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Inversion and Approximation of Laplace Transforms

Mission Planning and Analysis Division

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Lyndon B. Johnson Space Center
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SHUTTLE PROGRAM

INVERSION AND APPROXIMATION OF LAPLACE TRANSFORMS

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1.0 INTRODUCTION

Included in this report is a novel method of inverting Laplace transforms by using a new set of orthonormal functions. As a byproduct of the inversion, it is seen how to approximate very complicated Laplace transforms by a transform with a series of simple poles along the left-half plane real axis. The inversion and approximation process is simple enough to be put on a programable hand calculator.

2.0 INVERSION AND APPROXIMATION

Let $f(s)$ be a Laplace transform and $F(t)$ its exact inverse. ${}_NF(t)$ will be the approximate inverse, given by

$$ {}_NF(t) = A_1L_1(st) + A_2L_2(st) + \cdots + A_NL_N(st) \tag{1} $$

where the $L_n(st)$ are the new orthonormal functions (described below and in the appendix). The A_n values are the Fourier coefficients and are given by

$$ A_n(s) = \int_0^\infty F(t)L_n(st)dt \tag{2} $$

s is a free parameter chosen to produce the best approximation, as shown below.

The integral square approximation error is given by

$$ E(s) = \int_0^\infty \{{}_NF(t) - F(t)\}^2dt = \int_0^\infty F(t)^2dt - \sum_{n=1}^{N} A_n^2(s) \geq 0. \tag{3} $$

To minimize the integral square error, s is chosen such that

$$ C = \sum_{n=1}^{N} A_n^2(s) \quad \text{is maximum} \tag{4} $$

The new orthonormal functions are shown below.

$$L_n = {}_na_1e^{-st} + {}_na_2e^{-2st} + {}_na_3e^{-3st} + \dots + {}_na_ne^{-nst} \qquad (5)$$

The values of ${}_na_i$ are chosen such that

$$\int_0^\infty L_nL_m dt = 0 \quad \text{for } n \neq m$$
$$= 1 \quad \text{for } n = m \qquad (6)$$

The first 10 orthonormal functions are listed below.

$$L_1 = \sqrt{2s}e^{-st} \qquad s > 0$$

$$L_2 = \sqrt{4s}(-2e^{-st} + 3e^{-2st})$$

$$L_3 = \sqrt{6s}(3e^{-st} - 12e^{-2st} + 10e^{-3st})$$

$$L_4 = \sqrt{8s}(-4e^{-st} + 30e^{-2st} - 60e^{-3st} + 35e^{-4st})$$

$$L_5 = \sqrt{10s}(5e^{-st} - 60e^{-2st} + 210e^{-3st} - 280e^{-4st} + 126e^{-5st})$$

$$L_6 = \sqrt{12s}(-6e^{-st} + 105e^{-2st} - 560e^{-3st} + 1260e^{-4st}$$
$$- 1260e^{-5st} + 462e^{-6st})$$

$$L_7 = \sqrt{14s}(7e^{-st} - 168e^{-2st} + 1260e^{-3st} - 4200e^{-4st} + 6930e^{-5st}$$
$$- 5544e^{-6st} + 1716e^{-7st})$$

$$L_8 = \sqrt{16s}(-8e^{-st} + 252e^{-2st} - 2520e^{-3st} + 11550e^{-4st} - 27720e^{-5st}$$
$$+ 36036e^{-6st} - 24024e^{-7st} + 6435e^{-8st})$$

$$L_9 = \sqrt{18s}(9e^{-st} - 360e^{-2st} + 4620e^{-3st} - 27720e^{-4st} + 90090e^{-5st}$$
$$- 168168e^{-6st} + 180180e^{-7st} - 102960e^{-8st} + 24310e^{-9st})$$

$$L_{10} = \sqrt{20s}(-10e^{-st} + 495e^{-2st} - 7920e^{-3st} + 60060e^{-4st} - 252252e^{-5st} + 630630e^{-6st} - 960960e^{-7st} + 875160e^{-8st} - 437580e^{-9st} + 92378e^{-10st})$$

Figure 1 shows plots of the first four L_n.

The values of the ${}_na_i$ coefficients are given by

$$ {}_na_i = (-1)^{n+i}\sqrt{2sn}\frac{(n + i - 1)!}{i!(i - 1)!(n - i)!} \tag{7}$$

or

$$ {}_na_i = (-1)^{n+i}\sqrt{2sn}\frac{n}{i!(i - 1)!}\prod_{j=1}^{i-1}(n^2 - j^2) \tag{8}$$

where ${}_na_1 = (-1)^{n+1}\sqrt{2ns}\, n$ (9)

The recursion relationship for the ${}_na_i$ is given by

$$ {}_na_1 = (-1)^{n+1}\sqrt{2ns}\, n \qquad n = 1, 2, \cdots, N \tag{10}$$

$$ {}_na_i = -\frac{n^2 - (i - 1)^2}{i(i - 1)}\, {}_na_{i-1} \qquad i = 2, 3, \cdots, n \quad n = 2, 3, \cdots, N \tag{11}$$

The recursion relationship for the L_n is given by

$$U_n = 2\frac{2n - 1}{\sqrt{n(n - 1)}} \qquad n = 2, 3, \cdots, N \tag{12}$$

$$V_n = \frac{n(n - 1)}{2n - 1} \qquad n = 2, 3, \cdots, N \tag{13}$$

$$L_1 = \sqrt{2s}\, e^{-st} \tag{14}$$

$$L_2 = \sqrt{s}\, e^{-st}(6e^{-st} - 4) \tag{15}$$

$$L_n = U_n\left((e^{-st} - V_n + V_{n-1})L_{n-1} - L_{n-2}/U_{n-1}\right) \quad n = 3, 4, \cdots, N \tag{16}$$

This is the relationship that should be used to compute the L_n in a computer program. It is simple, fast, and accurate.

Equation 2 gave the Fourier coefficients in terms of F(t). In terms of the Laplace transform, f(s), they are given by

$$A_n = \sum_{i=1}^{n} {}_n a_i f(is) \tag{17}$$

Note that as n increases, so does the magnitude of the ${}_n a_i$, which has an oscillating sign. This can cause serious roundoff error problems in computing the A_n. It is speculated that the maximum value of $n = N$ be limited to approximately the number of significant decimal digits of accuracy used by a particular computer. One way to evaluate this problem for a particular computer is to set[a]

$$f(s) = \frac{1}{s + 1}$$

Let $s = 1$ and compute the A_n. Theoretically

$$A_1 = 1/\sqrt{2}$$

$$A_n = 0 \qquad \text{for} \quad n > 1$$

[a]Also see theorem 15 in the ap ix.

and

$$C = \sum_{n=1}^{N} A_n^2 = 0.5$$

Due to roundoff error, the theoretical values will not be achieved for N large.

Perhaps a better way of computing the A_n (which may be slightly less affected by roundoff error) is to use the algorithm shown below, which also computes C.

```
        C = 0

        DO  c  n = 1, N

        A_n = f(ns)

        IF (n.EQ.1) GOTO b

        δ = 1

        DO  a  i = 1, n - 1

                     i(2n - i)
        A_n = ---------------------- A_n - δf((n - i)s)
              (n + 1 - i)(n - i)

a       δ = -δ

b       A_n = √(2ns) nA_n

        C = C + A_n^2

c       PRINT C
```

Note that s should be chosen such that C is maximum.

