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Maximum Likelihood Method for Estimating Airplane Stability and Control Parameters From Flight Data in Frequency Domain

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SUMMARY

A frequency domain maximum likelihood method is developed for the estimation of airplane stability and control parameters from measured data. The model of an airplane is represented by a discrete-type steady-state Kalman filter with time variables replaced by their Fourier series expansions. The likelihood function of innovations is formulated, and by its maximization with respect to unknown parameters the estimation algorithm is obtained. This algorithm is then simplified to the output error estimation method with the data in the form of transformed time histories, frequency response curves, or spectral and cross-spectral densities. The development is followed by a discussion on the equivalence of the cost function in the time and frequency domains, and on advantages and disadvantages of the frequency domain approach. The algorithm developed is applied in four examples to the estimation of longitudinal parameters of a general aviation airplane using computer-generated and measured data in turbulent and still air. The cost functions in the time and frequency domains are shown to be equivalent; therefore, both approaches are complementary and not contradictory. Despite some computational advantages of parameter estimation in the frequency domain, this approach is limited to linear equations of motion with constant coefficients.

INTRODUCTION

The early approaches to the extraction of airplane stability and control parameters from flight data were based on simple semigraphical or analytical methods. Some of these methods used measured frequency response curves which provided good insight into the physics of the system and reduced data processing to the use of simple algebra. One of the first attempts to analyze measured data in the frequency domain for obtaining the characteristics of the short-period longitudinal motion of an airplane was made in reference 1. In reference 2 the same characteristics were estimated either by fitting the measured frequency response curves or by substituting the measured data in the transfer function equation and minimizing the resulting error. In both cases the least-squares technique was applied. The same technique was used for the direct estimation of the longitudinal and lateral aerodynamic parameters in references 3 and 4, respectively.

The regression with complex variables was developed in reference 5 and applied to the estimation of airplane transfer function coefficients from measured frequency response curves. A more general formulation of the regression in the frequency domain was introduced in reference 6 and extended to the maximum likelihood method in reference 7. In both cases the procedure was used for the design of an optimal input for system identification rather than for parameter estimation.

With the availability of modern digital computers, the frequency domain for airplane parameter estimation was almost forgotten and the measured data

have been mostly analyzed in the time domain. However, some further research and applications in this area have appeared. New frequency domain methods for system identification based on the equation-error formulation were introduced in reference 8. Frequency domain data were used for the extraction of parameters of an elastic airplane in reference 9, of parameters of an airplane with nonsteady aerodynamics in references 10 and 11, and of flying qualities criteria in reference 12.

The material contained in this report is an extension of the research initiated in reference 5 and continued in references 6 and 7. Also included in this report are some of the developments and results from references 13 and 14, respectively. The purpose of this report is to present a rigorous development of an algorithm for the maximum likelihood estimation of airplane parameters in the frequency domain. The report also briefly points out the relationships between the estimation in the time and frequency domains, and the advantages and disadvantages of the frequency domain approach, mainly in terms of applicability, computing complexity, and accuracy of final results. The development starts with the formulation of a steady-state Kalman filter for a linear dynamical system. Before the log-likelihood function of the innovations is formulated, the basic properties of a complex random number and random sequence are presented. The log-likelihood function is minimized by using the modified Newton-Raphson technique. The maximum likelihood algorithm is then simplified by neglecting external disturbances to the airplane. Following the discussion, four examples are presented. They deal with the simplified longitudinal motion of a general aviation airplane and use both computer-generated and real-flight data.

SYMBOLS

A	sensitivity matrix
a_z	reading of vertical accelerometer, g units
B	covariance matrix of residuals
C_m	pitching-moment coefficient, $M_Y/\bar{q}S\bar{c}$
C_Z	vertical-force coefficient, $F_Z/\bar{q}S$
$\bar{c}$	wing mean aerodynamic chord, m
c_1, c_2	constants in differential equation for Gauss-Markoff process
D	matrix of transformed-system equations
E{ }	expected value
F	matrix of continuous system
F_Z	force along vertical body axis, N

G	control matrix of continuous system
G_w	process-noise distribution matrix of continuous system
g	acceleration due to gravity, m/sec^2
H	transformation matrix
I	identity matrix
I_Y	moment of inertia about lateral body axis, $kg-m^2$
$J(\Theta)$	log-likelihood function
j	$= \sqrt{-1}$
K	Kalman-filter-gain matrix
$K_{11,12,\ldots,33}$	elements of K matrix
$k_{1,2,\ldots,10}$	constants in equations of motion
M	Fisher information matrix
M_Y	pitching moment, N-m
m,n	transformed quantity at mth and nth interval, respectively
m	mass, kg (in appendix D)
N	number of data points
P	covariance matrix of state variables
P{ }	probability
p[ ]	probability density
Q	process-noise covariance matrix
q	rate of pitch, rad/sec
$\bar{q}$	kinetic pressure, $\frac{1}{2}\rho v^2$, N/m^2
R	measurement-noise covariance matrix
R_{zz}	correlation function of z
r	number of output variables

S	wing area, m^2
S_{yu}	cross-spectral density of y and u
S_{zz}	spectral density of z
s, t, τ	quantity at sth, tth, and τth interval, respectively
T	transfer-function matrix
u	control vector
V	true airspeed, m/sec
v	measurement-noise vector
w	process-noise vector
w_g	vertical component of turbulence velocity, m/sec
x	state vector
y	measurement vector
z	random variable real or complex
z_n	$= \exp(jn\omega_0)$
α	angle of attack, rad
α_v	angle of attack measured by wind vane, rad
Γ	control matrix of discrete system
Γ_w	process-noise distribution matrix of discrete system
δ_e	elevator deflection, rad
$\left.\begin{array}{l}\delta_{m,n} \\ \delta_{t,\tau}\end{array}\right\}$	Kronecker delta
ϵ	arbitrary small number
Θ	vector of unknown parameters
θ	pitch angle, rad
ν	innovation vector
ρ	air density, kg/m^3
σ^2	variance (σ is standard deviation)

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Φ transition matrix

φ_y phase angle of complex variable y, deg

φ_{yu} phase-angle characteristics relating y and u variables, deg

ω angular frequency, rad/sec

ω_0 $= 2\pi/N$

Aerodynamic derivatives (referenced to a system of body axes with the origin at the airplane center of gravity);

$$C_{m_q} = \frac{\partial C_m}{\partial \dfrac{q\bar{c}}{2V_0}} \qquad C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha} \qquad C_{m_{\dot\alpha}} = \frac{C_m}{\partial \dfrac{\dot\alpha\bar{c}}{2V_0}}$$

$$C_{m_{\delta e}} = \frac{\partial C_m}{\partial \delta_e} \qquad C_{z_q} = \frac{\partial C_z}{\partial \dfrac{q\bar{c}}{2V_0}} \qquad C_{z_\alpha} = \frac{\partial C_z}{\partial \alpha}$$

$$C_{z_{\delta e}} = \frac{\partial C_z}{\partial \delta_e}$$

$$\left. \begin{array}{l} C'_{m_\alpha}, C'_{m_q}, \\[6pt] C^o_{m_q}, C'_{m_{\delta e}} \end{array} \right\} \quad \text{defined in appendix D (eqs. (D5) to (D8))}$$

Subscripts:

c   continuous system

E   measured quantity

g   gust

k   kth element of vector or kth column of matrix

ℓ ℓth element of vector or ℓth row of matrix

m   vector consisting of all elements up to and including m

0   initial value

Superscripts:

T   transpose matrix

-1   inverse matrix

^   estimated value

~   transformed variable

.   derivative with respect to time

*   transpose complex conjugate matrix

R   real part

I   imaginary part

Mathematical notation:

Tr   trace of matrix

Re   real part of complex number

$|\ |$ determinant

$\left|\dfrac{\tilde{y}}{\tilde{u}}\right|$ amplitude-ratio characteristic relating $\tilde{y}$ and $\tilde{u}$ variables

Δ increment

ESTIMATION ALGORITHM

For the development of the estimation algorithm it is necessary first to postulate the model of an airplane and then to transform this model into the frequency domain. The next step is the formulation of the likelihood function of innovations and its maximization with respect to the unknown parameters. This step leads to the iterative scheme for parameter estimation, which updates the previous estimates by employing the second- and first-order gradients of the log-likelihood function.

The linear airplane equations of motion are assumed in discrete-time form to be

$$x(t+1) = \Phi\, x(t) + \Gamma\, u(t) + \Gamma_w\, w(t) \qquad (t = 0,1,\ \ldots,\ N - 1) \qquad (1)$$

$$y(t) = H\, x(t) + v(t) \qquad (t = 0,1,\ \ldots,\ N - 1) \qquad (2)$$

$$E\{x(0)\} = 0$$
$$E\{x(0)\ x^T(0)\} = P_0$$

(3)

where $x(t)$ is a state vector, $u(t)$ is a control vector, $w(t)$ is a process-noise vector, $y(t)$ is an output vector, and $v(t)$ is a measurement-noise vector.

It is assumed that

(a) Φ, Γ, Γ_w, and H are constant matrices

(b) Φ is stable

(c) (Φ,Γ) and (Φ,Γ_w) are controllable pairs

(d) (Φ,H) is observable

(e) w and v are stationary, Gaussian uncorrelated noise sequences with

$$E\{w(t)\} = 0$$
$$E\{w(t)\ w^T(\tau)\} = Q\delta_{t,\tau}$$

(4)

$$E\{v(t)\} = 0$$
$$E\{v(t)\ v^T(\tau)\} = R\delta_{t,\tau}$$

(5)

$$E\{v(t)\ w^T(\tau)\} = 0 \quad \text{for all } t \text{ and } \tau$$

(f) N is even

In the general case the unknown parameters will occur in the matrices Φ, Γ, Γ_w, H, Q, R, $x(0)$, and P_0. Their estimation may be extremely difficult because of the algorithm complexity (see ref. 15) and possible identifiability problems (see ref. 16). The system parameter estimation will be simplified by formulating a steady-state Kalman-filter representation of equations (1) and (2) and by considering the unknown parameters in this representation.

The conditional expected value of the state vector is defined as

$$\bar{x}(t) = E\{x(t)\,|\,y(0) \quad y(1) \quad \cdots \quad y(N-1)\}$$

(6)

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The innovations are defined as

$$\nu(t) = y(t) - H \bar{x}(t) \tag{7}$$

and the covariance matrix of state variables is defined as

$$P = E\{\lceil x(t) - \bar{x}(t) \rceil \lceil x(t) - \bar{x}(t) \rceil^T\} \tag{8}$$

Then the steady-state Kalman-filter representation of the system described by equations (1) and (2) is

$$\bar{x}(t+1) = \Phi \bar{x}(t) + \Gamma u(t) + K \nu(t) \tag{9}$$

$$y(t) = H \bar{x}(t) + \nu(t) \tag{10}$$

Reference 17 shows that the innovations $\nu(t)$ form a sequence of independent Gaussian vectors with

$$\left. \begin{array}{l} E\{\nu(t)\} = 0 \\ E\{\nu(t) \ \nu^T(\tau)\} = B\delta_{t,\tau} \end{array} \right\} \tag{11}$$

The definition of the gain matrix K in equation (9) can be found in reference 15 or 16 in the form

$$K = \Phi PH^TB^{-1} \tag{12}$$

$$B = HPH^T + R \tag{13}$$

$$P = \Phi P \Phi^T - KBK^T + \Gamma_w Q \Gamma_w^T \tag{14}$$

For the further development of the identification algorithm all time functions in equations (9) and (10) are written in terms of their Fourier series expansions. As stated in appendix A, the Fourier series expansion of random variables holds in the mean-square sense. If the Fourier series component of $x(t)$ is defined as

$$\tilde{x}(n) = \frac{1}{N} \sum_{t=0}^{N-1} x(t) \exp(-jn\omega_0 t) \tag{15}$$

for

$$n = -\frac{N}{2}, \ -\frac{N}{2} + 1, \ \ldots, \ 0, 1, \ \ldots, \ \frac{N}{2} - 1$$

where $\omega_0 = \dfrac{2\pi}{N}$, and similarly for the other variables, equations (9) and (10) transformed from the time to the frequency domain have the form

$$z_n \tilde{x}(n) = \Phi \tilde{x}(n) + \Gamma \tilde{u}(n) + K \tilde{\nu}(n) \tag{16}$$

$$\tilde{y}(n) = H \tilde{x}(n) + \tilde{\nu}(n) \tag{17}$$

In equation (16) $z_n = \exp(jn\omega_0)$, which follows from the relationship

$$\frac{1}{N} \sum_{t=0}^{N-1} x(t+1) \exp(-jn\omega_0 t) = \tilde{x}(n) \exp(jn\omega_0)$$

assuming that $x(0) = x(N) = 0$. As proved in appendix A, the transformed innovations $\tilde{\nu}(n)$ are uncorrelated, orthogonal, and Gaussian random variables with

$$\left. \begin{array}{l} E\{\tilde{\nu}(n)\} = 0 \\[2mm] E\{\tilde{\nu}(n) \ \tilde{\nu}^*(n)\} = \dfrac{S_{\nu\nu}}{N} \end{array} \right\} \tag{18}$$

where $S_{\nu\nu}$ is the spectral density of $\nu(t)$, and $\tilde{\nu}^*(n)$ is the complex conjugate of $\tilde{\nu}^T(n)$. It follows from equations (16) and (17) that

$$\tilde{y}(n) = H(z_n I - \Phi)^{-1} \Gamma \tilde{u}(n) + [H (z_n I - \Phi)^{-1} K + I] \tilde{\nu}(n)$$

$$= T_1(n, \Theta) \ \tilde{u}(n) + T_2(n, \Theta) \ \tilde{\nu}(n) \tag{19}$$

where I is the identity matrix, T_1 and T_2 are the system transfer functions defined as

$$T_1(n,\Theta) = H(z_n I - \Phi)^{-1}\Gamma \tag{20}$$

$$T_2(n,\Theta) = H(z_n I - \Phi)^{-1}K + I \tag{21}$$

and Θ is the vector of unknown parameters in equations (9) and (10). Equation (19) is invertible in the sense that $\tilde{y}(n)$ can be solved for directly in terms of $\tilde{\upsilon}(n)$ and $\tilde{\upsilon}(n)$ in terms of $\tilde{y}(n)$ (see ref. 18). This implies that T_2 is nonsingular. Therefore, from equation (19)

$$\tilde{\upsilon}(n) = T_2^{-1}\,\tilde{y}(n) - T_2^{-1}T_1\,\tilde{u}(n) \tag{22}$$

To obtain the likelihood function, i.e., the joint probability density of the transformed innovations $\tilde{\upsilon}(n)$ (assuming that all parameters are known), a vector $\tilde{\upsilon}_m$ consisting of all innovations up to and including frequency m is introduced. Therefore

$$\tilde{\upsilon}_m = \left[\tilde{\upsilon}\left(\frac{N}{2}\right) \quad \tilde{\upsilon}\left(\frac{N}{2}+1\right) \quad \cdots \quad \tilde{\upsilon}(m) \right]^T \tag{23}$$

Assuming that the probability distribution of $\tilde{\upsilon}_m$ has a density $p[\tilde{\upsilon}_m]$, then it follows from the definition of conditional probabilities that

$$p[\tilde{\upsilon}_m] = p[\tilde{\upsilon}(m)\,|\,\tilde{\upsilon}_{m-1}]\ p[\tilde{\upsilon}_{m-1}] \tag{24}$$

Repeated use of this formula gives the expression for the likelihood function as

$$p[\tilde{\upsilon}_m] = p[\tilde{\upsilon}(m)\,|\,\tilde{\upsilon}_{m-1}]\ p[\tilde{\upsilon}(m-1)\,|\,\tilde{\upsilon}_{m-2}]\ \cdots\ p\left[\tilde{\upsilon}\left(\frac{N}{2}+1\right)\,\bigg|\,\tilde{\upsilon}\left(\frac{N}{2}\right)\right] p\left[\tilde{\upsilon}\left(\frac{N}{2}\right)\right] \tag{25}$$

Because the distribution of $\tilde{\upsilon}(m)$ is Gaussian, then the distribution of $\tilde{\upsilon}(m)$ given $\tilde{\upsilon}(m-1)$ is also Gaussian; i.e.,

$$p[\tilde{v}(m)|\tilde{v}(m)-1)] = \frac{\exp[-N\ \tilde{v}^*(m)\ S_{\nu\nu}^{-1}\ \tilde{v}(m)]}{\pi^r|S_{\nu\nu}/N|} \qquad (26)$$

as follows from the definition of a complex multivariate distribution in appendix B. In equation (26) r is the dimension of the innovation vector.

