Research on the Application of a Decoupling Algorithm

for Structure Analysis

Final Report

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Symbols

A	$mn \times mn$ matrix
$A(\lambda)$	$\lambda I - A$
$A(\lambda)$	lambda matrix or matrix polynomial
J	Jordan form matrix
J_i	pseudo-Jordan block
$\tilde{J}_i$	$n \times n$ matrix of Jordan blocks
P_{i0}	primary eigenprojector or matrix residue of $[A(\lambda)]^{-1}$
$\hat{P}_{i0}$	primary latent projector or matrix residue of $[A(\lambda)]^{-1}$
P_{ij}	$j>0$ secondary eigenprojector
$\hat{P}_{ij}$	$j>0$ secondary latent projector
A_i	coefficients of lambda matrix or matrix polynomial
q	number of pseudo-Jordan blocks in A
r_i	multiplicity of repeated eigenvalue λ_i
ℓ_i	number of generalized eigenvectors for a repeated eigenvalue λ_i
S	sign matrix
F_{i0}	$mn \times mn$ matrix with sign{diag[0 0 ... abs(λ_i) ... 0]}
F_{ij}	$mn \times mn$ matrix of superdiagonal elements for J_i
R_i	ith solvent
E	diagonal matrix with ± 1 on diagonals
Q	right eigenvector matrix
P^+	positive projector equal to sum of P_{i0} with $Re(\lambda_i)>0$
P^-	negative projector equal to sum of P_{i0} with $Re(\lambda_i)<0$
I	identity matrix
y_i	right eigenvector for λ_i
z_i	left eigenvector for λ_i

$\hat{y}_i$	right latent vector for λ_i
$\hat{z}_i$	left latent vector for λ_i
T	transformation matrix
ρ	scalar parameter
adj	adjoint of matrix
Badj	block adjoint of block matrix
det	determinant of matrix
Tr	trace of matrix
$\cdot\rangle\langle\cdot$	outer product
$\langle\cdot,\cdot\rangle$	inner product
s	Laplace variable
$\Theta(t,0)$	state transition matrix

Abstract

The mathematical theory for decoupling mth-order matrix differential equations is presented. It is shown that the decoupling procedure can be developed from the algebraic theory of matrix polynomials. The report discusses the role of eigenprojectors and latent projectors in the decoupling process and develops the mathematical relationships between eigenvalues, eigenvectors, latent roots and latent vectors. It is shown that the eigenvectors of the companion form of a matrix contains the latent vectors as a subset. The spectral decomposition of a matrix and the application to differential equations is given.

1. Introduction

The purpose of the material in this report is to formulate the algebraic theory of systems and application to spectral decomposition and decoupling of differential equations. The relationship between eigenvalues, eigenvectors, latent roots and latent vectors of matrix polynomials will be given. Since most of the equations of motion of vibrating systems are cast in second-order form, the algebraic properties of second-order matrix polynomials have an important role in the determination of solutions of vibration problems. Although the mathematical development will be in general form, the analysis includes second-order matrix polynomials.

The concept of scalar residues is well understood in complex variable theory and the inversion of Laplace transforms. The theory and use of matrix residues is not widely used and the relationship to eigenvectors and latent vectors has received little attention. Matrix residues, eigenprojectors and latent projectors are useful in analyzing matrix polynomials and time domain solutions to differential equations. Several papers have been published in recent years on matrix polynomials, see Dennis, Traub and Weber, [1], as well as a short paper by Denman, [2]. Some material on matrix residues has been given by Zadeh and Desoer, [3], and on projectors, Cullen [4]. Lancaster's book, [5], is an excellent source on latent roots and latent vectors of matrix polynomials which he denotes as lambda matrices.

It will be shown that eigenvalues and eigenvectors of the matrix companion form and latent roots and latent vectors of a matrix polynomial are related. The matrix residues of the inverse of a matrix polynomial $\Lambda(\lambda) = I\lambda^m+A_1\lambda^{m-1}+\ldots+A_m$, which will be called latent projectors, are submatrices of the matrix residues of the inverse of $(\lambda I-A_c)$ where A_c is in block companion form. The latter residues will be referred to as eigenprojectors. It will be shown that latent projectors and eigenprojectors are useful for solving simultaneous differential equations.

The theory of Laplace transforms will be useful in introducing the concepts that are to follow. The material on Laplace transforms in most textbooks is limited to scalar problems and functions which is unfortunate since modern engineering problems are likely to be formulated as matrix problems due to the complexities of the systems to be analyzed. The extension of Laplace transforms to matrix functions is a simple task provided that the development of Laplace theory is based on algebraic functions rather than scalar functions.

Let f(t) be a scalar function for which the one-sided Laplace transform is given by

$$(1.1) \qquad L[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

with the usual assumption that the integral exists. The inverse transform is defined as

$$(1.2) \qquad L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds$$

where c is properly defined to enclose all singularities of the integrand. If F(s) has the property that

$$(1.3) \qquad \lim_{s\to\infty} |F(s)| = 0$$

the inverse transform of F(s) is

$$(1.4) \qquad L^{-1}[F(s)] = f(t) = \sum_{i=1}^{n} \text{residues of } [F(s)e^{st}]\Big|_{s=s_i} = \sum_{i=1}^{n} P_{i0} e^{s_i t}$$

where F(s) is a ratio of two scalar polynomials.

The Laplace transform method is valid for vector and matrix functions provided that certain restrictions are satisfied. Let A(s) be the matrix polynomial

$$(1.5) \qquad A(s) = Is^m + A_1 s^{m-1} + \ldots + A_m$$

where all coefficients A_i are n×n. The inverse of A(s) is in general form

$$(1.6) \qquad [A(s)]^{-1} = \frac{\text{adj}[A(s)]}{\det[A(s)]} = \frac{B(s)}{d(s)}$$

with adj the adjoint and det the determinant of A(s) respectively. The characteristic equation of A(s) is given by d(s) = det A(s) and will have a maximum of mn roots, Lancaster [5] calls these latent roots. The inverse transform of $F(s) = [A(s)]^{-1}$ when the roots are distinct are

$$(1.7) \qquad L^{-1}[F(s)] = f(t) = L^{-1}[(A(s))^{-1}] = \sum_{i=1}^{mn} \text{residues of } [F(s)e^{st}\Big|_{s=s_i}$$

which can be expressed as the matrix analog to (1.4) with

$$(1.8) \qquad f(t) = \sum_{i=1}^{mn} \hat{P}_{i0} \exp(s_i t)$$

where the matrices $\hat{P}_{i0}$ are matrix residues or latent projectors. It is obvious that the latent projectors are coefficients of the partial fraction expansion

$$(1.9) \qquad [A(s)]^{-1} = \sum_{i=1}^{mn} \frac{P_{i0}}{s-s_i}.$$

The usefulness of the above approach to Laplace transforms can be illustrated by considering n simultaneous differential equations of m-th order, i.e.

$$(1.10) \qquad A_0 \frac{d^m x}{dt^m} + A_1 \frac{d^{m-1} x}{dt^{m-1}} + \ldots + A_m x = 0 \qquad x(0) = c$$

where x(t) is a nth-order vector. It follows that

$$(1.11) \qquad A(s)X(s) = A_m c \qquad \dot{x}(0) = \ddot{x}(t_0) = \ldots x^{(m)}(0) = 0$$

or

$$(1.12) \qquad X(s) = [A(s)]^{-1} A_m c.$$

The time domain solution to (1.10) is then given by

$$(1.13) \qquad x(t) = \sum_{i=1}^{mn} \hat{P}_{i0} A_m c \exp(s_i t)$$

when the latent roots are distinct.

The matrix polynomial given in (1.5) can also arise from the canonical form or companion matrix. It is not difficult to show that the matrix $A(\lambda)$ given by

$$(1.14)\qquad \mathcal{A}(\lambda) = \begin{bmatrix} \lambda I & -I & 0 & \cdots & 0 & 0 \\ 0 & \lambda I & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda I & -I \\ A_m & A_{m-1} & A_{m-2} & \cdots & A_2 & \lambda I + A_1 \end{bmatrix}$$

has the same characteristic equation as (1.5) when $A_0 = I$. The root of the characteristic equation obtained from $\det[\mathcal{A}(\lambda)] = 0$ will be the eigenvalues of $\mathcal{A}$ and are equal to the latent roots of $A(\lambda)$ when $A_0 = I$. The eigenvectors of $\mathcal{A}$ must be related to the latent vectors of $A(\lambda)$; that relationship will be given later.

If $z(t)$ is defined as the vector

$$(1.15)\qquad [z(t)]^T = \{[x(t)]^T\ [\dot{x}(t)]^T\ \ldots\ [x^{(m)}(t)]^T\}$$

then $z(t)$ satisfies the equation

$$(1.16)\qquad \dot{z}(t) = \mathcal{A}\, z(t)$$

with

$$(1.17)\qquad Ix^{(m)}(t) + A_1 x^{(m-1)}(t) + \ldots + A_m x(t) = 0.$$

The solution vector $z(t)$ is given by

$$(1.18)\qquad z(t) = \sum_{i=1}^{mn} P_{i0}\ C \exp(\lambda_i t)$$

where P_{i0} are the matrix residues of $[A(\lambda)]^{-1}$ which will be called eigenprojectors. It is assumed that (1.18) is for distinct eigenvalues of A. Since the eigenvalues of A and the latent roots of $A(\lambda)$ are the same and the eigenvectors of A and the latent vectors of $A(\lambda)$ are related, the eigenprojectors P_{i0} and the latent projectors $\hat{P}_{i0}$ must be related. The vector C in (1.18) is obtained from (1.12) and the definition of the canonical form for the system.

2. Eigenprojectors of Matrices

Let A be defined as a $mn \times mn$ matrix with eigenvalues λ_i, right eigenvectors y_i and left eigenvectors z_i. Define Q as a $mn \times mn$ matrix constructed from the eigenvectors y_i such that

$$(2.1) \qquad Q = [y_1\ y_2\ y_3 \ldots y_{mn}]$$

where it is assumed that the columns of Q are linearly independent and spans the $C^{mn \times mn}$ space. The matrix Q has the property that a similarity transformation on A with Q will reduce A to the Jordan form

$$(2.2) \qquad J = Q^{-1}AQ$$

with $J = \mathrm{diag}[J_1, J_2 \ldots J_p]$ with J_i a Jordan block. The Jordan form will be diagonal if A has mn distinct eigenvalues or if A has mn linearly independent eigenvectors satisfying $[\lambda_i I - A]y_i = 0$. If A has repeated eigenvalues and is defective, the Jordan blocks leading to the defectiveness of A will have one or more plus ones on the superdiagonal on a Jordan block. It will be necessary to utilize the chain rule for generating the generalized eigenvectors for the defective Jordan block.

The mathematical analysis will be simplified if all eigenvalues with the same values are considered as a pseudo-Jordan block with the plus ones on the superdiagonal. Assume that J_i and J_{i+1} are as shown with J_{i+1} having the same eigenvalues as J_i but where J_{i+1} has the plus ones on the superdiagonal. Although the definition

$$(2.3)\qquad \begin{bmatrix} J_i & 0 \\ 0 & J_{i+1} \end{bmatrix} = \begin{bmatrix} \lambda_i & 0 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 \\ 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \lambda_i \end{bmatrix}$$

is not conventional. The two blocks J_i and J_{i+1} will be considered as a pseudo-Jordan block.

Let A have q values o. λ_i with q pseudo-Jordan blocks as defined. Assume that mn-k of the eigenvalues are distinct and q-mn+k are repeated. Each repeated eigenvalues will have multiplicity r_i and the number of generalized eigenvectors for the repeated eigenvalues will be ℓ_i, the number of plus ones on the superdiagonal of J_i. It will be assumed that the ones are located in the last ℓ_i rows of the repeated eigenvalue pseudo-Jordan block. The term pseudo-Jordan block will be dropped in the following discussion and the term Jordan block will be utilized with J denoting a pseudo-Jordan block.

In addition to the above assumptions, let F_{ij} denote a mn×mn matrix. The first subscript denotes the eigenvalue to which the F_{ij} matrix belongs and the second subscript is an index which has a maximum value equal to the number of generalized eigenvectors required for the eigenvalue λ_i; this will be ℓ_i. If ℓ_i is the number of generalized eigenvectors for λ_i, then $j = 0,1,2,\ldots,\ell_i$ with $j = 0$ for zero generalized eigenvectors. If λ_i is a distinct eigenvalue, then F_{i0} will be defined as

$$(2.4)\qquad F_{i0} = \mathrm{diag}[0,0,\ldots 0,1,0\ldots 0]$$

with the one located in the same row and column as λ_i is in J. The matrix

F_{i0} for a repeated eigenvalue λ_i with multiplicity $r_i = 3$ will have the form

$$(2.5) \qquad F_{i0} = \mathrm{diag}[0,0,\ldots 0,1,1,1,0\ldots 0]$$

with the one located in the same rows and columns as λ_i is in J.