All the L_n approach zero as t approaches infinity. Therefore, the approximations work well only when $F(t) \to 0$ as t approaches infinity. This will be the case for stable system weighting functions - an important application. An example of what to do when $F(t)$ does not decay to zero is shown below. Let

$$g(s) = \frac{1 - e^{-2s}}{s^2}$$

Apply the final value theorem.

$$G(\infty) = \lim_{s \to 0} sg(s) = 2$$

So instead of inverting $g(s)$, invert

$$f(s) = g(s) - \frac{2}{s}$$

Now $F(t) \to 0$ as t approaches infinity and $G(t) = F(t) + 2$. Thus

$$_N G(t) = 2 + {_N F(t)}$$

3.0 EXAMPLES

As the first example, let

$$f(s) = \frac{s + 1}{(s + 1)^2 + \pi^2} \tag{18}$$

The exact inverse is

$$F(t) = e^{-t} \cos (\pi t) \tag{19}$$

Figures 2 through 9 show the values of

$$C = \sum_{n=1}^{N} A_n^2 \tag{20}$$

versus s for values of N from 1 to 14. The maximum value that C can obtain (neglecting roundoff errors) is 0.27300 since

$$\int_0^\infty F(t)^2 dt = 0.27300 \tag{21}$$

It is seen that each value of N has its own optimum value of s, and the choice of s can greatly influence the accuracy of the fit.

Figure 10 shows plots of $F(t)$, ${}_3F(t)$, and ${}_6F(t)$. For $N = 3$ the optimum value of s was $s = 2.2$. In this case

$A_1 = 0.33378\ 95910$

$A_2 = 0.28719\ 57089$

$A_3 = -0.21431\ 58487$

$${}_3F(t) = -3.345e^{-2.2t} + 11.921e^{-4.4t} - 7.805e^{-6.6t} \tag{22}$$

The approximate Laplace transform is thus seen to be

$${}_3f(s) = -\frac{3.345}{s + 2.2} + \frac{11.921}{s + 4.4} - \frac{7.805}{s + 6.6} \tag{23}$$

For $N = 6$ the optimum value of $s = 0.9$ and

$A_1 = 0.18930\ 92215$ $\qquad$ $A_2 = 0.36507\ 5176$[illegible]

$A_3 = 0.22324\ 71254$ $\qquad$ $A_4 = -0.14742\ 23756$

$A_5 = -0.11047\ 63746$ $\qquad$ $A_6 = 0.11492\ 41046$

$$_6F(t) = -1.916e^{-0.9t} + 43.527e^{-1.8t} - 252.178e^{-2.7t} + 554.831e^{-3.6t} - 517.636e^{-4.5t} + 174.488e^{-5.4t} \qquad (24)$$

From figure 8 it is seen that $N = 10$ and $s = 0.65$ will give an excellent fit. For this case

$A_1 = 0.14940\ 23073$ $\qquad$ $A_2 = 0.31134\ 50651$

$A_3 = 0.31771\ 10153$ $\qquad$ $A_4 = 0.03684\ 84058$

$A_5 = -0.18661\ 71768$ $\qquad$ $A_6 = -0.03590\ 26947$

$A_7 = 0.10304\ 31558$ $\qquad$ $A_8 = -0.02596\ 24761$

$A_9 = -0.03656\ 09018$ $\qquad$ $A_{10} = 0.04125\ 86606$

and

$$\begin{aligned} _{10}F(t) = & -0.832e^{-0.65t} + 59.853e^{-1.3t} - 1195.825e^{-1.95t} \\ & + 10\ 138.374e^{-2.6t} - 44\ 250.068e^{-3.25t} + 110\ 056.528e^{-3.9t} \\ & - 169\ 440.633e^{-4.55t} + 142\ 526.134e^{-5.2t} \\ & - 68\ 134.644e^{-5.85t} + 13\ 742.171e^{-6.5t} \end{aligned} \qquad (25)$$

For the next example

$$f(s) = \ln\left(\frac{s + 2}{s + 1}\right) - \frac{1}{s + 2} \qquad (26)$$

The exact inverse is

$$F(t) = \frac{1}{t}(e^{-t} - e^{-2t}) - e^{-2t} \tag{27}$$

Figure 11 shows plots of $C = \sum_{n=1}^{N} A_n^2$ versus s for $N = 4$, 8, and 12. It is clear that for $s = 0.5$, only four terms are needed to give an excellent fit. In this case

$$_4F(t) = -0.00261e^{-0.5t} + 0.17291e^{-t} + 0.66316e^{-1.5t} - 0.83353e^{-2t} \tag{28}$$

Figure 11 shows plots of $F(t)$ and $_4F(t)$. There is no visible difference between $F(t)$ and $_4F(t)$.

Let

$$g(s) = \ln\left(\frac{s + 2}{s + 1}\right) \tag{29}$$

Then from equations 26 and 28, $g(s)$ is approximated by

$$_4g(s) = -\frac{0.00261}{s + 0.5} + \frac{0.17292}{s + 1} + \frac{0.66316}{s + 1.5} + \frac{0.16647}{s + 2} \tag{30}$$

For $s > 0$, $_4g(s)$ is an excellent approximation of $g(s)$, as seen below.

s	g(s)	${}_4g(s)$
0	0.69315	0.69304
0.1	.64663	.64660
1	.40547	.40547
2	.28768	.28769
5	.15415	.15415
10	.08701	.08701

Note

$$ {}_4G(t) = -0.00261e^{-0.5t} + 0.17292e^{-t} + 0.66316e^{-1.5t} + 0.16647e^{-2t} \quad (31) $$

where

$$ G(t) = \frac{1}{t}(e^{-t} - e^{-2t}) \quad (32) $$

Note $G(0) = 1$ and ${}_4G(0) = 0.99994$.

For the final example

$$ f(s) = e^{-\sqrt{s}} \quad (33) $$

which has an exact inverse of

$$ F(t) = \frac{1}{2\sqrt{\pi t^3}} \exp\left(-\frac{1}{4t}\right) \quad (34) $$

Figure 13 shows plots of $C = \sum_{n=1}^{N} A_n^2$ versus s for values of $N = 6$, 10, and 14.

For $N = 6$ the optimum value of $s = 0.8$, and

$$ {}_6F(t) = 1.4551e^{-0.8t} - 15.6761e^{-1.6t} + 83.8937e^{-2.4t} $$
$$ - 204.8879e^{-3.2t} + 232.5782e^{-4t} - 97.7713e^{-4.8t} \quad (35) $$

As seen from figure 14, ${}_6F(t)$ is a very good approximation of $F(t)$, which is remarkable since $F(t)$ is a complicated function of time that is very dissimilar to a power series in $e^{-0.8t}$.