Using equation (26) the logarithm of equation (25) can be written as

$$J(\Theta) = -N \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{v}^*(n)\ S_{\nu\nu}^{-1}\ \tilde{v}(n) - N\ \log\ |S_{\nu\nu}| - \text{Constant} \qquad (27)$$

In the log-likelihood function given by equation (27), the unknown parameters are the elements of the matrices Φ, Γ, H, K, and $S_{\nu\nu}$. An estimate of the unknown parameters is obtained by minimizing the log-likelihood function from the feasible set of parameter values. Optimizing the log-likelihood function for parameters in $S_{\nu\nu}$ gives

$$\hat{S}_{\nu\nu} = \Sigma_n\ \tilde{v}(n)\ \tilde{v}^*(n) \qquad (28)$$

where $\Sigma_n = \displaystyle\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1}$. The estimates of the remaining unknown parameters are given

by the root of the equation

$$\left.\frac{\partial J(\Theta)}{\partial \Theta}\right|_{\Theta=\hat{\Theta}} = 0 \qquad (29)$$

for $S_{\nu\nu}$ replaced by $\hat{S}_{\nu\nu}$. This root can be found by a modified Newton-Raphson iteration (e.g., ref. 19) as

$$\hat{\Theta} = \Theta_0 + \Delta\hat{\Theta} \qquad (30)$$

where the step size $\Delta\hat{\Theta}$ for parameter estimates is given by

$$\Delta\hat{\Theta} = -M_0^{-1} \left. \frac{\partial J(\Theta)}{\partial\Theta} \right|_{\Theta=\Theta_0} \tag{31}$$

The index 0 at the matrix M indicates that its elements were computed for $\Theta = \Theta_0$. In equation (31) M is the Fisher information matrix

$$M = -E\left\{ \frac{\partial^2 J(\Theta)}{\partial\Theta\ \partial\Theta^T} \right\} \tag{32}$$

Because the step size for parameter estimates is a vector with real elements only, and the log-likelihood function is real, the expressions for the first- and second-order gradients of $J(\Theta)$ are also real; i.e.,

$$\frac{\partial J(\Theta)}{\partial\Theta_k} = -2N\ \text{Re}\ \Sigma_n\ \tilde{\upsilon}^*(n)\ S_{\nu\nu}^{-1}\ \frac{\partial\tilde{\upsilon}(n)}{\partial\Theta_k} \tag{33}$$

and

$$\frac{\partial^2 J(\Theta)}{\partial\Theta_k\ \partial\Theta_\ell} = -2N\ \text{Re}\ \Sigma_n\ \frac{\partial\tilde{\upsilon}^*(n)}{\partial\Theta_\ell}\ S_{\nu\nu}^{-1}\ \frac{\partial\tilde{\upsilon}(n)}{\partial\Theta_k} \tag{34}$$

The expressions for the elements of the information matrix and the gradient of the log-likelihood function are developed in appendix C in the form

$$M_{k\ell} = 2\ \text{Re}\ \Sigma_n \left\{ \text{Tr}\left[\frac{\partial T_1^*}{\partial\Theta_\ell}\left(T_2^*\right)^{-1} S_{\nu\nu}^{-1} T_2^{-1} \frac{\partial T_1}{\partial\Theta_k} S_{uu}(n) \right] + \text{Tr}\left[\frac{\partial T_2^*}{\partial\Theta_\ell}\left(T_2^*\right)^{-1} T_2^{-1} \frac{\partial T_2}{\partial\Theta_k} \right] \right\} \tag{35}$$

and

$$\frac{\partial J(\Theta)}{\partial\Theta_k} = -2\ \text{Re}\ \Sigma_n \left\{ \text{Tr}\left[\frac{\partial T_1^*}{\partial\Theta_k}\left(T_2^*\right)^{-1} S_{\nu\nu}^{-1} T_2^{-1}\left(\hat{S}_{yu}(n) - T_1\ \hat{S}_{uu}(n)\right) \right] - \text{Tr}\left[\frac{\partial T_2^*}{\partial\Theta_k}\left(T_2^*\right)^{-1} \right] \right\} \tag{36}$$

where

$$\left.\begin{array}{l} \hat{S}_{uu}(n) = N \; \tilde{u}(n) \; \tilde{u}^*(n) \\[20pt] \hat{S}_{yu}(n) = N \; \tilde{y}(n) \; \tilde{u}^*(n) \end{array}\right\} \qquad\qquad (37)$$

are the estimates of input spectral densities and cross-spectral densities, respectively.

The final estimates of unknown parameters have the following properties (see refs. 20 and 21):

They are consistent; i.e.,

$$\lim_{N\to\infty} P\{|\hat{\Theta} - \Theta| \leq \varepsilon\} = 1$$

(with ε arbitrarily small)

They are asymptotically unbiased; i.e.,

$$\lim_{N\to\infty} E\{\hat{\Theta}\} = \Theta$$

and they are asymptotically efficient with

$$E\{(\hat{\Theta} - \Theta)(\hat{\Theta} - \Theta)^T\} \geq -E\left\{\frac{\partial^2 J(\Theta)}{\partial\Theta \; \partial\Theta^T}\right\} \qquad\qquad (38)$$

Because of equations (32) and (38), the inverse of the information matrix provides the Cramér-Rao lower bounds on the variance and covariance of errors in the estimated parameters.

OUTPUT ERROR METHOD

If the process noise is zero and the initial states are assumed to be equal identically to zero, i.e., $w(t) = 0$, $x(0) = 0$, and $P(0) = 0$, the state-covariance matrix is also zero. Then, as follows from equations (12) and (21), the Kalman gains are zero and $T_2 = I$. The innovations are reduced to output errors

$$\tilde{\mathbb{v}}(n) = \tilde{y}(n) - T(n,\Theta_0)\ \tilde{u}(n) \tag{39}$$

where $T(n,\Theta_0)$ is equal to $T_1(n,\Theta)$ defined by equation (20). For $\Theta_0 \rightarrow \Theta$ the innovations $\tilde{\mathbb{v}}(n) \rightarrow \tilde{v}(n)$ and $S_{\mathbb{v}\mathbb{v}} \rightarrow S_{vv}$. The expressions for the elements of the information matrix and gradient of the log-likelihood function are obtained by simplifying equations (35) and (36) as

$$M_{k\ell} = 2\ \mathrm{Re}\ \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1}\ \mathrm{Tr}\left[\frac{\partial T^*}{\partial \Theta_\ell}\ S_{vv}^{-1}\ \frac{\partial T}{\partial \Theta_k}\ \hat{S}_{uu}(n)\right] \tag{40}$$

and

$$\frac{\partial J(\Theta)}{\partial \Theta_k} = -2\ \mathrm{Re}\ \Sigma_n\ \mathrm{Tr}\left[\frac{\partial T^*}{\partial \Theta_k}\ S_{vv}^{-1}\left(\hat{S}_{yu}(n) - T\ \hat{S}_{uu}(n)\right)\right] \tag{41}$$

The expressions for the information matrix and the gradient of the log-likelihood function can also be easily derived from the simplified log-likelihood function, which takes the form of the output error cost function

$$J(\Theta) = -N\Sigma_n\ \tilde{\mathbb{v}}^*(n)\ S_{vv}^{-1}\ \tilde{\mathbb{v}}(n) \tag{42}$$

These expressions are

$$M(\Theta) = 2N\ \mathrm{Re}\ \Sigma_n\ A^*(n)\ S_{vv}^{-1}\ A(n) \tag{43}$$

$$\frac{\partial J(\Theta)}{\partial \Theta} = -2N\ \mathrm{Re}\ \Sigma_n\ A^*(n)\ S_{vv}^{-1}\ \tilde{\mathbb{v}}(n) \tag{44}$$

where $A(n)$ is the sensitivity matrix whose elements are equal to $\partial[T(n,\Theta_0)\ \tilde{u}(n)]/\partial \Theta_\ell$.

In some experiments airplane transfer functions are measured directly using a harmonic input or are determined from measured input-output time histories. Then the cost function includes a transfer function error rather than an output error. The cost function is therefore formed as

$$J(\Theta) = -N\Sigma_n[T_E(n) - T(n,\Theta_0)]^*\ S_{vv}^{-1}\ [T_E(n) - T(n,\Theta_0)] \tag{45}$$

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where T is a vector which includes system transfer functions as elements. These transfer functions are computed from equation (20) for a given Θ_0.

Both cost functions (42) and (45) can be minimized with respect to unknown parameters in Φ, G, and H or with respect to transfer function coefficients in T. The estimates are obtained from equations (31), (43), and (44); the spectral densities are given by equation (28) using pertinent residuals.

For a system with a single input, the output error cost function with measured transfer functions (frequency response curves) is defined as

$$J(\Theta) = -N\Sigma_n\{\tilde{u}(n) \ [T_E(n) - T(n,\Theta_0)]\}^* \ S_{vv}^{-1} \ \tilde{u}(n) \ [T_E(n) - T(n,\Theta_0)] \qquad (46)$$

In this formulation the scalar variable $\tilde{u}(n)$ may be interpreted as a weighting function expressing the reliability of the measured data according to the harmonic content of an input.

DISCUSSION

The frequency domain identification has several features which are distinct from the time domain approach. They are mainly associated with the model representation and estimation algorithm. There is, however, the equivalence in the cost function used in the time and frequency domains as expressed by Parceval's theorem. This theorem postulates the relationship between the squared magnitudes of the Fourier transform pairs. It therefore states that the time domain cost function,

$$J_{TD} = \Sigma_t \ v^T(t) \ s_{vv}^{-1} \ v(t) \qquad (47)$$

where $\Sigma_t = \displaystyle\sum_{t=0}^{N-1}$, is equal to the frequency domain cost function,

$$J_{FD} = \frac{1}{N} \ \Sigma_n \ \tilde{v}^*(n) \ s_{vv}^{-1} \ \tilde{v}(n) \qquad (48)$$

Using equation (15) the frequency domain cost function can be written as

$$J_{FD} = \frac{1}{N} \ \Sigma_n\Sigma_t \ v^T(t) \ \exp(jn\omega_0 t) \ S_{vv}^{-1} \ \Sigma_\tau \ v(\tau) \ \exp(-jn\omega_0\tau)$$

$$= \frac{1}{N} \ \Sigma_t\Sigma_\tau \ v^T(t) \ s_{vv}^{-1} \ v(\tau) \ \Sigma_n \ \exp\lceil jn\omega_0(t - \tau) \rceil$$

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where $\Sigma_\tau = \displaystyle\sum_{\tau=0}^{N-1}$. But according to appendix A,

$$\Sigma_n \; \exp[\,jn\omega_0(t - \tau)\,] = N \qquad\qquad (\text{for} \quad t = \tau)$$

$$= 0 \qquad\qquad (\text{for} \quad t \neq \tau)$$

Therefore,

$$J_{FD} = \Sigma_t \; \nu^T(t) \; S_{\nu\nu}^{-1} \; \nu(t) = J_{TD}$$

The equivalence of both approaches is no longer valid if the frequency domain cost function is restricted to a given frequency range. Such a restriction is not necessary, but it is an option which is a strong point in favor of frequency domain analysis with respect to time domain analysis. The selected frequency range of interest was used, for example, in reference 9, where airplane rigid modes were separated from elastic ones. For similar results in the time domain the data must be filtered accordingly.

The early airplane estimation techniques in the frequency domain were using measured frequency response curves only. This approach could have an advantage when repeated measurements under the same conditions are available. A hypothesis concerning the model adequacy can be tested using the variance estimates from scatter around the mean and from residuals (ref. 5). On the other hand the simultaneous analysis of repeated maneuvers for obtaining a single set of estimates with increased accuracy can also be applied to directly measured or transformed time histories. In general, transformed input-output time histories are preferred in frequency domain parameter estimation. The inaccuracies of frequency response curves computed from transformed inputs and outputs can be quite pronounced for frequencies in which the harmonic content of an input is close to zero.

The transformation of model equations into the frequency domain replaces differentiation and convolution with multiplication. As a result the sensitivity equations in the nonlinear estimation algorithm are reduced to uncoupled algebraic expressions. This simplification can be appreciated mainly in cases for which convolution integrals are included in the equations of motion (ref. 11).

The computational differences between the time and frequency domains discussed so far could be viewed as advantages of the frequency domain analysis. There is, however, a substantial disadvantage of the airplane identification in the frequency domain. This approach is limited, for practical reasons, to only linear equations of motion with constant coefficients. The computing time needed for parameter estimation in the frequency domain (transformation of measured data included) is about 50 percent more than in the time domain. The assessment was obtained from the number of equations used in both domains for one iteration when the algorithms were applied to the system of equations without convolutions and process noise.

The estimation algorithm was developed for a linear discrete-time model. The airplane equations of motion are, however, usually given in a continuous form as

$$x = Fx + Gu + G_w w \qquad (49)$$

where the unknown parameters can be in the matrices F, G, and G_w. For the continuous model (eq. (49)), the expressions for the information matrix and gradient of the log-likelihood function remain the same as equations (35) and (36). But now the transfer functions are defined as

$$T_1(\omega,\theta) = H(j\omega I - F)^{-1}G \qquad (50)$$

$$T_2(\omega,\theta) = H(j\omega I - F)^{-1}K_c + I \qquad (51)$$

where the Kalman-filter-gain matrix is obtained from the relationships (see, e.g., ref. 22)

$$K_c = PH^T R^{-1}$$

and

$$FP + PF^T - PH^T R^{-1} HP + G_w Q_c G_w^T = 0$$

In the model formulation it was assumed that the initial conditions were equal to zero, that the model described a stable motion of an airplane, that there were no a priori known values of stability and control parameters, and that the measurement noise was Gaussian and uncorrelated. If the initial conditions differ from zero, the additional term $H(z_n I - \Phi)^{-1} x(0)$ would have to be included in equation (19) or the new transfer function $H(j\omega I - F)^{-1} x(0)$ would have to be added to those defined by equations (50) and (51). Then the vector of unknown parameters can be augmented by the vector of initial conditions.

If the airplane motion includes an unstable mode, the parameter estimation still can proceed provided that the degree of instability is not high. A large instability, on the other hand, can result in excessive transient motion due to nonzero initial conditions and/or the input and thus limit the validity of the linear equations of motion. If the a priori mean values and variances of some parameters are known, they can be included in the estimation procedure. In this case the cost function must be expanded in a similar way as indicated in reference 19.

The maximum likelihood method developed earlier assumed a Gaussian, uncorrelated measurement noise. If the random sequence representing this noise is correlated, the estimation algorithm does not change. The constant values of

spectral densities $S_{\nu\nu}$ or S_{vv} are merely replaced by the frequency dependent values estimated from expressions similar to equation (37).

EXAMPLES

As examples the parameters of a small general aviation airplane were estimated from computer-generated data and from measured data in still and turbulent air. (Some of the data for examples 1, 3, and 4 are from ref. 14.) For all examples the model of the airplane was based on simplified longitudinal equations of motion with the atmospheric turbulence (gusts) approximated as a Gauss-Markoff process of first order. The model equations (continuous form) are developed in appendix D. When the state and output equations (D4) and (D9) are transformed into the frequency domain and rearranged, they have the form

$$\tilde{x}D = G\tilde{\delta}_e + G_w\tilde{w} \tag{52}$$

$$\tilde{y} = H\tilde{x} + \tilde{v} \tag{53}$$

The state and output vectors are specified as

$$\tilde{x} = [\tilde{\alpha} \quad \tilde{q} \quad \tilde{w}_g]^T$$

$$\tilde{y} = [\tilde{\alpha}_v \quad \tilde{q} \quad \tilde{a}_z]^T$$

where the random input is assumed to be a Gaussian, uncorrelated noise process with $E\{w\} = 0$ and $E\{w^2\} = \sigma_g^2$. The matrices in equations (52) and (53) are formulated as

$$D = \begin{bmatrix} j\omega - k_1 C_{z_\alpha} & -\left(1 + k_2 C_{z_q}\right) & -k_3 C_{z_\alpha} \\[2ex] -k_5 C'_{m_\alpha} & j\omega - k_6 C'_{m_q} & -k_7 C'_{m_\alpha} - k_8 c_1 C^0_{m_q} \\[2ex] 0 & 0 & j\omega + c_1 \end{bmatrix}$$

$$G = \begin{bmatrix} k_1 C_{z_{\delta e}} & k_5 C'_{m_{\delta e}} & 0 \end{bmatrix}^T$$

$$G_w = \begin{bmatrix} 0 & -k_8 c_2 C^0_{m_q} & c_2 \end{bmatrix}^T$$

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$$H = \begin{bmatrix} k_9 & -k_{10} & \dfrac{k_9}{v_0} \\[2ex] 0 & 1 & 0 \\[2ex] \dfrac{v_0}{g}\,j\omega & -\dfrac{v_0}{g} & 0 \end{bmatrix}$$

The known constants k_1, k_2, . . ., k_{10}, c_1, and c_2, and the pitching-moment derivatives C'_{m_α}, C'_{m_q}, $C^O_{m_q}$, and $C'_{m_{\delta_e}}$ are defined in appendix D.