To complete the definition of F_{ij}, assume that λ_i has multiplicity r_i with $r_i - \ell_i$ linearly independent eigenvectors and ℓ_i generalized eigenvectors. The associated Jordan block will have ℓ_i ones on the superdiagonal with the ones located in the last ℓ_i rows of J_i. The matrix F_{i0} will be as given in (2.5) but the set of matrices $F_{i1}, F_{i2}, \ldots F_{i\ell_i}$ will now exist with F_{i1} having only the ones of the superdiagonal of J_i located on the superdiagonal of F_{i1}. The next matrix in the sequence F_{ij} will be generated by moving the ones on the superdiagonal of F_{i1} up one diagonal position by moving to the next columns of F_{i1}. To illustrate the construction of F_{ij}, let J be defined as in (2.3), then F_{i0}, F_{i1} and F_{i2} are

$$(2.6) \qquad F_{i0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad F_{i1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad F_{i2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenprojectors of A, [6], can now be defined using the established notation. Assume that A has mn-k distinct eigenvalues and q-mn+k repeated eigenvalues. The primary eigenprojectors will be defined as

$$(2.7) \qquad P_{i0} = Q\, F_{i0}\, Q^{-1} \qquad i = 1,2,\ldots q$$

with the primary eigenprojectors having the properties

(2.8a) $$\sum_{i=1}^{q} P_{i0} = I$$

(2.8b) $$P_{i0}P_{i0} = P_{i0}$$

(2.8c) $$P_{i0}P_{j0} = 0 \qquad i \neq j$$

These properties follow directly from (2.6) since $\sum_{i=1}^{q} F_{i0} = I$, $F_{i0}F_{i0} = F_{i0}$ and $F_{i0}F_{j0} = 0$. The primary eigenprojectors are idempotent matrices, i.e. $P_{i0}^{\alpha} = P_{i0}$ where α is a positive integer.

The spectral decomposition for A which is not defective is given by

(2.9) $$A = \sum_{i=1}^{q} \lambda_i P_{i0}$$

which follows directly from the definition of the primary eigenprojectors.

If A is defective, a set of secondary eigenprojectors will be defined as the projectors constructed from the eigenvectors and the sequence of F_{ij} matrices. Let the secondary eigenprojectors be defined by

(2.10) $$P_{ij} = Q\, F_{ij}\, Q^{-1} \qquad j = 1,2,\ldots \ell_i$$

with i set by the associated eigenvalue which is repeated but defective. It follows from the definition of F_{ij} that

(2.11a) $$P_{i0}P_{ij} = P_{ij} \qquad j = 1,2,\ldots \ell_i$$

(2.11b) $$P_{ij}P_{ij} = P_{i,j+1} \qquad j \neq 0$$

(2.11c) $$P_{ij}P_{i,j+1} = 0 \qquad j \neq 0$$

The secondary eigenprojectors P_{i1} are required for the spectral decomposition of the most general matrix A.

If A has general form and is defective then

(2.12) $$A = Q\ J\ Q^{-1} = Q[\Lambda + \sum_{i=1}^{q} F_{ir}]Q^{-1}$$

but this is equal to

(2.13) $$A = Q\{\sum_{i=1}^{q} [F_{i0}\lambda_i + F_{i1}]\}Q^{-1}$$

or finally

(2.14) $$A = \sum_{i=1}^{q} [P_{i0}\lambda_i + P_{i1}]$$

Although the secondary eigenprojectors P_{ij} with $j>1$ are not necessary for the spectral decomposition, it will be shown later that the partial traction expansion of $[A(\lambda)]^{-1} = [\lambda I - A]^{-1}$ can be expressed in terms of the eigenprojectors.

The procedure given for computing the eigenprojectors for a matrix has been based on the assumption that the right eigenvector matrix, Q, is known completely. The inverse of Q will have row vectors that are the left eigenvecto. of A thus P_{i0} depends on the right and left eigenvectors, y_i and z_i respectively. The right and left eigenvectors for distinct eigenvalues are determined from the equations

(2.15) $$[\lambda_i I - A]y_i = 0$$

(2.16) $$z_i[\lambda_i I-A] = 0$$

or equivalently $[\lambda_i I-A^T]z_i^T = 0$ for the left eigenvectors. If A is defective for an eigenvalue λ_i, the chain rules, [7]

(2.17) $$[\lambda_i I-A]y_i^{k+1} = -y_i^k \qquad k = 1,2,\ldots \ell_i$$

(2.18) $$[\lambda_i I-A^T](z_i^{k+1})^T = -(z_i^k)^T \qquad k = 1,2,\ldots \ell_i$$

are used for the generalized eigenvectors where y_i^1 and z_i^1 are any one of the linearly independent eigenvectors for the repeated eigenvalues.

The primary eigenprojectors P_{i0} were defined earlier and are given by

(2.19) $$P_{i0} = Q\, F_{i0}\, Q^{-1}$$

Let $Q = Q_r$ be the matrix of right eigenvectors and Q_ℓ be the matrix of left eigenvectors for distinct eigenvalues with

(2.20) $$Q_r = [\bar{y}_1\ \bar{y}_2\ \ldots\ \bar{y}_{mn}] \qquad Q_\ell = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_{mn} \end{bmatrix}$$

and let Q_ℓ be scaled such that $Q_\ell Q_r = I$. It then follows for a distinct eigenvalue λ_i that

(2.21) $$P_{i0} = Q_r\, F_{i0}\, Q_\ell = \bar{y}_i{>}{<}\bar{z}_i = \bar{y}_i\bar{z}_i\ .$$

Since $Q_r F_{i0} Q_\ell$ will be given by the outer product of $\bar{y}_i$ and $\bar{z}_i$, the scaling of Q_ℓ is equivalent to scaling $\bar{y}_i$ or $\bar{z}_i$ such that $\langle \bar{z}_i \bar{y}_i \rangle = \bar{z}_i \cdot \bar{y}_i = 1$. Since any eigenvector can be multiplied by a constant then any set of arbitrary scaled eigenvectors can be used to compute P_{i0} provided that the arbitrary constants are removed by dividing by the scaling factor. If y_i and z_i are arbitrary eigenvectors the eigenprojectors for the distinct eigenvalues are given by

$$(2.22) \qquad P_{i0} = \frac{y_i z_i}{z_i y_i}$$

The eigenprojectors for the repeated eigenvalues when A is not defective are determined from a simple extension of (2.21). Since the eigenvectors are linearly independent, P_{i0} is given by

$$(2.23) \qquad P_{i0} = Q_r F_{i0} Q_\ell = \sum_{j=1}^{r_i} \bar{y}_i^j \bar{z}_i^j = \sum_{j=1}^{r_i} \frac{y_i^j z_i^j}{z_i^j y_i^j}$$

where the superscript denotes the jth eigenvector belonging to the pseudo-Jordan block.

The eigenprojectors for the repeated eigenvalues when A is defective are computed in a similar manner to the repeated eigenvalues for the nondefective case. The primary eigenprojector for a defective Jordan block is given by (2.23). The secondary eigenprojectors can be computed from the eigenvectors by considering (2.10). Let $\hat{F}_{ij}$ denote the subblock of F_{ij} for a Jordan block J_i where $\hat{F}_{ij}$ is $r_i \times r_i$. Let $\bar{Y}_i$ and $\bar{Z}_i$ denote the rectangular matrices of right and left eigenvectors respectively of λ_i. Equation (2.10) can be rewritten as

(2.24) $$P_{ij} = \overline{Y}_i \hat{F}_{ij} \overline{Z}_i = [\overline{y}_i^1 \overline{y}_i^2 \ \dots \ \overline{y}_i^{r_i}] \hat{F}_{ij} \begin{bmatrix} \overline{z}_i^1 \\ \overline{z}_i^2 \\ \vdots \\ \overline{z}_i^{r_i} \end{bmatrix}$$

where Q_r and Q_ℓ have been properly scaled. Let f_{st} denote the elements of $\hat{F}_{ij}$ with $f_{st} = 1$ or 0 depending on the ith row and jth column of $\hat{F}_{ij}$. Equation (2.24) can then be written as

(2.25) $$P_{ij} = \sum_{s}^{r_i} \sum_{t}^{r_i} \overline{y}_i^s \, f_{ij} \, \overline{z}_i^t = \sum_{s=1}^{r_i} \sum_{t=1}^{r_i} \left[\frac{y_i^s \, f_{st} \, z_i^t}{z_i^s \, y_i^s}\right]$$

Since $\hat{F}_{ij}$ will be sparse, only a few terms of the summation are required.

An example will now be given to illustrate the computational procedure using the eigenprojectors. Let A be given by

$$A = \frac{1}{2} \begin{bmatrix} 5 & -1 & 0 & -2 \\ 0 & 4 & -1 & -1 \\ -2 & 0 & 5 & -1 \\ -1 & -1 & -2 & 6 \end{bmatrix}$$

with Jordan form

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \begin{array}{ll} \lambda_1 = 1 & \\ \lambda_2 = 3 & \\ r_2 = 3 & \ell_2 = 2 \end{array}$$

The right eigenvector matrix Q_r is

$$Q_r = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = [y_1\ y_2^1\ y_2^2\ y_2^3]$$

and the left eigenvector matrix by

$$Q_\ell = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2^1 \\ z_2^2 \\ z_2^3 \end{bmatrix}$$

The eigenprojector P_{10} is found from y_1 and z_1 and is

$$P_{10} = \frac{y_1\ z_1}{z_1\ y_1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \hat{F}_{10} = 1$$

with the primary eigenprojector P_{20} for $\lambda_2 = 3$ given by

$$P_{20} = \sum_{j=1}^{z} \frac{y_2^j\ z_2^j}{z_2^j\ y_2^j} = \frac{y_2^1\ z_2^1}{z_2^1\ y_2^1} + \frac{y_2^2\ z_2^2}{z_2^2\ y_2^2} + \frac{y_2^3\ z_2^3}{z_2^3\ y_2^3}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \qquad \hat{F}_{20} = I_{3\times 3}$$

As a check on the two eigenprojectors, note that $P_{10}+P_{20} = I$. The first secondary eigenprojector P_{21} is given by

$$P_{21} = \sum_{s=1}^{z} \sum_{t=1}^{z} \frac{y_2^s\, f_{ij}\, z_2^t}{z_2^s\, y_2^t} = \frac{y_2^1\, z_2^2}{z_2^1\, y_2^1} + \frac{y_2^2\, z_2^3}{z_2^2\, y_2^2}$$

$$= \frac{1}{4}\begin{bmatrix} 0 & 0 & 2 & -2 \\ 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix} \qquad f_{12} = 1 \qquad f_{23} = 1$$

The second secondary eigenprojector is determined from (2.25) with $f_{13} = 1$ as the only nonzero element in $\hat{F}_{22}$. The eigenprojector P_{22} is then

$$P_{22} = \frac{y_2^1\, z_2^3}{z_2^1\, y_2^1} = \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

A check will show that $P_{20}\, P_{21} = P_{21}, P_{20}\, P_{22} = P_{22}$ and $P_{21}P_{22} = 0$.

The spectral decomposition of A is given by $\lambda_1 P_{10} + \lambda_2 P_{20} + P_{21}$ which is

$$A = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 & 0 & 2 & -2 \\ 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 10 & -2 & 0 & -4 \\ 0 & 8 & -2 & -2 \\ -4 & 0 & 10 & -2 \\ -2 & -2 & -4 & 12 \end{bmatrix}$$

which agrees with A as given.

The eigenprojectors can also be computed from the inverse of $A(\lambda)$ where $A(\lambda) = \lambda I - A$. The most general partial fraction expansion of $[A(\lambda)]^{-1}$ is given by

$$(2.26) \qquad [A(\lambda)]^{-1} = \sum_{i=1}^{q} \{\frac{P_{i0}}{\lambda-\lambda_i} + \sum_{j=1}^{\ell_i} \frac{P_{ij}}{(\lambda-\lambda_i)^{j+1}}\}$$

The three cases, distinct eigenvalue, repeated eigenvalues with multiplicity r_i but not defective and the defective matrix must be discussed.