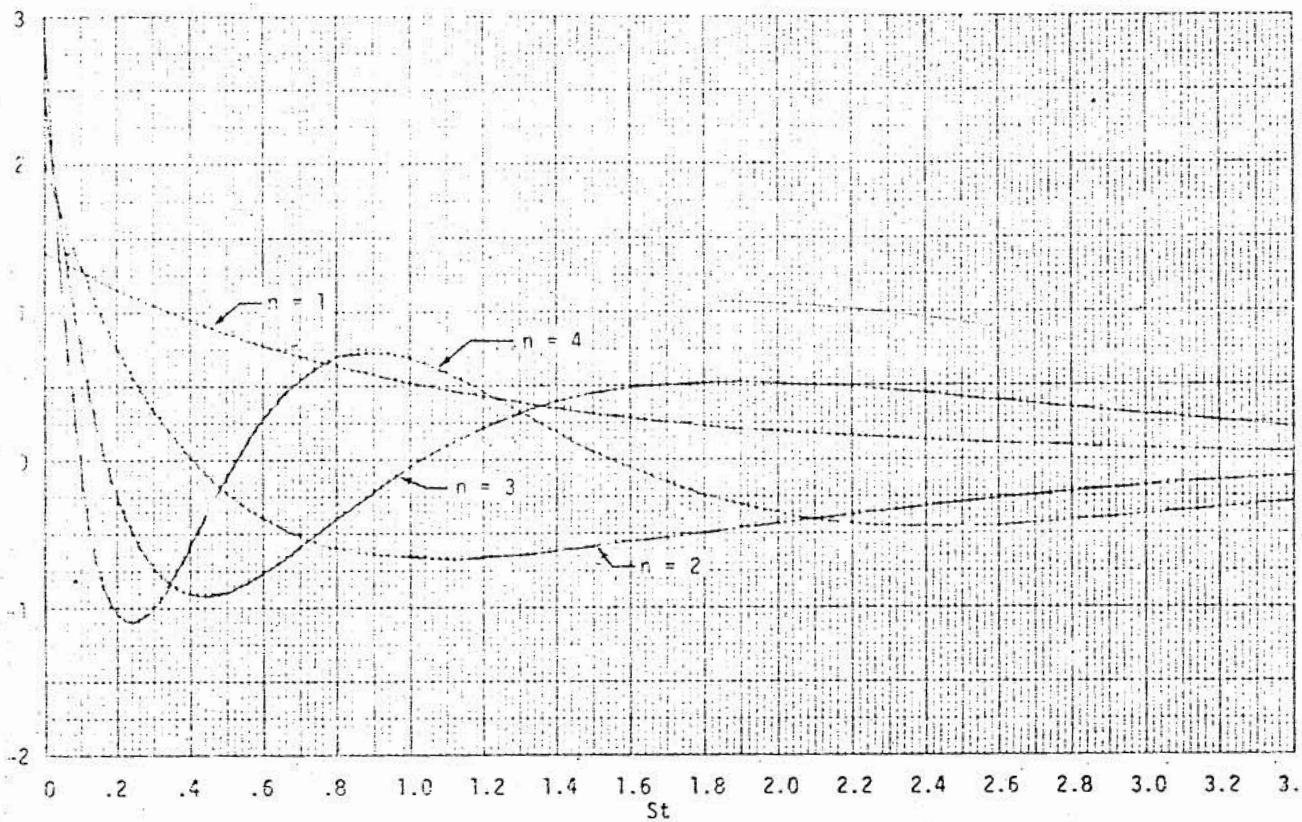

Figure 1.- Plot of first four orthonormal functions.

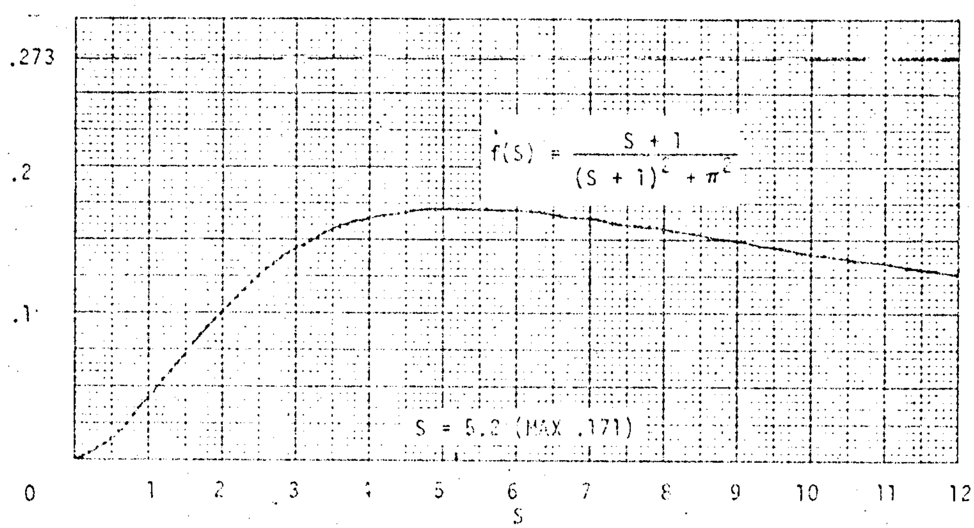

Figure 2.- $\sum_{n=1}^{1} A_n^2$ for $F(t) = e^{-t}\cos(\pi t)$

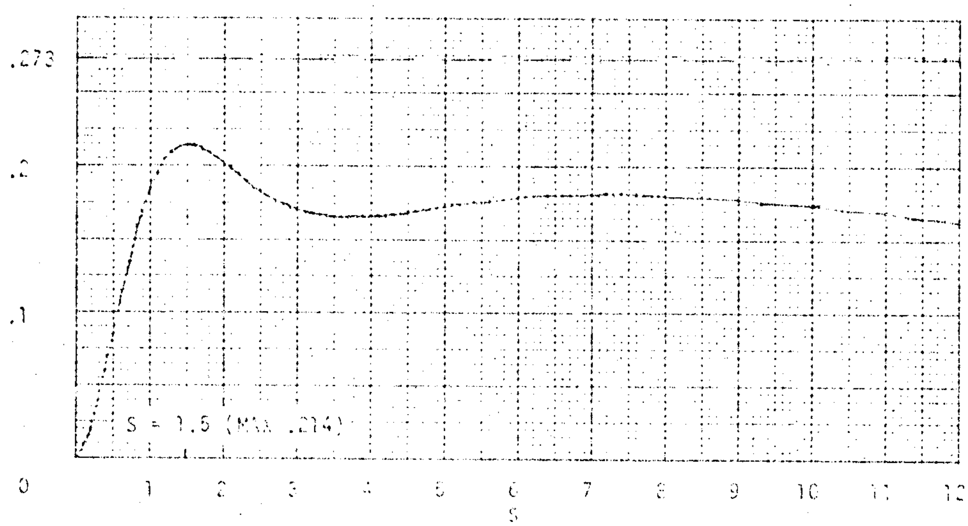

Figure [illegible].- $\sum_{n=1} A_n^2$ for F(t) [illegible]

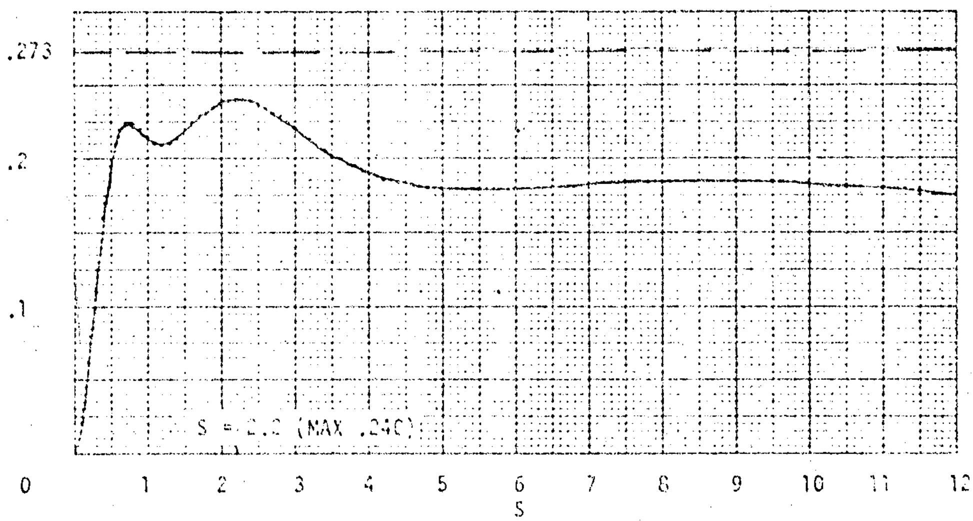

Figure 4.- $\sum_{n=1}^{3} A_n^2$ for $F(t) = e^{-t} \cos(\pi t)$.