Example 1

The output data were computed from equations (D4) and (D9) for given inputs δ_e and w (with $\sigma_g = 1$ m/sec) and for a given set of parameters. To the computed time histories of output variables, an uncorrelated and Gaussian measurement noise was added. The measurement-noise standard errors were selected as

$$\sigma_\alpha = 0.0028 \text{ rad} \qquad \sigma_q = 0.0063 \text{ rad/sec} \qquad \sigma_{a_z} = 0.02g$$

The time histories of the input and output variables are plotted in figure 1. For the estimation algorithm these data were transformed into the frequency domain using the Filon integration formula. The sampling interval of the transformed data was 1.047 rad/sec. The transformed data were truncated at the frequency interval ±20.944 rad/sec because outside this interval their amplitudes were very small.

The steady-state Kalman filter representation of the airplane motion described by equations (52) and (53) is

$$\tilde{x}D = G\tilde{\delta}_e + K_c\tilde{\vartheta} \tag{54}$$

$$\tilde{y} = H\tilde{x} + \tilde{\vartheta} \tag{55}$$

It was assumed that the parameters c_1 and c_2, the initial conditions, and the variances of the process and measurement noise were known. Also assumed as known was the parameter $C^O_{m_q}$ because of the identifiability problem.

The last assumption is substantiated by the small effect of the term $k_8 c_1 C_{m_q}^o$ on the airplane motion. The vector of unknown parameters was therefore formed as

$$\Theta = \begin{bmatrix} C_{z_\alpha} & C_{z_{\delta_e}} & C_{z_q} & C'_{m_\alpha} & C'_{m_q} & C'_{m_{\delta_e}} & K_{11} & K_{12} & \cdots & K_{33} \end{bmatrix}^T$$

where K_{11}, K_{12}, . . ., and K_{33} are the elements of the Kalman-filter-gain matrix K_c. The initial values of these elements were computed from equations (50) and (51).

First, the unknown parameters were determined by the maximum likelihood method developed. In table I the estimated stability and control parameters are compared with their true values, and the estimated Kalman gains are compared with their initial values. The agreement between the first set of parameters is, in general, very good. The estimated values of the Kalman gains differ significantly from their initial values, and the standard errors (lower bounds) of these parameters are quite high. This indicates low accuracy of these estimates. When, however, the Kalman gains were fixed on their initial value, the estimates of the airplane parameters were farther from the true value, as indicated by results in the fourth column of table I. The last set of airplane parameters was obtained by considering no process noise effect on the output data. These estimates are also less accurate than those obtained by the maximum likelihood method with all 15 unknown parameters. Table I also includes the variance estimates of the residuals. The limited experience obtained from this example indicates that for stability and control parameter estimation from data with pronounced effect of the process noise (i.e., $\sigma_g \geq 1$ m/sec), the algorithm in its complete form should be used and the Kalman filter gains should be treated as the additional unknown parameters.

Example 2

In this example the measured data in turbulent air were used in the same model as in the previous example. The measured input-output time histories are presented in figure 2. For the parameter estimation the transformed data were taken from the frequency interval ±9.817 rad/sec. The standard error of the vertical gust velocity was determined from the part of the measured data with δ_e = Constant to be σ_g = 1.12 m/sec. The values for measurement-noise standard errors were taken from the results in reference 23 as

$$\sigma_{\alpha_v} = 0.0017 \text{ rad} \quad \sigma_q = 0.005 \text{ rad/sec} \quad \sigma_{a_z} = 0.01g$$

The estimated stability and control parameters are given in the third and fourth columns of table II. In the first case (fourth column) the Kalman gains were treated as unknown parameters; in the second case (third column) they were set equal to zero (assumption of no process noise). The inclusion of the process noise in the model resulted in better accuracy of the parameter C_{z_α}, as

indicated by its comparison with the average value obtained from the estimates in the time domain (see ref. 23). On the other hand, the process-noise consideration in the estimation process degraded the estimates of the parameters C_{Z_q} and C'_{m_q}. No explanation for this degradation could be found.

Example 3

From the measured time histories in still air which are presented in figure 3, the transformed input and output data and the frequency response curves relating all three outputs to the elevator deflection were obtained. By setting $w = 0$ the state vector in equations (52) and (53) was changed to $x = [\alpha, q]^T$ and the matrices D, H, and G were simplified accordingly. The unknown parameters were estimated from the minimization of the cost function, given by equation (42) for the transformed data and by equation (45) for the frequency response curves.

The estimated parameters are given in the sixth and seventh columns of table II, and they are compared with the results from the time domain estimation given in the fifth column of the same table. The three sets of estimates from the same flight agree well. The standard errors of the estimates in the frequency domain are, however, higher than those in the time domain. This could be due to truncation of the transformed data and additional inaccuracies in measured frequency response curves caused by taking the ratios of two complex numbers. The transformed data and those computed are plotted in figure 4; the measured and computed frequency response curves are plotted in figure 5. Both figures indicate some modeling errors in the equation for α_v. It is also apparent from figure 5 that the measured frequency response curves are inaccurate around the frequency 6.4 rad/sec as a result of the low harmonic content of the input at the same frequency.

Example 4

The response of the airplane to turbulence was measured in two flights (designated run 1 and run 2 in table II) with the minimum pilot interference ($\delta_e \sim 0$). From the time histories of the measured output variables the spectral density of the vertical gust velocity $S_{w_g w_g}$ and the cross-spectral densities $S_{w_g q}$ and $S_{w_g a_z}$ were computed. They are related by the airplane transfer function resulting from equations (52) and (53).

The state and output equations were modified in the following way:

(a) In equation (52) w and δ_e were set equal to zero

(b) w_g was assumed as a known input

(c) The term $k_8 c_1 C_{m_q}^O w_g$ was replaced by $-j\omega k_8 C_{m_q}^O w_g$

(d) In equation (53) α_v was set equal to zero

The matrices D, G, and H were therefore changed as

$$D = \begin{bmatrix} j\omega - k_1 C_{Z_\alpha} & -\left(1 + k_2 C_{Z_q}\right) \\ -k_5 C'_{m_\alpha} & j\omega - k_6 C'_{m_q} \end{bmatrix}$$

$$G = \begin{bmatrix} k_3 C_{Z_\alpha} & k_7 C'_{m_\alpha} - j\omega k_8 C^O_{m_q} \end{bmatrix}^T$$

$$H = \begin{bmatrix} 0 & 1 \\ \dfrac{V_0}{g} j\omega & -\dfrac{V_0}{g} \end{bmatrix}$$

and the model was formulated as

$$S_{w_g x} D = G S_{w_g w_g} \tag{56}$$

$$S_{w_g y} = H S_{w_g x} + \tilde{v} \tag{57}$$

where

$$S_{w_g x} = \begin{bmatrix} S_{w_g \alpha} & S_{w_g q} \end{bmatrix}^T$$

$$S_{w_g y} = \begin{bmatrix} S_{w_g q} & S_{w_g a_z} \end{bmatrix}^T$$

The vector of unknown parameters in these equations is formed as

$$\Theta = \begin{bmatrix} C_{Z_\alpha} & C_{Z_q} & C'_{m_\alpha} & C'_{m_q} & C^O_{m_q} \end{bmatrix}^T$$

The estimated values of the first four parameters are given in the last two columns of table II. From the estimates of C'_{m_q} and $C^O_{m_q}$ the value of C_{m_α}

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was computed from equations (D6) and (D7) and included among the unknown parameters. The agreement between the results from both runs is very good. The parameters also agree with the estimates from the still air measurement with the exception of the parameter C_{z_α}. This parameter has a smaller value than expected, probably because of some modeling errors in equations (56) and (57).

The measured spectral and cross-spectral densities from run 1 and those computed by using the estimated parameters are plotted in figure 6. It was verified that the large fit error in the phase of the cross-spectrum $S_{w_g q}$ did not affect the values of the estimated parameters significantly. The estimate from turbulence and measurement demonstrates a possibility for using these data also for airplane stability and control parameter estimation. For certain model formulations the derivative of pitching-moment coefficient with respect to the rate of change in angle of attack can be estimated explicitly.

CONCLUDING REMARKS

A frequency domain maximum likelihood method has been developed for the estimation of airplane parameters from measured flight data. A discrete-type steady-state Kalman filter was used in the derivation of the computing algorithm. The time variables in the model equations were transformed into the frequency domain by using a Fourier series expansion. If the initial data were Gaussian and uncorrelated, the transformed data formed a complex random sequence which was uncorrelated, orthogonal, and Gaussian. Then, the likelihood function could be formulated as a multivariate distribution of complex innovations.

The connection between the continuous form of airplane equations of motion and the developed algorithm is easily established. The algorithm can be simplified to the output error method with the measured data in the form of transformed time histories, frequency response curves, or spectral and cross-spectral densities. In general, transformed input-output time histories are preferred in frequency domain estimation. The inaccuracies of frequency response curves computed from transformed inputs and outputs can be quite pronounced for frequencies in which the harmonic content of an input is close to zero. The frequency domain approach simplifies the estimation procedure by reducing the sensitivity equations to simple algebraic expressions. It also provides an easier way than the time domain for using the data within a frequency range of interest. The serious disadvantage of the frequency domain identification is in its practical limitation to a system described by linear equations of motion with constant coefficients. It was shown that the cost functions in time domain and frequency domain approaches are equivalent. It is therefore necessary to consider both approaches as complementary and not contradictory.

The maximum likelihood method has been applied to computer-generated and real flight data for the longitudinal motion of a small general aviation airplane. In the first case the estimates obtained were more accurate than those from the simplified output error method, which did not consider the effect of the process noise. In the second case the results were inconclusive because of an insufficient amount of measured data. Then, the simplified algorithm was used with the flight data from measurements in still air and turbulent air with

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no pilot input. The first set of results for deterministic input showed the expected similarity in parameter values obtained from the time and frequency domains. The estimates from turbulence measurements demonstrated a possibility for using these data also for airplane parameter estimation and for explicit estimation of the pitching-moment derivative with respect to the rate of change in angle of attack.

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FOURIER TRANSFORM OF A STOCHASTIC PROCESS

Let $z(t)$, where $t = 0, 1, \ldots,$ and $N - 1$, be a real random sequence with

$$
\left.
\begin{aligned}
E\{z(t)\} &= 0 \\[2em]
E\{z(t)\,z(\tau)\} &= R_{zz}(t-\tau)
\end{aligned}
\right\}
\tag{A1}
$$

<u>Theorem 1</u>: If $z(t)$ is periodic in the mean-square sense, then it can be expanded into a Fourier series:

$$
z(t) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{z}(n)\, e^{jn\omega_0 t}
\tag{A2}
$$

where $\omega_0 = 2\pi/N$; and the coefficients $\tilde{z}(n)$, given as

$$
\tilde{z}(n) = \frac{1}{N} \sum_{t=0}^{N-1} z(t)\, e^{-jn\omega_0 t}
\tag{A3}
$$

are uncorrelated and orthogonal random variables such that

$$
\left.
\begin{aligned}
E\{\tilde{z}(n)\} &= 0 \\[2em]
E\{\tilde{z}(m)\,\tilde{z}^*(n)\} &= \frac{S_{zz}(n)}{N}\, \delta_{m,n}
\end{aligned}
\right\}
\tag{A4}
$$

where $S_{zz}(n)$ is the spectral density of $z(t)$.

<u>Proof</u>: From equations (A1) and (A3) the expected value of $\tilde{z}(n)$ is found as

$$
E\{\tilde{z}(n)\} = \frac{1}{N} \sum_{t=0}^{N-1} E\{z(t)\}\, e^{-jn\omega_0 t} = 0
$$

APPENDIX A

To prove the second part of equation (A4), the conjugate of $\tilde{z}(n)$ is first multiplied by $z(t)$ and then the expected value is taken; i.e.,

$$E\{\tilde{z}^*(n)\ z(t)\} = \frac{1}{N}\ E\left\{\sum_{s=0}^{N-1} z(s)\ e^{jn\omega_0 s}\ z(t)\right\}$$

$$= \frac{1}{N}\sum_{s=0}^{N-1} E\{z(t)\ z(s)\}\ e^{jn\omega_0 s}$$

$$= \frac{1}{N}\sum_{s=0}^{N-1} R_{zz}(t-s)\ e^{jn\omega_0 s}$$

The new variable $\tau = t - s$ and the relationships

$$R_{zz}(\tau) = \frac{1}{N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} S_{zz}(n)\ e^{jn\omega_0 \tau} \tag{A5}$$

$$S_{zz}(n) = \sum_{\tau=-(N-1)}^{N-1} R_{zz}(\tau)\ e^{-jn\omega_0 \tau} \tag{A6}$$

are introduced (see, e.g., ref. 6). Then

$$E\{\tilde{z}(n)\ z(t)\} = \frac{1}{N}\sum_{\tau=-(N-1)}^{N-1} R_{zz}(\tau)\ e^{jn\omega_0(t-\tau)} = \frac{1}{N}\ e^{jn\omega_0 t}\ S_{zz}(n) \tag{A7}$$

Using equations (A3) and (A7), the expected value of $\tilde{z}(m)\ \tilde{z}^*(n)$ can be formulated as

$$E\{\tilde{z}(m)\ \tilde{z}^*(n)\} = \frac{1}{N}\sum_{t=0}^{N-1} E\{\tilde{z}^*(n)\ z(t)\}\ e^{-jm\omega_0 t} = \frac{S_{zz}}{N^2}\sum_{t=0}^{N-1} e^{-j(m-n)\omega_0 t} \tag{A8}$$

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For $m = n$ there is

$$\sum_{t=0}^{N-1} e^{-j(m-n)\omega_0 t} = N$$

For $m \neq n$ the difference $m - n = a$, where a is an integer; therefore,

$$\sum_{t=0}^{N-1} e^{-ja\omega_0 t} = \frac{1 - e^{-jaN\omega_0 t}}{1 - e^{-j\omega_0 t}} = \frac{1 - \cos 2a\pi + j \sin 2a\pi}{1 - e^{-j\omega_0 t}} = 0$$

and the proof of equation (A4) is thus completed.

To prove equation (A2), it is sufficient to show that the sequence $z(t)$ tends to $\sum_n \tilde{z}(n) \exp(jn\omega_0 t)$ in the mean-square sense; i.e.,

$$E\left\{ \left| z(t) - \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{z}(n) e^{jn\omega_0 t} \right|^2 \right\} = 0 \tag{A9}$$

The square in equation (A9) can be written as a product of itself by its conjugate. Equation (A9) is therefore changed to

$$E\{|z(t)|^2\} - \sum_n E\{\tilde{z}^*(n) z(t)\} e^{-jn\omega_0 t} - \sum_n E\{\tilde{z}(n) z^*(t)\} e^{jn\omega_0 t}$$

$$+ \sum_n \sum_m E\{\tilde{z}(m) \tilde{z}^*(n)\} e^{j(m-n)\omega_0 t} = 0$$

In the double summation all the terms with $m \neq n$ are equal to zero. Using equations (A1), (A2), and (A7), the previous equation can be expressed as

$$R_{zz}(0) - \frac{1}{N} \sum_n S_{zz}(n) e^{jn\omega_0 t} e^{-jn\omega_0 t} - R_{zz}(0) + \frac{1}{N} \sum_n S_{zz}(n) = 0$$

Each sum above equals $R_{zz}(0)$ (see eq. (A5)); hence the whole expression equals zero and equation (A9) is proved.

