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Discrete Tchebycheff Orthonormal Polynomials and Applications

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SHUTTLE PROGRAM

DISCRETE TCHEBYCHEFF ORTHONORMAL POLYNOMIALS AND APPLICATIONS

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CONTENTS

TABLES

FIGURES

1.0 INTRODUCTION

Discrete Tchebycheff orthonormal polynomials offer a convenient way to make least-squares polynomial fits of uniformly spaced discrete data. Computer programs to do so are simple and fast, and appear to be less affected by computer roundoff error, for the higher order fits, than conventional least-squares programs.

P. Tchebycheff, a Russian mathematician, developed these polynomials about 100 years ago.[a] Their application to modern computer technology is very germane, but few people know about them. They are useful for any application of polynomial least-squares fits: approximation of mathematical functions, noise analysis of radar data, and real time smoothing (filtering) of noisy data, to name a few.

Uniformly spaced, discrete data, as shown below

x	n	y*
x_1	1	y_1^*
x_2	2	y_2^*
.	.	.
.	.	.
.	.	.
x_N	N	y_N^*

has an Ith order least-squares polynomial fit given by

$$y(n) = \hat{A}_0\overline{T}_0(n) + \hat{A}_1\overline{T}_1(n) + \hat{A}_2\overline{T}_2(n) + \cdots + \hat{A}_I\overline{T}_I(n) \qquad (1.1)$$

where the $\overline{T}_i(n)$ are the discrete Tchebycheff orthonormal polynomials of n.

The $\hat{A}_i$ coefficients are called the Fourier coefficients and are given by

$$\hat{A}_i = \sum_{n=1}^{N} y_n^*\overline{T}_i(n) \qquad (1.2)$$

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[a] Erdélyi, A., ed.: Higher Transcendental Functions, volume II, McGraw-Hill Book Company, 1953, section 10.22.

A simple recursion formula is used to generate the $\overline{T}_i(n)$ polynomials. Note an important fact: the $\hat{A}_i$ coefficients are independent of the order of the polynomial fit. That is, an $I + 1$ order fit uses the same values of $\hat{A}_i$ as an Ith order fit, only the $\hat{A}_{I+1}\,\overline{T}_{I+1}(n)$ term is added to increase the fit order. Let

$$y_n^* = y_n + \epsilon_n \tag{1.3}$$

where y_n is the error-free value of y, and ϵ_n is the error (noise) adding to y_n. Equations will be developed to estimate the statistics of ϵ, and equations will be derived showing how accurately y_n and its derivatives can be predicted.

When ϵ_n is highly correlated, a surprising result is shown. The fit polynomial at the end points of the fit can be less accurate than the raw data. This newly discovered information bears reemphasizing. There are instances where the raw (noisy) data is more accurate than that of the polynomial fit (filtered) data.

2.0 THE DISCRETE TCHEBYCHEFF POLYNOMIALS

First shown will be a set of Tchebycheff orthogonal polynomials.

$$T_0 = 1$$

$$T_1 = 1 + a_{11}n$$

$$T_2 = 1 + a_{21}n + a_{22}n^2$$

$$T_3 = 1 + a_{31}n + a_{32}n^2 + a_{33}n^3$$

$$\vdots$$

$$T_i = 1 + a_{i1}n + a_{i2}n^2 + a_{i3}n^3 + \cdots + a_{ii}n^i$$

The scalar product of the two variables $f(n)$ and $g(n)$ is defined by

$$(f,\ g) = \sum_{n=1}^{N} f(n)g(n) \tag{2.1}$$

Since the polynomials are orthogonal, then by definition

$$(T_i, T_j) = 0 \quad \text{for} \quad i \neq j \tag{2.2}$$

This equation may be used to determine the a_{ij} coefficients. That is

$$\sum_{n=1}^{N} T_i(n)T_j(n) = 0 \quad i \neq j \tag{2.3}$$

Note: the original discrete Tchebycheff polynomials used a summation of $n = 0$ to $n = N - 1$, op cit.

It has been determined that the norm of T_i is given by

$$(T_i, T_i) = \sum_{n=1}^{N} T_i^2 = \frac{N}{2i + 1} \frac{(N - 1)(N - 2)\cdots(N - i)}{(N + 1)(N + 2)\cdots(N + i)} \tag{2.4}$$

The <u>orthonormal</u> polynomials are given by

$$\overline{T}_i = T_i/\sqrt{(T_i, T_i)} \tag{2.5}$$

where

$$\overline{T}_i = \overline{a}_{i0} + \overline{a}_{i1}n + \overline{a}_{i2}n^2 + \overline{a}_{i3}n^3 + \cdots + \overline{a}_{ii}n^i \tag{2.6}$$

Note that

$$\begin{aligned}(\overline{T}_i, \overline{T}_j) &= 0 \quad \text{for} \quad i \neq j \\ &= 1 \quad \text{for} \quad i = j\end{aligned}$$

The first eight sets of the a_{ij} coefficients are shown as follows.

$$a_{11} = -\frac{2}{N + 1}$$

$$a_{21} = -6\,\frac{N + 1}{(N + 1)(N + 2)}$$

$$a_{22} = \frac{6}{(N + 1)(N + 2)}$$

$$a_{31} = -2\,\frac{6N^2 + 15N + 11}{(N + 1)(N + 2)(N + 3)}$$

$$a_{32} = 30\,\frac{N + 1}{(N + 1)(N + 2)(N + 3)}$$

$$a_{33} = -\frac{20}{(N + 1)(N + 2)(N + 3)}$$

$$a_{41} = -10\,\frac{2N^3 + 9N^2 + 17N + 10}{(N + 1)(N + 2)(N + 3)(N + 4)}$$

$$a_{42} = 10\,\frac{9N^2 + 21N + 17}{(N + 1)(N + 2)(N + 3)(N + 4)}$$

$$a_{43} = -140\,\frac{N + 1}{(N + 1)(N + 2)(N + 3)(N + 4)}$$

$$a_{44} = \frac{70}{(N + 1)(N + 2)(N + 3)(N + 4)}$$

$$a_{51} = -2 \frac{15N^4 + 105N^3 + 365N^2 + 525N + 274}{(N+1)(N+2)(N+3)(N+4)(N+5)}$$

$$a_{52} = 210 \frac{N^3 + 4N^2 + 8N + 5}{(N+1)(N+2)(N+3)(N+4)(N+5)}$$

$$a_{53} = -140 \frac{4N^2 + 9N + 8}{(N+1)(N+2)(N+3)(N+4)(N+5)}$$

$$a_{54} = 630 \frac{N + 1}{(N+1)(N+2)(N+3)(N+4)(N+5)}$$

$$a_{55} = - \frac{252}{(N+1)(N+2)(N+3)(N+4)(N+5)}$$

$$a_{61} = -42 \frac{N^5 + 10N^4 + 55N^3 + 140N^2 + 178N + 84}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)}$$

$$a_{62} = 42 \frac{10N^4 + 60N^3 + 215N^2 + 315N + 178}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)}$$

$$a_{63} = -420 \frac{4N^3 + 15N^2 + 32N + 21}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)}$$

$$a_{64} = 210 \frac{15N^2 + 33N + 32}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)}$$

$$a_{65} = -2772 \frac{N + 1}{(N+1)(N+2)(N+3)(N+4)(N+5)(N+6)}$$

$$a_{66} = \frac{924}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)}$$

$$a_{71} = -4 \frac{14N^6 + 189N^5 + 1505N^4 + 5880N^3 + 13223N^2 + 14847N + 6534}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)}$$

$$a_{72} = 42 \frac{18N^5 + 150N^4 + 835N^3 + 2100N^2 + 2811N + 1414}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)}$$

$$a_{73} = -84 \frac{50N^4 + 275N^3 + 1030N^2 + 1540N + 937}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)}$$

$$a_{74} = 2310 \frac{5N^3 + 18N^2 + 41N + 28}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)}$$

$$a_{75} = -924 \frac{18N^2 + 39N + 41}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)}$$

$$a_{76} = 12012 \frac{N + 1}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)}$$

$$a_{77} = -\frac{3432}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)}$$

$$a_{81} = -12 \frac{6N^7 + 105N^6 + 1141N^5 + 6300N^4 + 21763N^3 + 42399N^2 + 44158N + 18264}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)}$$

$$a_{82} = 6 \frac{210N^6 + 2310N^5 + 18375N^4 + 69300N^3 + 160951N^2 + 187110N + 88316}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)}$$

$$a_{83} = -4620 \frac{2N^5 + 15N^4 + 66N^3 + 216N^2 + 305N + 162}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)}$$

$$a_{84} = 2310 \frac{15N^4 + 78N^3 + 307N^2 + 468N + 305}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)}$$

$$a_{85} = -36036 \frac{2N^3 + 7N^2 + 17N + 12}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)}$$

$$a_{86} = 12012 \frac{7N^2 + 15N + 17}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)}$$

$$a_{87} = -51480 \frac{N + 1}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)}$$

$$a_{88} = \frac{12870}{(N + 1)(N + 2)(N + 3)(N + 4)(N + 5)(N + 6)(N + 7)(N + 8)}$$

Note that all the odd order polynomials in the numerators of the above coefficients have a factor of $N + 1$.

Several interesting relationships concerning the a_{ij} coefficients have been observed. Some of them are

$$T_i(1) = 1 + a_{i1} + a_{i2} + \cdots + a_{ii} = \frac{(N - 1)(N - 2)\cdots(N - i)}{(N + 1)(N + 2)\cdots(N + i)} \tag{2.7}$$

$$T_i(N) = 1 + a_{i1}N + a_{i2}N^2 + \cdots + a_{ii}N^i = (-1)^i \frac{(N - 1)(N - 2)\cdots(N - i)}{(N + 1)(N + 2)\cdots(N + i)} \tag{2.8}$$

For i odd, the midpoint value of T_i is

$$T_i\left(\frac{N+1}{2}\right) = 1 + a_{i1}\frac{N+1}{2} + a_{i2}\left(\frac{N+1}{2}\right)^2 + \cdots + a_{ii}\left(\frac{N+1}{2}\right)^i = 0 \tag{2.9}$$

For i even[a]

$$T_i\left(\frac{N+1}{2}\right) = \frac{1 \times 3 \times 5 \times \cdots \times (i-1)}{2 \times 4 \times 6 \times \cdots \times i} \; \frac{(N-1)(N-3)\cdots(N-(i-1))}{(N+2)(N+4)\cdots(N+i)} \tag{2.10}$$

Other, more complex relations are shown below. Let

$$a_{i0} = \frac{(N+1)(N+2)\cdots(N+i)}{(N+1)(N+2)\cdots(N+i)} = 1 \tag{2.11}$$

$$a^*_{ij} = a_{ij} \text{ with } N = -N \text{ in the numerator.} \tag{2.12}$$

The for $k = 0, 1, 2, \cdots, i$

$$\sum_{j=k}^{i} \frac{j!}{(j-k)!} N^{j-k} a_{ij} = k! a^*_{ik} \tag{2.13}$$

$$\sum_{j=k}^{i} \frac{j!}{(j-k)!} a_{ij} = (-1)^{i+k} k! a^*_{ik} \tag{2.14}$$

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[a] $T_0\left(\frac{N+1}{2}\right) = 1.$

From these two equations, it is easily seen that

$$\sum_{j=k}^{i} \frac{j!}{(j-k)!}\left(N^{j-k} - (-1)^{i+k}\right)a_{ij} = 0 \tag{2.15}$$

Also observed were the two important relationships

$$a_{ii} = -\frac{2}{i}(2i-1)\frac{a_{i-1,\, i-1}}{N+i} \tag{2.16}$$

$$a_{i,i-1} = -\frac{i}{2}(N+1)a_{ii} \tag{2.17}$$

The $T_i(n)$ polynomials have the following symmetry properties about $n = (N + 1)/2$.

$$T_i\left(\frac{N+1}{2} + m\right) = (-1)^i T_i\left(\frac{N+1}{2} - m\right) \tag{2.18}$$

Other interesting relationships are

$$(n^j, T_i) = \sum_{n=1}^{N} n^j T_i = 0 \quad \text{for} \quad 0 \leq j < i \tag{2.19}$$

$$\sum_{n=1}^{N} n^i T_i = \frac{1}{a_{i+1,i}} \frac{(N+1)N(N-1)(N-2)\cdots(N-i)}{(N+1)(N+2)\cdots(N+i+1)} \tag{2.20}$$

$$\sum_{n=1}^{N} n^{i+1} T_i = \frac{1}{a_{i+1,i+1}} \frac{(N+1)N(N-1)(N-2)\cdots(N-i)}{(N+1)(N+2)\cdots(N+i+1)} \tag{2.21}$$

Note from equation 2.19 that T_i $(i > 0)$ has an average value of zero.