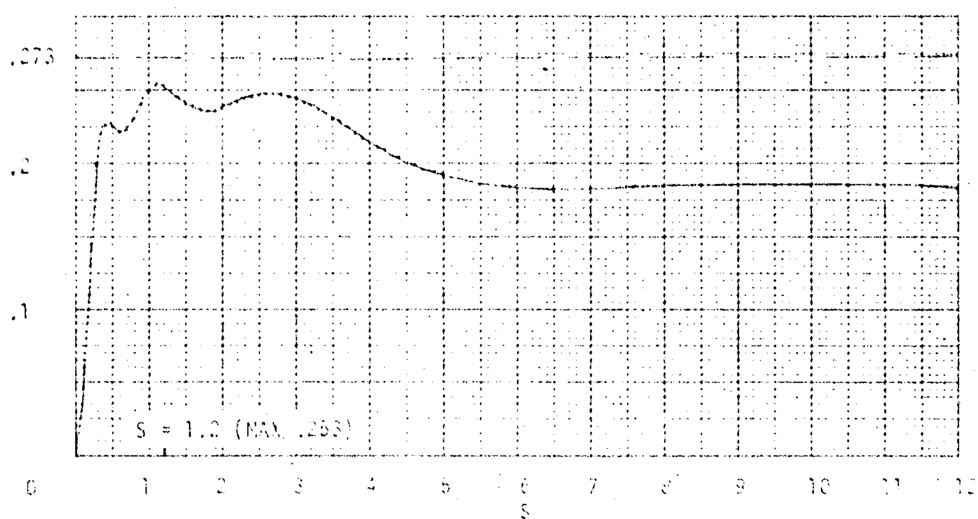

Figure 5.- $\sum A_n^2$ for $F(t) = e^{-t} \cos(\pi t)$.

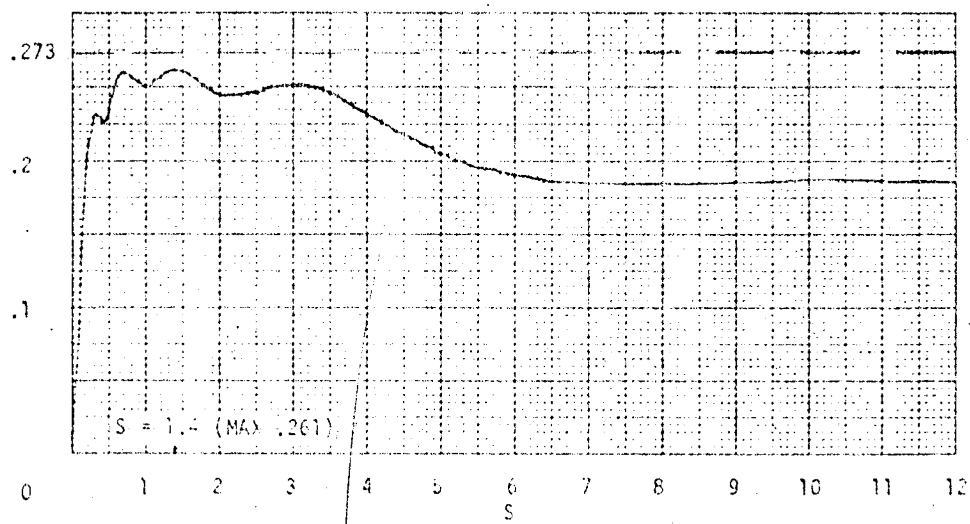

Figure [illegible].- $\sum_{n=1}^{5} A_n^2$ for $F(t) = e^{-t} \cos(\pi t)$.

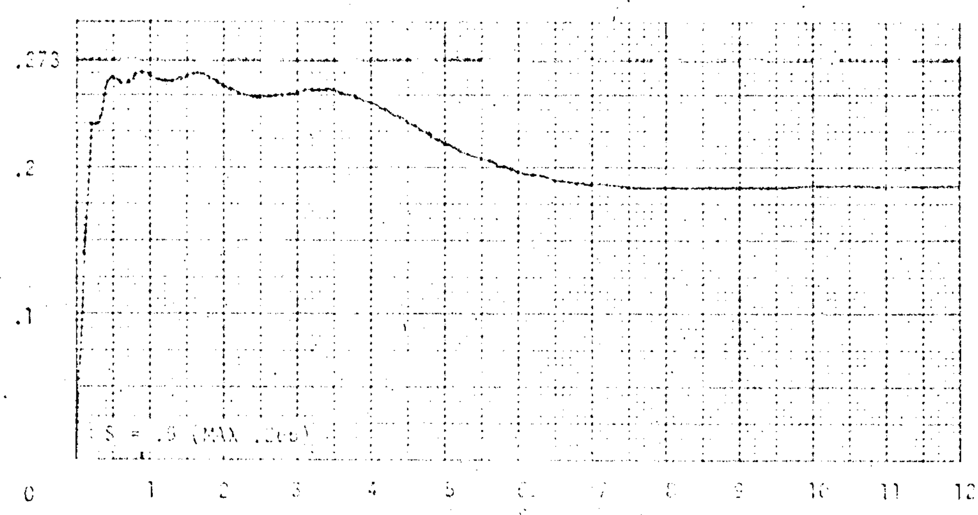

Figure [illegible].- $\sum$ [illegible] for [illegible] e^{-t} [illegible].

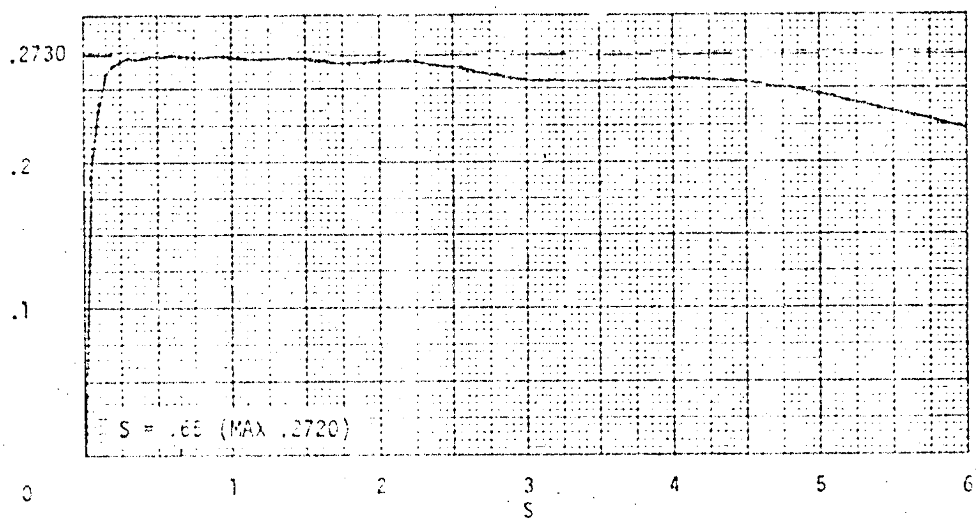

Figure 8.- $\sum_{n=1}^{10} A_n^2$ for $F(t) = e^{-t}\cos(\pi t)$.

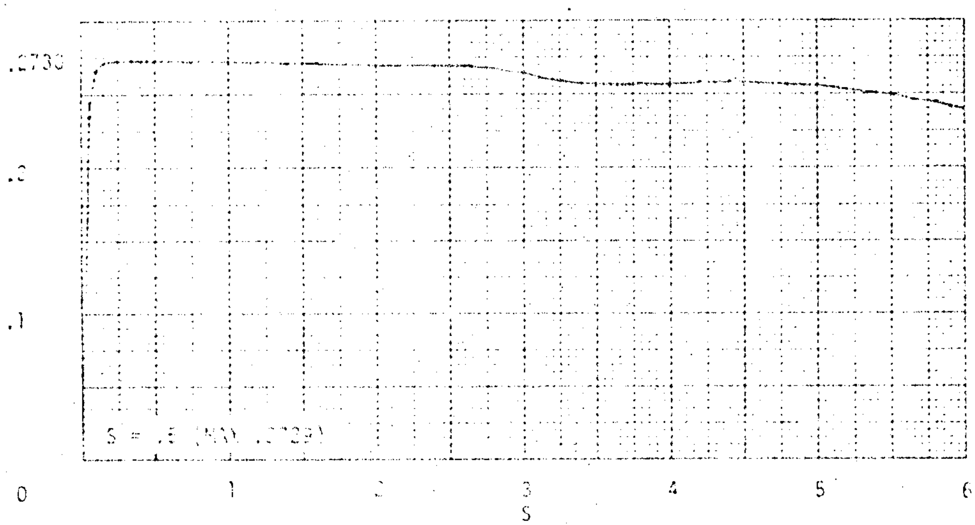

Figure [illegible].- $\sum A_n^2$ for $F(t) = e^{-t}\cos(\pi t)$.