The three cases can be analyzed by considering the Jordan blocks for the three different eigenvalue cases. Rather than consider the mixed Jordan forms, consider the three individual Jordan forms. Assume first that $J = \mathrm{diag}[\lambda_1\ \lambda_2\ \lambda_3]$ thus

$$(2.27) \qquad A = QJQ^{-1} = Q \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} Q^{-1}$$

with $[A(\lambda)]^{-1}$ given by

$$(2.28) \qquad [A(\lambda)]^{-1} = [\lambda I - A]^{-1} = Q \begin{bmatrix} (\lambda-\lambda_1)^{-1} & 0 & 0 \\ 0 & (\lambda-\lambda_2)^{-1} & 0 \\ 0 & 0 & (\lambda-\lambda_3)^{-1} \end{bmatrix} Q^{-1}$$

It follows directly from $(\lambda-\lambda_i)\ [A(\lambda)]^{-1}$ evaluated at λ_i that

$$(2.29) \qquad (\lambda-\lambda_i)[\lambda I-A]^{-1}\Big|_{\lambda=\lambda_i} = Q\ F_{i0}\ Q^{-1} = P_{i0}$$

Consider the Jordan form with λ_i repeated 3 times but where A is not defective. The inverse of $A(\lambda)$ is

$$\text{(2.30)} \qquad [A(\lambda)]^{-1} = [\lambda I - A]^{-1} = Q \begin{bmatrix} (\lambda-\lambda_1)^{-1} & 0 & 0 \\ 0 & (\lambda-\lambda_1)^{-1} & 0 \\ 0 & 0 & (\lambda-\lambda_1)^{-1} \end{bmatrix} Q^{-1}$$

Since λ_i has multiplicity 3 then $(\lambda-\lambda_1)^3[A(\lambda)]^{-1}$ will be

$$\text{(2.31)} \qquad (\lambda-\lambda_1)^3[A(\lambda)]^{-1} = Q \begin{bmatrix} (\lambda-\lambda_1)^2 & 0 & 0 \\ 0 & (\lambda-\lambda_1)^2 & 0 \\ 0 & 0 & (\lambda-\lambda_1)^2 \end{bmatrix} Q^{-1}$$

It is obvious that (2.31) will be zero when evaluated at $\lambda = \lambda_1$. The first derivative of (2.31) with respect to λ will also be zero at $\lambda = \lambda_1$ with the second derivative of (2.31) given by

$$\text{(2.32)} \qquad \frac{d^2}{d\lambda^2} \{(\lambda-\lambda_1)^3[A(\lambda)]^{-1}\} = Q \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} Q^{-1} = 2QF_{10}Q^{-1}$$

The eigenprojector for the repeated eigenvalue matrix with multiplicity r_i but for A not defective is given by

$$\text{(2.33)} \qquad P_{i0} = QF_{i0}Q^{-1} = \frac{1}{(r_i-1)!} \frac{d^{r_i-1}}{d\lambda^{r_i-1}} \{(\lambda-\lambda_i)^{r_i}[A(\lambda)]^{-1}\}\Big|_{\lambda=\lambda_i}$$

The defective matrix will be analyzed by considering the Jordan block with λ_1 of multiplicity 3 and 2 generalized eigenvectors. Let $[A(\lambda)]^{-1}$ be given by

$$\text{(2.34)} \qquad [A(\lambda)]^{-1} = Q \begin{bmatrix} \lambda-\lambda_1 & -1 & 0 \\ 0 & \lambda-\lambda_1 & -1 \\ 0 & 0 & \lambda-\lambda_1 \end{bmatrix}^{-1} Q^{-1} = Q \begin{bmatrix} (\lambda-\lambda_1)^{-1} & (\lambda-\lambda_1)^{-2} & (\lambda-\lambda_1)^{-3} \\ 0 & (\lambda-\lambda_1)^{-1} & (\lambda-\lambda_1)^{-2} \\ 0 & 0 & (\lambda-\lambda_1)^{-1} \end{bmatrix} Q^{-1}$$

The sequence of evaluations of $(\lambda-\lambda_1)^3\,[A(\lambda)]^{-1}$ and the derivatives with respect at $\lambda = \lambda_1$ will be

$$(2.35)\qquad (\lambda-\lambda_1)^3[A(\lambda)]^{-1}\Big|_{\lambda=\lambda_1} = Q\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}Q^{-1} = QF_{12}Q^{-1} = P_{12}$$

$$(2.36)\qquad \frac{d}{d\lambda}\{(\lambda-\lambda_1)^3[A(\lambda)]^{-1}\}\Big|_{\lambda=\lambda_1} = Q\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}Q^{-1} = QF_{11}Q^{-1} = P_{11}$$

$$(2.37)\qquad \frac{1}{2}\,\frac{d^2}{d\lambda^2}\{(\lambda-\lambda_1)^3[A(\lambda)]^{-1}\}\Big|_{\lambda=\lambda_1} = Q\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}Q^{-1} = QF_{10}Q^{-1} = P_{10}$$

The computational procedure for finding the eigenprojectors, usually referred to as matrix residues in the above, from the partial fraction expansion can now be summarized. If λ_i is distinct then

$$(2.38)\qquad P_{i0} = (\lambda-\lambda_i)[A(\lambda)]^{-1}\Big|_{\lambda=\lambda_i}$$

with the primary eigenprojectors for the repeated eigenvalues λ_i of multiplicity r_i given by

$$(2.39)\qquad P_{i0} = \frac{1}{(r_i-1)!}\,\frac{d^{r_i-1}}{d\lambda^{r_i-1}}\{(\lambda-\lambda_i)^{r_i}[A(\lambda)]^{-1}\}\Big|_{\lambda=\lambda_i}$$

The secondary eigenprojectors are defined only for repeated eigenvalues with multiplicity r_i with A defective and requiring ℓ_i generalized eigenvectors. The secondary eigenprojectors are given by

$$P_{i,\ell_i-j} = \frac{1}{j!}\frac{d^j}{d\lambda^j}\{(\lambda-\lambda_i)^{\hat{r}_i}[A(\lambda)]^{-1}\} \qquad j = 0,1,\ldots\ell_i \tag{2.40}$$

where $[A(\lambda)]^{-1}$ has all common factors of adj$[A(\lambda)]$ and det$[A(\lambda)]$ cancelled so that $[A(\lambda)]^{-1}$ is a minimum polynomial with $\hat{r}_i$ equal to the power of $(\lambda-\lambda_i)^{\hat{r}_i}$ in the denominator.

The computation of the eigenprojectors by the residue method will be illustrated with the previous example. The inverse of $[A(\lambda)]^{-1}$ is given by

$$[A(\lambda)]^{-1} = \frac{1}{(\lambda-1)(\lambda-3)^3}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\lambda^3 + \begin{bmatrix} -7.5 & -0.5 & 0 & -1 \\ 0 & -8 & -0.5 & -0.5 \\ -1 & 0 & -7.5 & -0.5 \\ -0.5 & -0.5 & -1 & -7 \end{bmatrix}\lambda^2\right.$$

$$\left. + \begin{bmatrix} 17.75 & 3.25 & 1.25 & 4.75 \\ 0.75 & 20.25 & 3.25 & 2.75 \\ 5.25 & 0.75 & 17.75 & 3.25 \\ 3.25 & 2.75 & 4.75 & 16.25 \end{bmatrix}\lambda + \begin{bmatrix} -13.25 & -4.75 & -3.25 & -5.75 \\ -2.75 & -15.25 & -4.75 & -4.25 \\ -6.25 & -2.75 & -13.25 & -4.75 \\ -4.75 & -4.25 & -5.75 & -12.25 \end{bmatrix}\right\}$$

The eigenprojector for $\lambda = 1$ is

$$P_{10} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = (\lambda-1)[A(\lambda)]^{-1}\Big|_{\lambda=1}$$

The eigenvalue $\lambda = 3$ will have a primary eigenprojector and two secondary eigenprojectors. The eigenprojectors P_{22}, P_{21} and P_{20} are

$$P_{22} = (\lambda-3)^3[A(\lambda)]^{-1}\Big|_{\lambda=3} = \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$P_{21} = \frac{d}{d\lambda}\{(\lambda-3)^3[A(\lambda)]^{-1}\}\Big|_{\lambda=3} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 2 & -2 \\ 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

$$P_{20} = \frac{1}{2}\frac{d}{d\lambda^2}\{(\lambda-3)^3[A(\lambda)]^{-1}\}\Big|_{\lambda=3} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

The eigenprojectors agree with the previous values found from the eigenvectors.

The partial fraction expansion of $[A(\lambda)]^{-1}$ is

$$[A(\lambda)]^{-1} = \frac{P_{10}}{\lambda-1} + \frac{P_{20}}{\lambda-3} + \frac{P_{21}}{(\lambda-3)^2} + \frac{P_{22}}{(\lambda-3)^3} \tag{2.41}$$

Two methods of computing the eigenprojectors (or matrix residues) have been discussed in this section. The first procedure given was based on the eigenvectors with the second method requiring the inversion of $[A(\lambda)]^{-1}$ and evaluation of the residues. It has been shown that the two methods are equivalent although the numerical computations may not necessarily be comparable.

3. Lambda Matrices and Latent Projectors

The analysis in Section 2 was based on the assumption that the A matrix was in general form with mn rows and columns. This assumption is valid when the differential equation describing the dynamics of a system are in the state variable form, first-order differential equations. The system equation may not always be in first-order form as it is common practice in some engineering disciplines to write the differential equations in mth-order form. If such is the practice, then lambda matrices will be encountered. This section considers lambda matrices or matrix polynomials, Gantmacher, [8].

Assume that $\bar{A}(\lambda)$ is a matrix polynomial in λ of mth-order with n×n coefficients of the form

$$\bar{A}(\lambda) = \bar{A}_0\lambda^m+\bar{A}_1\lambda^{m-1} +\ldots+ \bar{A}_{m-1}\lambda+\bar{A}_m \tag{3.1}$$

which Lancaster, [1], calls a lambda matrix. Dennis, Traub and Weber [3], make a distinction between A(λ) and A(X) where X is n×n by calling the latter a matrix polynomial. The polynomial in (3.1) is commonly referred to as a matrix polynomial in control theory and that designation will be followed here.

The roots of $\det[\bar{A}(\lambda)]$ are called latent roots and the vectors that satisfy $[\bar{A}(\lambda_i)]\hat{y}_i$ are referred to as latent vector. This terminology will be followed in this work to avoid confusion with eigenvalues and eigenvectors. The concept of latent projectors will be introduced in this section with the latent projectors having an analogous role to eigenprojectors. It will be assumed that a latent root may have multiplicity r_i and that $\bar{A}(\lambda)$ will be defective requiring ℓ_i generalized latent vectors where a defective lambda matrix has the same meaning as A being defective, there will not be r_i linearly latent vectors for the latent root λ_i.

If $\bar{A}_0$ is invertible, then (3.1) can be written as

$$(3.2) \qquad \overline{A}(\lambda) = \overline{A}_0[I\lambda^m + A_1\lambda^{m-1} + \ldots + A_{m-1}\lambda + A_m] = \overline{A}_0 A(\lambda)$$

where $A_i = \overline{A}_0^{-1}\overline{A}_i$. The discussion that follows will focus on $A(\lambda)$ although a complete treatment of lambda matrices should include the case when $\bar{A}_0$ is singular.

The latent roots of $A(\lambda)$ will be denoted by λ_i with the right and left latent vectors, denoted by $\hat{y}_i$ and $\hat{z}_i$ respectively. The latent vectors for the latent roots λ_i satisfy

$$(3.3a) \qquad A(\lambda_i)\hat{y}_i = 0$$

$$(3.3b) \qquad \hat{z}_i\, A(\lambda_i) = 0$$

for the right and left latent vectors respectively when λ_i is distinct or $A(\lambda_i)$ is not defective. If $A(\lambda)$ is defective for λ_i then a chain rule must be employed. Lancaster and Webber, [9], have given the chain rule as

$$(3.4) \qquad A(\lambda_i)\hat{y}_i^{\ell} + \frac{dA(\lambda_i)}{d\lambda}\hat{y}_i^{\ell-1} + \frac{1}{2}\frac{d^2A(\lambda_i)}{d\lambda^2}\hat{y}_i^{\ell-2} + \ldots + \frac{1}{\ell_i - 1}\frac{d^{\ell-1}A(\lambda_i)}{d\lambda^{\ell-1}}\hat{y}_i^1 = 0$$

with $\hat{y}_i$ as a linearly independent latent vector. If $\hat{y}_i^1$ is a linear independent latent vector than $\hat{y}^2$ is given by

$$(3.5a) \qquad A(\lambda_i)\hat{y}_i^2 = -\frac{dA(\lambda_i)}{d\lambda}\hat{y}_i^1$$

and $\hat{y}_i^3$ by

$$A(\lambda_i)\hat{y}_i^3 = -\frac{dA(\lambda_i)}{d\lambda}\hat{y}_i^2 - \frac{1}{2}\frac{d^2A(\lambda_i)}{d\lambda^2}\hat{y}_i^1 \tag{3.5b}$$

with all others computed by recursive use of (3.4). The chain rule for the left latent vector is similar except that $\hat{z}_i^j$ is a premultiplier of the terms in (3.4).

The computation of the latent vectors is described in the example. Let $A(\lambda)$ be

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\lambda^2 + \begin{bmatrix} -4.5 & 1.5 \\ 1.5 & -4.5 \end{bmatrix}\lambda + \begin{bmatrix} 5.5 & -3.5 \\ -3.5 & 5.5 \end{bmatrix}$$

which has latent roots $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ and $\lambda_4 = \lambda_3 = 3$ with $\ell_3 = 1$. Let $\lambda = 1$ then

$$A(1)\hat{y}_1 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\hat{y}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \hat{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with $\hat{y}_1$ as given for the latent vector. The latent vector for $\lambda = 2$ is obtained from

$$A(2)\hat{y}_2 = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\hat{y}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \hat{y}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

The linear independent latent vector for $\lambda = 3$ is found from

$$A(3)\hat{y}_3^1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\hat{y}_3^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \hat{y}_3^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the generalized latent vector for $\lambda = 3$, $\ell_3 = 1$, is computed from the chain rule. Using (3.5a)

$$A(3)\hat{y}_3^2 = -\frac{dA(3)}{d\lambda}\hat{y}_3^1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\hat{y}_3^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $(\hat{y}_3^2)^T = [-1\ 1]$. The latent vectors $\hat{y}_2$ and $\hat{y}_3^2$ are chosen with signs opposite to $\hat{y}_1$ and $\hat{y}_3^1$ respectively for convenience. Since y_i is in a two-dimensional space, only two of the latent vectors are necessary to span the space. It should be noted that the term linearly independent latent vector is not proper terminology and will be dropped in favor of latent vector hereafter if $\hat{y}_i$ satisfies $A(\lambda_i)\hat{y}_i = 0$.