The relationship to Legendre polynomials is shown below. Let

$$n = \frac{N}{2}(x + 1) \tag{2.22}$$

Then for $N \to \infty$

$$T_i(n) \to (-1)^i P_i(x) \tag{2.23}$$

where the $P_i(x)$ are the well known Legendre polynomials.

The equations for $\overline{T}_0$ through $\overline{T}_4(n)$ are shown below.

$$\overline{T}_0 = \frac{1}{\sqrt{N}} \tag{2.24}$$

$$\overline{T}_1(n) = \frac{\sqrt{3}}{\sqrt{N}} \frac{1}{\sqrt{N^2 - 1^2}}(N + 1 - 2n) \tag{2.25}$$

$$\overline{T}_2(n) = \frac{\sqrt{5}}{\sqrt{N}} \frac{1}{\sqrt{(N^2 - 1^2)(N^2 - 2^2)}}\left[(N + 1)(N + 2) - 6(N + 1)n + 6n^2\right] \tag{2.26}$$

$$\overline{T}_3(n) = \frac{\sqrt{7}}{\sqrt{N}} \frac{1}{\sqrt{(N^2 - 1^2)(N^2 - 2^2)(N^2 - 3^2)}}\left[(N + 1)(N + 2)(N + 3) - 2(6N^2 + 15N + 11)n + 30(N + 1)n^2 - 20n^3\right] \tag{2.27}$$

$$\overline{T}_4(n) = \frac{\sqrt{9}}{\sqrt{N}} \frac{1}{\sqrt{(N^2 - 1^2)(N^2 - 2^2)(N^2 - 3^2)(N^2 - 4^2)}}\left[(N + 1)(N + 2)(N + 3)(N + 4) - 10(2N^3 + 9N^2 + 17N + 10)n + 10(9N^2 + 21N + 17)n^2 - 140(N + 1)n^3 + 70n^4\right] \tag{2.28}$$

Plots of $\overline{T}_0$ through $\overline{T}_4(n)$ are shown in figure 1. The value of N was 119. Note the symmetry about the midpoint of $n = (N + 1)/2 = 60$. Also note, from equation 2.19, that for $i > 0$ the mean value of $\overline{T}_i(n)$ is zero.

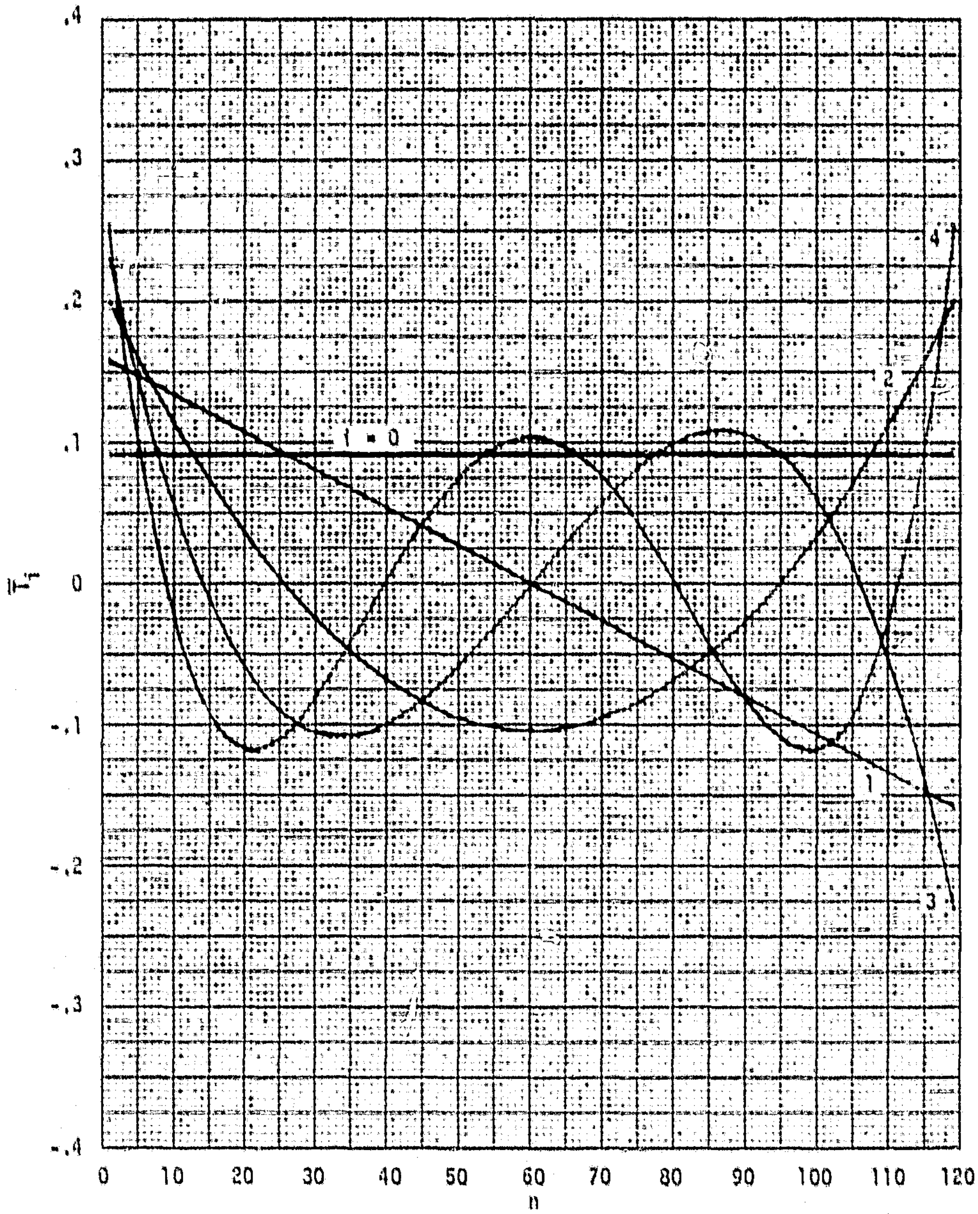

Figure 1.- Plots of the orthonormal functions, $\bar{T}_i(n)$, (N = 119 points).

3.0 RECURSION EQUATIONS

The recursion equations are the best way to generate the Tchebycheff polynomials and the a_{ij} and $\overline{a}_{ij}$ coefficients.

$$T_i(n) = \frac{1}{i(N + i)}\Big((2i - 1)(N + 1 - 2n)T_{i-1}(n)$$

$$- (i - 1)(N + 1 - i)T_{i-2}(n)\Big) \tag{3.1}$$

where

$$T_0(n) = 1 \tag{3.2}$$

$$T_1(n) = 1 - \frac{2}{N + 1}n \tag{3.3}$$

The recursion relationship for the Tchebycheff orthonormal polynomials, $\overline{T}_i(n)$, is given by

$$\overline{T}_0 = 1/\sqrt{N} \tag{3.4}$$

$$\overline{T}_1 = \left(1 - \frac{2}{N + 1}n\right)\sqrt{\frac{3(N + 1)}{N(N - 1)}} \tag{3.5}$$

$$S_i = \frac{1}{i}\sqrt{\frac{4i^2 - 1}{N^2 - i^2}} \tag{3.6}$$

$$\overline{T}_i(n) = S_i \left[(N + 1 - 2n)\overline{T}_{i-1}(n) - \frac{1}{S_{i-1}}\overline{T}_{i-2}(n)\right] \tag{3.7}$$

To generate the a_{ij} coefficients in

$$T_i(n) = a_{i0} + a_{i1}n + a_{i2}n^2 + \cdots + a_{ii}n^i \tag{3.8}$$

use

$$a_{i0} = 1 \qquad i = 0, 1, \cdots, I \tag{3.9}$$

$$a_{ii} = -\frac{2(2i - 1)}{i(N + i)}a_{i-1,i-1} \qquad i = 1, 2, \cdots, I \tag{3.10}$$

$$a_{i,i-1} = -\frac{i}{2}(N + 1)a_{ii} \qquad i = 2, 3, \cdots, I \tag{3.11}$$

$$a_{ij} = \frac{1}{i(N + i)}\Big((2i - 1)((N + 1)a_{i-1,j} - 2a_{i-1,j-1}) - (i - 1)(N + 1 - i)a_{i-2,j}\Big) \tag{3.12}$$

where $i = 3, 4, \cdots, I$ and $j = 1, 2, \cdots, i - 2$.

To generate the $\bar{a}_{ij}$ coefficients in

$$\bar{T}_i(n) = \bar{a}_{i0} + \bar{a}_{i1}n + \bar{a}_{i2}n^2 + \cdots + \bar{a}_{ii}n^i \tag{3.13}$$

use

$$S_i = \frac{1}{i}\sqrt{\frac{4i^2 - 1}{N^2 - i^2}} \qquad i = 2, 3, \ldots, I \tag{3.14}$$

$$\bar{a}_{00} = \frac{1}{\sqrt{N}} \tag{3.15}$$

$$\bar{a}_{10} = \frac{3(N + 1)}{N(N - 1)} \tag{3.16}$$

$$\bar{a}_{11} = -\frac{2}{N + 1}\bar{a}_{10} \tag{3.17}$$

$$\bar{a}_{i0} = \sqrt{\frac{2i + 1}{N}\frac{(N + 1)(N + 2)\cdots(N + i)}{(N - 1)(N - 2)\cdots(N - i)}} \qquad i = 2, 3, \cdots, I \tag{3.18}$$

$$\bar{a}_{ii} = -2S_i\,\bar{a}_{i-1,i-1} \qquad i = 2, 3, \cdots, I \tag{3.19}$$

$$\bar{a}_{i,i-1} = -\frac{i}{2}(N + 1)\bar{a}_{ii} \qquad i = 2, 3, \cdots, I \tag{3.20}$$

$$\bar{a}_{ij} = S_i\left[(N + 1)\bar{a}_{i-1,j} - 2\bar{a}_{i-1,j-1} - \frac{1}{S_{i-1}}\bar{a}_{i-2,j}\right] \tag{3.21}$$

for $i = 3, 4, \cdots, I$ and $j = 1, 2, \cdots, i - 2$.