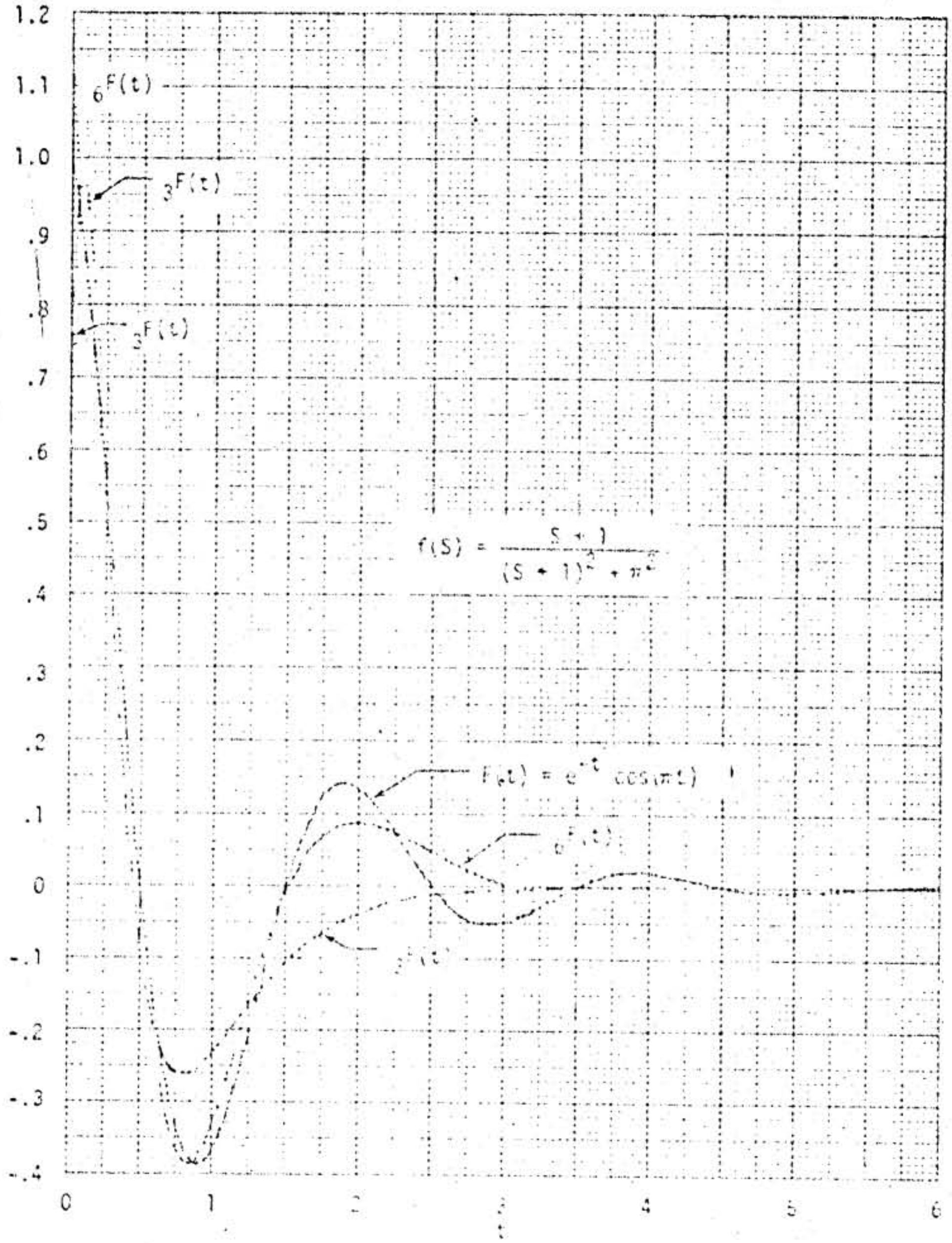

Figure 10.- Approximations of $F(t) = e^{-t}\cos(\pi t)$.

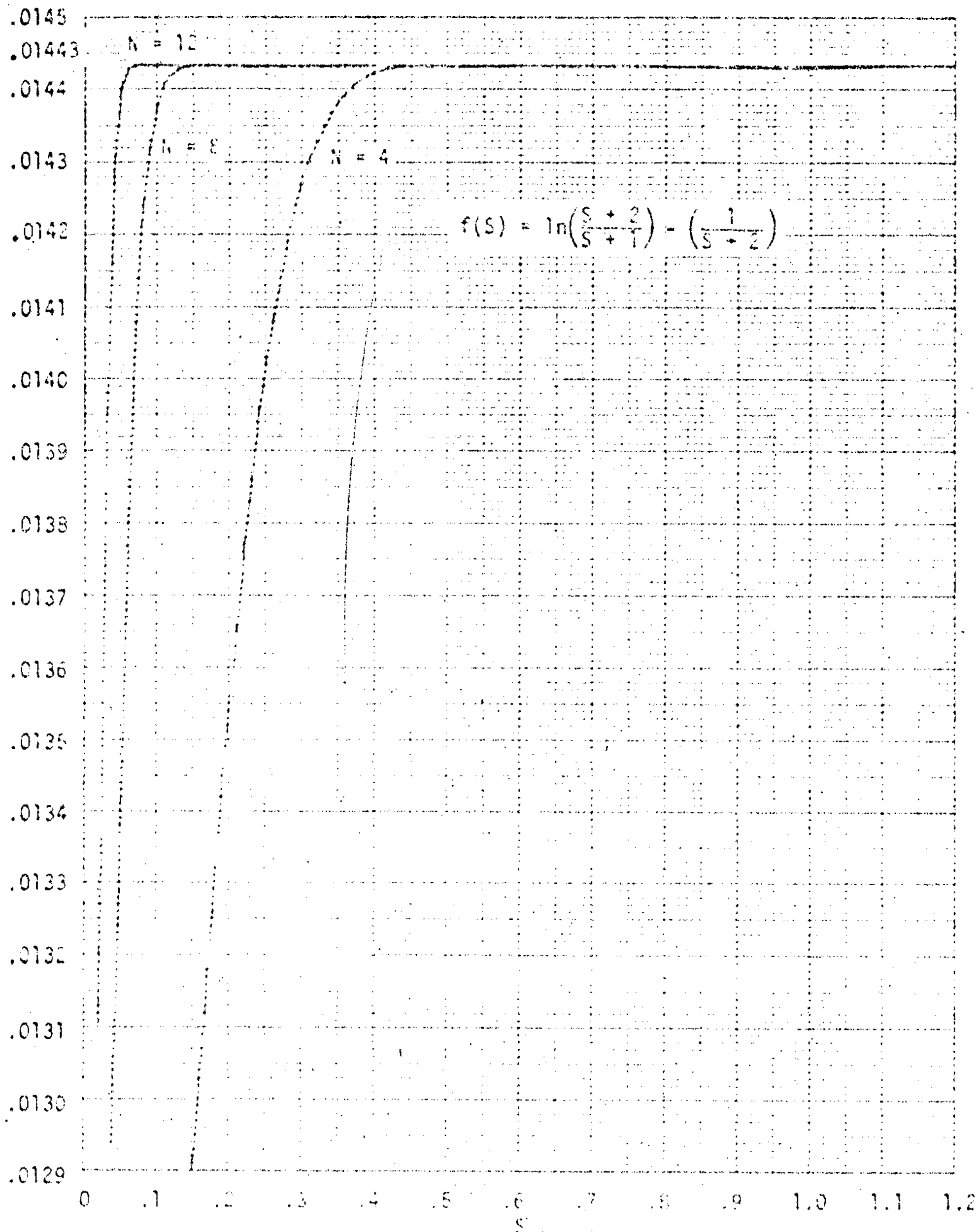

Figure 11.- $\sum A_n$ for $F(t) = \frac{1}{2}(e^{-t}$ - [illegible]