Theorem 2: Let $z(t)$, $t = 0$, . . ., and $N - 1$, be mutually stochastically independent random variables having, respectively, Gaussian distributions with $E\{z(t)\} = 0$ and $E\{z^2(t)\} = \sigma^2$. Then the sequence

$$\tilde{z}(n) = \frac{1}{N} \sum_{t=0}^{N-1} z(t) \, e^{-jn\omega_0 t} \tag{A3}$$

where

$$n = -\frac{N}{2}, \ldots, 0, 1, \ldots, \frac{N}{2} - 1$$

consisting of real parts $\tilde{z}^R(n)$ and imaginary parts $\tilde{z}^I(n)$ is a complex Gaussian and uncorrelated sequence with

$$E\{z^R(n)\} = E\{z^I(n)\} = 0 \tag{A10}$$

and

$$E\left\{ \begin{bmatrix} \tilde{z}^R(m) \\ \tilde{z}^I(m) \end{bmatrix} \begin{bmatrix} \tilde{z}^R(n) \\ \tilde{z}^I(n) \end{bmatrix}^T \right\} = \frac{1}{2N} \begin{bmatrix} S_{zz} & 0 \\ 0 & S_{zz} \end{bmatrix} \delta_{m,n} \tag{A11}$$

Proof: It has already been proved that

$$E\{\tilde{z}(n)\} = 0$$

This result combined with the definition of the expected value of a complex random variable (eq. (B1)) implies that

$$E\{\tilde{z}^R(n)\} = E\{\tilde{z}^I(n)\} = 0$$

It has also been shown that (in eq. (A8))

$$E\{\tilde{z}(m) \, \tilde{z}^*(n)\} = \frac{S_{zz}(n)}{N^2} \sum_{t=0}^{N-1} e^{-j(m-n)\omega_0 t} = \frac{S_{zz}(n)}{N} \delta_{m,n} \tag{A12}$$

Using the same approach as that for the development of equation (A8), it can be shown that

$$E\{\tilde{z}(m)\ \tilde{z}^*(n)\} = E\{\tilde{z}^R(m)\ \tilde{z}^R(n)\} + E\{\tilde{z}^I(m)\ \tilde{z}^I(n)\}$$

$$+ jE\{\tilde{z}^I(m)\ \tilde{z}^R(n)\} - jE\{\tilde{z}^R(m)\ \tilde{z}^I(n)\} \tag{A13}$$

and

$$E\{\tilde{z}(m)\ \tilde{z}(n)\} = E\{\tilde{z}^R(m)\ \tilde{z}^R(n)\} - E\{\tilde{z}^I(m)\ \tilde{z}^I(n)\}$$

$$+ jE\{\tilde{z}^R(m)\ \tilde{z}^I(n)\} + jE\{\tilde{z}^I(m)\ \tilde{z}^R(n)\} \tag{A14}$$

From equations (A8) and (A12) to (A14), two sets of equations can be formed as

$$\left.\begin{array}{l} E\{\tilde{z}^R(m)\ \tilde{z}^R(n)\} + E\{\tilde{z}^I(m)\ \tilde{z}^I(n)\} = \dfrac{S_{zz}(n)}{N}\ \delta_{m,n} \\[3ex] E\{\tilde{z}^R(m)\ \tilde{z}^R(n)\} - E\{\tilde{z}^I(m)\ \tilde{z}^I(n)\} = 0 \end{array}\right\} \tag{A15}$$

and

$$\left.\begin{array}{l} E\{\tilde{z}^I(m)\ \tilde{z}^R(n)\} - E\{\tilde{z}^R(m)\ \tilde{z}^I(n)\} = 0 \\[3ex] E\{\tilde{z}^R(m)\ \tilde{z}^I(n)\} + E\{\tilde{z}^I(m)\ \tilde{z}^R(n)\} = 0 \end{array}\right\} \tag{A16}$$

These two sets give the solution

$$E\{\tilde{z}^R(m)\ \tilde{z}^R(n)\} = E\{\tilde{z}^I(m)\ \tilde{z}^I(n)\} = \dfrac{S_{zz}(m)}{2N}\ \delta_{m,n} \tag{A17}$$

$$E\{\tilde{z}^R(m)\ \tilde{z}^I(n)\} = E\{\tilde{z}^I(m)\ \tilde{z}^R(n)\} = 0 \tag{A18}$$

which proves the validity of equation (A11). Equations (A15) and (A16) have been developed for m and n being positive. In the case where $(-N/2) \leqq m,n \leqq (N/2 - 1)$ the proof based on these equations is still valid.

The only change might occur in equations (A15), where the right-hand sides would be interchanged. The form of the solution given by equations (A17) and (A18) does not change.

Finally, it is necessary to prove that $\tilde{z}(n)$ is Gaussian. For this proof the concept of the moment-generating function is used. When $\tilde{z}(n)$ is expressed as

$$\tilde{z}(n) = \sum_{t=0}^{N-1} c_t \, z(t)$$

where

$$c_t = \frac{1}{N} \, e^{-jn\omega_0 t}$$

then the moment-generating function of the variable $c_t \, z(t)$ is given according to reference 24 as

$$E\left\{ e^{bc_t z(t)} \right\} = \exp\left[\frac{\sigma^2}{2} (bc_t)^* (bc_t) \right] \tag{A19}$$

where b is a constant independent of $c_t \, z(t)$. Thus, the moment-generating function of $\tilde{z}(n)$ is

$$E\{\exp b\lceil c_0 \, z(0) + c_1 \, z(1) + \ldots + c_{N-1} \, z(N-1)\rceil\}$$

$$= E\{\exp bc_0 \, z(0)\} \, E\{\exp bc_1 \, z(1)\} \, . \, . \, . \, E\{\exp bc_{N-1} \, z(N-1)\}$$

$$= \prod_{t=0}^{N-1} \exp\left[\frac{\sigma^2}{2} (bc_t)^* (bc_t) \right] = \exp\left[\frac{b^2\sigma^2}{2} \sum_{t=0}^{N-1} c_t^* c_t \right] = \exp\left[\frac{b^2\sigma^2}{2N} \right] \tag{A20}$$

which means that $\tilde{z}(n)$ is Gaussian with zero mean and variance σ^2/N or, using equation (A4), with variance S_{zz}/N. The sequence $\tilde{z}(n)$ is formed by a collection of uncorrelated Gaussian random variables.

COMPLEX RANDOM VARIABLE

A complex random variable z is a complex number $z(\zeta)$ determined by an outcome ζ; i.e.,

$$z(\zeta) = z^R(\zeta) + jz^I(\zeta)$$

such that the functions z^R and z^I are random variables. By definition, the expected value of a complex random variable is

$$E\{z\} = E\{z^R\} + jE\{z^I\} \tag{B1}$$

the variance is

$$\sigma_z^2 = E\{|z-E\{z\}|^2\} \tag{B2}$$

and the covariance is

$$E\{[z_k-E\{z_k\}]\ [z_\ell^*-E\{z_\ell^*\}]\} \tag{B3}$$

If

$$E\{z_k z_\ell^*\} = E\{z_k\}\ E\{z_\ell^*\} \qquad \text{(for } k \neq \ell\text{)}$$

then the complex random variables z_1, z_2, . . ., and z_n are uncorrelated. They are orthogonal if

$$E\{z_k z_\ell^*\} = 0$$

If the complex random variable $z = z^R + jz^I$ has its real and imaginary parts normally distributed with

$$E\{z^R\} = E\{z^I\} = 0$$

$$E\{(z^R)^2\} = E\{(z^I)^2\} = \frac{1}{2}\sigma_z^2$$

and these parts are stochastically independent, then the distribution of a complex random variable z will be defined as a joint distribution of the independent variables z^R and z^I such that

$$p[z] = p[z^R, z^I] = p[z^R]\, p[z^I] = \frac{1}{\sqrt{\pi}\sigma_z} \exp\left[-(z^R)^2/\sigma_z^2\right] \frac{1}{\sqrt{\pi}\sigma_z} \exp\left[-(z^I)^2/\sigma_z^2\right]$$

$$= \frac{1}{\pi\sigma_z^2} \exp\left(-z^* z/\sigma_z^2\right) \tag{B4}$$

If z is a complex vector of dimension r, then the Gaussian distribution is defined as

$$p[z] = \frac{1}{\pi^r |\Sigma|} \exp(-z^* \Sigma^{-1} z) \tag{B5}$$

where Σ is the covariance matrix of z. This is a Hermitian nonnegative square matrix of dimension r. If z has components that are stochastically independent, then Σ is a diagonal matrix. Equation (B5) agrees with the definition of the Gaussian distribution presented in references 24 and 25.

INFORMATION MATRIX AND GRADIENT OF LOG-LIKELIHOOD FUNCTION

From equations (32) and (34) the elements of the information matrix are given as

$$M_{k\ell} = 2N \text{ Re } E\left\{ \Sigma_n \frac{\partial \tilde{\upsilon}^*}{\partial \Theta_k} S_{\upsilon\upsilon}^{-1} \frac{\partial \tilde{\upsilon}}{\partial \Theta_\ell} \right\} \tag{C1}$$

where the expected value can be written as

$$E\left\{ \frac{\partial \tilde{\upsilon}^*}{\partial \Theta} S_{\upsilon\upsilon}^{-1} \frac{\partial \tilde{\upsilon}}{\partial \Theta_\ell} \right\} = \text{Tr}\left[E\left\{ \frac{\partial \tilde{\upsilon}}{\partial \Theta_k} \frac{\partial \tilde{\upsilon}^*}{\partial \Theta_\ell} \right\} S_{\upsilon\upsilon}^{-1} \right] \tag{C2}$$

The measured outputs and innovations are given by equations (19) and (22) as

$$\tilde{y}(n) = T_1 \tilde{u}(n) + T_2 \tilde{\upsilon}(n) \tag{C3}$$

and

$$\tilde{\upsilon}(n) = T_3 \tilde{y}(n) - T_4 \tilde{u}(n) \tag{C4}$$

where $T_3 = T_2^{-1}$, $T_4 = T_2^{-1}T_1$, and T_1 and T_2 are defined by equations (20) and (21). Therefore,

$$E\left\{ \frac{\partial \tilde{\upsilon}}{\partial \Theta_k} \frac{\partial \tilde{\upsilon}^*}{\partial \Theta_\ell} \right\} = E\left\{ \left(T_{3_k}\tilde{y} - T_{4_k}\tilde{u} \right)\left(T_{3_\ell}\tilde{y} - T_{4_\ell}\tilde{u} \right)^* \right\}$$

$$= T_{3_k} E\{\tilde{y}\tilde{y}^*\} T_{3_\ell}^* - T_{4_k} E\{\tilde{u}\tilde{y}^*\} T_{3_\ell}^*$$

$$- T_{3_k} E\{\tilde{y}\tilde{u}^*\} T_{4_\ell}^* + T_{4_k} E\{\tilde{u}\tilde{u}^*\} T_{4_\ell}^* \tag{C5}$$

where

$$T_{3_k} = \frac{\partial T_3}{\partial \Theta_k} \quad \cdots \quad T_{4_\ell}^* = \frac{\partial T_4^*}{\partial \Theta_\ell}$$

Because $u(t)$ and $\nu(t)$ are uncorrelated, the transformed variables $\tilde{u}(n)$ and $\tilde{\nu}(n)$ are also uncorrelated. Using equations (C3) and (B4) the expected values in (C5) can be expressed as

$$E\{yy^*\} = E\left\{T_1 uu^* T_1^* + T_2\tilde{\nu}\tilde{\nu}^*T_2^*\right\} = \frac{1}{N}(T_1 S_{uu}T_1^* + T_2 S_{\nu\nu}T_2^*)$$

$$E\{\tilde{u}\tilde{y}^*\} = E\{\tilde{u}\tilde{u}^*T_1^* + \tilde{u}\tilde{\nu}^*T_2^*\} = \frac{1}{N} S_{uu}T_1^*$$

$$E\{\tilde{y}\tilde{u}^*\} = E\{T_1\tilde{u}\tilde{u}^* + T_2\tilde{\nu}\tilde{u}^*\} = \frac{1}{N} T_1 S_{uu}$$

$$E\{\tilde{u}\tilde{u}^*\} = \frac{1}{N} S_{uu}$$

$$(C6)$$

After substituting equation (C6) into equation (C5) and some tedious manipulation,

$$E\left\{\frac{\partial\tilde{\nu}^*}{\partial\Theta_k}\frac{\partial\tilde{\nu}}{\partial\Theta_\ell}\right\} = T_2^{-1}T_{2_k}S_{\nu\nu}T_{2_\ell}^*(T_2^*)^{-1} + T_2^{-1}T_{1_k}S_{uu}T_{1_\ell}^*(T_2^*)^{-1} \tag{C7}$$

From equations (C2) and (C7),

$$E\left\{\frac{\partial\tilde{\nu}^*}{\partial\Theta_k} S_{\nu\nu}^{-1} \frac{\partial\tilde{\nu}}{\partial\Theta_\ell}\right\} = \frac{1}{N}\left[Tr\left(T_{1_\ell}^*(T_2^*)^{-1}S_{\nu\nu}^{-1}(T_2^*)^{-1}T_{1_k}S_{uu}\right) + Tr\left(T_2^{-1}T_{2_k}T_{2_\ell}^*(T_2^*)^{-1}\right)\right]$$

$$(C8)$$

Substituting equation (C8) into equation (C1), the final form for the elements of the information matrix is obtained as

$$M_{k\ell} = 2 \, \text{Re} \, \Sigma_n \left[\text{Tr}\left(\frac{\partial T_1^*}{\partial \Theta_\ell} (T_2^*)^{-1} S_{\nu\nu}^{-1} T_2^{-1} \frac{\partial T_1}{\partial \Theta_k} S_{uu} \right) + \text{Tr}\left(\frac{\partial T_2^*}{\partial \Theta_\ell} (T_2^*)^{-1} T_2^{-1} \frac{\partial T_2}{\partial \Theta_k} \right) \right] \qquad (C9)$$

From equation (33) the element of the gradient of the log-likelihood function is

$$\frac{\partial J(\Theta)}{\partial \Theta_k} = -2N \, \text{Re} \, \Sigma_n \, \tilde{\nu}^* S_{\nu\nu}^{-1} \frac{\partial \tilde{\nu}}{\partial \Theta_k} = -2N \, \text{Re} \, \Sigma_n \, \frac{\partial \tilde{\nu}^*}{\partial \Theta_k} S_{\nu\nu}^{-1} \tilde{\nu} \qquad (C10)$$

Using equation (C4)

$$\frac{\partial \tilde{\nu}}{\partial \Theta_k} = T_{2_k}^{-1} T_2 \tilde{\nu} - T_2^{-1} T_{1_k} \tilde{u} \qquad (C11)$$

Then

$$\frac{\partial \tilde{\nu}^*}{\partial \Theta_k} S_{\nu\nu}^{-1} \tilde{\nu} = \left(T_{2_k}^{-1} T_2 \tilde{\nu} \right)^* S_{\nu\nu}^{-1} \tilde{\nu} - \left(T_2^{-1} T_{1_k} \tilde{u} \right)^* S_{\nu\nu}^{-1} \tilde{\nu} \qquad (C12)$$

where

$$\left(T_{2_k}^{-1} T_2 \tilde{\nu} \right)^* S_{\nu\nu}^{-1} \tilde{\nu} = \text{Tr}\left[\tilde{\nu} \tilde{\nu}^* S_{\nu\nu}^{-1} T_{2_k}^* (T_2^*)^{-1} \right] \qquad (C13)$$

and

$$\left(T_2^{-1} T_{1_k} \tilde{u} \right)^* S_{\nu\nu}^{-1} \tilde{\nu} = \text{Tr}\left[\tilde{y} \tilde{u}^* T_{1_k}^* (T_2^*)^{-1} S_{\nu\nu}^{-1} T_2^{-1} \right] - \text{Tr}\left[uu^* T_{1_k} (T_2^*)^{-1} S_{\nu\nu}^{-1} T_2^{-1} T_1 \right] \qquad (C14)$$