Let

$$\bar{T}_i'(n) = \frac{d\bar{T}_i}{dn} \tag{3.22}$$

The recursion relationship for the first derivative is obtained by differentiating equation 3.7.

$$\bar{T}_i'(n) = S_i\left[-2\bar{T}_{i-1} + (N + 1 - 2n)\bar{T}_{i-1}'(n) - \frac{1}{S_{i-1}}\bar{T}_{i-2}'(n)\right] \tag{3.23}$$

The following interesting relationships can be obtained for derivatives.

For i even,

$$\sum_{n=1}^{N} \frac{d^k \overline{T}_i}{dn^k} = 0 \qquad \text{for} \quad k = 1, 3, 5, 7, \cdots \tag{3.24}$$

For i odd,

$$\sum_{n=1}^{N} \frac{d^k \overline{T}_i}{dn^k} = 0 \qquad \text{for} \quad k = 0, 2, 4, 6, \cdots \tag{3.25}$$

4.0 LEAST-SQUARES FITS AND ACCURACY

Consider a uniformly spaced set of data to be fit, y^*, as shown below.

x	n	y^*
x_1	1	y_1^*
x_2	2	y_2^*
.	.	.
.	.	.
.	.	.
x_N	N	y_N^*

where $\Delta x = x_i - x_{i-1}$ is constant, and

$$\boxed{n = \frac{x - x_1}{\Delta x} + 1} \tag{4.1}$$

For an Ith order polynomial fit, let

$$\boxed{\hat{y}_n = \hat{A}_0 \overline{T}_0 + \hat{A}_1 \overline{T}_1 + \hat{A}_2 \overline{T}_2 + \cdots + \hat{A}_I \overline{T}_I} \tag{4.2}$$

where

$$\boxed{\hat{A}_i = \sum_{n=1}^{N} y_n^* \bar{T}_i(n)} \tag{4.3}$$

are called the Fourier coefficients. As will be seen, this will result in a conventional least-squares fit.

The residual is defined by

$$\boxed{r_n = y_n^* - \hat{y}_n} \tag{4.4}$$

For a least-squares fit, the $\hat{A}_i$ are chosen such that

$$\sum_{n=1}^{N} r_n^2 = 0 \quad \text{is minimized}$$

That is,

$$\frac{\partial}{\partial \hat{A}_i} \sum_{n=1}^{N} r_n^2 = 0 \qquad \text{for} \quad i = 0, 1, 2, \cdots, I \tag{4.5}$$

It is easily seen that

$$\sum_{n=1}^{N} r_n^2 = \sum_{n=1}^{N} (y_n^*)^2 + \hat{A}_0^2 + \hat{A}_1^2 + \hat{A}_2^2 + \cdots + \hat{A}_I^2$$

$$- 2\hat{A}_0 \sum_{n=1}^{N} y_n^* \overline{T}_0 - 2\hat{A}_1 \sum_{n=1}^{N} y_n^* \overline{T}_1 - \cdots$$

$$- 2\hat{A}_I \sum_{n=1}^{N} y_n^* \overline{T}_I \qquad (4.6)$$

Differentiating with respect to $\hat{A}_i$ gives

$$2\hat{A}_i - 2\sum_{n=1}^{N} y_n^* \overline{T}_i = 0 \qquad (4.7)$$

which yields equation 4.3 for $\hat{A}_i$.

An interesting aspect of a least-squares fit is that

$$\boxed{\sum_{n=1}^{N} r_n = 0} \qquad (4.8)$$

This is seen from

$$\frac{\partial}{\partial \hat{A}_i} \sum_{n=1}^{N} r_n^2 = 2\sum_{n=1}^{N} r_n \frac{\partial r_n}{\partial \hat{A}_i} = -2\sum_{n=1}^{N} r_n \frac{\partial \hat{y}_n}{\partial \hat{A}_i} = 0 \qquad (4.9)$$

or

$$\boxed{\sum_{n=1}^{N} r_n \overline{T}_i(n) = 0} \qquad i = 0, 1, 2, \cdots, I \tag{4.10}$$

But $\overline{T}_0$ is the constant $1/\sqrt{N}$, which will factor out. Hence, the sum of the residuals will be zero. Equation 4.10 indicates that the Fourier coefficients of the residuals will all be zero. That is, an Ith order polynomial fit of the residuals will be zero, and so will an I - 1 order fit, an I - 2 order fit, etc.

The y_n^* values can be represented as

$$\boxed{y_n^* = y_n + \varepsilon_n} \tag{4.11}$$

where y_n is the error-free value and ε_n is the error (noise) adding to y_n. It will be assumed tht ε_n is zero mean. That is, the expected value of ε_n is

$$E(\varepsilon_n) = 0 \tag{4.12}$$

The autocorrelation function of ε_n is defined by

$$\phi_i = E(\varepsilon_n \varepsilon_{n+i}) \tag{4.13}$$

ϕ_i is assumed independent of n, stationary process, and ϕ_i will be symmetric.

$$\phi_i = \phi_{-i} \tag{4.14}$$

The standard deviation of ε_n is defined by

$$\sigma_\varepsilon = \sqrt{\phi_0} \tag{4.15}$$

If the error is "uncorrelated" in time, then

$$\phi_i = 0 \qquad \text{for} \quad i \neq 0 \tag{4.16}$$

The Fourier coefficients of ε are given by

$$\boxed{E_i = \sum_{n=1}^{N} \varepsilon_n \overline{T}_i(n)} \tag{4.17}$$

With a little effort, it can be shown that

$$\boxed{E\left(E_i E_j\right) = \sigma_\varepsilon^2 \delta_{ij} + \sum_{k=1}^{N-1} \phi_k \sum_{n=1}^{N-k} \left(\overline{T}_i(n)\overline{T}_j(n+k) + \overline{T}_j(n)\overline{T}_i(n+k)\right)} \tag{4.18}$$

where

$$\begin{aligned} \delta_{ij} &= 1 \qquad \text{for} \quad i = j \\ &= 0 \qquad \text{for} \quad i \neq j \end{aligned}$$

Note that for uncorrelated noise

$$E\left(E_i^2\right) = \sigma_\varepsilon^2 \tag{4.19}$$

$$E\left(E_i E_j\right) = 0 \qquad \text{for} \quad i \neq j \tag{4.20}$$

It will be assumed that y_n is adequately represented by the Ith order polynomial

$$y_n = A_0\overline{T}_0 + A_1\overline{T}_1 + A_2\overline{T}_2 + \cdots + A_I\overline{T}_I \tag{4.21}$$

where

$$A_i = \sum_{n=1}^{N} y_n \overline{T}_i(n) \tag{4.22}$$

"Adequately" means that the error in equation 4.21 is small compared to σ_ε. The Fourier coefficients for y_n^* were

$$\hat{A}_i = \sum_{n=1}^{N} y_n^* T_i(n)$$

But $y_n^* = y_n + \varepsilon_n$, so

$$\hat{A}_i = A_i + E_i \tag{4.23}$$

And

$$\hat{y}_n = \hat{A}_0 \overline{T}_0 + \hat{A}_1 \overline{T}_1 + \cdots + \hat{A}_I \overline{T}_I$$

becomes

$$\hat{y}_n = y_n + E_0 \overline{T}_0 + E_1 \overline{T}_1 + E_2 \overline{T}_2 + \cdots + E_I \overline{T}_I \tag{4.24}$$

The mean value of the error in $\hat{y}_n$ is easily seen to be

$$E(\hat{y}_n - y_n) = 0 \tag{4.25}$$

The variance of the error for an Ith order fit is

$$\boxed{E((\hat{y}_n - y_n)^2) = \sum_{i=0}^{I} \overline{T}_i^2 E(E_i^2) + 2 \sum_{j=1}^{I} \sum_{i=0}^{I-j} \overline{T}_i \overline{T}_{i+j} E(E_i E_{i+j})} \tag{4.26}$$

For uncorrelated noise, equations 4.19 and 4.20,

$$\boxed{E\left((\hat{y}_n - y_n)^2\right) = \sigma_\varepsilon^2 \sum_{i=0}^{I} \overline{T}_i^2} \tag{4.27}$$

From equations 2.4, 2.7 and 2.8, the endpoint values of $\overline{T}_i^2$ are

$$\boxed{\overline{T}_i^2(1) = \overline{T}_i^2(N) = \frac{2i + 1}{N} \frac{(N - 1)(N - 2)\cdots(N - i)}{(N + 1)(N + 2)\cdots(N + i)}} \tag{4.28}$$

For N large and $n = 1$ or N

$$E\left((\hat{y}_n - y_n)^2\right) = \frac{1}{n}\sigma_\varepsilon^2\left(1 + 3 + 5 + 7 + \cdots + (2I + 1)\right) \tag{4.29}$$

or

$$\boxed{E\left((\hat{y}_1 - y_1)^2\right) = E\left((\hat{y}_N - y_N)^2\right) = (I + 1)^2\sigma_\varepsilon^2/N} \tag{4.30}$$

where I is the order of the fit.

From equations 2.4, 2.9, and 2.10, for the midpoint value of $n = (N + 1)/2$

$$\overline{T}_i^2\left(\frac{N + 1}{2}\right) = 0 \qquad \text{for } i \text{ odd} \tag{4.31}$$

$$\overline{T}_i^2\left(\frac{N + 1}{2}\right) = \frac{1}{N} \qquad \text{for } i = 0 \tag{4.32}$$

And for i even

$$\overline{T}_i^2\left(\frac{N+1}{2}\right) = \frac{2i+1}{N}\left(\frac{1 \times 3 \times 5 \times \cdots \times (i-1)}{2 \times 4 \times 6 \times \cdots \times i}\right)^2 \frac{(N^2-1)(N^2-3^2)\cdots(N^2-(i-1)^2)}{(N^2-2^2)(N^2-4^2)\cdots(N^2-i^2)} \tag{4.33}$$

For N large and $n = (N+1)/2$

$$E\left((\hat{y}_n - y_n)^2\right) = \frac{1}{N}\sigma_\epsilon^2 \left[1 + 0 + \frac{5}{4} + 0 + \frac{81}{64} + 0 + \frac{2925}{2304} + \cdots\right] \tag{4.34}$$

$$I = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

The residual is $r_n = y_n^* - \hat{y}_n$. Substituting equations 4.11 and 4.24 into this yields

$$r_n = \epsilon_n - E_0\overline{T}_0 - E_1\overline{T}_1 - E_2\overline{T}_2 - \cdots - E_I\overline{T}_I \tag{4.35}$$

The sum of the residuals squared is

$$\sum_{n=1}^{N} r_n^2 = \sum_{n=1}^{N} \epsilon_n^2 - 2\sum_{n=1}^{N}(E_0\epsilon_n\overline{T}_0 + E_1\epsilon_n\overline{T}_1 + \cdots + E_I\epsilon_n\overline{T}_I)$$

$$+ \sum_{n=1}^{N}(E_0\overline{T}_0 + E_1\overline{T}_1 + E_2\overline{T}_2 + \cdots + E_I\overline{T}_I)^2$$

which becomes

$$\sum_{n=1}^{N} r_n^2 = \sum_{n=1}^{N} \epsilon_n^2 - 2(E_0^2 + E_1^2 + \cdots + E_I^2) + E_0^2 + E_1^2 + \cdots + E_I^2$$

or

$$\boxed{\sum_{n=1}^{N} r_n^2 = \sum_{n=1}^{N} \epsilon_n^2 - E_0^2 - E_1^2 - E_2^2 - \cdots - E_I^2} \qquad (4.36)$$