The concept of distinct and repeated latent roots as well as a defective matrix polynomial will be clarified by relating $A(\lambda)$ to the companion form of the mn×mn matrix $\mathcal{A}$. It is well known that the matrix

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I \\ -A_m & -A_{m-1} & -A_{m-2} & \cdots & -A_1 \end{bmatrix} \tag{3.6}$$

will have eigenvalues λ_i that are the same as the latent roots of $A(\lambda)$ that is $\det[\mathcal{A}(\lambda)] = \det[A(\lambda)]$. Furthermore, it can be shown that the latent vectors are subvectors of the eigenvectors of $\mathcal{A}$. If y_i is an eigenvector of $\mathcal{A}$ for an eigenvalue λ_i then $\hat{y}_i$ is a subvector of y_i. The eigenvector y_i of $\mathcal{A}$ is given by

$$(3.7)\qquad \mathcal{A}(\lambda_i)y_i = \begin{bmatrix} \lambda_i I & -I & 0 & \cdots & 0 \\ 0 & \lambda_i I & -I & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -I \\ A_m & A_{m-1} & A_{m-2} & \cdots & \lambda_i I + A_1 \end{bmatrix} \begin{bmatrix} \hat{y}_i \\ \lambda_i \hat{y}_i \\ \vdots \\ \lambda_i^{m-1}\hat{y}_i \end{bmatrix}$$

when λ_i is a distinct eigenvalue or $\mathcal{A}$ is not defective. It follows that the first n elements of y_i is a latent vector of $A(\lambda)$. Similarly if z_i is a left eigenvector of $\mathcal{A}$ then the left latent vector of $A(\lambda)$ will be the last n-elements of the row vector z_i under the same restriction on λ_i.

There is a second relationship between $\mathcal{A}(\lambda)$ and $A(\lambda)$ that will be useful in the development that follows. The inverse of $\mathcal{A}(\lambda)$ in companion form is given by

$$(3.8)\qquad [\mathcal{A}(\lambda)]^{-1} = \begin{bmatrix} \lambda I & -I & 0 & \cdots & 0 \\ 0 & \lambda I & -I & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -I \\ A_m & A_{m-1} & A_{m-2} & \cdots & \lambda I + A_1 \end{bmatrix}^{-1} = [A(\lambda)]^{-1} B_{adj}[\mathcal{A}(\lambda)]$$

where $\mathrm{Badj}[\mathcal{A}(\lambda)]$ denotes the block adjoint of $\mathcal{A}(\lambda)$. The block adjoint is defined as the adjoint matrix of $\mathcal{A}(\lambda)$ with each block matrix of $\mathcal{A}(\lambda)$ treated as a scalar element. As an example the block adjoint of $\mathcal{A}(\lambda)$ with m = 3 is

$$(3.9)\qquad \mathrm{Badj}[\mathcal{A}(\lambda)] = \mathrm{Badj}\begin{bmatrix} \lambda I & -I & 0 \\ 0 & \lambda I & -I \\ A_3 & A_2 & \lambda I + A_1 \end{bmatrix} = \begin{bmatrix} \lambda^2 I + A_1\lambda + A_2 & \lambda I + A_1 & I \\ -A_3 & \lambda^2 I + \lambda A_1 & \lambda I \\ -\lambda A_3 & -\lambda A_2 - A_3 & \lambda^2 I \end{bmatrix}$$

The last column of $B_{adj}[A(\lambda)]$ will always have the form shown with the last block equal to $\lambda^{m-1}I$.

Consider the inverse of $[A(\lambda)]$ and let $A(\lambda)$ have distinct latent roots. The partial fraction expansion of $[A(\lambda)]$ was given by

$$[A(\lambda)]^{-1} = \sum_{i=1}^{mn} \frac{P_{i0}}{\lambda-\lambda_i} \tag{3.10}$$

where $P_{i0} = (\lambda-\lambda_i)\ [A(\lambda)]^{-1}$ evaluated at $\lambda = \lambda_i$. It follows from (3.9) and (3.10) that the eigenprojector P_{i0} and the latent projector $\hat{P}_{i0}$ are related since

$$P_{i0} = \hat{P}_{i0}\ B_{adj}[A(\lambda_i)] = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \hat{P}_{i0} \\ \cdot & \cdot & \cdot & \cdot & \lambda_i\hat{P}_{i0} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \lambda_i^{m-1}\hat{P}_{i0} \end{bmatrix}. \tag{3.11}$$

where the first m-1 columns of (3.11) are not important to the development. It was shown earlier that $\sum_{i=1}^{mn} P_{i0} = I$ thus it follows from (3.11) that the latent projectors have the properties that

$$\sum_{i=1}^{mn} \hat{P}_{i0} = 0 \tag{3.12a}$$

with

$$\sum_{i=1}^{mn} \lambda_i^j\ \hat{P}_{i0} = 0 \qquad j = 1,2,\ldots,m-2 \tag{3.12b}$$

$$\sum \lambda_i^{m-1}\hat{P}_{i0} = I \tag{3.12c}$$

where $\hat{P}_{i0}$ are the primary latent projectors. If P_{i0} is a matrix residue of $[A(\lambda)]^{-1}$ then P_{i0} is a matrix of $[A(\lambda)]^{-1}$.

The partial fraction expansion of $[A(\lambda)]^{-1}$ for distinct latent roots can be obtained from (3.10) and (3.11) and is

$$(3.13)\qquad [A(\lambda)]^{-1} = \sum_{i=1}^{mn} \frac{\hat{P}_{i0}}{\lambda-\lambda_i}$$

where

$$(3.14)\qquad \hat{P}_{i0} = (\lambda-\lambda_i)[A(\lambda)]^{-1}\Big|_{\lambda=\lambda_i}$$

with $\hat{P}_{i0}$ being primary latent projectors.

It was shown in Section 2 that the eigenprojectors for the A matrix with repeated eigenvalues are given by

$$(3.15)\qquad P_{i,\ell_i-j} = \frac{1}{(\ell_i-r_i+j+1)!}\frac{d^j}{d\lambda^j}\{(\lambda-\lambda_i)^{r_i}[A(\lambda_i)]^{-1}\}$$

with $j = 0,1,2,\ldots,r_i$ and $(\cdot)! = 1$ for $(\cdot)\leq 0$.

Using (3.11), it follows that

$$(3.16)\qquad P_{i,\ell_i-j} = \frac{1}{j!}\frac{d^j}{d\lambda^j}\{(\lambda-\lambda_i)^{\hat{r}_i}[A(\lambda)]^{-1}\}$$

Equation (3.16) agrees with the usual partial fraction expansion formula provided that $[A(\lambda)]^{-1}$ is a minimum polynomial, factors common to adj$[A(\lambda)]$ and det$[A(\lambda)]$ have been cancelled.

The numerical procedure for computing the latent projectors by the residues will be given for the matrix polynomial

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -5 & 1 \\ 1 & -5 \end{bmatrix} \lambda + \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$$

which has latent roots $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 3$ and $\ell_2 = 1$. The inverse of $A(\lambda)$ is

$$[A(\lambda)]^{-1} = \frac{1}{(\lambda-1)(\lambda-3)^3} \begin{bmatrix} \lambda^2-5\lambda+6 & -\lambda+3 \\ -\lambda+3 & \lambda^2-5\lambda+6 \end{bmatrix} = \frac{1}{(\lambda-1)(\lambda-3)^2} \begin{bmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{bmatrix}$$

thus

$$\hat{P}_{10} = (\lambda-1)[A(\lambda)]^{-1}\Big|_{\lambda=1} = \frac{1}{4} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Since $\lambda = 3$ appears twice in the minimum form of $[A(\lambda)]^{-1}$ and $\ell = 1$, there will be one primary and one secondary latent projector. The latent projectors are

$$P_{21} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$P_{20} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The eigenprojector of A were constructed from the eigenvectors of A in Section 2. Since the eigenvectors of A have the latent vectors as subvectors, the latent projectors of $A(\lambda)$ can be constructed from the latent vectors. Consider the distinct eigenvalues λ_i for which the eigenprojectors P_{i0} are given by

$$P_{i0} = \frac{y_i\ z_i}{z_i\ y_i} \tag{3.17}$$

Let the order of the matrix polynomial m be 2 then y_i satisfies the equation

$$\begin{bmatrix} \lambda_i I & -I \\ A_2 & \lambda_i I+A_1 \end{bmatrix} y_i = \begin{bmatrix} \lambda_i I & -I \\ A_2 & \lambda_i I+A_1 \end{bmatrix} \begin{bmatrix} \hat{y}_i \\ \lambda_i \hat{y}_i \end{bmatrix} \tag{3.18}$$

The left eigenvector z_i must satisfy

$$z_i \begin{bmatrix} \lambda_i I & -I \\ A_2 & \lambda_i I+A_1 \end{bmatrix} = \hat{z}_i[-\beta_i\ \ I] \begin{bmatrix} \lambda_i I & -I \\ A_2 & \lambda_i I+A_1 \end{bmatrix} \tag{3.19}$$

from which it follows that

$$z_i = \hat{z}_i[\lambda_i I+A_1\ \ I] \tag{3.20}$$

thus the numerator of (3.17) is

$$\begin{aligned} y_i\ z_i &= \begin{bmatrix} I \\ \lambda_i \end{bmatrix} \hat{y}_1\ \hat{z}_i[\lambda_i I+A_1 I] \\ &= \begin{bmatrix} \hat{y}_i\hat{z}_i(\lambda_i I+A_1) & \hat{y}_i\hat{z}_i \\ -\hat{y}_i\hat{z}_i A_2 & \lambda_i\hat{y}_i\hat{z}_i \end{bmatrix} \end{aligned} \tag{3.21}$$

The outer product $z_i\ y_i$ is

$$z_i y_i = \hat{z}_i[\lambda_i I+A_1 I] \begin{bmatrix} I \\ \lambda_i \end{bmatrix} \hat{y}_i = \hat{z}_i\ \frac{dA(\lambda_i)}{d\lambda}\ \hat{y}_i \tag{3.22}$$

which gives for the distinct eigenvalue or latent root the eigenprojector

(3.23) $$P_{i0} = \frac{1}{\hat{z}_i \frac{dA(\lambda_i)}{d\lambda} \hat{y}_i} \begin{bmatrix} \hat{y}_i\hat{z}_i(\lambda_i I + A_1) & \hat{y}_i\hat{z}_i \\ -\hat{y}_i\hat{z}_i A_2 & \lambda_i\hat{y}_i\hat{z}_i \end{bmatrix}$$

The latent projector, as given in (3.11) is the (1,2) block of P_{i0} or

(3.24) $$\hat{P}_{i0} = \frac{\hat{y}_i \; \hat{z}_i}{\hat{z}_i \frac{dA(\lambda_i)}{d\lambda} \hat{y}_i}$$

for the distinct latent root. The eigenprojector for the repeated eigenvalue nondefective case was given as

(3.25) $$P_{i0} = \sum_{j=1}^{r_i} \frac{y_i^j \; z_i^j}{z_i^j \; y_i^j}$$

with the obvious extension to the latent projectors as given in (3.26)

(3.26) $$\hat{P}_{i0} = \sum_{j=1}^{r_i} \frac{\hat{y}_i^j \; \hat{z}_i^j}{\hat{z}_i^j \frac{dA(\lambda_i)}{d\lambda} \hat{y}_i^j}$$

Consider the matrix polynomial

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -4.5 & 0.5 \\ 0.5 & -4.5 \end{bmatrix} \lambda + \begin{bmatrix} 4.5 & -1.5 \\ -1.5 & 4.5 \end{bmatrix}$$

for which the latent projectors are to be found. The latent vectors for the latent roots $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = \lambda_4 = 3$ are

$$\hat{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \hat{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \hat{y}_3^1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \hat{y}_3^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\hat{z}_1 = [-1\ -1] \qquad \hat{z}_2 = [1\ -1] \qquad \hat{z}_3^1 = [-1\ -1] \qquad \hat{z}_3^2 = [1\ -1]$$

The two distinct latent projectors are

$$\hat{P}_{10} = \frac{\hat{y}_1\, \hat{z}_1}{\hat{z}_1 \frac{dA(1)}{d\lambda} \hat{y}_1} = \frac{1}{4} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

and

$$\hat{P}_{20} = \frac{\hat{y}_2\, \hat{z}_2}{\hat{z}_2 \frac{dA(2)}{d\lambda} \hat{y}_2} = \frac{1}{4} \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$$

The latent projector for the repeated latent root is

$$\hat{P}_{30} = \frac{\hat{y}_3^1\, \hat{z}_3^1}{\hat{z}_3^1 \frac{dA(3)}{d\lambda} \hat{y}_3^1} + \frac{\hat{y}_3^2\, \hat{z}_3^2}{\hat{z}_3^2 \frac{dA(3)}{d\lambda} \hat{y}_3^2} = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

The partial traction expansion of $[A(\lambda)]^{-1}$ is then

$$[A(\lambda)]^{-1} = \frac{1}{4(\lambda-1)} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} + \frac{1}{4(\lambda-2)} \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} + \frac{1}{4(\lambda-3)} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

with $\hat{P}_{10}+\hat{P}_{20}+\hat{P}_{30} = 0$, and $\lambda_1\hat{P}_{10}+\lambda_2\hat{P}_{20}+\lambda_3\hat{P}_{30} = I$ as required.

The formulation of the latent projectors for the repeated latent root polynomial when A is defective in terms of the latent vectors remains as a problem. Attempts to formulate the latent projectors for the defective case have been unsuccessful. Future work will be devoted to this problem.

4. Projectors and the Sign Matrix

It was shown in Section 2 that the primary eigenprojectors for the mn×mn A matrix are given by

$$(4.1) \qquad P_{i0} = Q\, F_{i0}\, Q^{-1} \qquad i = 1,2,\ldots,q$$

where F_{i0} is a diagonal matrix with ones along the diagonal of the ith pseudo-Jordan block. The secondary eigenprojectors were defined as

$$(4.2) \qquad P_{ij} = Q\, F_{ij}\, Q^{-1} \qquad j = 1,2,\ldots,\ell_i$$

when A has ℓ_i generalized eigenvectors for the eigenvalue λ_i.