Introducing

$$\tilde{v}\tilde{v}^* = \frac{1}{N}\hat{S}_{vv}$$

$$\tilde{y}\tilde{u}^* = \frac{1}{N}\hat{S}_{yu}$$

$$\tilde{u}\tilde{u}^* = \frac{1}{N}\hat{S}_{uu}$$

and approximating $\hat{S}_{vv}S_{vv}^{-1} = I$, equation (C12) is changed as

$$\frac{\partial \tilde{v}^*}{\partial \Theta_k} S_{vv}^{-1}\tilde{v} = \frac{1}{N}\left\{ \text{Tr}\left[T_{1_k}^*(T_2^*)^{-1}S_{vv}^{-1}T_2^{-1}(\hat{S}_{yu} - T_1\hat{S}_{uu})\right] - \text{Tr}\left[T_{2_k}^*(T_2^*)^{-1}\right]\right\} \qquad \text{(C15)}$$

Finally, substituting equation (C15) into equation (C10), the elements in the gradient vector are obtained as

$$\frac{\partial J(\Theta)}{\partial \Theta_k} = -2 \text{ Re } \Sigma_n \left\{ \text{Tr}\left[\frac{\partial T_1^*}{\partial \Theta_k}(T_2^*)^{-1}S_{vv}^{-1}T_2^{-1}(\hat{S}_{yu} - T_1\hat{S}_{uu})\right] - \text{Tr}\left[\frac{\partial T_2^*}{\partial \Theta_k}(T_2^*)^{-1}\right]\right\} \qquad \text{(C16)}$$

APPENDIX D

EQUATIONS OF LONGITUDINAL MOTION OF AN AIRPLANE IN PRESENCE OF TURBULENCE

The airplane equations of motion are referred to the body axes. They are perturbed equations for datum conditions corresponding to steady horizontal flight. The equations are based on the following assumptions:

(1) The airplane is a rigid body

(2) The elevator deflection and turbulence excite the longitudinal motion during which the airspeed remains constant

(3) The turbulence is approximated by a one-dimensional gust field, and the angle-of-attack and pitch-rate perturbations due to turbulence are given as

$$\alpha_g = \frac{w_g}{v_0}$$

and

$$q_g = -\dot{\alpha}_g$$

(4) The aerodynamic model equations for the increments in C_Z and C_m have the form

$$\Delta C_Z = C_{Z_\alpha}(\alpha + \alpha_g) + C_{Z_q}(q - \dot{\alpha}_g)\frac{\bar{c}}{2v_0} + C_{Z_{\delta_e}}\delta_e$$

$$\Delta C_m = C_{m_\alpha}(\alpha + \alpha_g) + C_{m_{\dot{\alpha}}}(\dot{\alpha} + \dot{\alpha}_g)\frac{\bar{c}}{2v_0} + C_{m_q}(q - \dot{\alpha}_g)\frac{\bar{c}}{2v_0} + C_{m_{\delta_e}}\delta_e$$

where the input and output variables are the increments with respect to the initial steady flight.

Using these assumptions, the longitudinal equations of motion can be expressed as

$$\dot{\alpha} = q + \frac{\bar{q}S}{mV_0}\left(C_{z_\alpha}\alpha + C_{z_q}\frac{q\bar{c}}{2V_0} + C_{z_{\delta e}}\delta_e + C_{z_\alpha}\alpha_g - C_{z_q}\frac{\dot{\alpha}_g\bar{c}}{2V_0}\right) - g\sin\theta_0\theta \qquad (D1)$$

$$\dot{q} = \frac{\bar{q}S\bar{c}}{I_Y}\left(C_{m_\alpha}\alpha + C_{m_{\dot\alpha}}\frac{\dot{\alpha}\bar{c}}{2V_0} + C_{m_q}\frac{q\bar{c}}{2V_0} + C_{m_{\delta e}}\delta_e + C_{m_\alpha}\alpha_g - C_{m_q}\frac{\dot{\alpha}_g\bar{c}}{2V_0} + C_{m_{\dot\alpha}}\frac{\dot{\alpha}_g\bar{c}}{2V_0}\right) \qquad (D2)$$

where $g\sin\theta_0\theta$ in equation (D1) is negligible.

In equations (D1) and (D2) α_g is a stochastic variable. Its spectral density can be modeled, for example, by a Dryden formula (see ref. 26). In the further development it will be assumed that the turbulence velocity component w_g is a Gauss-Markoff process of first order governed by the differential equation

$$\dot{w}_g = -c_1 w_g + c_2 w \qquad (D3)$$

where

$$c_1 = 2.4\frac{V_0}{L_g}$$

$$c_2 = V_0\sqrt{\frac{4.8}{\pi L_g}}$$

L_g is the scale of turbulence, and w is the uncorrelated noise process with $E\{w\} = 0$ and $E\{w^2\} = \sigma_g^2$.

When equation (D1) is substituted into equation (D2) and equation (D3) is considered, then the state equations for the longitudinal motion of the airplane will have the form

APPENDIX D

$$
\begin{bmatrix} \dot{\alpha} \\[2ex] \dot{q} \\[2ex] \dot{w}_g \end{bmatrix} = \begin{bmatrix} k_1 C_{z_\alpha} & \left(1 + k_2 C_{z_q}\right) & k_3 C_{z_\alpha} + k_4 c_1 C_{z_q} \\[2ex] k_5 C'_{m_\alpha} & k_6 C'_{m_q} & k_7 C'_{m_\alpha} + k_8 c_1 C^o_{m_q} \\[2ex] 0 & 0 & -c_1 \end{bmatrix} \begin{bmatrix} \alpha \\[2ex] q \\[2ex] w_g \end{bmatrix} + \begin{bmatrix} k_1 C_{z_{\delta e}} \\[2ex] k_5 C'_{m_{\delta e}} \\[2ex] 0 \end{bmatrix} \delta_e + \begin{bmatrix} -k_4 c_2 C_{z_q} \\[2ex] -k_8 c_2 C^o_{m_q} \\[2ex] c_2 \end{bmatrix} w
$$

(D4)

where

$$k_1 = \frac{\rho S V_0}{2m} \qquad\qquad k_2 = \frac{\rho S \bar{c}}{4m}$$

$$k_3 = \frac{\rho S}{2m} \qquad\qquad k_4 = \frac{\rho S \bar{c}}{4m V_0}$$

$$k_5 = \frac{\rho S V_0^2 \bar{c}}{2 I_Y} \qquad\qquad k_6 = \frac{\rho S V_0 \bar{c}^2}{4 I_Y}$$

$$k_7 = \frac{\rho S V_0 \bar{c}}{2 I_Y} \qquad\qquad k_8 = \frac{\rho S \bar{c}^2}{4 I_Y}$$

(Note: $k_4 c_2 C_{z_q} \approx 0$).

$$C'_{m_\alpha} = C_{m_\alpha} + \frac{\rho S \bar{c}}{4m} C_{m_{\dot{\alpha}}} C_{z_\alpha}$$

(D5)

$$C'_{m_q} = C_{m_q} + C_{m_{\dot{\alpha}}} \left(1 + \frac{\rho S \bar{c}}{4m} C_{z_q}\right)$$

(D6)

39

$$C_{m_q}^O = C_{m_q} - C_{m_{\dot{\alpha}}}\left(1 + \frac{\rho S \bar{c}}{4m} C_{z_q}\right) \tag{D7}$$

$$C_{m_{\delta e}}' = C_{m_{\delta e}} + \frac{\rho S \bar{c}}{4m} C_{m_{\dot{\alpha}}} C_{z_{\delta e}} \tag{D8}$$

For the parameter estimation from measured data, state equations (D4) are completed by the output equations

$$
\begin{bmatrix} \alpha_v \\ q \\ a_z \end{bmatrix} = \begin{bmatrix} k_9 & -k_{10} & \dfrac{k_9}{V_0} \\ 0 & 1 & 0 \\ \dfrac{V_0}{g} k_1 C_{z_\alpha} & \dfrac{V_0}{g} k_2 C_{z_q} & \dfrac{V_0}{g} k_3 C_{z_\alpha} \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ w_g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{V_0}{g} k_1 C_{z_{\delta e}} \end{bmatrix} \delta_e \tag{D9}
$$

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TABLE I.- ESTIMATED AIRPLANE PARAMETERS FROM COMPUTER-GENERATED DATA

| Parameter | Initial value | Estimate of parameter (a) | | Assumed no process noise |
| | | With process noise | | |
		K estimated	K fixed	
C_{z_α}	-5.0	-5.07 (.091)	-5.72 (.20)	-5.2 (.10)
C_{z_q}	-20	-23.8 (2.3)	-17 (2.7)	-15 (4.0)
$C_{z_{\delta e}}$	-1.0	-1.17 (.071)	-1.1 (.17)	-.57 (.31)
C'_{m_α}	-.80	-.82 (.015)	-.94 (.031)	-.83 (.014)
C'_{m_q}	-24	-24.0 (.52)	-25.5 (.38)	-22.1 (.65)
$C'_{m_{\delta e}}$	-3.3	-3.21 (.062)	-3.16 (.080)	-3.18 (.050)
K_{11}	-.0563	-.44 (.011)	-.0563	0
K_{12}	2.544	2.31 (.010)	2.544	0
K_{13}	.0025	-.21 (.010)	.0025	0
K_{21}	2.438	1.84 (.044)	2.438	0
K_{22}	9.7	10.8 (.14)	9.7	0
K_{23}	-.018	.09 (.97)	-.018	0
K_{31}	213.0	156 (8.9)	213.0	0
K_{32}	-5.065	-8 (2.1)	-5.065	0
K_{33}	-12.63	-15 (6.2)	-12.63	0
$\hat{\sigma}^2_{\tilde{v}_\alpha} \times 10^5$	--------	.67	.72	.70
$\hat{\sigma}^2_{\tilde{v}_q} \times 10^4$	--------	.30	.60	.82
$\hat{\sigma}^2_{\tilde{v}_{a_z}}$	--------	.50	1.1	1.6

[a]Numbers in parentheses are Cramér-Rao lower bounds on standard errors.

TABLE II.- ESTIMATED AIRPLANE PARAMETERS FROM MEASUREMENTS IN TURBULENT AND STILL AIR

Parameter	Average value and its standard error (b)	Estimate of parameter (a)						
		Example 2 (turbulent air)		Example 3 (still air)			Example 4 (turbulent air) Frequency domain (e)	
		Assumed no process noise	With process noise	Time domain	Frequency domain (c)	Frequency domain (d)	Run 1	Run 2
c_{z_α}	-5.3 ± 0.1	-6.4 (.68)	-5.1 (.20)	-5.67 (.057)	-5.70 (.086)	-5.6 (.19)	-4.2 (.47)	-4.4 (.61)
c_{z_q}	-19 ± 3	-10 (6.1)	-44 (8.4)	-10.3 (.92)	-8 (3.5)	-4 (4.7)	-9 (8.0)	-4 (10)
$c_{z_{\delta e}}$	-1.2 ± 0.2	-.8 (.32)	-1.4 (.47)	-.6 (.16)	-.6 (.28)	-.6 (.33)	-------	-------
c'_{m_α}	-.80 ± 0.02	-.88 (.070)	-.84 (.050)	-.783 (.0074)	-.81 (.011)	-.82 (.027)	-.9 (.16)	-.9 (.17)
c'_{m_q}	-24.2 ± 0.4	-28 (2.2)	-32 (2.8)	-26.6 (.40)	-28.6 (.50)	-26.3 (.92)	-24 (4.3)	-26 (4.6)
$c_{m_{\dot\alpha}}$	-8.0	--------	--------	---------	--------	--------	f-7 (2.6)	f-5 (2.5)
$c'_{m_{\delta e}}$	-3.32 ± 0.05	-3.3 (.12)	-3.19 (.080)	-3.21 (.030)	-3.54 (.040)	-3.38 (.060)	-------	-------

[a] Numbers in parentheses are Cramér-Rao lower bounds on standard errors.
[b] From the maximum likelihood estimates in the time domain (ref. 23).
[c] Transformed input and output time histories.
[d] Frequency response curves.
[e] Spectral and cross-spectral densities.
[f] Computed from estimated C'_{m_q} and $C^o_{m_q}$.