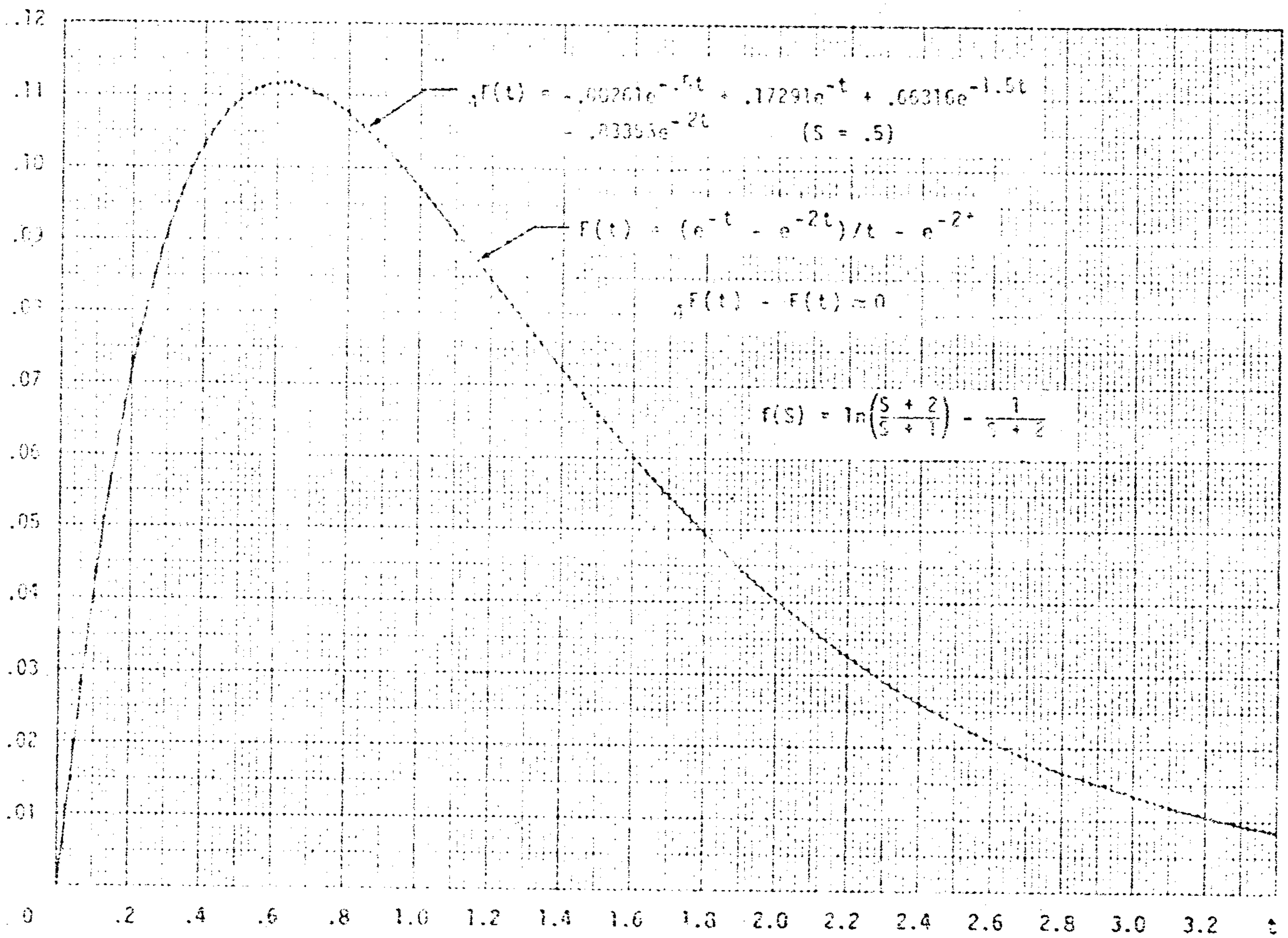

Figure 12.- Approximation of $F(t) = \frac{1}{t}(e^{-t} - e^{-2t}) - e^{-2t}$.

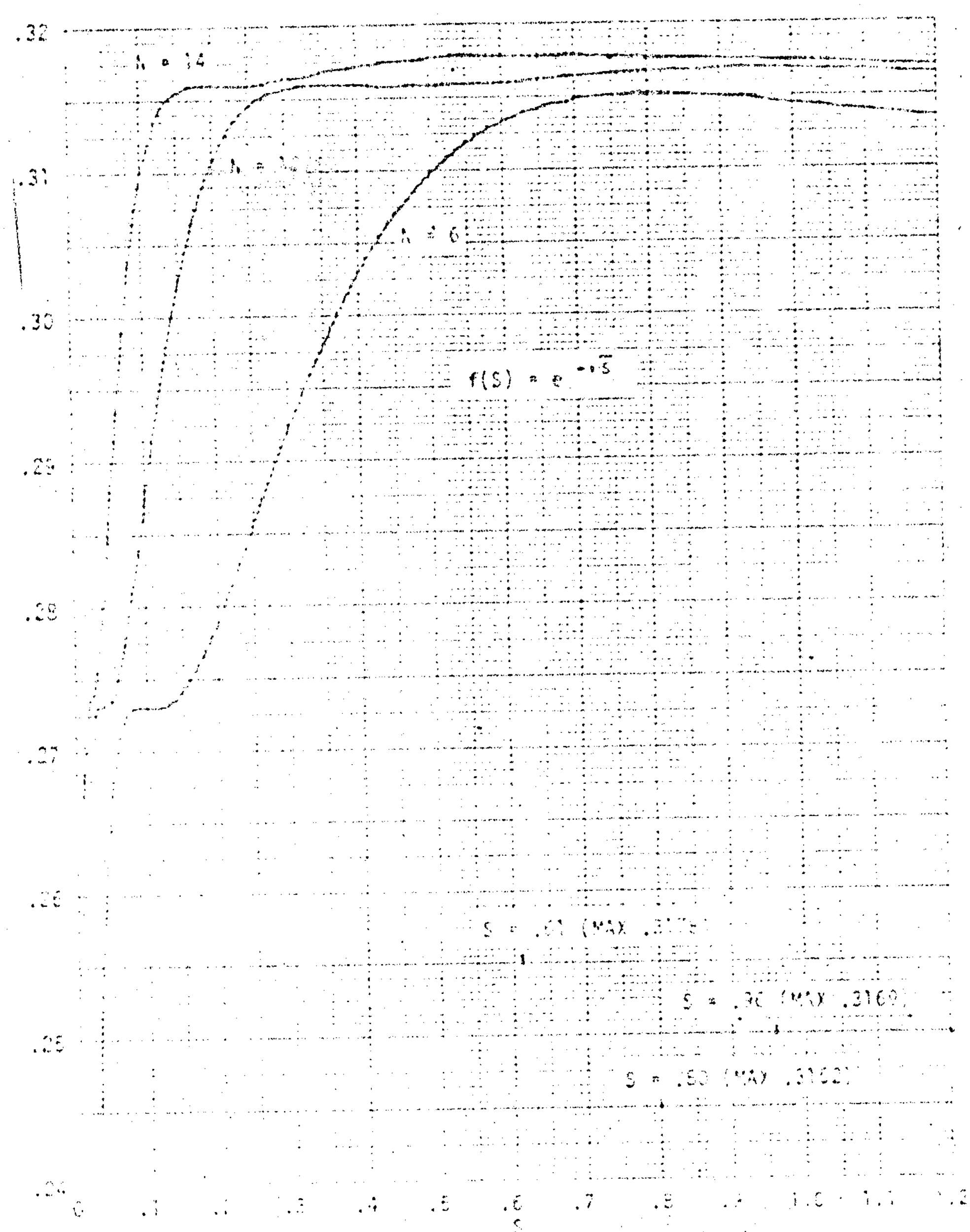

Figure [illegible] $\sum$ [illegible] for [illegible]