Assume that A has q_1 eigenvalueswith $Re(\lambda_i)>0$ and q_2 eigenvalues having $Re(\lambda_i)<0$ and no eigenvalues with $Re(\lambda_i) = 0$ so that $q_1+q_2 = q$. Let P^+ be denoted as the sum of the eigenprojectors with $Re(\lambda_i)>0$ and P^- the sum of the eigenprojectors with $Re(\lambda_i)<0$; that is

$$(4.3) \qquad P^+ = \sum_{i=1}^{q_1} P_{i0} = Q \sum_{i=1}^{q_1} F_{i0}\, Q^{-1}$$

$$(4.4) \qquad P^- = \sum_{i=q_1+1}^{q} P_{i0} = Q \sum_{i=q_1+1}^{q} F_{i0}\, Q^{-1}$$

where it has been assumed that the first q_i eigenvalues have $Re(\lambda_i)>0$.

The sign of a matrix, denoted by S, will be defined as the matrix

$$(4.5) \qquad S = Q\{\text{sign}[Re(\Lambda]\}Q^{-1}$$

where $Re(\Lambda)$ denotes the real part of the eigenvalues of A or the diagonal elements of the Jordan matrix J. Let E_1 be the mn×mn matrix with diagonal elements equal to 1 if $Re(\lambda_i)>0$ and zero for $Re(\lambda_i)<0$ and E_2 be the complement to E_1 such that $E_1+E_2 = I$. The sign matrix can then be defined as

$$S = Q\, E_1\, Q^{-1} - Q\, E_2\, Q^{-1} \tag{4.6}$$

$$= Q[\sum_{i=1}^{q_1} F_{i0} - \sum_{i=q_1+1}^{q} F_{i0}]Q^{-1}$$

therefore S is equal to

$$S = P^+ - P^- \tag{4.7}$$

Knowledge of the eigenprojectors is sufficient to construct the sign matrix, similarly it can be shown that knowledge of the sign matrix is sufficient to construct P^+ and P^-. Assume that P^+ is given as

$$P^+ = \frac{1}{2}\,(S+I) = \frac{1}{2}\,[QE_1Q^{-1}-QE_2Q^{-1}+QQ^{-1}] \tag{4.8}$$

but since $E_1+E_2 = I$ then

$$P^+ = \frac{1}{2}\,Q[E_1-E_2+E_1+E_2]Q^{-1} = QE_1Q^{-1} = Q\sum_{i=1}^{q_1} F_{i0}Q^{-1} \tag{4.9}$$

It is not difficult to show that P^- is given by

$$P^- = \frac{1}{2}\,[I-S] \tag{4.10}$$

The computation of the sign of A is a rather simple task. Roberts [10], gave an iterative algorithm to compute S which is based on Newton's method for computing the square root of $S^2 = I$. The algorithm is

$$(4.11) \qquad S(i+1) = \frac{1}{2}\{S(i)+[S(i)]^{-1}\} \qquad S(0) = A$$

where the index i denotes the ith iteration. The algorithm will converge quadratically to S provided that A has no eigenvalues on the $j\omega$ axis. The simplest test of convergence of (4.11) to the sign of A is to compute the trace of $S^2(i)$ at each iteration. Since $S^2(i)$ converges to I, then trace [S] will be mn.

Several accelerated versions of (4.11) have been described in the literature, Roberts [10], Hoskins and Walton, [11], and Mattheys, [12]. Numerous applications of the sign algorithm to system analysis have been given in the literature, [13]-[16].

The example below gives the sign of A where A is

$$A = \frac{1}{4} \begin{bmatrix} -1 & -1 & 9 & -3 \\ -3 & 1 & -1 & 7 \\ 9 & -3 & -1 & -1 \\ -1 & 7 & -3 & 1 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = \lambda_4 = -2$ and $\ell_3 = 1$. The sign of A is

$$S \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which converged in 5 iterations. The trace of $[S(i)]^2$ was

iteration	trace
1	6.90278
2	4.3857
3	4.01697
4	4.00006
5	4.00000

The positive and negative projectors, P^+ and P^-, were found to be

$$P^+ = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = P_{10} + P_{20}$$

$$P^- = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = -P_{30}$$

The sign matrix of A when all eigenvalues have $\mathrm{Re}(\lambda_i)>0$ will be I whereas the sign of A with $\mathrm{Re}(\lambda_i)<0$ will be $-I$. Eigenvalues along the $j\omega$ axis can be removed from the axis by an origin shift or by computing the sign of $(A+\rho I)$ where ρ is a real number. The eigenvalues of $A+\rho I$ will be $\lambda_i+\rho$ since $A+\rho I = Q(J+\rho I)Q^{-1}$. Eigenvalues belonging to a Jordan block cannot be separated nor can eigenvalues along the $j\omega$ axis be split by the described procedure.

A method of separating eigenvalues according to their magnitude is to compute a new matrix by the bilinear transformation

$$A_0 = (A-\rho I)(A+\rho I)^{-1} \tag{4.12}$$

where ρ has the same meaning as above. All of the eigenvalues with $|\lambda_i|<\rho$ will be mapped into the half plane $Re(\lambda_i)<0$ with others mapped into the plane with $Re(\lambda_i)>0$. This procedure is more general than the origin shifting method since the spectrum splitting will be according to the magnitudes of λ_i. The two methods, shifting and splitting can be combined if desired to isolate any circular region of the eigenvalue space. For example, the matrix A_0 given by

$$A_0 = (A+\rho_1 I-\rho_2 I)(A+\rho_1 I+\rho_2 I)^{-1} \tag{4.13}$$

can be used to isolate eigenvalues inside a circle of radius ρ_2 centered at ρ_1.

As an example of the bilinear transformation procedure, let

$$A = \frac{1}{2}\begin{bmatrix} 3 & -2 & -9 & 6 \\ -2 & 3 & 6 & -9 \\ -9 & 6 & 3 & -2 \\ 6 & -9 & -2 & 3 \end{bmatrix} \qquad \text{trace} = 6$$

which has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = -5$ and $\lambda_4 = 10$. If the value of $\rho = 4$ is selected then all eigenvalues inside of the circle $\rho = 4$ will be mapped to the left half plane and those outside the circle will be in the right half plane. The sign of A is

$$S = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

with projectors

$$P^+ = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$P^- = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The projectors P^+ and P^- are idempotent matrices and satisfy the properties of the eigenprojector given earlier. It is not difficult to show that the positive and negative projectors can be used for the spectral decomposition of A. Since $P^+ = Q\ E_1\ Q^{-1}$ with $E_1 = \mathrm{diag}[1\ 1\ 1\ 1\ .\ .\ 0\ 0\ 0]$ with the ones in the first q_1 locations. The product AP^+ will have the eigenvalues with $|\lambda_i|>\rho$ whereas AP^- will have eigenvalues $|\lambda_i|<\rho$ with all other eigenvalues zero. Using the example

$$A^+ = AP^+ = \frac{1}{4}\begin{bmatrix} 5 & 5 & -15 & 15 \\ -5 & 5 & 15 & -15 \\ -15 & 15 & 5 & -5 \\ 15 & -15 & -5 & 5 \end{bmatrix} \quad \text{trace} = 5 = \lambda_4-\lambda 3$$

$$A^- = AP^- = \frac{1}{4}\begin{bmatrix} 1 & 1 & -3 & -3 \\ 1 & 1 & -3 & -3 \\ -3 & -3 & 1 & 1 \\ 3 & -3 & 1 & 1 \end{bmatrix} \quad \text{trace} = 1 = \lambda_2-\lambda_1$$

The sum of AP^+ and AP^- must be A since $P^+ + P^- = I$.

It is obvious that P^+ and P^- can be decomposed into eigenprojectors for the eigenvalues λ_1, λ_2, λ_3 and λ_4. The details for the determination of the eigenprojectors have been covered in Section 2 and will not be covered at this point.

The positive and negative projectors, P^+ and P^-, have been defined in (4.3) and (4.4). It follows from (4.3) and (3.11) that P^+ is also given by

$$(4.13)\qquad P^+ = \sum_{i=1}^{q_1} P_{i0} = \begin{bmatrix} \sum_{i=1}^{q_1} \hat{P}_{i0}(\lambda_i I + A_1) & \sum_{i=1}^{q_1} \hat{P}_{i0} \\ -\sum_{i=1}^{q_1} \hat{P}_{i0}A_2 & \sum_{i=1}^{q_1} \lambda_i \hat{P}_{i0} \end{bmatrix}$$

for the companion form when $A(\lambda)$ is a second-order polynomial.

Similarily, $P^- = I - P^+$ thus

$$(4.14)\qquad P^- = -\sum_{i=q_1+1}^{q} P_{i0} = \begin{bmatrix} -\sum_{i=q_1+1}^{q} \hat{P}_{i0}(\lambda_i I + A_1) & -\sum_{i=q_1+1}^{q} \hat{P}_{i0} \\ \sum_{i=q_1+1}^{q} \hat{P}_{i0}A_2 & -\sum_{i=q_1+1}^{q} \lambda_i \hat{P}_{i0} \end{bmatrix}$$

The secondary eigenprojectors and latent projectors are not needed in the decomposition of A into A^+ and A^-. This can be shown from the definition of A^+ and A^-.

The individual eigenprojectors P_{i0} can be computed by repeated use of the sign algorithm. Assume that $n = 4$ with $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ and $\lambda_4 = 4$. The bilinear transformation can be carried out first with $\rho = 1.5$

to separate λ_1 from the other eigenvalues. If S_1 denotes the sign of A_{01} with

$$(4.14) \qquad A_{01} = (A-1.5I)(A+1.5I)^{-1}$$

then P_{10} will be given by

$$(4.15) \qquad P_{10} = \frac{1}{2}(I-S_1) = P_1^-$$

with $P^+ = \sum_{i=2}^{3} P_{i0}$. The next step in the procedure is to compute A_{02} with $\rho_2 = 2.5$ and compute the sign of A_{02}. The negative projector of A_{02} will be the sum of P_{10} and P_{20} or

$$P_2^- = P_{10} + P_{20} = \frac{1}{2}(I-S_2) \ .$$

Thus $P_{20} = P_2^- - P_{10}$. This process can be continued until each eigenprojector has been found. Since the eigenprojectors also give the latent projectors as the upper right block, the latent projectors will also be known when A is in companion form.

A method of computing the projectors of a matrix has been discussed in this section. It has been shown that the eigenprojector for any general matrix can be computed from the sign of a matrix. If A is in companion form, the latent projectors can also be found from the eigenprojectors.

5. Solvents of Matrix Polynomials

The concept of matrix polynomials was introduced in Section 1 of this report where it was shown that $A(\lambda)$ arises when n simultaneous equations of mth-order are used to define the time-behavior of a dynamic system. The eigenprojectors for a matrix in companion form was discussed in Section 2 and the latent projectors were described in Section 3. The application of lambda matrices to the dynamics of systems has been described by Frazer, Duncan and Collar, [17], in their book Elementary Matrices and Some Applications to Dynamics and Differential Equations. The concept of solvents or matrix roots of a matrix polynomial will be given in this section. It will be shown in the next section that solvents are useful in solving sets of differential equations.

Let $A(\lambda)$ be defined as a mth-order matrix polynomial with $n\times n$ matrix coefficients. The associated $mn\times mn$ A matrix, which will be called the block companion matrix, is given by

$$(5.1)\qquad A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_m & -A_{m-1} & -A_{m-2} & \cdots & -A_1 \end{bmatrix}$$

The eigenvector matrix Q for A will always have the form

$$(5.2)\qquad Q = \begin{bmatrix} Q_1 & Q_2 & \cdots & Q_m \\ Q_1\Lambda_1 & Q_2\Lambda_2 & \cdots & Q_m\Lambda_m \\ Q_1\Lambda_1^2 & Q_2\Lambda_2^2 & \cdots & Q_m\Lambda_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ Q_1\Lambda_1^{m-1} & Q_2\Lambda_2^{m-1} & \cdots & Q_m\Lambda_m^{m-1} \end{bmatrix}$$

when A has distinct eigenvalues or $A(\lambda)$ has distinct latent roots. The submatrices Q_j will be a matrix of latent vectors $\hat{y}_j$ for the latent roots λ_j. It will be assumed that Q_j exists and is invertible; under the above assumptions, $R_j = Q_j \Lambda_j Q_j^{-1}$ is a solvent of the matrix polynomial and satisfies the equation, [1],

$$R_j^m + A_1 R_j^{m-1} + \dots + A_m = 0 \qquad j = 1,2,\dots,m \tag{5.3}$$

The proof of this is straightforward if $A(\Lambda)$ is considered. Let $A(\Lambda)$ be defined as the block matrix $\Lambda I - \mathbf{A}$; it then follows that

$$\begin{bmatrix} \Lambda_j & -I & 0 & 0 & 0 \\ . & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \Lambda_j & -I \\ A_m & A_{m-1} & A_{m-2} & A_2 & \Lambda_j + A_1 \end{bmatrix} \begin{bmatrix} Q_j \\ Q_j \Lambda_j \\ . \\ . \\ Q_j \Lambda_j^{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \tag{5.4}$$

where (5.3) is given by the last row of (5.4) provided that $\Lambda_j Q_j = Q_j \Lambda_j$ which must hold from the first row of (5.4).