Taking the expected value gives

$$E\left[\sum_{n=1}^{N} r_n^2\right] = N\sigma_\varepsilon^2 - E(E_0^2) - E(E_1^2) - \cdots - E(E_I^2) \qquad (4.37)$$

Assuming the noise is uncorrelated yields

$$E\left[\sum_{n=1}^{N} r_n^2\right] = N\sigma_\varepsilon^2 - (I + 1)\sigma_\varepsilon^2$$

or

$$\boxed{\sigma_\varepsilon^2 = \frac{1}{N - I - 1} E\left[\sum_{n=1}^{N} r_n^2\right]} \qquad (4.38)$$

From equation 4.24, it is seen that

$$\frac{d\hat{y}_n}{dx} - \frac{dy_n}{dx} = \left(E_0\frac{d\overline{T}_0}{dn} + E_1\frac{d\overline{T}_1}{dn} + E_2\frac{d\overline{T}_2}{dn} + \cdots + E_I\frac{d\overline{T}_I}{dn}\right)\frac{dn}{dx} \tag{4.39}$$

where, from equation 4.1

$$\frac{dn}{dx} = \frac{1}{\Delta x} \tag{4.40}$$

Assuming uncorrelated noise

$$E\left[\left(\frac{d\hat{y}_n}{dx} - \frac{dy_n}{dx}\right)^2\right] = \frac{\sigma_\varepsilon^2}{\Delta x^2}\left[\left(\frac{d\overline{T}_0}{dn}\right)^2 + \left(\frac{d\overline{T}_1}{dn}\right)^2 + \cdots + \left(\frac{d\overline{T}_I}{dn}\right)^2\right] \tag{4.41}$$

where $d\overline{T}_0/dn = 0$. In general, for uncorrelated noise

$$\boxed{E\left[\left(\frac{d^k\hat{y}_n}{dx^k} - \frac{d^k y_n}{dx^k}\right)^2\right] = \frac{\sigma_\varepsilon^2}{\Delta x^{2k}}\left[\left(\frac{d^k\overline{T}_k}{dn^k}\right)^2 + \left(\frac{d^k\overline{T}_{k+1}}{dn^k}\right)^2 + \cdots + \left(\frac{d^k\overline{T}_I}{dn^k}\right)^2\right]} \tag{4.42}$$

Figures 2 through 5 show the accuracies of $\hat{y}_n$ and $d\hat{y}_n/dx$ versus N for uncorrelated noise for $n = 1$, N and $(N + 1)/2$. Figures 6 and 7 show the accuracies of $\hat{y}_n$ and $d\hat{y}_n/dx$ versus n for uncorrelated noise for $N = 99$. The figures are graphically informative. Clearly the midpoint estimates of $\hat{y}_n$ and $d\hat{y}_n/dx$ are much more accurate than the estimates at the end points. Also at the midpoint value of $n = (N + 1)/2$, it is seen that the accuracy of $\hat{y}_n$ is the same for $I = 0$ order fit and $I = 1$, $I = 2$ and $I = 3$, $I = 4$ and $I = 5$, etc. The reason for this is interesting. Let

${}_I\hat{y}_n$ = estimate using Ith order fit

Then for $n = (N + 1)/2$

$$\boxed{{}_{I}\hat{y}_{(N+1)/2} = {}_{I+1}\hat{y}_{(N+1)/2} \qquad \text{for} \quad I = 0, 2, 4, \cdots} \tag{4.43}$$

Likewise for the midpoint derivative $\hat{y}'_n = d\hat{y}_n/dx$

$$\boxed{{}_{I}\hat{y}'_{(N+1)/2} = {}_{I+1}\hat{y}'_{(N+1)/2} \qquad \text{for} \quad I = 1, 3, 5, \cdots} \tag{4.44}$$

The extension to higher-order derivatives is easily seen from the above.

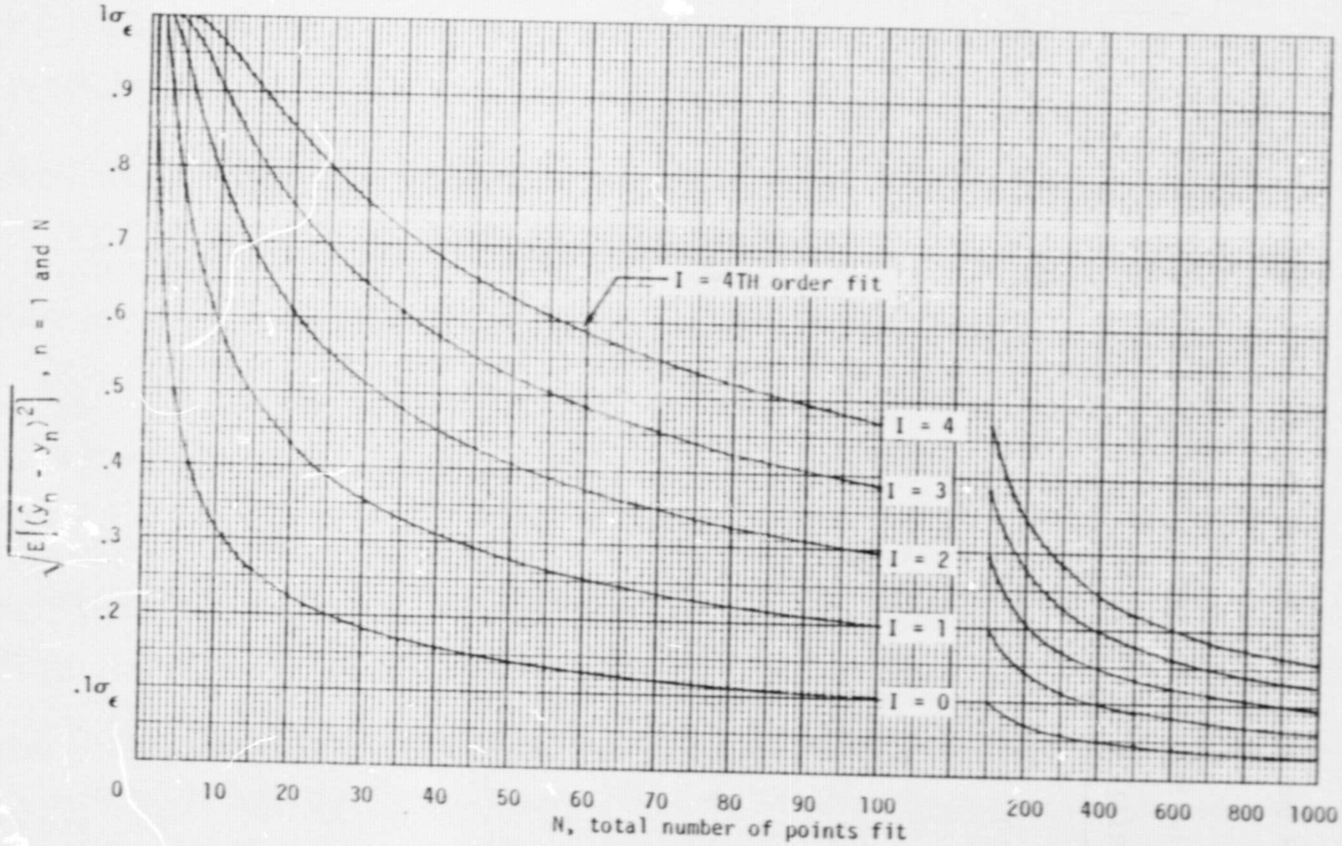

Figure 2.- Accuracy of $\hat{y}_n$ at end points versus N.

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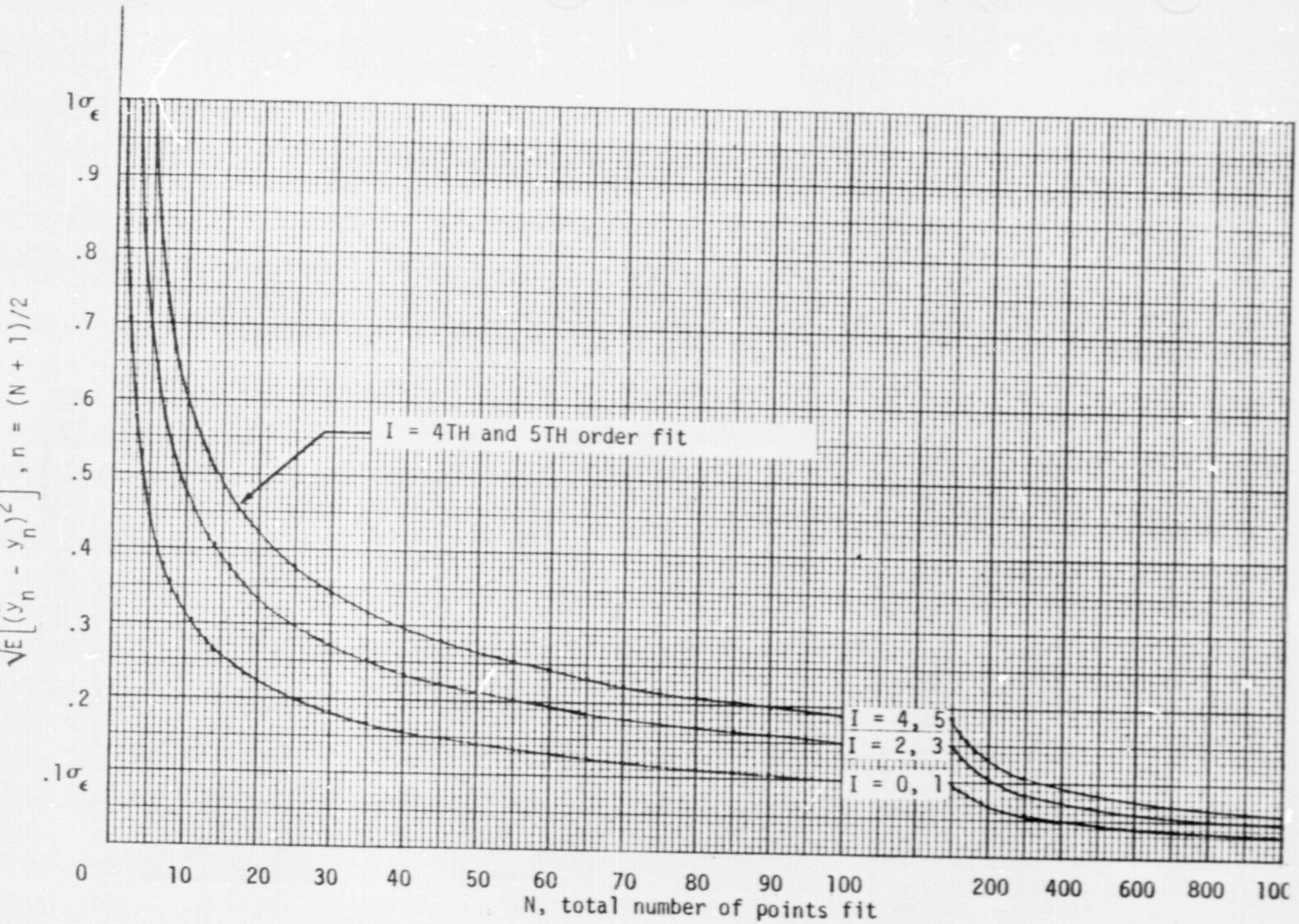

Figure 3.- Accuracy of $\hat{y}_n$ at midpoint of fit versus N.