.10

α_v, rad   0

-.10

.6

q, rad/sec   0

-.6

1.0

a_z, g units   0

-1.0

.1

δ_e, rad   0

-.1

0   1   2   3   4   5   6

Time, t, sec
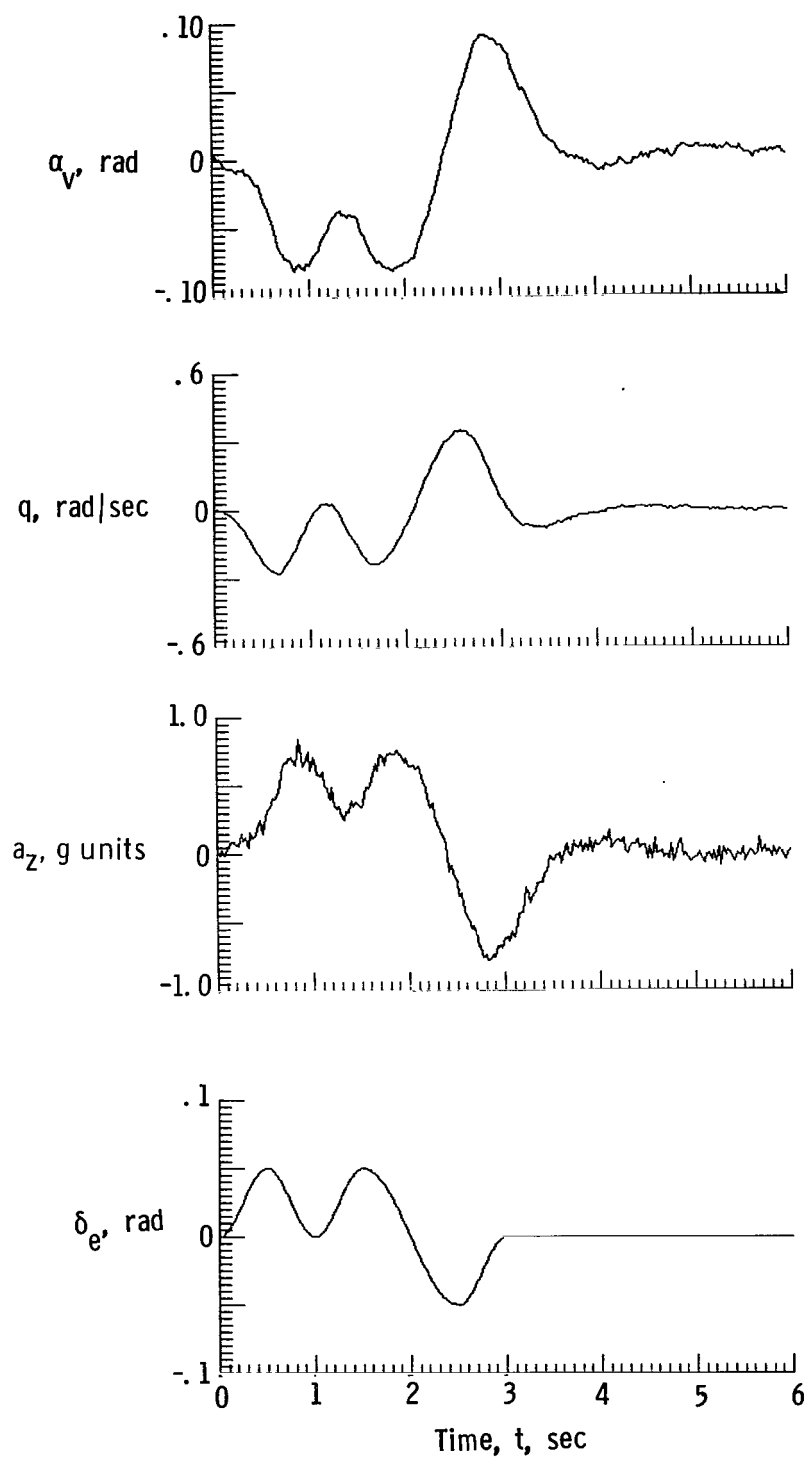
Figure 1.- Time histories of computer-generated output and input variables.
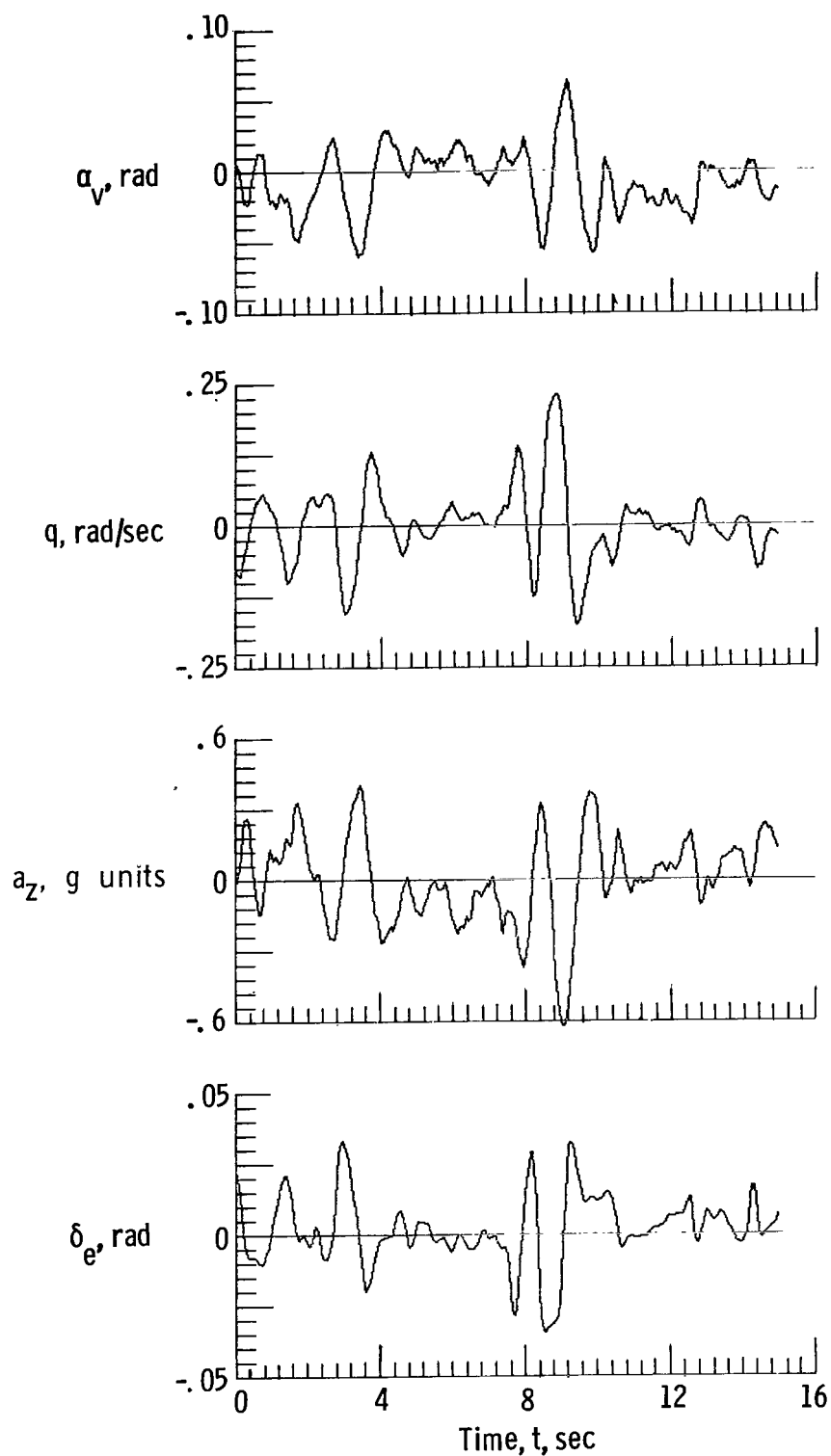
Figure 2.- Time histories of measured output and input variables.
Flight in turbulence.
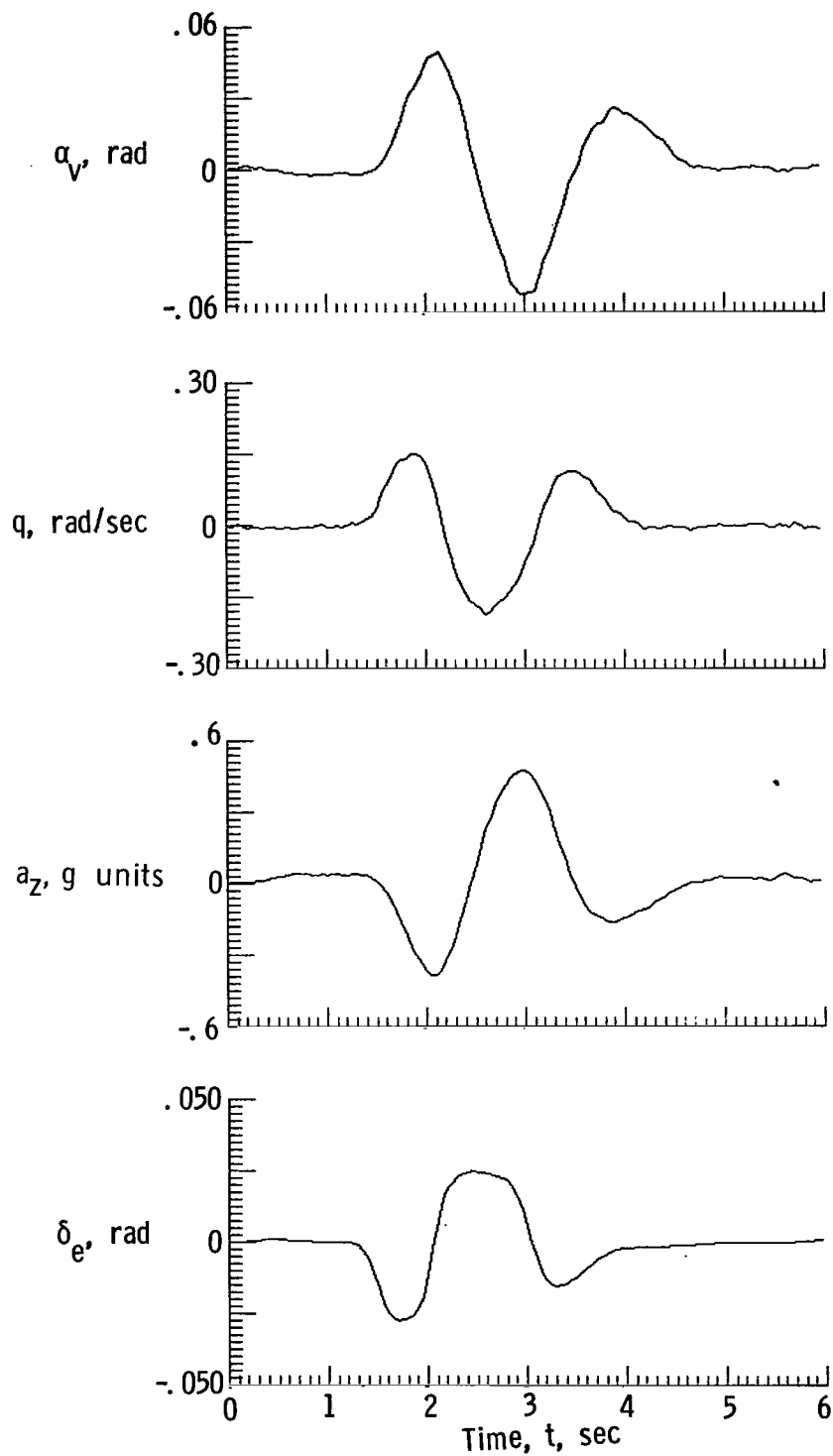
Figure 3.- Time histories of measured output and input variables.
Flight in still air.
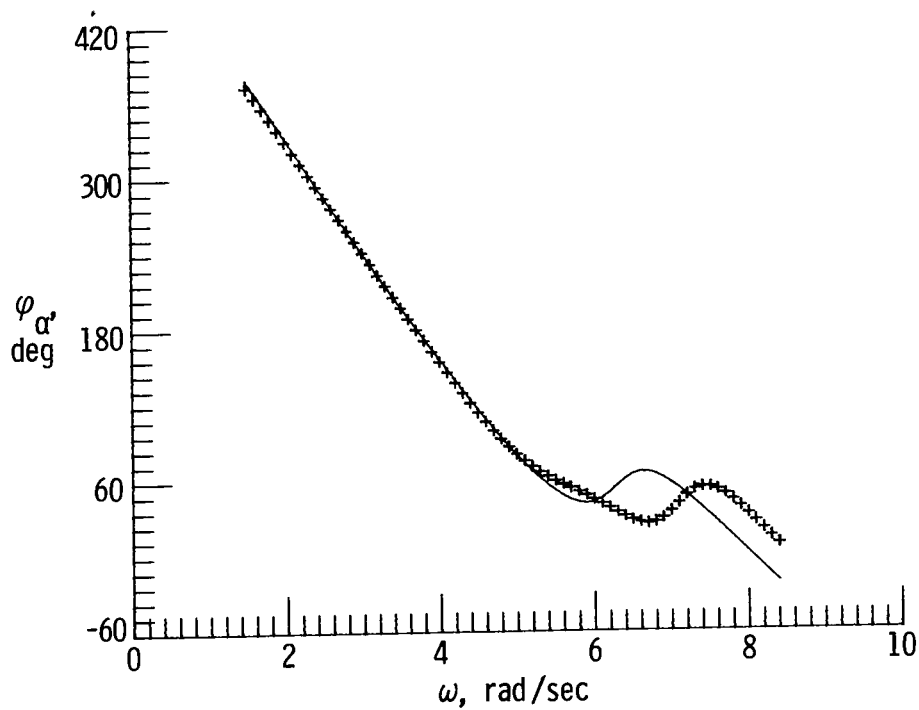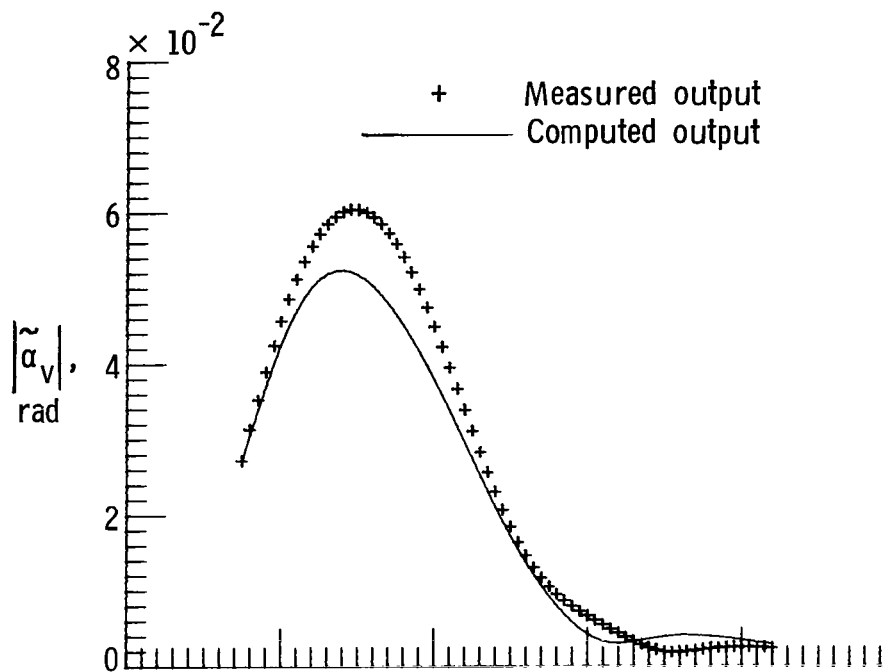
Figure 4.- Measured transformed time histories and those computed by using estimated parameters.
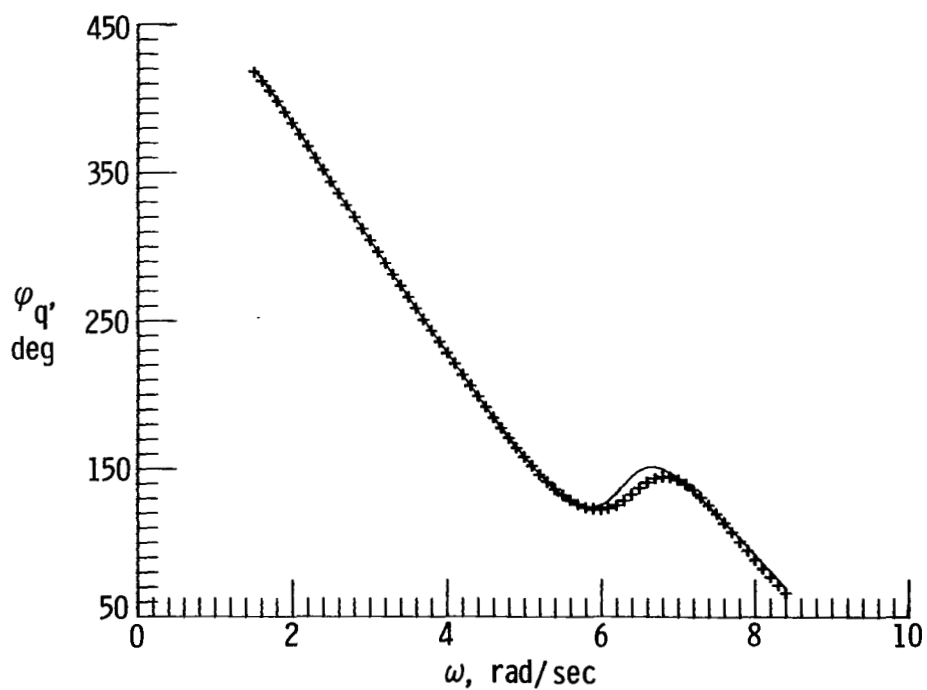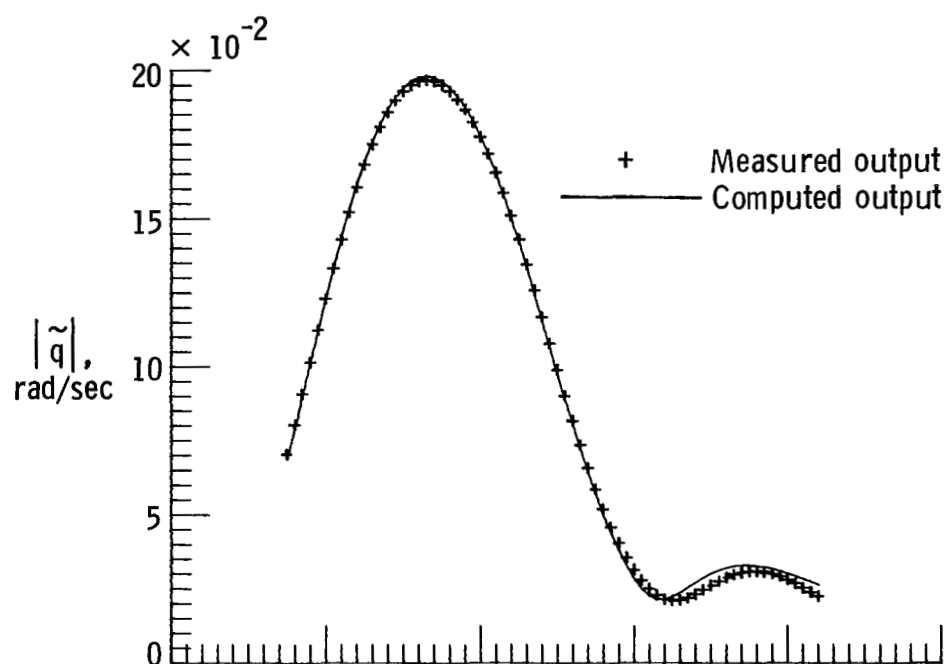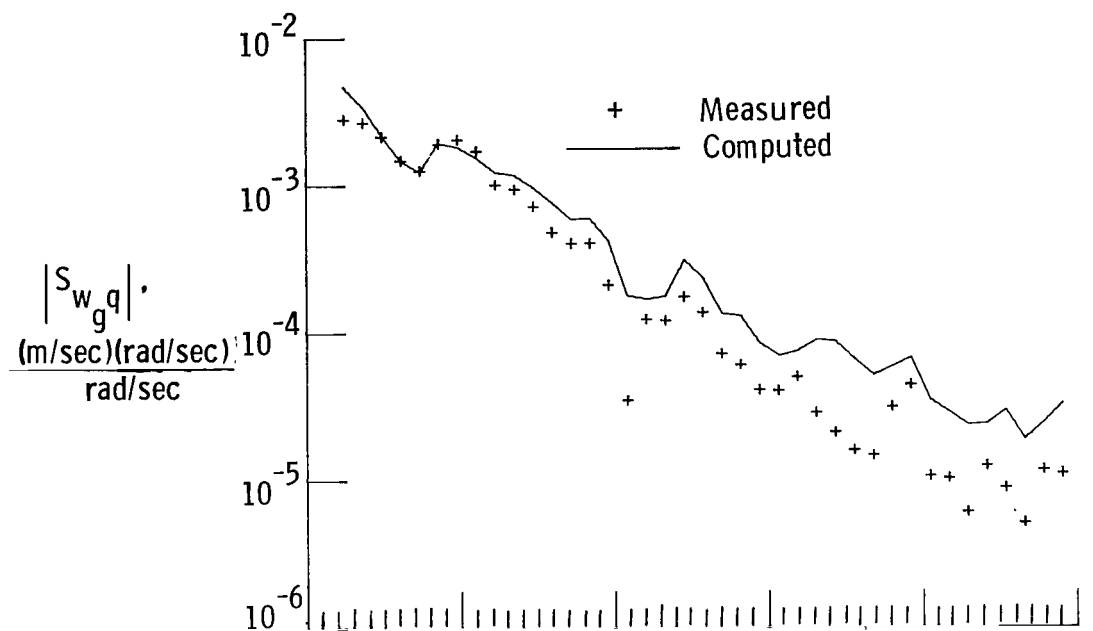