The block matrices Λ_j are n×n diagonal matrices constructed from a subset of the latent roots of $A(\lambda)$ or the eigenvalues of A. Each R_j is defined in terms of n latent roots λ_i and n latent vectors $\hat{y}_i$, as an example, consider the matrix polynomial $A(\lambda)$ with

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -5 & 2 \\ 2 & -5 \end{bmatrix} \lambda + \begin{bmatrix} 7 & -5 \\ -5 & 7 \end{bmatrix}$$

which has latent roots $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ and $\lambda_4 = 4$. The right latent vectors of $A(\lambda)$ are

$$\hat{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \hat{y}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \hat{y}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \hat{y}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The solvents are constructed from the latent roots and latent vectors by forming the 2×2 matrices $Q_j \Lambda_j Q_j^{-1}$ where $Q_j = \{\hat{y}_j\}$ such that Q_j is invertible with j indicating a subset of latent vectors. Noting that $\{y_1, y_2\}$ is singular, then

$$R_1 = [\hat{y}_1 \ \hat{y}_3]\mathrm{diag}[\lambda_1 \ \lambda_3][\hat{y}_1 \ \hat{y}_3]^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$R_2 = [\hat{y}_2 \ \hat{y}_4]\mathrm{diag}[\lambda_2 \ \lambda_4][\hat{y}_2 \ \hat{y}_4]^{-1} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

It can be shown that R_1 and R_2 satisfy (5.3).

The eigenprojectors for the companion matrix A are given by

$$(5.5) \qquad P_{i0} = \hat{P}_{i0} \begin{bmatrix} \cdots\cdot & I \\ \cdots\cdot & \lambda_i I \\ \cdots\cdot & \lambda_i^2 I \\ \cdots\cdot & \cdot \\ \cdots\cdot & \cdot \\ \cdots\cdot & \lambda_i^{m-1} I \end{bmatrix}$$

where the first m-1 block columns are not important in the development. Suppose that the latent projectors are known and A has distinct eigenvalues. It can be shown that if Q_j is defined as

$$(5.6) \qquad Q_{j+1} \equiv \sum_{i=jn+1}^{(j+1)n} \hat{P}_{i0} \qquad j = 0,1,2,\ldots,m-1$$

and

$$(5.7) \qquad Q_{j+1}\, \Lambda_{j+1} \equiv \sum_{i=jn+1}^{(j+1)n} \lambda_i\, \hat{P}_{i0}$$

then R_j is defined by the latent projectors of $A(\lambda)$ provided that Q_j is invertible. If Q_j is singular, the latent projectors are reordered until a set is found for Q_j^{-1} to exist.

The solvents for repeated roots can be defined by noting that Λ_j may include several Jordan blocks. Assuming that the multiplicity r_i is less than n, and that full Jordan blocks are included in Λ_j, Q_j is defined as in (5.6). The product $Q_j\Lambda_j$ must be modified since Λ_j is no longer diagonal but may include the ones on the super diagonal of Λ_j due to the included Jordan block. Assuming that the Jordan block is defective, then $Q_j\tilde{J}_j$ will be defined as

$$(5.8) \qquad Q_j\tilde{J}_j = \sum_{i=jn+1}^{(j+1)n} [\lambda_i \hat{P}_{i0} + \hat{P}_{i1}]$$

The matrix polynomial

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -3.5 & 0.5 \\ 1.5 & -4.5 \end{bmatrix} \lambda + \begin{bmatrix} 2.5 & -0.5 \\ -3.5 & 5.5 \end{bmatrix}$$

has latent roots $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 2$ and $\lambda_4 = 2$ with $\ell_3 = 1$. The Jordan block for $\lambda = 2$ is then

$$J_3 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

The latent vectors of $A(\lambda)$ are

$$\hat{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \hat{y}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \hat{y}_3^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \hat{y}_3^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\hat{z}_1 = [1\ 0] \quad \hat{z}_2 = [1\ -1] \quad \hat{z}_3^1 = [1\ -1] \quad \hat{z}_3^2 = [-1\ -1]$$

The first solvent is given by

$$R_1 = Q_1 \Lambda_1 Q_1^{-1} = \sum_{i=1}^{2} \hat{P}_{i0} \lambda_i [\sum_{i=1}^{2} \hat{P}_{i0}]^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

which gives $A(R_1) = 0$ as required. The second solvent requires the primary latent projector and the secondary latent projector which are

$$\hat{P}_{30} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = Q_2 \qquad \hat{P}_{31} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

with $\quad Q_2 J_2 = \hat{P}_{30} \lambda_3 + \hat{P}_{31}$ or

$$Q_2 J_2 = \frac{1}{2} \begin{bmatrix} 2 & 2 \\ 6 & -2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}$$

The second solvent is

$$R_2 = Q_2 J_2 Q_2^{-1} = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$$

which gives $A(R_2) = 0$ as required.

The extension to higher order polynomials is a simple matter and does not require additional analysis. Each solvent, $R_1, R_2, \ldots, R_m$, is found by the procedure given in the preceeding work.

The solvents can be determined by the sign algorithm provided that the spectrum of A has the required distribution. The first step in the procedure is to establish A for the matrix polynomial and assume that the multiplicity of the repeated eigenvalues are less than n. Let $\lambda_1|,\ldots,|\lambda_n|$ have magnitudes less than ρ_1, and compute the sign of A_{D1} where $A_{D1} = (A-\rho_1 I)(A+\rho_1 I)$. The sign of A_{D1} will have n eigenvalues of -1 and mn-n +1 eigenvalues. The sign of A_{D1} can be arranged in the form

$$(5.9) \qquad S_1 = \mathrm{Sign}\ (A_{D1}) = Q \begin{bmatrix} -I_{n\times n} & 0 \\ 0 & I_{mn-n\times mn-n} \end{bmatrix} Q^{-1}$$

by row-column interchanges of the sign as computed by (4.11). Let the eigenvalue matrix (5.9) be denoted by J_{I1}, then

$$(5.10) \qquad (S_1+J_{I1}) = [QJ_{I1}+J_{I1}+J_{I1}Q]Q^{-1}$$

Suppose that Q is partitioned as

$$(5.11) \qquad Q = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 & \cdots & Q_m \\ Q_1\Lambda_1 & Q_2\Lambda_2 & \cdots & Q_m\Lambda_m \\ \vdots & \vdots & \vdots & \vdots \\ Q_1\Lambda_1^{m-1} & Q_2\Lambda_2^{m-1} & \cdots & Q_m\Lambda_m^{m-1} \end{bmatrix}$$

which when substituted into (5.10) gives

$$(5.12) \qquad [S_1+J_{I1}] = 2 \begin{bmatrix} Q_1 & 0 \\ 0 & Q_{22} \end{bmatrix} Q^{-1}$$

The similarity transformation $[S_1+J_I]A[S_1+J_I]^{-1}$ then gives

$$(5.13)\qquad [S_1+J_{I1}]A[S_1+J_{I1}]^{-1} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_{22}^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} R_1 & 0 \\ 0 & \overline{R}_2 \end{bmatrix}$$

The similarity transformation T required to reduce A to the block diagonal form is given by

$$(5.17)\qquad T = \frac{1}{(2)^{m-1}} \begin{bmatrix} I & -R_2^{-1} & \cdots & (-1)^{m+1}R_m^{-(m+1)} \\ R_1 & -I & \cdots & (-1)^{m+1}R_m^{-(m+1)+1} \\ R_1^2 & R_2 & \cdots & (-1)^{m+1}R_m^{-(m+1)+2} \\ \vdots & \vdots & \cdots & \vdots \\ R_1^{m-1} & R_2^{m-2} & \cdots & (-1)^{m+1}I \end{bmatrix}$$

where R_i is a solvent of $A(\lambda)$.

The spectral decomposition of A will now be shown using the second example in this section. Let $A(\lambda)$ be defined as

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -3.5 & 0.5 \\ 1.5 & -4.5 \end{bmatrix} \lambda + \begin{bmatrix} 2.5 & -0.5 \\ -3.5 & 5.5 \end{bmatrix}$$

with $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = \lambda_4 = 2$ and $\ell_3 = 1$. The companion form A is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2.5 & 0.5 & 3.5 & -0.5 \\ 3.5 & -5.5 & -1.5 & 4.5 \end{bmatrix}$$

with solvents

$$R_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 2.5 & -0.5 \\ 0.5 & 1.5 \end{bmatrix}$$

From (5.17), T is

$$T = \frac{1}{2}\begin{bmatrix} 1 & 0 & 3/8 & 1/8 \\ 0 & 1 & -1/8 & 5/8 \\ 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \end{bmatrix}$$

with

$$A_m = A_2 = T\,A\,T^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2.5 & -0.5 \\ 0 & 0 & 0.5 & 1.5 \end{bmatrix}$$

The characteristic equation for the upper block is $\lambda^2-4\lambda+3$ and the lower block has $\lambda^2-4\lambda+4$ as its characteristic equation. The eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$ are in the upper block with $\lambda_3 = 2$ and $\lambda_3 = 2$ in the lower block. The computations for this example were checked by the sign algorithm with a shift of -1.8 and $\rho_1 = 0.5$. The two solvents, R_1 and R_2 computed by the sign algorithm, agree with the values given.

6. Solution of a System of Differential Equations

The mathematical tools developed in the previous sections will now be applied to the time domain analysis of systems. Assume that the system has been characterized in the first-order form, usually called the state variable form, with states x(t) such that x(t) satisfies the differential equation

(6.1) $$\frac{dx(t)}{dt} = Ax(t)\ Bu(t)$$

where A is a general mn×mn matrix, B is mn×kn, x(t) is a mn×1 vector and u(t) is kn×1. Let Z(t) denote the outvector with

(6.2) $$Z(t) = Cx(t)$$

where Z(t) is km×1 and C is km×mn. The vector u(t) will be considered as the input to the system or a control vector if the system is a control system. It will be assumed that A,B and C are constant matrices which will be referred to as the system triplet.

It will be assumed that the system is stable, all eigenvalues of A have $Re(\lambda_i)<0$ except for distinct eigenvalues along the $j\omega$ axis and multiple eigenvalues at the origin, $\lambda_i = 0$.

The solution to (6.1) can be expressed as

(6.3) $$x(t) = \Theta(t,t_0)x(t_0) + \int_{t_0}^{t} \Theta(t,\tau)Bu(\tau)d\tau$$

where $\Theta(t,t_0)$ denotes* the state transition matrix, (STM). The state-transition matrix satisfies the differential equation

$$\text{(6.4)} \qquad \frac{d\Theta(t,t_0)}{dt} = A\,\Theta(t,t_0) \qquad\qquad \Theta(t_0,t_0) = I = \Theta(t,t)$$

where $\Theta(t,t_0)$ is mn×mn and I is the identity matrix. There are numerous methods of obtaining $\Theta(t,t_0)$, several of the methods will be described.

Since A is a constant matrix, the Laplace transform of (6.4) can be taken with

$$\text{(6.5)} \qquad L[\tfrac{d}{dt}\,\Theta(t,0)] = s\Theta(s) - \Theta(0,0) = s\Theta(s) - I$$

thus the transform of (6.4) is

$$\text{(6.6)} \qquad \Theta(s) = [sI-A]^{-1} = Q[sI-J]^{-1}Q^{-1}$$

where J is the Jordan form. Noting that s in the Laplace domain is equivalent to λ in the eigenvalue domain, $\Theta(s)$ is equivalent to $[\lambda I-A]^{-1} = [A(\lambda)]^{-1}$. It therefore follows that $\Theta(s)$ can be expressed as

$$\text{(6.7)} \qquad \Theta(s) = \sum_{i=1}^{q} \left[\frac{P_{i0}}{s-s_i} + \sum_{j=1}^{\ell_i} \frac{P_{ij}}{(s-s_i)^j}\right]$$

The inverse Laplace transform of $\Theta(s)$ is $L^{-1}[\Theta(s)]$ or

$$\text{(6.8)} \qquad \Theta(t,0) = Q\{\exp(Jt)\}Q^{-1} = \sum_{i=1}^{q} \{P_{i0}\exp(s_i t) + \sum_{j=1}^{\ell_i} \frac{P_{ij}t^j}{j!}\exp(s_i t)\}$$

*The normal use of $\Phi(t,t_0)$ as the state-transition matrix will not be made due to the use of ϕ as the modal matrix, [17].

where s_i is a eigenvalue of A or a root of the characteristic equation $\det[sI-A] = 0$.