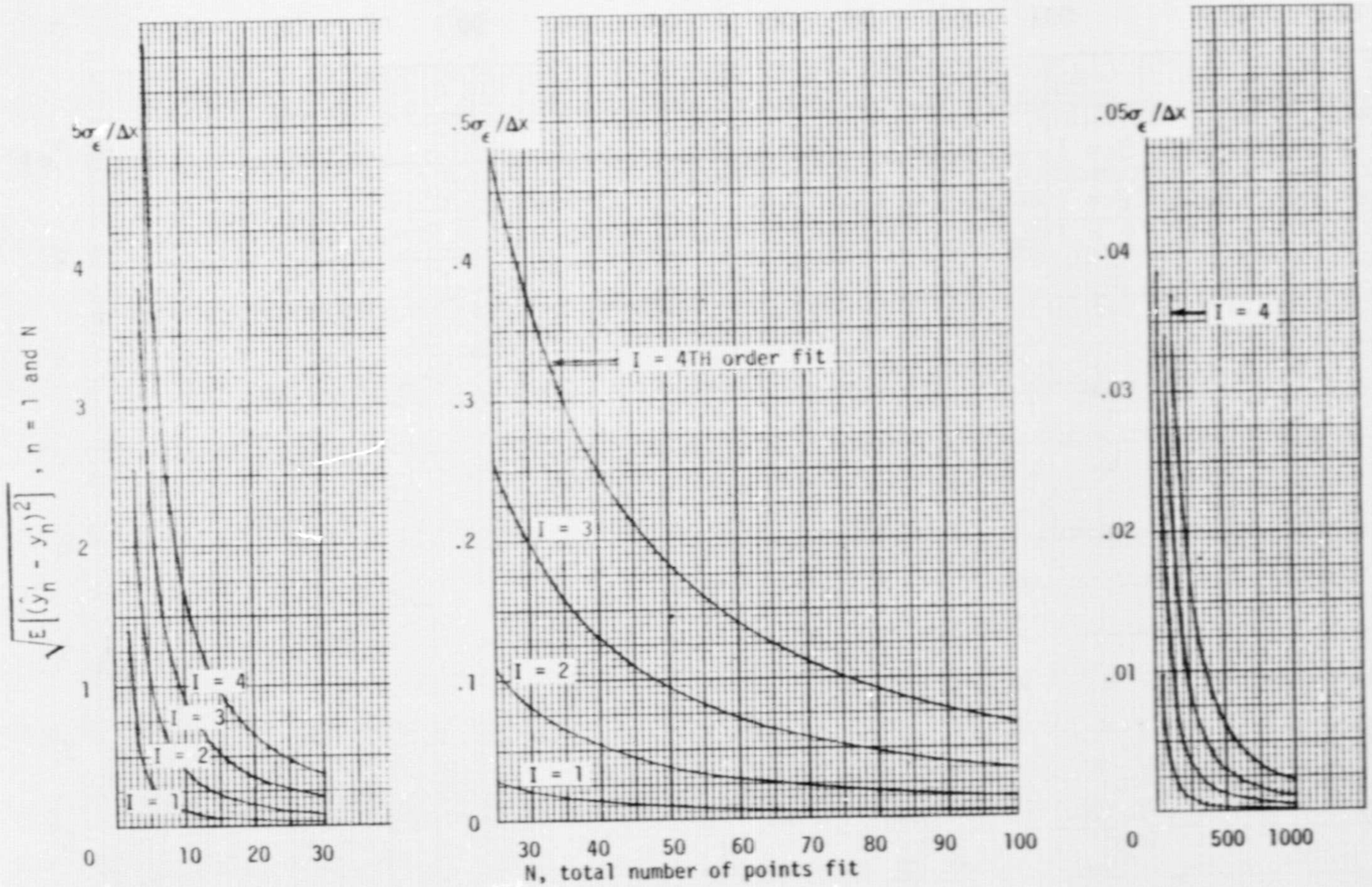

Figure 4.- Accuracy of $d\hat{y}_n/dx$ at fit end points versus N.

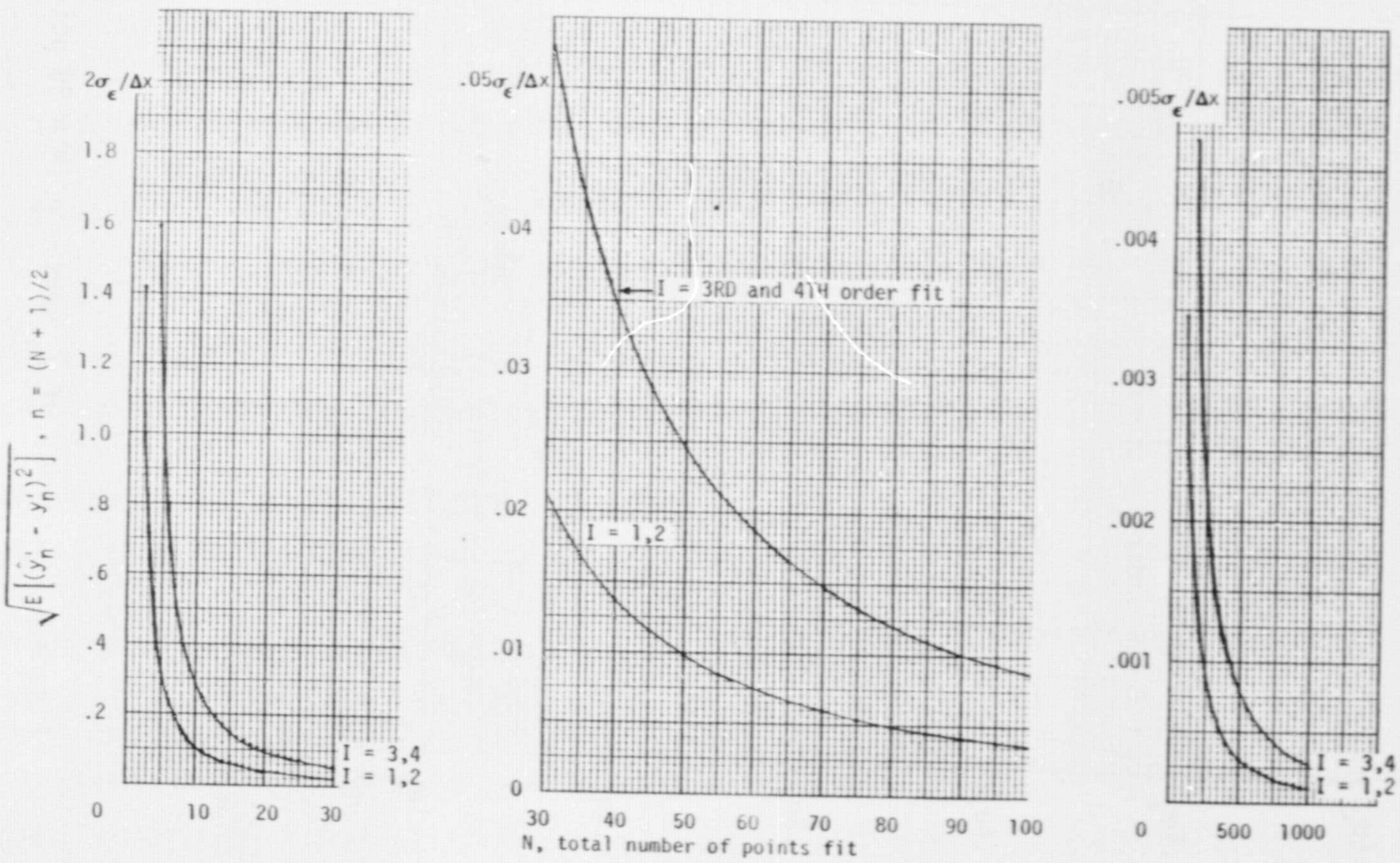

Figure 5.- Accuracy of $d\hat{y}_n/dx$ at fit midpoint versus N.

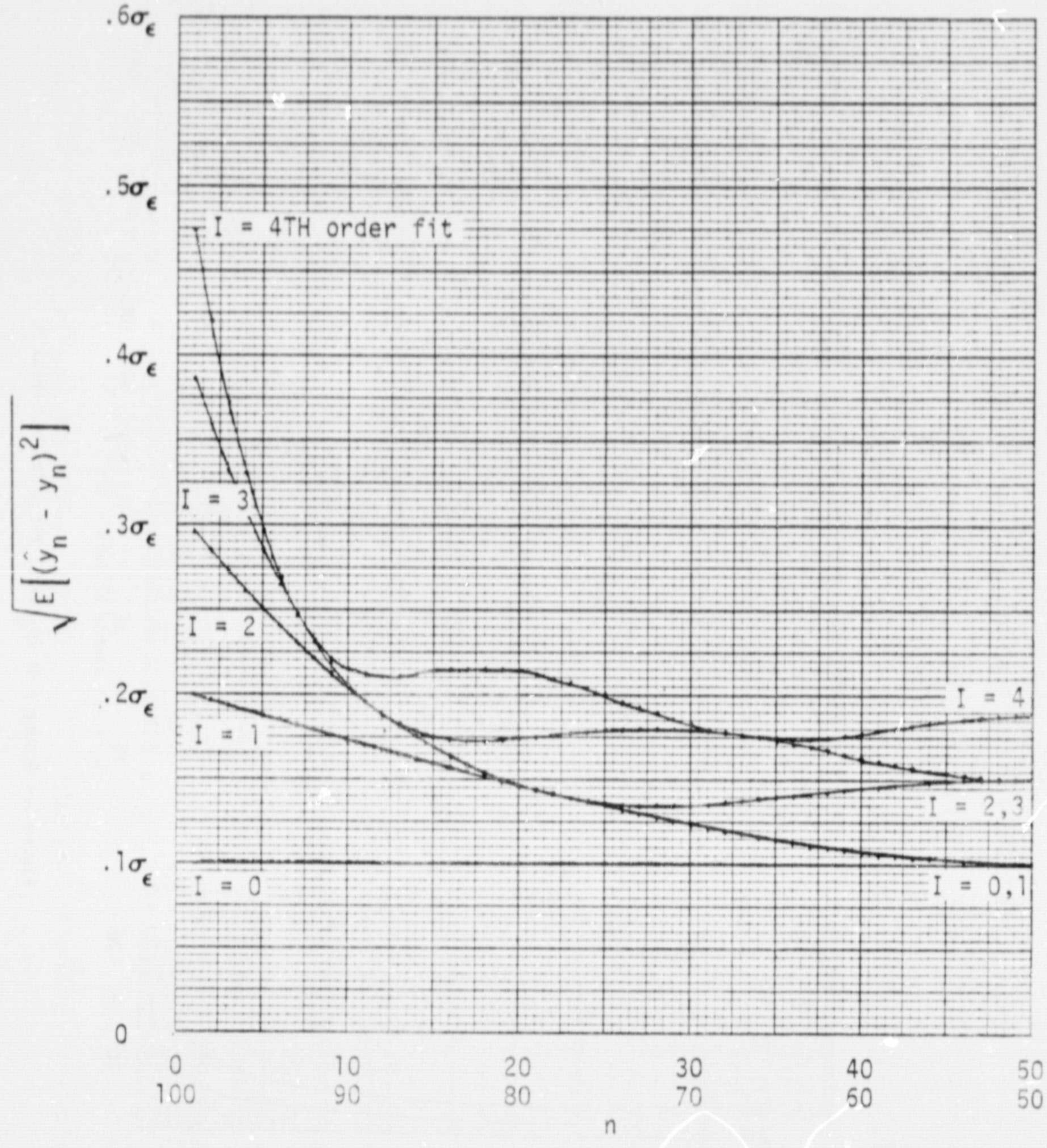

Figure 6.- Accuracy of $\hat{y}_n$ versus n (N = 99 points).