Figure 4.- Continued.
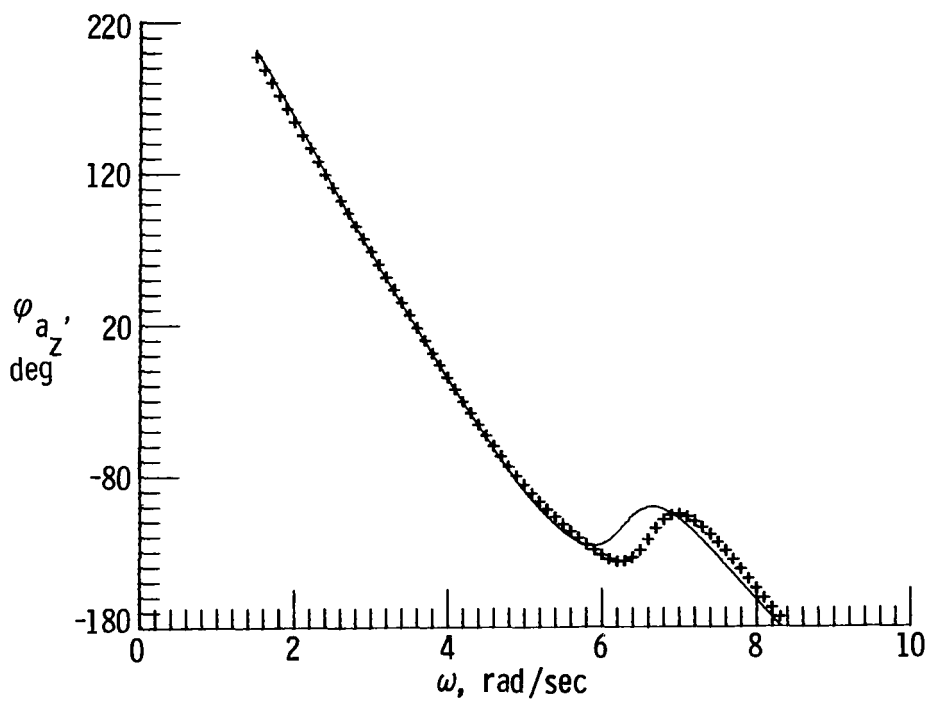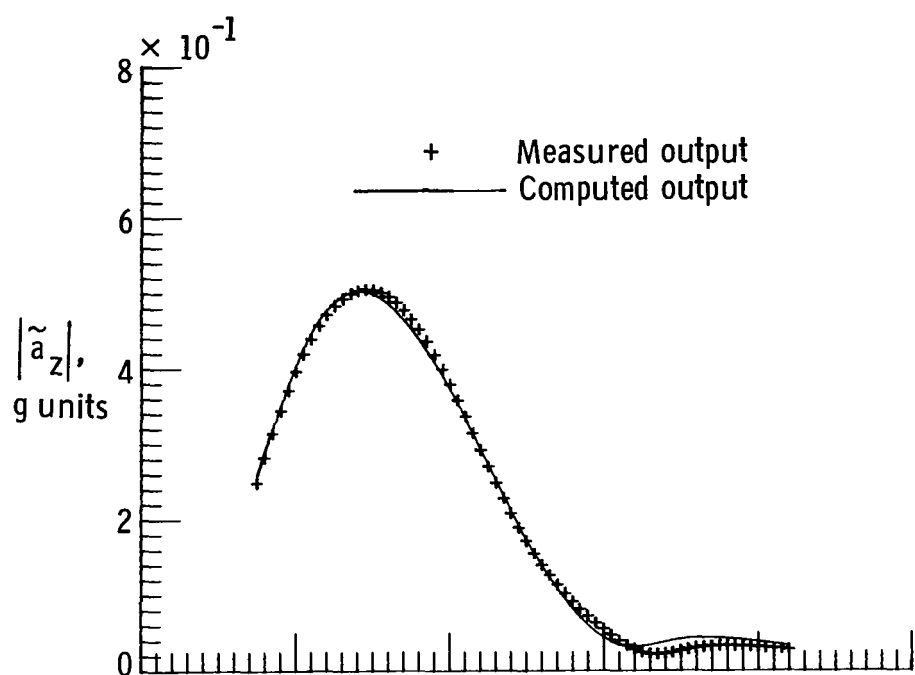
Figure 4.- Continued.
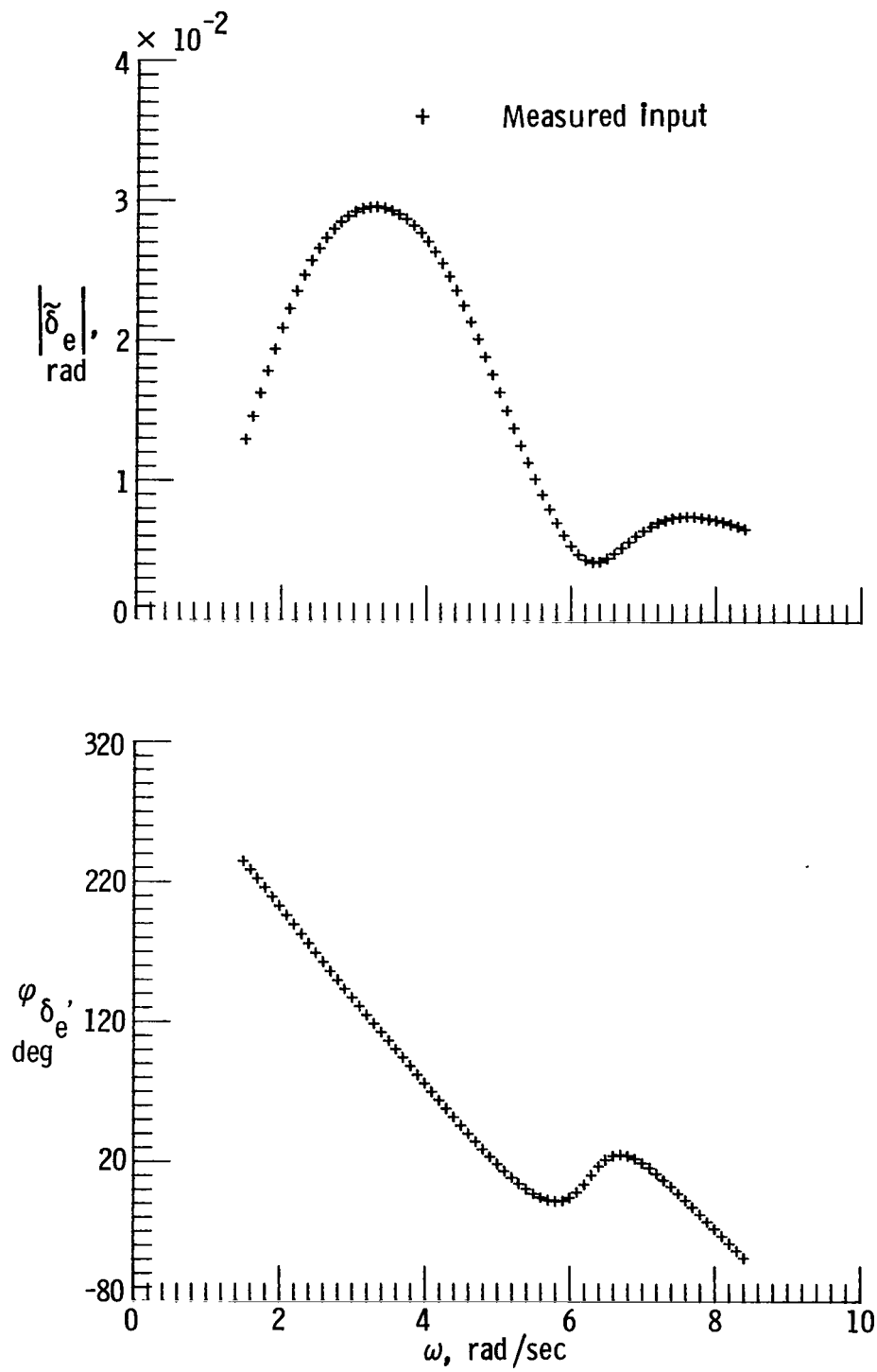
Figure 4.- Concluded.
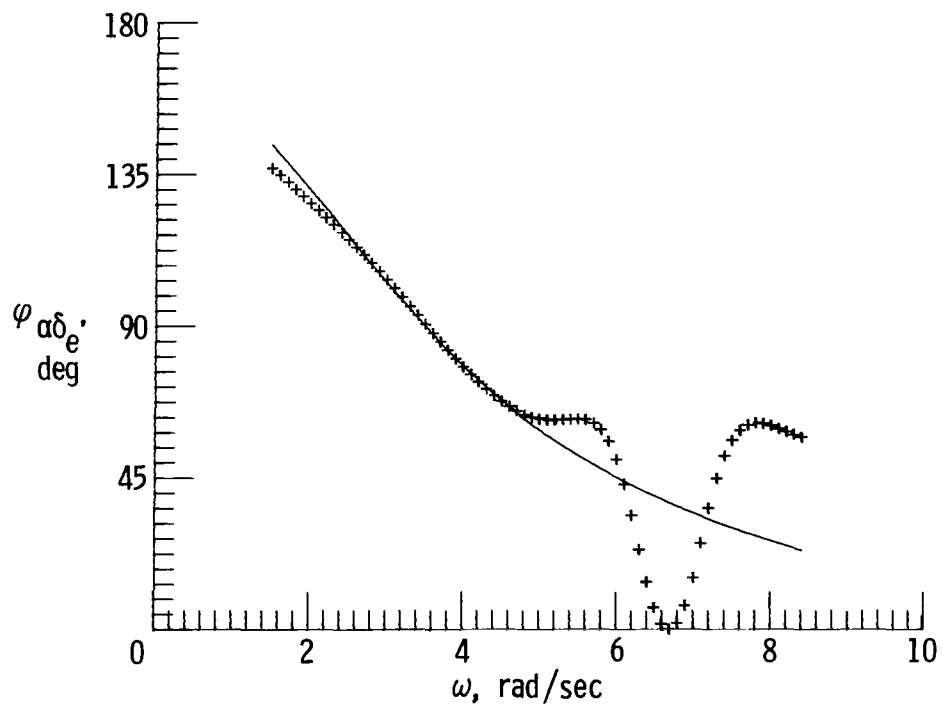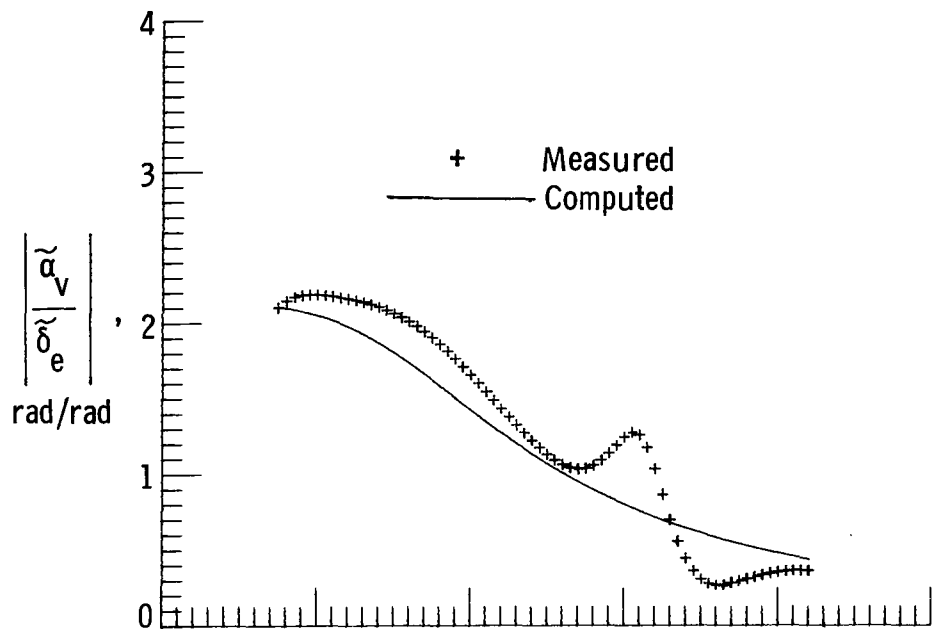
Figure 5.- Measured frequency response curves and those computed by
using estimated parameters.
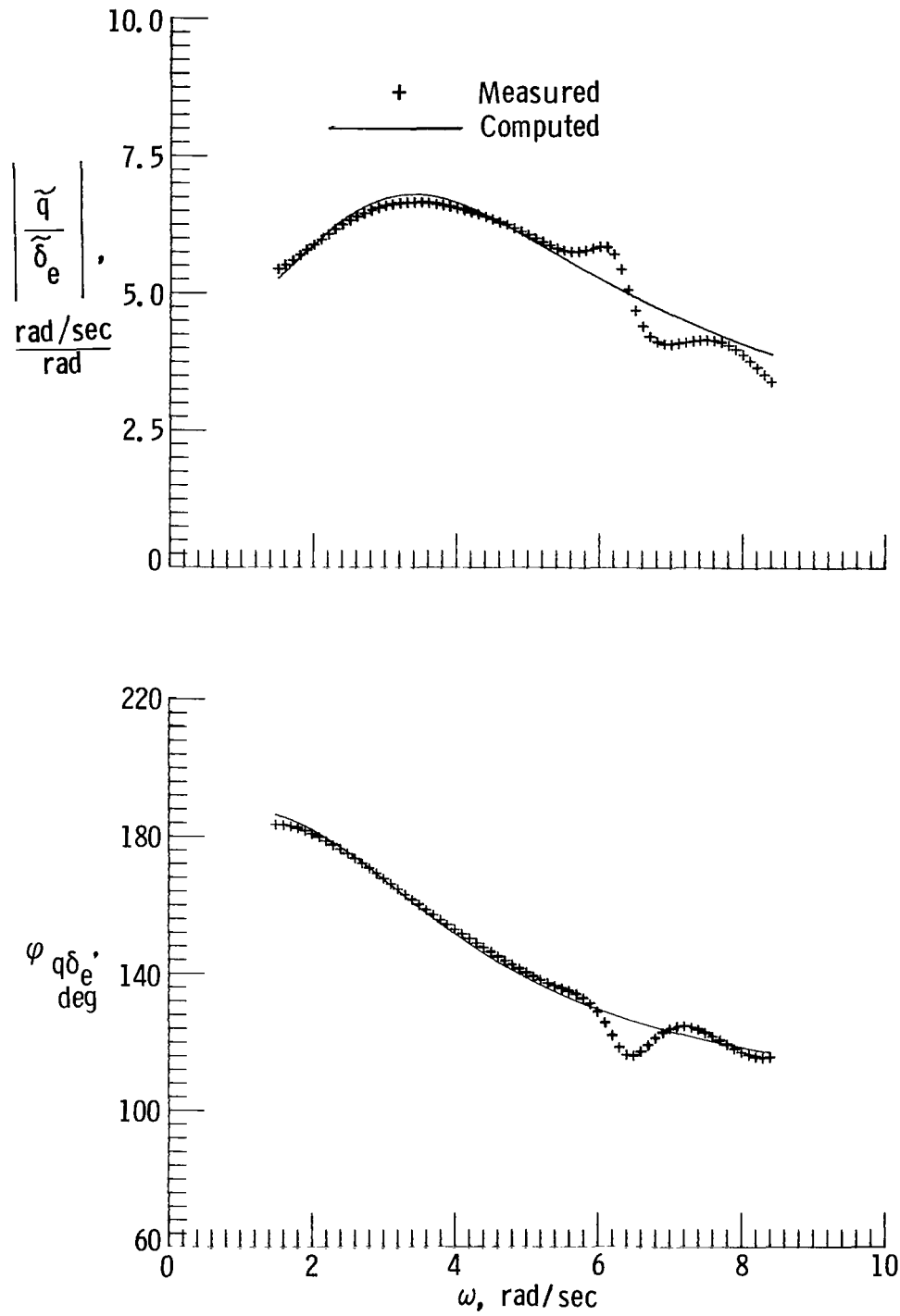
Figure 5.- Continued.
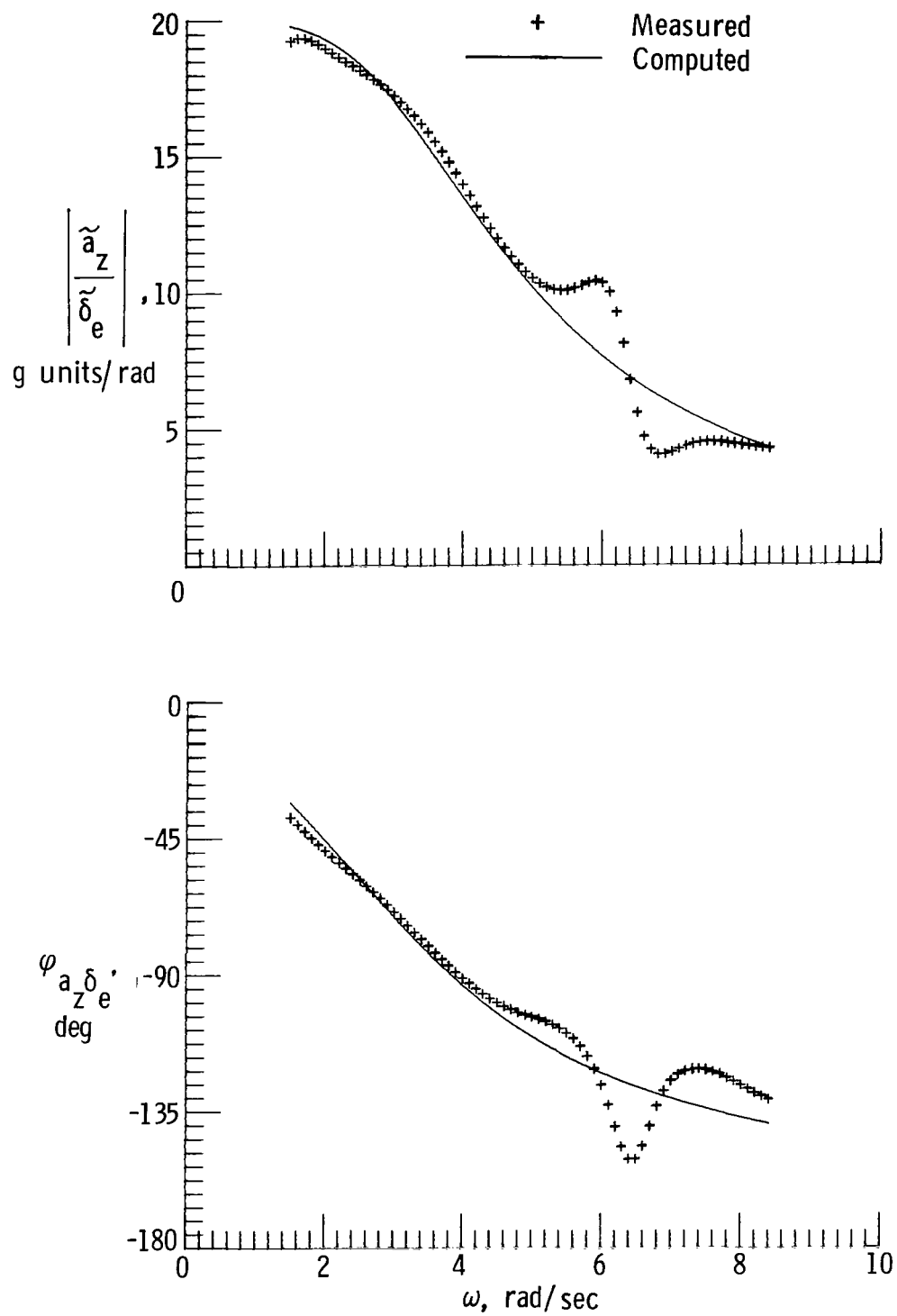
Figure 5.- Concluded.
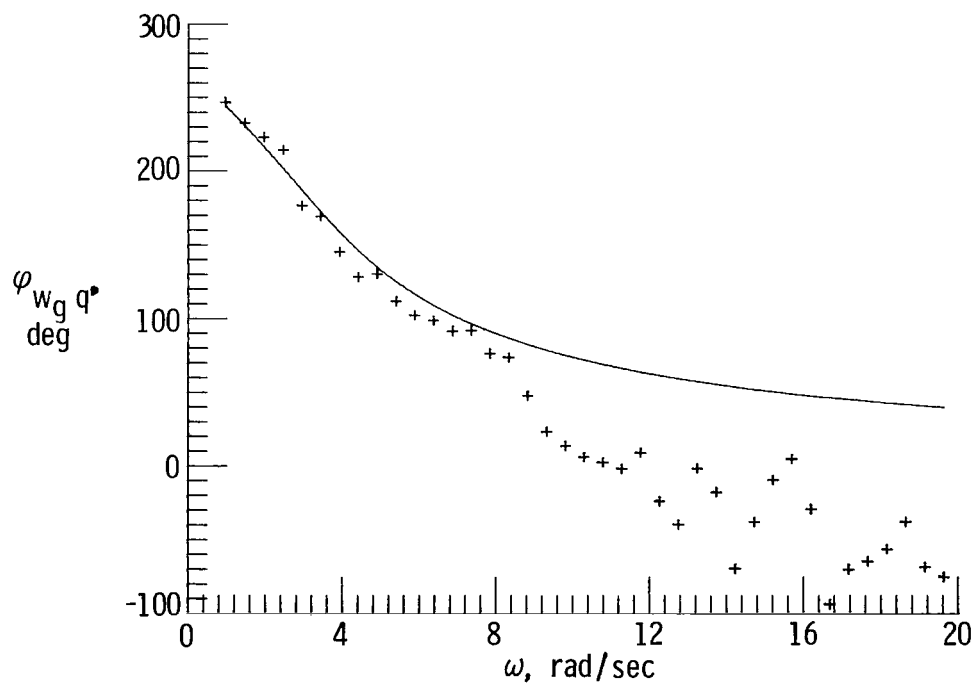
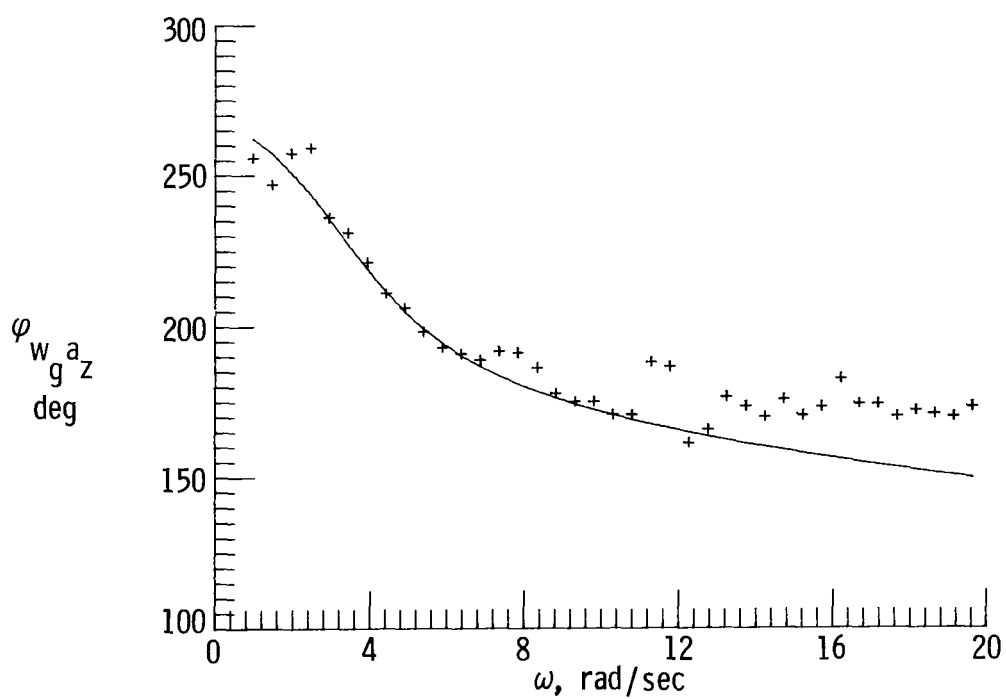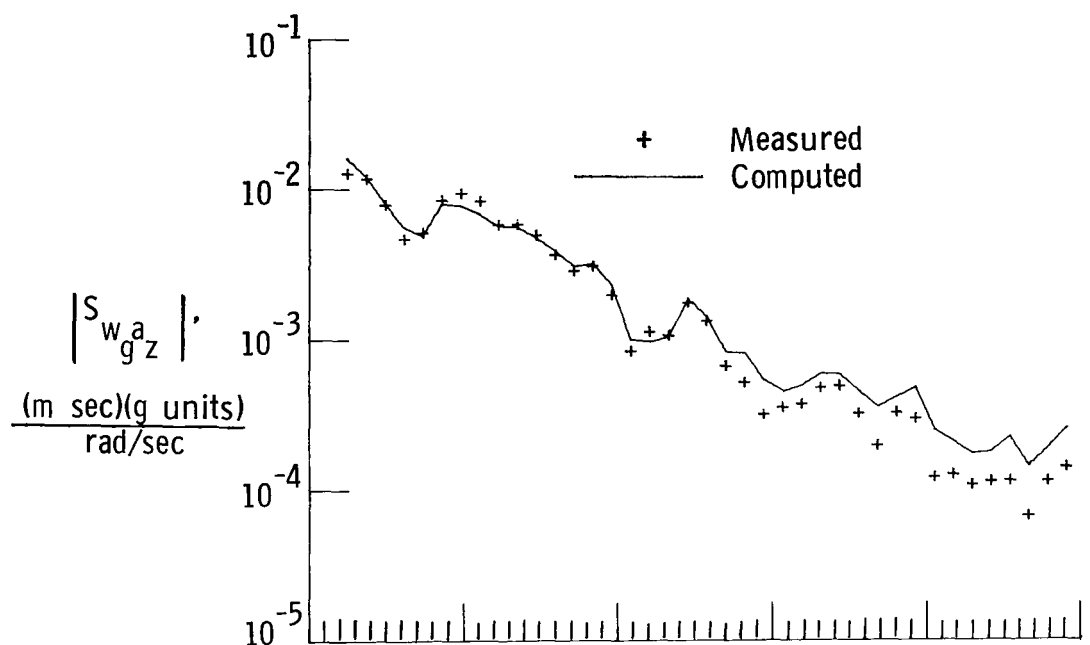Figure 6.- Measured cross-spectral and spectral densities and those computed by using estimated parameters.