The Laplace transform is usually taken with $t_0 = 0$ rather than on arbitrary value of t_0. The state transition matrix $\Theta(t,t_0)$ can be found by using the semigroup properly

$$\Theta(t,t_0) = \Theta(t,0)\Theta(0,t_0) \tag{6.9}$$

where $\Theta(0,t_0) = [\Theta(t_0,0]^{-1}$ if $t_0>0$. It follows that $\Theta(t,t_0)$ is given by

$$\Theta(t,t_0) = Q\{\exp[J(t-t_0)]\}Q^{-1} = \sum_{i=1}^{q} \{P_{i0} \exp(s_i(t-t_0)] + \sum_{j=1}^{\ell_i} \frac{P_{ij}(t-t_0)^j}{j!} \exp[s_i(t-t_0)]\} \tag{6.10}$$

The analytical solution to (6.1) can be expressed as

$$x(t) = Q \exp[J(t-t_0)]Q^{-1}x(t_0)+Q \int_{t_0}^{t} \exp[J(t-\tau)]Q^{-1}Bu(\tau)d\tau \tag{6.11}$$

where Q is the eigenvector matrix and J is the Jordan form. If the eigenprojectors P_{i0} and P_{ij} are used, then

$$x(t) = \sum_{i=1}^{q} \{P_{i0} \exp[s_i(t-t_0)] + \sum_{j=1}^{\ell_i} \frac{P_{ij}(t-t_0)^j}{j!} \exp[s_i(t-t_0)]x(t_0) + \int_{t_0}^{t} \sum_{i=1}^{q} \{\exp[s_i(t-\tau)]P_{i0}+ \sum_{j=1}^{\ell_i} \frac{(t-\tau)^d}{j!} \exp[s_i(t-\tau)P_{ij}\}Bu(\tau)d\tau \tag{6.12}$$

where the order of the operation under the integral has been reversed for convenience. This reordering is permissible since $\exp(\cdot)$ is a scalar.

The system defined in (6.1) is said to be controllable if the input $u(t)$ (or the control) can drive the initial states $x(t_0)$ to the origin $x(t) = 0$ (or to an arbitrary value $x(t) = x_f$). The usual test of controllability is to examine the matrix

$$Q_c = [B, AB, A^2B, \ldots, A^mB] \tag{6.13}$$

to determine if Q_c is invertible. The system is controllable if $\det Q_c \neq 0$ or if Q_c is invertible. Controllability of a system can also be measured by considering the product $P_{ij}B$ for $i = 1,2,\ldots,q$ and $j = 1,\ldots,\ell_i$ for the eigenprojectors. The system mode $\exp(\lambda_i t)$ is not controllable if

$$P_{ij}B = 0 \tag{6.14}$$

for the primary and secondary eigenprojectors of the eigenvalue λ_i. This test implies that $u(t)$ cannot drive the mode $\exp(\lambda_i t)$ as the mode is following the natural response rather than a forced response.

The partial fraction expansion method of determining $\Theta(t,t_0)$ for all t is not a computationally efficient process and would be used only when the analytical form of $\Theta(t,t_0)$ is desired. For computational purposes, the state-transition matrix $\Theta(t,0)$ can be determined more efficiently by expressing $\Theta(t,0)$ in the form

$$\Theta(t,0) = \exp At = I + \sum_{j=1}^{\infty} \frac{A^j t^j}{j!} \tag{6.15}$$

The series is then used to compute $\Theta(t,0)$ for a small value of t with truncation of the series when the change in $\Theta(t,0)$ is beyond the accuracy of the digital computer. Assume that the small t is taken as Δt, with $\Theta(\Delta t,0)$ determined from (6.15). The semigroup properties is utilized to find $\Theta(t,0)$ for $t = k\Delta t$ by the operation

$$\Theta(t,0) = [\Theta(\Delta t,0)]^k = \Theta(\Delta t,0)\ \Theta(t-\Delta t,0) \tag{6.16}$$

when A is constant.

The solution vector x(t) depends upon the integral as well as the state transition matrix. The expression in (6.3) can be given in a recursive form which is more convenient and which has computational advantages. Let (6.3) be written as

$$x(t) = \Theta(t,t_0)\ x(t_0) + \Gamma(t,t_0) \tag{6.17}$$

where $\Theta(t,t_0)$ has been computed by means of (6.10) and where $\Gamma(t,t_0)$ is the integral. The solution vector at $t+\Delta t$ can be expressed as

$$x(t+\Delta t) = \Theta(t+\Delta,t)x(t) + \Gamma(t+\Delta t,t) \tag{6.18}$$

Substituting (6.17) into (6.18),

$$x(t+\Delta t) = \Theta(t+\Delta t,t)\Theta(t,t_0)x(t_0)+\Gamma(t+\Delta t,t)+\Theta(t+\Delta t,t)\Gamma(t,t_0) \tag{6.19}$$

or

$$x(t+\Delta t) = \Theta(t+\Delta t,t_0)x(t_0) + \Gamma(t+\Delta t,t_0) \tag{6.20}$$

Knowledge of $\Gamma(t+\Delta t,t)$ is required along with $\Theta(t+\Delta t,t)$ or $\Theta(\Delta t,0)$, since A is constant, to find $x(t+\Delta t)$. The vector $\Gamma(t,t_0)$ satisfies the equation

$$(6.21) \qquad \frac{d\Gamma(t,t_0)}{dt} = A\,\Gamma(t,t_0) + Bu(t) \qquad \Gamma(t_0,t_0) = 0$$

and can be found a 4th-order Runge-Kutta algorithm. Since $\Gamma(t+\Delta t,t)$ is required for (6.20), the value is found from (6.21) by integrating (6.21) from t to $t+\Delta t$ for all t. Equation (6.19) is then used in a recursive manner with $\Theta(t+\Delta t,t) = \Theta(\Delta t,0)$ for constant A. The series given in (6.15) is used to compute $\Theta(\Delta t,0)$ which remains constant thereafter.

If $u(t)$ is slowly varying and Δt is small, a reasonably good approximation may be obtained from the z-transform, [3], [18], of (6.1). The Laplace transform of (6.1) is

$$(6.22) \qquad [sI-A]X(s) = Bu(s)$$

where the initial condition $x(0)$ has been neglected. Equation (6.22) can be written as

$$(6.23) \qquad x(s) = [sI-A]^{-1}Bu(s) = G(s)\,U(s)$$

Taking the z-transform of $G(s)$ with a zero-order-hold gives

$$(6.24) \qquad G(z) = Z\{\frac{1-e^{-sT}}{s}\,[sI-A]^{-1}\}B$$

which is found to be

(6.25) $$G(z) = [zI-\exp(AT)]^{-1}[\exp(AT)-I]A^{-1}B$$

where $T = \Delta t$. The associated difference equation of (6.1) is obtained from (6.25) with*

(6.26) $$x(k+1) = [\exp(AT)-I]A^{-1}B\ U(k) + \exp(AT)x(k)$$

with $t = kT$, $(k+1)T = t+\Delta t$ and $x(0)$ the initial condition vector. The validity of (6.26) can be shown by noting that

(6.27) $$\Theta(t,0) = \exp(AT)$$

(6.28) $$\int_0^T \Theta(t,\tau)B\ d\tau = \int_0^T \exp[A(t-\tau)]B\ d\tau = [\exp(AT)-I]A^{-1}B$$

Equation (6.26) is valid provided that the assumption of slowly varying $u(t)$ and small Δt is not too strict.

The eigenprojectors can also be utilized to separate the solutions into modes if desired. Recalling that $[sI-A]^{-1}$ is equivalent to $[\lambda I-A]^{-1}$ and that $[\lambda I-A]^{-1}$ can be expanded as a partial fraction, (6.24) can be reformulated as

(6.29) $$G(z) = [A(0)]^{-1} + \sum_{i=1}^{q} \frac{z-1}{s_i} \left\{\frac{P_{i0}}{z-e^{s_iT}} + \sum_{j=1}^{\ell_i} \frac{P_{ij}\ T^j}{(z-e^{s_iT})^{j+1}}\ e^{js_iT}\right\}$$

which can be written as a series of transfer functions G_{ij} with

*The argument of $x(\cdot)$ and $u(\cdot)$ is $(k+1)T$ with T being dropped for convenience hereafter.

$$G(z) = G_{00}(z) + \sum_{i=1}^{q} [G_{i0}(z) + \sum_{j=1}^{\ell_i} G_{ij}(z)] \tag{6.30}$$

The vector x(z) is then given by

$$x(z) = G_{00}(z)B\ U(z) + \sum_{i=1}^{q} [G_{i0}(z) + \sum_{j=1}^{\ell_i} G_{ij}(z)]B\ U(z) \tag{6.31}$$

Equation (6.31) defines a set of transfer functions that can be placed in parallel with each having u(z) as an input and each block can be implemented separately by a matrix difference equation.

Recalling that the eigenprojectors are defined by the right and left eigenvectors, the eigenprojectors need not be stored in the digital computer. The eigenprojectors are stored and used in (6.29) to construct (6.29) at each T.

The discussion in this section has been based on A having a general form. Assume that A is the companion form of the matrix polynomial $A(\lambda)$ with

$$I \frac{d^m \hat{x}}{dt^m} + A_1 \frac{d^{m-1}\hat{x}}{dt} + \dots + A_m \hat{x} = u(t) \tag{6.32}$$

The canonical form of x(t) is the same as given in (6.1) but with

$$\dot{x}(t) = \begin{bmatrix} \dot{\hat{x}}(t) \\ \ddot{\hat{x}}(t) \\ \vdots \\ \hat{x}^{(m)}(t) \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -A_m & -A_{m-1} & -A_{m-2} & \dots & -A_1 \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \dot{\hat{x}}(t) \\ \vdots \\ \hat{x}^{(m-1)}(t) \end{bmatrix} \tag{6.33}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I \end{bmatrix} u(t) = A\, x(t) + B\, u(t)$$

Rather than use (6.33), the solution vector $\hat{x}(t)$ can be determined directly from (6.32). Taking the Laplace transform of (6.32) gives

$$[Is^m + A_1 s^{m-1} + \ldots + A_{m-1} x + A_m]\hat{x}(s) = U(s) \tag{6.34}$$

where the initial conditions have been neglected. Equation (6.34) can be rewritten as

$$\hat{X}(s) = [A(s)]^{-1} U(s) = G(s)\ U(s) \tag{6.35}$$

which can be expanded into a partial fraction.

The z-transform of (6.35) can now be found provided that the zero-order-hold is included and u(t) is slowly varying. Letting

$$G(s) = \frac{1-e^{-sT}}{s}\, G(s) \tag{6.36}$$

gives

$$G(z) = Z\{\frac{1-e^{-sT}}{s}\ [A(s)]^{-1}\} = Z\{\frac{1-e^{-sT}}{s}\ G(s)\} \tag{6.37}$$

The z-transform of (6.37) is

$$G(z) = [A(0)]^{-1} + \sum_{i=1}^{q} \frac{(z-1)}{s_i} \left\{ \frac{\hat{P}_{i0}}{(z-e^{s_iT})} + \sum_{j=1}^{\ell_i} \frac{\hat{P}_{ij} T^j e^{js_iT}}{(z-e^{s_iT})^{j+1}} \right\} \tag{6.38}$$

from which the difference equation for $\hat{x}(k+1)$ can be determined.

The expression in (6.38) would be implemented on a digital machine in parallel form rather than as a series representation. As an example, let

$$A(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} s + \begin{bmatrix} 7 & -5 \\ -5 & 7 \end{bmatrix}$$

with roots $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$ and $\lambda_4 = -4$. The latent projectors are

$$\hat{P}_{10} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \hat{P}_{20} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \quad \hat{P}_{30} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \hat{P}_{40} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

with A(0) obtained from A(s). The transfer function G(z) is

$$G(z) = G_{00}(z)+G_{10}(z)+G_{20}(z)+G_{30}(z)+G_{40}(z)$$

where

$$G_{00}(z) = \frac{1}{24}\begin{bmatrix} 7 & 5 \\ 5 & 7 \end{bmatrix}$$

$$G_{10}(z) = \frac{z-1}{2(z-e^{-T})}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1-z^{-1}}{1-e^{-T}z^{-1}}$$

$$G_{20}(z) = \frac{z-1}{2(z-e^{-2T})}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \frac{1-z^{-1}}{1-e^{-2T}z^{-1}}$$

$$G_{30}(z) = \frac{z-1}{2(z-e^{-3T})} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{1-z^{-1}}{1-e^{-3T}z^{-1}}$$

$$G_{40}(z) = \frac{z-1}{2(z-e^{-4T})} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1-z^{-1}}{1-e^{-4T}z^{-1}}$$

The implementation as a difference equation would then be

$$\hat{X}_0(k+1) = \frac{1}{24} \begin{bmatrix} 7 & 5 \\ 5 & 7 \end{bmatrix} u(k+1)$$

$$\hat{X}_1(k+1) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (u(k)-u(k-1)) + e^{-T} \hat{X}_1(k)$$

$$\hat{X}_2(k+1) = \frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} (u(k)-u(k-1)) + e^{-2T} \hat{X}_2(k)$$

$$\hat{X}_3(k+1) = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (u(k)-u(k-1) + e^{-3T} \hat{X}_3(k)$$

$$\hat{X}_4(k+1) = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} (u(k)-u(k-1)) + e^{-4T} \hat{X}_4(k)$$

with

$$\hat{X}(k+1) = \sum_{k=0}^{4} \hat{X}_k(k+1)$$

The parameter T is the integration step size (or sampling rate) with $t = kT$. Different step sizes can be selected for the modes $\exp(-s_i T)$ if better accuracy is desired for the higher frequency modes.

The use of the eigenprojectors and the latent projector for solving the state equation of (6.1) has been described. The extension to the solution of n mth-order differential equations has been given.

7. Spectral Decomposition of Differential Equations

The increasing complexity of modern systems generally requires a large number of algebraic operations on system equations and the corresponding solutions to these equations to characterize the dynamics of the systems. It is therefore essential that the analysis of large scale systems be carried out on subsystems or subsets of the equations. A dissertation by K. S. Yoo, [15], used the concept of mode decoupling to analyze an optimal control system. The overhead for the decoupling procedure was moderately high but the total computational task for the analysis was decreased when compared to the usual procedure. Popeeva and Lupas, [16], published several papers describing the procedure, the referenced paper was the first reference to Yoo's work in the open literature. The original idea for decoupling, or order reduction, was probably due to Roberts, [10].