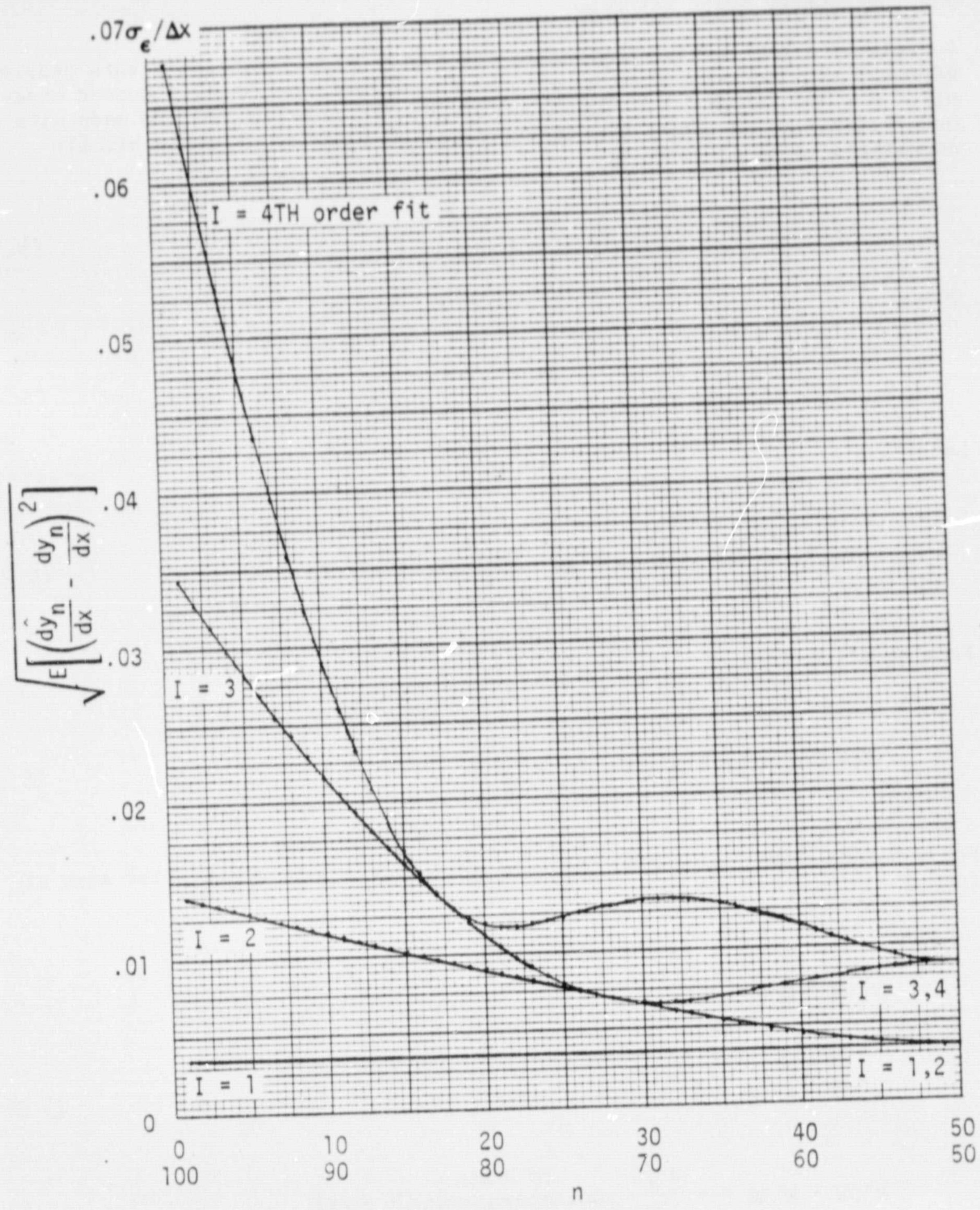

Figure 7.- Accuracy of $d\hat{y}_n/dx$ versus n (N = 99 points).

5.0 CORRELATED NOISE EXAMPLES

Correlated noise vastly complicates the accuracy equations. In this section, exponentially correlated noise will be considered when using a second order fit with $N = 101$ points. A comparison of the fit accuracy will be made with the uncorrelated noise case. The autocorrelation function of exponentially correlated noise is given by

$$\phi_k = \sigma_\varepsilon^2 a^k = E\left(\varepsilon_n \varepsilon_{n+k}\right) \tag{5.1}$$

where

$$0 \leq a \leq 1$$

Let

$$_k\tau_{ij} = \sum_{n=1}^{N-k} \left(\overline{T}_i(n)\overline{T}_j(n+k) + \overline{T}_j(n)\overline{T}_i(n+k)\right) \tag{5.2}$$

From equation 4.18

$$E\left(E_i E_j\right) = \sigma_\varepsilon^2 \,\delta_{ij} + \sigma_\varepsilon^2 \sum_{k=1}^{N-1} a^k \,{}_k\tau_{ij} \tag{5.3}$$

For a second-order polynomial fit, the following can be shown after some effort.

$$_k\tau_{00} = \frac{2}{N}(N - k) \tag{5.4}$$

$$_k\tau_{01} = {}_k\tau_{10} = 0 \tag{5.5}$$

$$_k\tau_{02} = {}_k\tau_{20} = -\frac{2\sqrt{5}}{N}\frac{(N-k)k}{\sqrt{(N^2-1)(N^2-4)}}(N - 2k) \tag{5.6}$$

$$k^{T}11 = \frac{2}{N}\,\frac{N-k}{N^2-1}\left(N^2 - 1 - 2(N+k)k\right) \tag{5.7}$$

$$k^{T}12 = k^{T}21 = 0 \tag{5.8}$$

$$k^{T}22 = \frac{2(N-k)}{N(N^2-1)(N^2-4)}\Big((N^2-1)(N^2-4) - (4N^2-10)(N+k)k + 6(N+k)k^3\Big) \tag{5.9}$$

<u>For a = 0 (uncorrelated noise)</u>

$$E(E_0{}^2) = \sigma_\varepsilon{}^2 \tag{5.10}$$

$$E(E_1{}^2) = \sigma_\varepsilon{}^2 \tag{5.11}$$

$$E(E_2{}^2) = \sigma_\varepsilon{}^2 \tag{5.12}$$

$$E(E_0E_2) = E(E_2E_0) = 0 \tag{5.13}$$

$$E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \tag{5.14}$$

The following results use $N = 101$.

<u>For a = 0.4</u>

$$E(E_0{}^2) = 2.3113\ 31133\ \sigma_\varepsilon{}^2 \tag{5.15}$$

$$E(E_1{}^2) = 2.2673\ 55494\ \sigma_\varepsilon{}^2 \tag{5.16}$$

$$E(E_2{}^2) = 2.2234\ 65949\ \sigma_\varepsilon{}^2 \tag{5.17}$$

$$E(E_0E_2) = E(E_2E_0) = -0.045873\ 81831\ \sigma_\varepsilon{}^2 \tag{5.18}$$

$$E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \tag{5.19}$$

For a = 0.8

$$E(E_0^2) = 8.6039\ 60396\ \sigma_\varepsilon^2 \tag{5.20}$$

$$E(E_1^2) = 7.8211\ 99767\ \sigma_\varepsilon^2 \tag{5.21}$$

$$E(E_2^2) = 7.0652\ 98249\ \sigma_\varepsilon^2 \tag{5.22}$$

$$E(E_0E_2) = E(E_2E_0) = -0.67000\ 73131\ \sigma_\varepsilon^2 \tag{5.23}$$

$$E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \tag{5.24}$$

For a = 0.9

$$E(E_0^2) = 17.217\ 86439\ \sigma_\varepsilon^2 \tag{5.25}$$

$$E(E_1^2) = 13.841\ 98613\ \sigma_\varepsilon^2 \tag{5.26}$$

$$E(E_2^2) = 10.933\ 27966\ \sigma_\varepsilon^2 \tag{5.27}$$

$$E(E_0E_2) = E(E_2E_0) = -2.1591\ 29562\ \sigma_\varepsilon^2 \tag{5.28}$$

$$E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \tag{5.29}$$

For a = 0.99

$$E(E_0^2) = 73.999\ 66291\ \sigma_\varepsilon^2 \tag{5.30}$$

$$E(E_1^2) = 13.714\ 80539\ \sigma_\varepsilon^2 \tag{5.31}$$

$$E(E_2^2) = 4.5863\ 43888\ \sigma_\varepsilon^2 \tag{5.32}$$

$$E(E_0E_2) = E(E_2E_0) = -4.6841\ 13901\ \sigma_\varepsilon^2 \tag{5.33}$$

$$E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \tag{5.34}$$

<u>For a = 0.999</u>

$$E(E_0^2) = 97.682\ 52696\ \sigma_\varepsilon^2 \tag{5.35}$$

$$E(E_1^2) = 1.9576\ 74533\ \sigma_\varepsilon^2 \tag{5.36}$$

$$E(E_2^2) = 0.48584\ 40305\ \sigma_\varepsilon^2 \tag{5.37}$$

$$E(E_0E_2) = E(E_2E_0) = -0.72319\ 19335\ \sigma_\varepsilon^2 \tag{5.38}$$

$$E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \tag{5.39}$$

<u>For a = 1</u>

$$E(E_0^2) = 101\ \sigma_\varepsilon^2 \tag{5.40}$$

$$E(E_1^2) = 0 \tag{5.41}$$

$$E(E_2^2) = 0 \tag{5.42}$$

$$E(E_0E_2) = E(E_2E_0) = 0 \tag{5.43}$$

$$E(E_0E_1) = E(E_1E_0) = E(E_1E_2) = E(E_2E_1) = 0 \tag{5.44}$$

From equation 4.26, for a second order fit, the accuracy of $\hat{y}_n$ is given by

$$E((\hat{y}_n - y_n)^2) = \overline{T}_0^2E(E_0^2) + \overline{T}_1^2E(E_1^2) + \overline{T}_2^2E(E_2^2)$$
$$+ 2\overline{T}_0\overline{T}_1E(E_0E_1) + 2\overline{T}_1\overline{T}_2E(E_1E_2) + 2\overline{T}_0\overline{T}_2E(E_0E_2)$$

But

$$E(E_0E_1) = E(E_1E_2) = 0$$

So

$$E((\hat{y}_n - y_n)^2) = \bar{T}_0^2 E(E_0^2) + \bar{T}_1^2 E(E_1^2) + \bar{T}_2^2 E(E_2^2) + 2\bar{T}_0\bar{T}_2 E(E_0 E_2) \tag{5.45}$$

where

$$\bar{T}_0 = 1/\sqrt{N} \tag{5.46}$$

$$\bar{T}_1 = \frac{\sqrt{3}}{\sqrt{N}} \frac{1}{\sqrt{N^2 - 1}}(N + 1 - 2n) \tag{5.47}$$

$$\bar{T}_2 = \frac{\sqrt{5}}{\sqrt{N}} \frac{1}{\sqrt{(N^2 - 1)(N^2 - 4)}}((N + 1)(N + 2) - 6(N + 1)n + 6n^2) \tag{5.48}$$

Table 5-I shows the accuracy of $\hat{y}_n$ at the end points of $n = 1$ and $n = N$ for various values of a. Note that $N = 101$ and a second-order fit is used. Table 5-II shows the accuracy of $\hat{y}_n$ at the midpoint of $n = (N + 1)/2 = 51$ for various values of a. Note that $a = 1$ is for a constant-bias error.