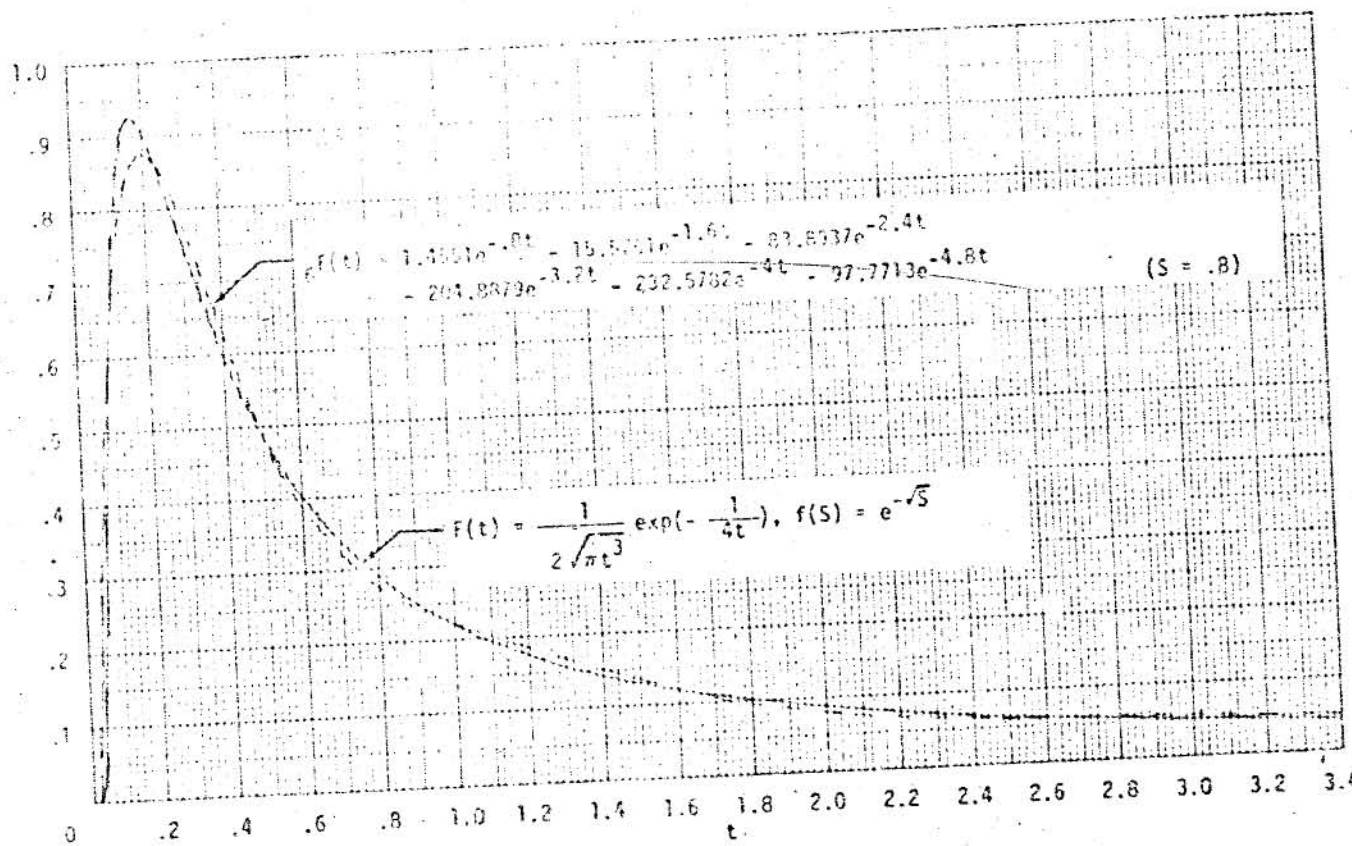

Figure 14.- Approximation of $F(t) = \frac{1}{2\sqrt{\pi t^3}} \exp(-\frac{1}{4t})$.

APPENDIX

THE L_n FUNCTIONS AND PROPERTIES

For brevity, the theorems and lemmas presented here will be shown without proof.

Definition 1:

The scalar product of $f(t)$ and $g(t)$ will be defined by

$$(f,g) = \int_0^\infty f(t)g(t)dt \tag{1}$$

Definition 2:

Define $L_n(st)$ by

$$L_n(st) = \sum_{i=1}^{n} {}_na_i e^{-ist} \tag{2}$$

where

$${}_na_i = (-1)^{n+i}\sqrt{2sn}\,\frac{(n+i-1)!}{i!(i-1)!(n-i)!} \tag{3}$$

Alternately

$${}_na_i = (-1)^{n+i}\sqrt{2sn}\,\frac{n}{i!(i-1)!}\prod_{j=1}^{i-1}(n^2 - j^2) \tag{4}$$

where

$${}_na_1 = (-1)^{n+1}\sqrt{2sn}\; n \tag{5}$$

Lemma 1:

For $n > 1$

$$\sum_{i=1}^{n} \frac{n^{a_i}}{x + i} = \sqrt{2sn} \frac{(x - 1)(x - 2)\cdots(x - (n - 1))}{(x + 1)(x + 2)\cdots(x + (n - 1))(x + n)} \qquad (6)$$

Corollary A:

$$\int_0^\infty L_n(st)dt = (-1)^{n+1} \frac{2}{\sqrt{2ns}} \qquad (7)$$

or

$$\sum_{i=1}^{n} \frac{n^{a_i}}{i} = (-1)^{n+1}\sqrt{2sn}\ \frac{1}{n} \qquad (8)$$

Corollary B:

$$L_n(0) = \sqrt{2ns} \qquad (9)$$

or

$$\sum_{i=1}^{n} n^{a_i} = \sqrt{2ns} \qquad (10)$$

Theorem 1:

The system of functions $L_n(st)$ are orthonormal. That is

$$\begin{aligned}(L_n, L_m) &= 0 \quad \text{for } n \neq m \\ &= 1 \quad \text{for } n = m\end{aligned} \qquad (11)$$

Definition 4:

The generating function $g(z,t)$ is defined as

$$g(z,t) = 1 - \frac{1}{\sqrt{1 + \dfrac{4z}{(1-z)^2} e^{-st}}} = g(1/z, t) \qquad (12)$$

Theorem 2:

Expansion of $g(z,t)$ into Maclaurin's series gives

$$g(z,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2ns}} z^n L_n(st) \qquad z^2 \leq 1 \qquad (13)$$

$$g(z,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2ns}} \frac{1}{z^n} L_n(st) \qquad z^2 \geq 1 \qquad (14)$$

Theorem 3:

The difference equation satisf· :d by $L_n(st)$ is

$$L_n = 2\,\frac{2n-1}{\sqrt{n(n-1)}}\left\{\left[e^{-st} - \frac{n(n-1)}{2n-1} + \frac{(n-1)(n-2)}{2n-3}\right]L_{n-1} - \frac{1}{2}\,\frac{\sqrt{(n-1)(n-2)}}{2n-3}L_{n-2}\right\} \tag{15}$$

Theorem 4:

The differential equation satisfied by L_n is

$$(e^{st} - 1)\ddot{L}_n + se^{st}\dot{L}_n + s^2n^2L_n = 0 \tag{16}$$

Also of interest is

$$L_n = (-1)^{n+1}\,\frac{\sqrt{2ns}}{(n-1)!}\;\frac{d^{n-1}}{d(e^{-st})^{n-1}}\left[e^{-nst}(1 - e^{-st})^{n-1}\right] \tag{17}$$

Theorem 5:

$$e^{-nst} = \frac{2}{s}\,n!(n-1)!\sum_{i=1}^{n}\frac{\sqrt{i}}{(n+i)!(n-i)!}L_i(st) \tag{18}$$

Definition 5:

Let

$$\int_0^{\infty} F(t)^2dt$$

be finite.

