.80030 87441 39140 31722900,  0.01951 90811 40145 02241000,
     . 0.81940 03107 37931 67553900,  0.01866 05796 27411 46738500,
     . 0.83762 35112 25127 12149400,  0.01778 25023 16045 26033800,
     . 0.85495 90334 34601 45546300,  0.01688 54798 64245 17245000,
     . 0.87138 85059 09296 50287400,  0.01597 05629 02562 29138100,
     . 0.88689 45174 02420 41605700,  0.01503 87210 26994 93800600,
     . 0.90145 05353 15352 34131900,  0.01409 09417 72314 86091600,
     . 0.91507 14231 20393 07420600,  0.01312 82295 66961 57263700,
     . 0.92771 24567 22303 59096500,  0.01215 16046 71088 31953500/
C****
      DATA((ARY(I,J),J=1,2),I=38,48)/
```

```
     .  0.93937 03397 52755 21593200,  0.01116 21020 99838 49859100,
     .  0.95003 27177 54437 53575600,  0.01016 07705 35008 41575800,
     .  0.95968 82914 48742 53930000,  0.00914 66712 30783 38663300,
     .  0.96832 63284 53264 21217400,  0.00812 68769 25698 75921700,
     .  0.97593 91745 85136 45645300,  0.00709 64707 91153 86526900,
     .  0.98251 72635 63014 67744700,  0.00605 85455 04235 96168300,
     .  0.98805 41263 29623 79948100,  0.00501 42027 42927 51769300,
     .  0.99254 39003 23762 62457200,  0.00396 45543 38444 68667400,
     .  0.99598 18429 87209 29065000,  0.00291 07318 17934 94640800,
     .  0.99836 43758 63181 67772400,  0.00185 39607 88946 92173200,
     .  0.99968 95038 83230 76682800,  0.00079 67920 65552 01242900/
C****
C****
C****               INTEGRATION DONE BY TRANSLATING F TO THE
C****               INTERVAL -1 TO 1
C****
      ANSWER = 0.D0
      DB = B
      DA = A
C****
      DO 1 I=1,48
      T = ((DB-DA)*ROOT(I) +(DB+DA))/2.D0
      ANSWER = ANSWER + WEIGHT(I) * F(T)
      T = ((DB-DA)*(-ROOT(I)) +(DB+DA))/2.D0
    1 ANSWER = ANSWER + WEIGHT(I) * F(T)
C****
      FINT2 = (DB-DA)*ANSWER/2.D0
C****
      RETURN
      END
```

```
      FUNCTION PHI(X)
C****
C****
C****                  NORMAL(0,1) DISTRIBUTION FUNCTION
C****                  PHI(X) = INTEGRAL OF NORMAL DENSITY
C****                  FROM -INFINITY TO X.
C****
C****
      LOGICAL FLAG
      IF(X .GT. -10.) GO TO 1
      PHI = 0.
      RETURN
C****
    1 IF(X .LT. 10.) GO TO 2
      PHI = 1.
      RETURN
C****
    2 FLAG = .T.
C****
C****                  DETERMINE IF X>0, SERIES EXPANSION IS FOR
C****                  POSITIVE VALUES OF X..
C****
      IF(X .GT. 0.) GO TO 3
      FLAG = .F.
C****
C****                  INITIALIZE VALUES FOR PARTIAL SUM OF THE SERIES...
C****
    3 Z = ABS(X)
      D = 1.
      SUM = 0.
      TOP = Z
      BOT = 1.
C****
    4 CONTINUE
      SAVE = SUM
      SUM = SUM + TOP/BOT
C****
C****                  CONTINUE TO SUM UNTIL MACHINE UNDERFLOWS...
C****
      IF(SAVE .EQ. SUM) GO TO 5
C****
C****                  UPDATE EXPRESSIONS FOR THE SUM...
C****
      TOP = TOP*Z*Z
      D = D + 2.
      BOT = BOT*D
      GO TO 4
C****
C****
C****                  DEPENDING UPON WHETHER ORIGINAL X>0 OR X<0,
C****                  GET APPROPRIATE INTEGRAL VALUE...
C****
    5 PHI = SUM/SQRT(6.283185308*EXP(X*X)) +.5
      IF(FLAG) RETURN
C****
      PHI = 1.-PHI
C****
```

```
C****
      RETURN
      END
```

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16. Abstract

Normal distribution is widely employed in numerous disciplines. Unfortunately, probabilities for the multivariate normal distribution are generally not available. This paper presents some results for the bivariate normal distribution. Computer programs for conditional normal probabilities, marginal probabilities as well as joint probabilities for rectangular regions are given: routines for computing fractile points and distribution functions are also presented. Some examples from a closed circuit television experiment are included.

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