TABLE 5-I.- ACCURACY OF $\hat{y}_1$ AND $\hat{y}_{101}$ FOR A SECOND-ORDER FIT AND N = 101

a	$\sqrt{E((\hat{y}_1 - y_1)^2)} = \sqrt{E((\hat{y}_{101} - y_{101})^2)}$
0	0.29270 σ_ε
0.4	.43665 σ_ε
.8	.78341 σ_ε
.9	.99538 σ_ε
.99	1.06989 σ_ε
.999	1.00784 σ_ε
1	1 σ_ε

TABLE 5-II.- ACCURACY OF $\hat{y}_{51}$ FOR A SECOND-ORDER FIT AND N = 101

a	$\sqrt{E((\hat{y}_{51} - y_{51})^2)}$
0	0.14927 σ_ε
0.4	.22677 σ_ε
.8	.43300 σ_ε
.9	.59467 σ_ε
.99	.94507 σ_ε
.999	.99458 σ_ε
1	1 σ_ε

The results shown in tables 5-I and 5-II indicate that increasingly correlated noise decreases the accuracy of the fit. This is not unexpected. However, the large amount of degradation is surprising. From table 5-I, it is seen that if $a > 0.9$, the fit error is greater than that using the raw data. This is so surprising that the equations leading to this result were double checked. No error was found. The calculator used (a Texas Instruments 59) had 13 decimal digits of internal accuracy, so it is believed that roundoff error was negligible.

Correlated error also affects the equation used to compute the standard deviation of the noise. From equation 4.37, for a second order fit,

$$E\left[\sum_{n=1}^{N} r_n^2\right] = N\sigma_\varepsilon^2 - E(E_0^2) - E(E_1^2) - E(E_2^2) \qquad (5.49)$$

If $a = 0$, uncorrelated noise,

$$E(E_0^2) = E(E_1^2) = E(E_2^2) = \sigma_\varepsilon^2$$

and

$$\sigma_\varepsilon^2 = E\left[\sum_{n=1}^{N} r_n^2\right] /(N - 3) \qquad (5.50)$$

The equations for other values of a, for $N = 101$, are shown in table 5-III. Note that for $a = 1$, theoretically

$$\sum_{n=1}^{N} r_n^2 = 0,$$

thus $\sigma_\varepsilon^2 = 0/0$, as shown in the table.

TABLE 5-III.- EQUATIONS FOR σ_ϵ^2 FOR A SECOND-ORDER FIT AND $N = 101$

a	σ_ϵ^2
0	$E\left[\sum_{n=1}^{N} r_n^2\right]/98$
0.4	$E\left[\sum_{n=1}^{N} r_n^2\right]/94.2$
.8	$E\left[\sum_{n=1}^{N} r_n^2\right]/77.5$
.9	$E\left[\sum_{n=1}^{N} r_n^2\right]/59.0$
.99	$E\left[\sum_{n=1}^{N} r_n^2\right]/8.70$
.999	$E\left[\sum_{n=1}^{N} r_n^2\right]/0.873$
1	$E\left[\sum_{n=1}^{N} r_n^2\right]/0$

6.0 NUMERICAL EXAMPLES

A numerical example with correlated noise will be considered first. A second-order fit will be made with $N = 101$ total fit points. The true value, y_n, will be chosen to be zero.

$$y_n = 0 \tag{6.1}$$

The noisy measurements, y_n^*, will be given by

$$y_n^* = y_n + \epsilon_n \tag{6.2}$$

Table 6-I shows n, y_n, $y_n^* = \epsilon_n$, $\hat{y}_n - y_n = \hat{y}_n$ and the residual $r_n = y_n^* - \hat{y}_n$. The actual statistics of ϵ_n are (the bar signifies average value)

$$\overline{\epsilon_n} = 0.027 \tag{6.3}$$

$$\sqrt{\overline{\epsilon_n{}^2}} = \sigma_\epsilon = 0.99 \tag{6.4}$$

$$a = \overline{\epsilon_n \epsilon_{n-1}}/\sigma_\epsilon{}^2 = 0.89 \tag{6.5}$$

The Fourier coefficients were

$$\hat{A}_0 = E_0 = 0.27273\ 96938$$

$$\hat{A}_1 = E_1 = -6.1441\ 75478$$

$$\hat{A}_2 = E_2 = -2.4988\ 18869$$

Higher-order fits were also made by the computer program and may be of interest.

$$\hat{A}_3 = E_3 = 4.3008\ 65931$$

$$\hat{A}_4 = E_4 = -0.47203\ 26435$$

$$\hat{A}_5 = E_5 = 1.7513\ 69813$$

$$\hat{A}_6 = E_6 = 1.1905\ 86888$$

$$\hat{A}_7 = E_7 = -0.49540\ 58757$$

$$\hat{A}_8 = E_8 = -0.18221\ 10594$$

From $\hat{A}_0$, $\hat{A}_1$ and $\hat{A}_2$ may be computed

$$\hat{y}_n = -1.6150\ 62680 + 0.05433\ 34609n - 0.00032\ 70951n^2 \tag{6.6}$$

The residual statistics are

$$\hat{a} = \overline{r_n r_{n-1}}/\overline{r_n^2} = 0.77 \qquad \text{(actual value 0.89)} \tag{6.7}$$

$$\hat{\sigma}_\varepsilon = \sqrt{\frac{1}{98} \sum r_n^2} = 0.75 \qquad (\text{for } a = 0) \tag{6.8}$$

$$\hat{\sigma}_\varepsilon = \sqrt{\frac{1}{78} \sum r_n^2} = 0.84 \qquad (\text{for } a = 0.77) \tag{6.9}$$

$$\hat{\sigma}_\varepsilon = \sqrt{\frac{1}{58} \sum r_n^2} = 0.97 \qquad (\text{for } a = 0.9) \tag{6.10}$$

It is seen from table 6-I that the residuals are, in general, not a very accurate representation of ε_n when the noise is strongly correlated. However, $\hat{\sigma}_\varepsilon = 0.97$ (actual 0.99) is an excellent estimate when using $a = 0.9$. However, one has no way of knowing that a is this large and must use $a = 0.77$, which gives $\hat{\sigma}_\varepsilon = 0.84$, which is low but is still a "ball park" answer.

TABLE 6-I.- SECOND-ORDER FIT RESULTS FOR $a = 0.9$

n	y_n	$y_n^* = y_n + \epsilon_n$	$\hat{y}_n = \hat{y}_n - y_n$	$r_n = y_n^* - \hat{y}_n$
1	0	0	-1.5611	1.5611
2	0	-0.3494	-1.5077	1.1583
3	0	-.3984	-1.4550	1.0566
4	0	-.4170	-1.4030	0.9860
5	0	-.1493	-1.3516	1.2023
6	0	-1.1816	-1.3008	.1192
7	0	-.9308	-1.2508	.3200
8	0	-1.3124	-1.2013	-.1111
9	0	-1.5577	-1.9628	-.4051
10	0	-1.2187	-1.1044	-.1143
11	0	-1.3775	-1.0570	-.3205
12	0	-1.9545	-1.0102	-.9443
13	0	-2.2164	-0.9640	-1.2524
14	0	-1.7793	-.9185	-.8608
15	0	-.9189	-.8737	-.0452
16	0	-.3002	-.8295	.5293
17	0	-.3919	-.7859	.3940
18	0	-.3126	-.7430	.4304
19	0	.0943	-.7008	.7951
20	0	-.7876	-.6592	-.1284
21	0	-1.1708	-.6183	-.5525
22	0	-1.1993	-.5780	-.6213
23	0	-1.9765	-.5384	-1.4381

TABLE 6-I.- Continued

n	y_n	$y_n^* = y_n + \epsilon_n$	$\hat{y}_n = \hat{y}_n - y_n$	$r_n = y_n^* - \hat{y}_n$
24	0	-1.9837	-0.4995	-1.4842
25	0	-2.3802	-.4612	-1.9190
26	0	-1.2493	-.4235	-0.8258
27	0	-1.8161	-.3865	-1.4296
28	0	-0.9532	-.3502	-.6030
29	0	-.2960	-.3145	.0185
30	0	.4600	-.2794	.7394
31	0	.1144	-.2451	.3595
32	0	.6317	-.2113	.8430
33	0	.7114	-.1783	.8897
34	0	.4601	-.1458	.6059
35	0	-.0068	-.1141	.1073
36	0	.3144	-.0830	.3974
37	0	.2466	-.0525	.2991
38	0	-.6742	-.0227	-.6515
39	0	-.3556	.0064	-.3620
40	0	.0802	.0349	.0453
41	0	-.3638	.0628	-.4266
42	0	-.9590	.0899	-1.0489
43	0	-.7855	.1165	-.9020
44	0	-.1741	.1424	-.3165
45	0	-.3331	.1676	-.5007
46	0	-.4284	.1921	-.6205

TABLE 6-I.- Continued

n	y_n	$y_n^* = y_n + \epsilon_n$	$\hat{y}_n = \hat{y}_n - y_n$	$r_n = y_n^* - \hat{y}_n$
47	0	-0.4398	0.2161	-0.6559
48	0	.1712	.2393	-.0681
49	0	-.4064	.2619	-.6683
50	0	.0252	.2839	-.2587
51	0	-.0035	.3052	-.3087
52	0	.6642	.3258	.3384
53	0	.5888	.3458	.2430
54	0	.5507	.3651	.1856
55	0	.9769	.3838	.5931
56	0	1.0063	.4018	.6045
57	0	.6205	.4192	.2013
58	0	1.0289	.4359	.5930
59	0	1.3180	.4520	.8660
60	0	.5744	.4674	.1070
61	0	-.2250	.4822	-.7072
62	0	-.5212	.4963	-1.0175
63	0	-.8818	.5097	-1.3915
64	0	-.1793	.5225	-.7018
65	0	.4494	.5346	-.0852
66	0	.8959	.5461	.3498
67	0	1.0487	.5569	.4918
68	0	.8538	.5671	.2867
69	0	1.6344	.5766	1.0578

TABLE 6-I.- Continued

n	y_n	$y_n^* = y_n + \epsilon_n$	$\hat{y}_n = \hat{y}_n - y_n$	$r_n = y_n^* - \hat{y}_n$
70	0	1.2763	0.5855	0.6908
71	0	0.9610	.5937	.3673
72	0	1.5397	.6013	.9384
73	0	1.5644	.6082	.9562
74	0	1.0447	.6144	.4303
75	0	1.0691	.6200	.4491
76	0	1.0992	.6250	.4742
77	0	1.5997	.6293	.9704
78	0	1.3414	.6329	.7085
79	0	1.7311	.6359	1.0952
80	0	1.6476	.6382	1.0094
81	0	.7782	.6399	.1383
82	0	1.0475	.6409	.4066
83	0	1.0863	.6413	.4450
84	0	.9385	.6410	.2975
85	0	1.1995	.6400	.5595
86	0	1.4800	.6384	.8416
87	0	1.1382	.6362	.5020
88	0	.8963	.6333	.2630
89	0	.9855	.6297	.3558
90	0	.9773	.6255	.3518
91	0	.8827	.6206	.2621
92	0	.4054	.6151	-.2097