Let $_NF(t)$ be an approximation of $F(t)$. The integral square error is defined by

$$E = \int_0^\infty ({}_NF(t) - F(t))^2 dt \qquad (19)$$

Theorem 6:

The best approximation of F(t) in the integral square error sense (E minimized) is given by

$$_NF(t) = \sum_{n=1}^{N} A_n L_n(st) \qquad (20)$$

where

$$A_n(s) = \int_0^\infty F(t) L_n(st) dt \qquad (21)$$

The integral square error is now given by

$$E = \int_0^\infty F(t)^2 dt - \sum_{n=1}^{N} A_n^2 \geq 0 \qquad (22)$$

E is minimized by choosing s such that $\sum_{n=1}^{N} A_n^2$ is maximum.

Theorem 7, completeness theorem:

If

$$\int_0^\infty F(t)^2 dt$$

is finite, and the Laplace transform of $F(t)$, $f(s)$ exists, then

$$E \to 0 \quad \text{as} \quad N \to \infty$$

Theorem 8:

Let the Laplace transform of $F(t)$ be

$$f(s) = \int_0^\infty F(t) e^{-st} dt \tag{23}$$

Then

$$A_n(s) = \sum_{i=1}^{n} {}_n a_i f(is) \tag{24}$$

Theorem 9:

${}_N F(t)$ can be written as

$$_N F(t) = \sum_{n=1}^{N} {}_N B_n e^{-nst} \tag{25}$$

where

$$_N B_n = 2s \left[{}_N b_{n1} f(s) + {}_N b_{n2} f(2s) + {}_N b_{n3} f(3s) + \cdots + {}_N b_{nN} f(Ns) \right] \tag{26}$$

and where

$$ {}_Nb_{ij} = {}_Nb_{ji} = \frac{1}{2s} \sum_{k=1}^{N} {}_ka_i \; {}_ka_j \qquad ({}_ka_m = 0 \quad \text{for} \quad m > k) \tag{27} $$

or

$$ {}_Nb_{ij} = {}_Nb_{ji} = \frac{(-1)^{i+j}}{2(i+j)} \frac{1}{i!(i-1)!} \frac{1}{j!(j-1)!} \frac{(N+i)!}{(N-i)!} \frac{(N+j)!}{(N-j)!} \tag{28} $$

Lemma 2:

$$ \sum_{i=1}^{N} \frac{{}_Nb_{ij}}{x+i} = \frac{(-1)^{N-j}}{2} \frac{(N+j)!}{(N-j)!j!(j-1)!} $$

$$ \frac{(x-1)(x-2)\cdots(x-N)}{(x+1)(x+2)\cdots(x+N)} \frac{1}{x-j} \tag{29} $$

Theorem 10:

$$ f(is) = \frac{1}{s} \sum_{n=1}^{N} \frac{{}_NB_n}{i+n} \tag{30} $$

Theorem 11:

$$ \sum_{n=1}^{N} A_n^2 = \sum_{n=1}^{N} {}_NB_n f(ns) \tag{31} $$

where ${}_NB_n$ was given by equation 26.

Theorem 12:

$$_NF(0) = \sum_{n=1}^{N} \sqrt{2ns}\, A_n \tag{32}$$

Theorem 13:

$$\int_0^\infty G(t)L_m(st)L_n(st)dt = \sum_{j=1}^{m} {}_ma_j\, {}_ng_j \tag{33}$$

where

$$_ng_j = \sum_{i=1}^{n} {}_na_i g\big((i + j)s\big) \tag{34}$$

where $g(s)$ is the Laplace transform of $G(t)$.

Theorem 14:

The best approximation to the jth derivative of $F(t)$ is

$$_NF^{(j)}(t) = \sum_{n=1}^{N} {}_jA_nL_n(st) \tag{35}$$

where

$$ {}_jA_n = \sum_{i=1}^{n} {}_na_i(is)^j f(is) - F(+0) \sum_{i=1}^{n} {}_na_i(is)^{j-1} \qquad (36) $$

$$ - \left.\frac{dF}{dt}\right|_{t=+0} \sum_{i=1}^{n} {}_na_i(is)^{j-2} - \left.\frac{d^2F}{dt^2}\right|_{t=+0} \sum_{i=1}^{n} {}_na_i(is)^{j-3} $$

$$ - \cdots - \left.\frac{d^{j-1}F}{dt^{j-1}}\right|_{t=+0} \sum_{i=1}^{n} {}_na_i $$

Note

$$ {}_NF^{(j)}(t) \neq \frac{d^j {}_NF(t)}{dt^j} \qquad (37) $$

For example, if $j = 1$, the first derivative, then

$$ {}_1A_n = \sum_{i=1}^{n} {}_na_i(is)f(is) - \sqrt{2sn}\, F(+0) \qquad (38) $$

Note equation 10, corollary B,

$$ \sum_{i=1}^{n} {}_na_i = \sqrt{2ns} $$

was used to obtain equation 38. The value of $F(+0)$ can be obtained from the initial value theorem.

$$ F(+0) = \lim_{s \to \infty} sf(s) \qquad (39) $$

If j = 2, the second derivative, then

$$_2A_n = \sum_{i=1}^{n} n^{a_i}(is)^2 f(is) - \sqrt{2sn}\, n^2 sF(+0) - \sqrt{2sn}\, \frac{dF}{dt}\bigg|_{t=+0} \tag{40}$$

If j = 3

$$_3A_n = \sum_{i=1}^{n} n^{a_i}(is)^3 f(is) - \sqrt{2sn}\, \frac{n^2}{2}(n^2 + 1)s^2 F(+0)$$

$$- \sqrt{2sn}\, n^2 s \frac{dF}{dt}\bigg|_{t=+0} - \sqrt{2sn}\, \frac{d^2F}{dt^2}\bigg|_{t=+0} \tag{41}$$

For j = 4

$$_4A_n = \sum_{i=1}^{n} n^{a_i}(is)^4 f(is) - \sqrt{2sn}\, \frac{n^2}{6}(n^4 + 4n^2 + 1)s^3 F(+0)$$

$$- \frac{1}{2}\sqrt{2sn}\, n^2(n^2 + 1)s^2 \frac{dF}{dT}\bigg|_{t=+0} - \sqrt{2sn}\, n^2 s \frac{d^2F}{dt^2}\bigg|_{t=+0}$$

$$- \sqrt{2sn}\, \frac{d^3F}{dt^3}\bigg|_{t=+0} \tag{42}$$

Theorem 15:

If

$$f(s) = \frac{A}{s + a} \tag{43}$$

then

$$A_n = (-1)^{n+1} A \sqrt{2ns} \frac{(s - a)(2s - a)\cdots\left((n - 1)s - a\right)}{(s + a)(2s + a)\cdots(ns + a)} \tag{44}$$

$$A_1 = A \sqrt{2s} \frac{1}{s + a} \tag{45}$$

Note the results for $A = 1$ and $a = 0$, $F(t)$ a unit step function. In this case

$$A_n = (-1)^{n+1} \frac{2}{\sqrt{2ns}} \tag{46}$$

Hence

$$_N F(t) = 2 \sum_{n=1}^{N} \frac{(-1)^{n+1}}{\sqrt{2ns}} L_n(st) \tag{47}$$

From corollary B, $L_n(0) = \sqrt{2ns}$, hence

$$\begin{aligned} {}_N F(0) &= 0 \qquad N \text{ even} \\ &= 2 \qquad N \text{ odd} \end{aligned} \tag{48}$$

The equations shown in theorem 15 are useful for testing the accuracy of computer computations.

END

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