Figure 6.- Continued.

Figure 6.- Concluded.
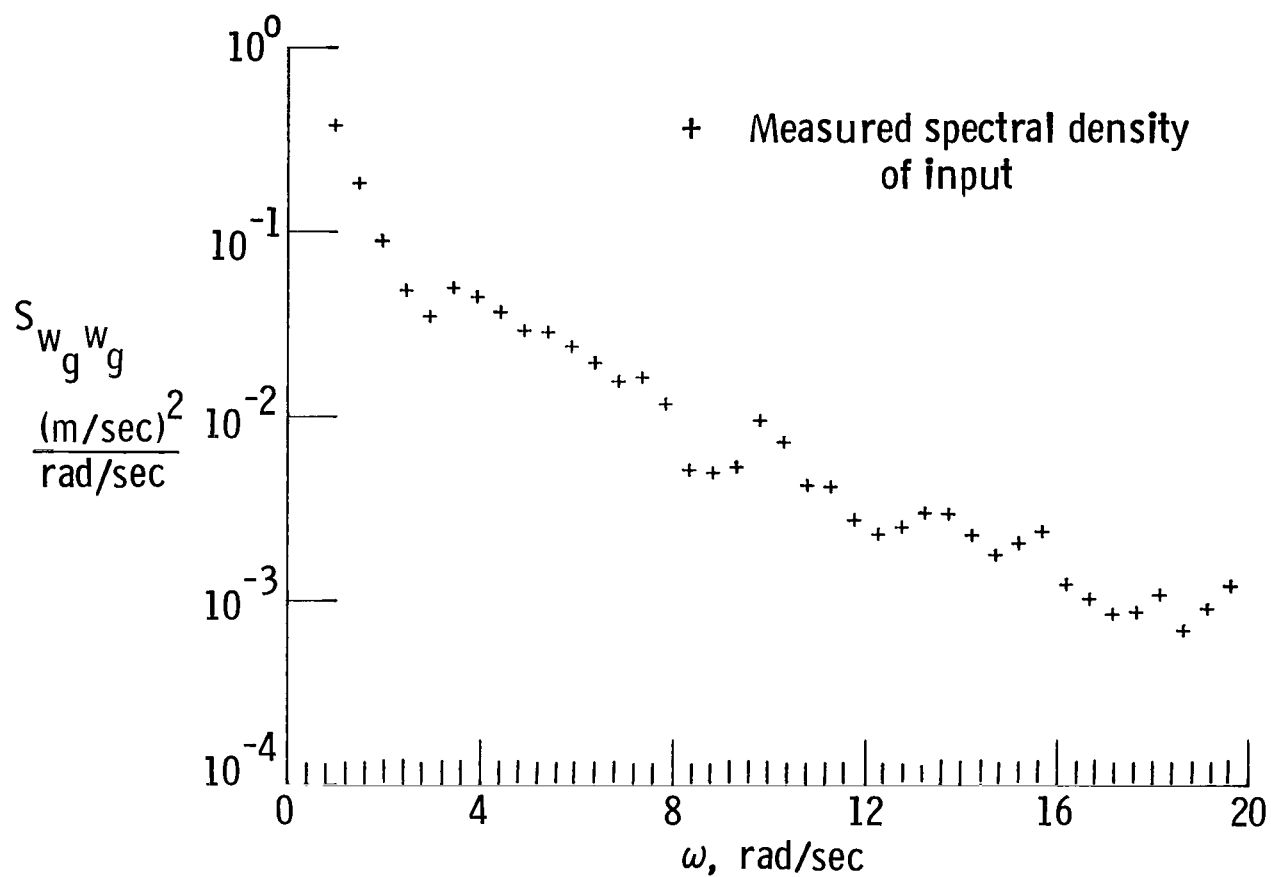
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15. Supplementary Notes

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16. Abstract

A frequency domain maximum likelihood method is developed for the estimation of airplane stability and control parameters from measured data. The model of an airplane is represented by a discrete-type steady-state Kalman filter with time variables replaced by their Fourier series expansions. The likelihood function of innovations is formulated, and by its maximization with respect to unknown parameters the estimation algorithm is obtained. This algorithm is then simplified to the output error estimation method with the data in the form of transformed time histories, frequency response curves, or spectral and cross-spectral densities. The development is followed by a discussion on the equivalence of the cost function in the time and frequency domains, and on advantages and disadvantages of the frequency domain approach. The algorithm developed is applied in four examples to the estimation of longitudinal parameters of a general aviation airplane using computer-generated and measured data in turbulent and still air. The cost functions in the time and frequency domains are shown to be equivalent; therefore, both approaches are complementary and not contradictory. Despite some computational advantages of parameter estimation in the frequency domain, this approach is limited to linear equations of motion with constant coefficients.

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