TABLE 6-I.- Concluded

n	y_n	$y_n^* = y_n + \epsilon_n$	$\hat{y}_n = \hat{y}_n - y_n$	$r_n = y_n^* - \hat{y}_n$
93	0	-0.2991	0.6089	-0.9080
94	0	.1809	.6021	-.4212
95	0	-.3407	.5946	-.9353
96	0	-.1370	.5864	-.7234
97	0	-.1366	.5776	-.7142
98	0	.4626	.5682	-.1056
99	0	-.0457	.5581	-.6038
100	0	-.8477	.5473	-1.3950
101	0	-.9879	.5359	-1.5238

Table 6-II shows the results of a second-order fit for $N = 101$ when $a = 0$, uncorrelated noise. The actual statistics of ϵ_n were (the bar signifies average value)

$$\overline{\epsilon_n} = 0 \tag{6.11}$$

$$\sqrt{\overline{\epsilon_n^2}} = \sigma_\epsilon = 1.01 \tag{6.12}$$

$$a = \overline{\epsilon_n\,\epsilon_{n-1}}/\sigma_\epsilon^2 = -0.02 \tag{6.13}$$

The Fourier coefficients were

$$\hat{A}_0 = E_0 = 0$$

$$\hat{A}_1 = E_1 = -0.74789\ 29138$$

$$\hat{A}_2 = E_2 = -1.8824\ 00360$$

$$\hat{A}_3 = E_3 = 0.84175\ 25686$$

$$\hat{A}_4 = E_4 = -0.61177\ 60429$$

$$\hat{A}_5 = E_5 = 0.22788\ 77925$$

$$\hat{A}_6 = E_6 = 0.39771\ 81088$$

$$\hat{A}_7 = E_7 = 0.45247\ 92538$$

$$\hat{A}_8 = E_8 = -0.19762\ 51796$$

Let the left subscript indicate the order of the fit. Then

$$_I\hat{\sigma}_\epsilon = \sqrt{\frac{1}{N - I - 1}\sum {}_I r_n^2} \qquad (\text{for } a = 0) \tag{6.14}$$

$$ {}_I\hat{a} = \overline{{}_I r_n \, {}_I r_{n-1}} / \overline{{}_I r_n^2} \qquad (6.15)$$

Based on the residuals from various order fits,

$${}_1\hat{\sigma}_\varepsilon = 1.01 \qquad {}_1\hat{a} = -0.03$$

$${}_2\hat{\sigma}_\varepsilon = 1.00 \qquad {}_2\hat{a} = -0.06$$

$${}_3\hat{\sigma}_\varepsilon = 1.00 \qquad {}_3\hat{a} = -0.07$$

$${}_4\hat{\sigma}_\varepsilon = 1.01 \qquad {}_4\hat{a} = -0.08$$

$${}_5\hat{\sigma}_\varepsilon = 1.01 \qquad {}_5\hat{a} = -0.08$$

$${}_6\hat{\sigma}_\varepsilon = 1.02 \qquad {}_6\hat{a} = -0.08$$

$${}_7\hat{\sigma}_\varepsilon = 1.02 \qquad {}_7\hat{a} = -0.08$$

$${}_8\hat{\sigma}_\varepsilon = 1.03 \qquad {}_8\hat{a} = -0.08$$

There are very good estimates of the statistics of ε. As can be seen from table 6-II, when $a = 0$, the values of the residuals are a better estimate of ε_n than when $a = 0.9$, as in table 6-I. Using figure 6 for a second-order fit, and using table 6-II, it is seen that

$$\sqrt{E\left((\hat{y}_1 - y_1)^2\right)} = 0.295 \qquad \text{(actual error = -0.53)}$$

$$\sqrt{E\left((\hat{y}_{101} - y_{101})^2\right)} = 0.295 \qquad \text{(actual error = -0.28)}$$

$$\sqrt{E\left((\hat{y}_{51} - y_{51})^2\right)} = 0.15 \qquad \text{(actual error = 0.21)}$$

$$\sqrt{E\left((\hat{y}_{27} - y_{27})^2\right)} = 0.135 \qquad \text{(actual error = 0.01)}$$

$$\sqrt{E\left((\hat{y}_{74} - y_{74})^2\right)} = 0.135 \qquad \text{(actual error = 0.14)}$$

TABLE 6-II.- SECOND-ORDER FIT RESULTS FOR $a = 0$

n	y_n	$y_n^* = y_n + \epsilon_n$	$\hat{y}_n = \hat{y}_n - y_n$	$r_n = y_n^* - \hat{y}_n$
1	0	0	-0.5342	0.5342
2	0	-0.7645	-.5072	-.2573
3	0	-.1547	-.4808	.3261
4	0	-.0955	-.4548	.3593
5	0	.5421	-.4294	.9715
6	0	-2.3196	-.4044	-1.9152
7	0	.3672	-.3799	.7471
8	0	-1.0032	-.3559	-.6473
9	0	-.7666	-.3324	-.4342
10	0	.4976	-.3094	.8070
11	0	-.5562	-.2869	-.2693
12	0	-1.5204	-.2649	-1.2555
13	0	-.9188	-.2434	-.6754
14	0	.5991	-.2223	.8214
15	0	1.6247	-.2018	1.8265
16	0	1.2377	-.1817	1.4194
17	0	-.2415	-.1622	-.0793
18	0	.1250	-.1431	.2681
19	0	.8721	-.1246	.9967
20	0	-1.9373	-.1065	-1.8308
21	0	-.9830	-.0889	-.8941
22	0	-.2559	-.0718	-.1841
23	0	-1.9367	-.0552	-1.8815

TABLE 6-II.- Continued

n	y_n	$y_n^* = y_n + \epsilon_n$	$\hat{y}_n = \hat{y}_n - y_n$	$r_n = y_n^* - \hat{y}_n$
24	0	-0.3525	-0.0391	-0.3134
25	0	-1.2250	-.0235	-1.2015
26	0	1.9110	-.0084	1.9194
27	0	-1.4747	.0063	-1.4810
28	0	1.6237	.0204	1.6033
29	0	1.3178	.0340	1.2838
30	0	1.6566	.0472	1.6094
31	0	-.6736	.0598	-.7334
32	0	1.1957	.0720	1.1237
33	0	.3091	.0837	.2254
34	0	-.4176	.0948	-.5124
35	0	-.9450	.1055	-1.0505
36	0	.7353	.1157	.6196
37	0	-.0781	.1254	-.2035
38	0	-1.9991	.1346	-2.1337
39	0	.6100	.1433	.4667
40	0	.9293	.1516	.7777
41	0	-.9619	.1593	-1.1212
42	0	-1.3799	.1665	-1.5464
43	0	.2342	.1733	.0609
44	0	1.2452	.1795	1.0657
45	0	-.3694	.1853	-.5547
46	0	-.2553	.1905	-.4458

TABLE 6-II.- Continued

n	y_n	$y_n^* = y_n + \epsilon_n$	$\hat{y}_n = \hat{y}_n - y_n$	$r_n = y_n^* - \hat{y}_n$
47	0	-0.0848	0.1953	-0.2801
48	0	1.3064	.1996	1.1068
49	0	-1.2447	.2034	-1.4481
50	0	.9108	.2066	.7042
51	0	-.0421	.2094	-.2515
52	0	1.5114	.2118	1.2996
53	0	-.0323	.2136	-.2459
54	0	.0377	.2149	-.1772
55	0	1.0702	.2157	.8545
56	0	.2583	.2160	.0423
57	0	-.6658	.2159	-.8817
58	0	1.0427	.2152	.8275
59	0	.8489	.2141	.6348
60	0	-1.4108	.2125	-1.6233
61	0	-1.6689	.2103	-1.8792
62	0	-.6857	.2077	-.8934
63	0	-.8829	.2046	-1.0875
64	0	1.4321	.2010	1.2311
65	0	1.3926	.1969	1.1957
66	0	1.0972	.1923	.9049
67	0	.5201	.1872	.3329
68	0	-.2309	.1816	-.4125
69	0	1.9175	.1756	1.7419

TABLE 6-II.- Continued

n	y_n	$y_n^* = y_n + \epsilon_n$	$\hat{y}_n = \hat{y}_n - y_n$	$r_n = y_n^* - \hat{y}_n$
70	0	-0.4912	0.1690	-0.6602
71	0	-.4598	.1619	-.6217
72	0	1.4848	.1544	1.3304
73	0	.3486	.1463	.2023
74	0	-.8655	.1378	-1.0033
75	0	.2591	.1288	.1303
76	0	.2764	.1193	.1571
77	0	1.3345	.1092	1.2253
78	0	-.2742	.0987	-.3729
79	0	1.1301	.0877	1.0424
80	0	.1405	.0762	.0643
81	0	-1.6334	.0643	-1.6977
82	0	.7596	.0518	.7078
83	0	.2921	.0388	.2533
84	0	-.1190	.0253	-.1443
85	0	.7698	.0114	.7584
86	0	.8602	-.0031	.8633
87	0	-.4826	-.0180	-.4646
88	0	-.3199	-.0334	-.2865
89	0	.3776	-.0494	.4270
90	0	.1758	-.0658	.2416
91	0	-.0192	-.0827	.0635
92	0	-.8927	-.1001	-.7926

TABLE 6-II.- Concluded

n	y_n	$y_n^* = y_n + \epsilon_n$	$\tilde{y}_n = \hat{y}_n - y_n$	$r_n = y_n^* - \hat{y}_n$
93	0	-1.4867	-0.1180	-1.3687
94	0	1.0384	-.1364	1.1748
95	0	-1.1175	-.1553	-0.9622
96	0	.4124	-.1747	.5871
97	0	-.0061	-.1945	.1884
98	0	1.3344	-.2149	1.5493
99	0	-1.0373	-.2358	-.8015
100	0	-1.7857	-.2571	-1.5286
101	0	-.4479	-.2789	-.1690

7.0 CONCLUSIONS

Discrete Tchebycheff orthonormal polynomials offer a convenient way to make least-squares polynomial fits of uniformly spaced discrete data. Computer programs to do so are simple and fast, and appear to be less affected by computer roundoff error, for the higher-order fits, than conventional least-squares programs.

When analyzing noisy data to obtain the error statistics, use the lowest-order fit that is "adequate." One way to ascertain this order is to examine ${}_I\hat{\sigma}_\varepsilon$ to find the first two values of I (the fit order) that give similar results. The lower value will be adequate. Probably the best way to obtain accurate residuals for generating error statistics is to use the midpoint residuals of a sliding polynomial fit of order 0, 2, 4, or 6, etc. Note that the results are identical to those with fits of 1, 3, 5, or 7, etc. For example, if a second-order midpoint fit is not adequate, then a fourth-order fit must be tried since a third-order fit will not alter the value of a midpoint residual.

For real-time polynomial smoothing (filtering), try to avoid using the endpoint ($n = N$) estimate. Estimates a short way from the end of the interval can double the accuracy, particularly if derivatives are needed. Again, the lowest-order adequate fit should be used.

In section 5.0, it was seen that exponentially correlated noise can greatly decrease the fit accuracy. In fact, a very surprising result was obtained. If the errors adding to the data are very strongly correlated, it is possible that the end point fit error standard deviation can be worse than that using the raw data.

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