Consider the system equation given in Section 3 with

$$(7.1) \qquad \frac{dx(t)}{dt} = Ax(t)\ Bu(t)$$

Let T denote the mn×mn transformation matrix

$$(7.2) \qquad T = \frac{1}{(2)^{m-1}} \begin{bmatrix} I & -R_2^{-1} & R_3^{-2} & \cdots & (-1)^m R_m^{-m+1} \\ R_1 & -I & R_3^{-1} & \cdots & (-1)^m R_m^{-m+2} \\ R_1^2 & R_2 & I & \cdots & (-1)^m R_m^{-m+3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ R_1^{m-1} & R_2^{m-2} & R_3^{m\cdot 3} & \cdots & I \end{bmatrix}$$

where I is the n×n identity matrix and R_i is the solvent of

$$R_i^m + A_1 R_i^{m-1} + \ldots + A_{m-1} R_i + A_m = 0 \tag{7.3}$$

Define the vector v(t) by the transformation

$$v(t) = T^{-1} x(t) \tag{7.4}$$

or $x(t) = T\, v(t)$. Assume that A is the companion form for the mth-order differential equation, it follows that v(t) must satisfy

$$\frac{dv(t)}{dt} = T^{-1} A\, T\, v(t) + T^{-1}\, B\, u(t) \tag{7.5}$$

Consider the matrix $T^{-1}AT$ when $m = 3$ and T is as given in (7.2). It follows that

$$A_B = T^{-1}AT = \begin{bmatrix} A_{B1} & 0 & 0 \\ 0 & A_{B2} & 0 \\ 0 & 0 & A_{B3} \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{bmatrix} \tag{7.6}$$

or in general for the mth-order differential equation

$$A_B = \mathrm{diag}[R_1, R_2, R_3, \ldots, R_m] \tag{7.7}$$

Recalling that the solvents R_k were given by

$$R_k = Q_k \check{J}_k Q_k^{-1} = \{\sum_{i=jn+1}^{(j+1)n} [\lambda_i \hat{P}_{i0} + \hat{P}_{i1}]\}\{\sum_{i=jn+1}^{(j+1)n} \hat{P}_{i0}\}^{-1} \tag{7.8}$$

A_B and T are completely known provided that the latent projectors are known.

The solution to (7.5) is obtained from the m equations

$$\frac{dv_i(t)}{dt} = R_i v(t) + B_i u(t) \tag{7.9}$$

where

$$\overline{B} = T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ B_m \end{bmatrix} \tag{7.10}$$

with initial conditions $v_i(t_0)$ obtained from (7.4) with $x(t_0)$ given. The mn×1 vector x(t) as given in (6.33) is

$$x(t) = \begin{bmatrix} \hat{x}(t) \\ \dot{\hat{x}}(t) \\ \cdot \\ \cdot \\ \hat{x}^{(m-1)}(t) \end{bmatrix} \tag{7.11}$$

and $x(t) = T\,v(t)$. The solution to the mth-order differential equation of (6.32) is

$$\hat{x}(t) = \frac{1}{(2)^{m-1}} \{v_1(t)-R_2^{-1}v_2(t) + R_3^{-2}v_3(t) - \ldots\} \tag{7.12}$$

The decoupling procedure described is useful in solving a large set of differential equations as it allows the set to be decoupled into a number of equations. A mth-order differential equations with n×n coefficients can be solved by generating m first-order equations with $n \times n$ coefficients. These equations can be reduced further if desired by decoupling any of the m first-order to several first order differential equations with coefficients less than $n \times n$. The limit to the decoupling procedure will be mn first-order equations.

As an example of the above let A be the general matrix

$$A = \begin{bmatrix} -168 & 48 & 279 & -218 & 8 & 50 \\ -136 & 32 & 233 & -180 & 14 & 36 \\ -80 & 16 & 95 & -58 & 8 & 18 \\ -60 & 12 & 75 & -50 & 8 & 14 \\ -40 & 8 & 50 & -32 & 3 & 10 \\ -20 & 4 & 25 & -16 & 2 & 4 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = -5$, $\lambda_4 = -12$, $\lambda_5 = -24$ and $\lambda_6 = -40$. This matrix can be reduced to the companion form by the Krylov transformation

$$A_c = K\,A\,K^{-1} \tag{7.13}$$

where the structure of K is given in the Appendix. The associated matrix polynomial $A(\lambda)$ is

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda^3 + \begin{bmatrix} 702.051 & -734.624 \\ 591.229 & -618.051 \end{bmatrix} \lambda^2 + \begin{bmatrix} 7688.79 & -6761.64 \\ 6568.29 & -5763.7 \end{bmatrix} \lambda$$

$$+ \begin{bmatrix} 9481.76 & -8521.03 \\ 8095.49 & -7263.08 \end{bmatrix}$$

The solvents for $A(\lambda)$ are

$$R_1 = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \quad \det[\lambda I - R_1] = \lambda^2+3\lambda+2 = (\lambda+1)(\lambda+2)$$

$$R_2 = \begin{bmatrix} -15 & 6 \\ -5 & -2 \end{bmatrix} \det[\lambda I - R_2] = \lambda^2+17\lambda+60 = (\lambda+5)(\lambda+12)$$

$$R_3 = \begin{bmatrix} 360 & -480 \\ 320 & -424 \end{bmatrix} \quad \det[\lambda I - R_3] = \lambda^2+64\lambda+960 = (\lambda+24)(\lambda+40)$$

The system equation $\dot{x}(t) = Ax(t)+Bu(t)$ can be solved by considering the reduced equations

(7.14) $$\mathring{v}_i(t) = R_i v(t) + B_i u_i(t)$$

provided that the overhead for computing the Krylov transformation and that for finding the solvents are acceptable as to efficiency and accuracy.

8. Summary

The mathematical analysis in this report had two objectives: the first to bring together the mathematical tools for understanding matrix polynomials, with the second of applying these tools to decoupling of system equations. The study of matrix polynomials, or lambda matrices, is justified as vibrating systems are generally defined with second-order matrix polynomials, see [19]. Although there are several books and journal articles on matrix polynomials, a library search did not reveal a single source of the material that is complete. It is essential that the mathematics of matrix polynomials be understood before efficient algorithms for analyzing vibrating systems can be developed. Present algorithms are not capable of handling large space systems--systems with the number of modes greater than 1000.

The report is incomplete in several areas. The algorithm for determining latent projectors from latent vectors for matrix polynomials with repeated roots was not fully developed. Algorithms for efficient computation of latent roots and latent vectors have not been devised. A thorough literature search has not revealed the availability of a computer program for that purpose. In addition, the decoupling scheme was presented but the computational algorithm for that task was not described. Work will continue in these areas.

The damping of large-space structures is an important engineering design task that must be addressed and algorithms must be developed for that purpose. The analysis of matrix polynomials as well as a comprehensive understanding of how damping affects the overall mathematical structure is an integral part of designing large space structures. Modification of matrix polynomials, which is necessary for the inclusion of damping, was not considered in the report. Studies will begin in that area with the emphasis on second order polynomials.

This report is only a beginning for the several tasks described above. The development of algorithms for analysis, and design of large space structures will be addressed during the next year of work.

Appendix Krylov Transformation

The Krylov transformation is a useful algorithm that transforms a general $mn \times mn$ matrix A to the companion form. Let A_c denote the companion form, then

$$A_c = K\,A\,K^{-1} \tag{A.1}$$

The Krylov transformation is a similarity transformation that leaves the eigenvalues invariant but changes the eigenvectors of A from Q to Q_c, where

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1m} \\ Q_{21} & Q_{22} & \cdots & Q_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ Q_{m1} & Q_{m2} & \cdots & Q_{mm} \end{bmatrix} \tag{A.2}$$

and

$$Q_c = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{m} \\ Q_{11}\Lambda_1 & Q_{12}\Lambda_2 & \cdots & Q_{1m}\Lambda_m \\ \vdots & \vdots & \cdots & \vdots \\ Q_{11}\Lambda_1^{m-1} & Q_{12}\Lambda_2^{m-1} & \cdots & Q_{1m}\Lambda_m^{m-1} \end{bmatrix} \tag{A.3}$$

when A has distinct eigenvalues. The similarity transformation in (A.1) and the structure of (A.2) and (A.3) requires that K satisfy

$$Q_c = KQ \tag{A.4}$$

Now from (A.2) - (A.4), if m = 3

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{11}\Lambda_1 & Q_{12}\Lambda_2 & Q_{13}\Lambda_3 \\ Q_{11}\Lambda_1^2 & Q_{12}\Lambda_2^2 & Q_{13}\Lambda_3^2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \tag{A.5}$$

The n×n matrix $Q_{11}\Lambda_1$ is from (A.5)

$$Q_{11}\Lambda_1 = K_{21}Q_{11}+K_{22}Q_{21}+K_{23}Q_{31} \tag{A.6}$$

or

$$Q_{11}\Lambda_1 Q_{11}^{-1} = K_{21}+K_{22}Q_{21}Q_{11}^{-1}+K_{23}Q_{31}Q_{11}^{-1} = A_{B1} \tag{A.7}$$

where A_{B1} is the matrix that would be in the upper diagonal from the similarity transformation $T^{-1}AT$ given in Section 7. The matrices $Q_{21}Q_{11}^{-1}$ and $Q_{31}Q_{11}^{-1}$ are $-R_{12}$ and R_{13} as defined in Section 7 but for the general matrix with

$$T = \frac{1}{(2)^2}\begin{bmatrix} I & -R_{12} & R_{13} \\ R_{21} & -I & R_{23} \\ R_{31} & -R_{32} & I \end{bmatrix} = \frac{1}{4}\begin{bmatrix} I & -Q_{12}Q_{22}^{-1} & Q_{13}Q_{33}^{-1} \\ Q_{21}Q_{11}^{-1} & -I & Q_{23}Q_{33}^{-1} \\ Q_{31}Q_{11}^{-1} & -Q_{32}Q_{22}^{-1} & I \end{bmatrix} \tag{A.8}$$

Analysis of the other equations in (A.5) gives the set of equations

$$\begin{aligned} A_{B1} &= K_{21}+K_{22}R_{21}+K_{23}R_{31} \\ R_{12}A_{B2} &= K_{21}R_{12}+K_{22}+K_{23}R_{32} \\ R_{13}A_{B3} &= K_{21}R_{13}+K_{22}R_{23}+K_{23} \end{aligned} \tag{A.9}$$

$$
\text{(A.10)}\qquad
\begin{aligned}
I &= K_{11}+K_{12}R_{21}+K_{13}R_{31}\\
R_{12} &= K_{11}R_{12}+K_{12}+K_{13}R_{31}\\
R_{13} &= K_{11}R_{13}+K_{22}R_{23}+K_{23}
\end{aligned}
$$

$$
\text{(A.11)}\qquad
\begin{aligned}
A_{B1}^2 &= K_{31}+K_{32}R_{21}+K_{33}R_{31}\\
R_{12}A_{B2}^2 &= K_{31}R_{12}+K_{32}+K_{33}R_{32}\\
R_{13}A_{B3}^2 &= K_{31}R_{13}+K_{32}R_{23}+K_{33}
\end{aligned}
$$

The Krylov matrix is given from (A.9)-(A.11) as

$$
\text{(A.12)}\qquad
\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}
\begin{bmatrix} I & R_{12} & R_{13} \\ R_{21} & I & R_{23} \\ R_{31} & R_{31} & I \end{bmatrix}
=
\begin{bmatrix} I & R_{12} & R_{13} \\ A_{B1} & R_{12}A_{B2} & R_{13}A_{B3} \\ A_{B1}^2 & R_{12}A_{B2}^2 & R_{13}A_{B3}^2 \end{bmatrix}
$$

or

$$
\text{(A.13)}\qquad
\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}
=
\begin{bmatrix} I & R_{12} & R_{13} \\ A_{B1} & R_{12}A_{B2} & R_{13}A_{B3} \\ A_{B1}^2 & R_{12}A_{B2}^2 & R_{13}A_{B3}^2 \end{bmatrix}
\begin{bmatrix} I & R_{12} & R_{13} \\ R_{21} & I & R_{23} \\ R_{31} & R_{32} & I \end{bmatrix}^{-1}
$$

$$
=\begin{bmatrix} I & 0 & 0 \\ A_{11} & A_{12} & A_{13} \\ A_{11}^2+A_{12}A_{21}+A_{13}A_{31} & A_{11}A_{12}+A_{12}A_{22}+A_{13}A_{32} & A_{11}A_{13}+A_{12}A_{23}+A_{13}A_{33} \end{bmatrix}
$$

$$
=\begin{bmatrix} E_R\ I \\ E_R\ A \\ E_R\ A^2 \end{bmatrix} = K
$$

where E_R = [I 0 0]. The similarity transformation given in (A.1) is required to show that the Krylov matrix is as given in (A.13).

The general form of (A.13) is

$$K = \begin{bmatrix} E_R I \\ E_R A \\ \cdot \\ \cdot \\ E_R A^{m-1} \end{bmatrix} \tag{A.14}$$

where A is the general matrix to be reduced to the companion form, A_c.

The inverse of all Q_{ii} matrices must exist. Row-column interchanges of A can be made in most cases to assure the existence of A_c. The algorithm will not have good accuracy when m is large particularly when the matrix is stiff; the eigenvalues have a large spread in magnitude.

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