THERMAL FLICKERS: A SEMI-ANALYTICAL APPROACH

J. Perdang†
Institut d'Astrophysique, Université de Liège
Cointe-Ougrée, Belgium, B-4200

and

J. Robert Buchler††
Department of Physics and Astronomy
University of Florida, Gainesville, FL 32611

Abstract

With the purpose of enhancing our physical insight into the nature of thermal oscillations arising from a thin helium burning shell we analyze the behaviour in its "phase-plane" of a simple two-zone model which, however, contains all the relevant physics. This simple model very naturally reproduces thermal flickers and is relatively insensitive to all but two parameters.

†Chercheur Qualifié du FNRS

††Supported in part by the National Science Foundation

It has been known since the pioneering work of Schwarzschild and Härm [1] that thin helium burning shells can give rise to thermal oscillations which have been studied by a number of authors [1 - 9]. A systematic exploration of this and similar evolutionary phases has been hampered by the enormous amount of computing time which is required to follow but a few of the thermal cycles. At the same time the interplay between the zones responsible for the thermal throbs is somewhat obscured by the complexity of the evolution codes. In order to enhance our understanding of the mechanism of the oscillations we have constructed a simple two-zone model appropriate for shell helium burning and radiation conduction with a Thomson opacity. The remainder of the star merely acts to impose boundary conditions. The behaviour of this model can readily be analyzed in the two-dimensional "phase-plane" of the temperatures and one is not limited as in a multi-zone model to a linear analysis of the neighbourhood of the equilibrium point. In fact, in our model, a linear stability analysis yields two real (unstable) roots while nevertheless a stable limit cycle can exist.

The simplest stellar configuration in which thermal oscillations have been reported is that of helium stars [3] in which energy generation takes place in a narrow shell. In order to mimick this situation we consider therefore a star with an inner, inert core, a narrow, energy generating first zone (shell), a second buffer zone on top of the shell and an outer region, e.g. the atmosphere; subscripts c, 1, 2, and a refer to these various regions. The equations governing the evolution can then be written in the form (see [11] for details) if we assume radiative energy transport:

$$\frac{dT_1}{dt} = \frac{1}{c_{p1}} \{\varepsilon(T_1) - \frac{1}{\Delta m_1} [K_1 (T_1^4 - T_2^4) - K_c (T_c^4 - T_1^4)]\} \equiv F_1(T_1,T_2) \qquad (1a)$$

$$\frac{dT_2}{dt} = \frac{1}{c_{p2}} \left[-\frac{T_2}{T_{2h}}\right] \frac{1}{\Delta m_2} [K_1(T_1^4 - T_2^4) - K_2(T_2^4 - T_a^4)] \equiv F_2(T_1,T_2) \quad (1b)$$

where the coefficients $K_i = (4\pi R_i^2)^2 \, ac/(3\kappa(T)\Delta m_i)$ are connected with the radiation transport and where $\kappa(T)$ is the opacity. The other symbols have their usual meaning. The first coefficient in square brackets, $-T_2/T_{h2}$ is a hydrostatic adjustment factor:

$$T_h = (.1 \text{ to } 1.) \times \frac{GM}{Rc_p} \frac{M}{\Delta m} \quad (2)$$

For the buffer zone $T_{h2} = 10^6 - 10^7$ K $< T_2$, whereas for the shell, in view of the smallness of $\Delta m_1/M$, we have $T_{h1} >> T_1$ and hydrostatic adjustment can be neglected. For the energy generation the expression appropriate for the triple-alpha reaction is given by

$$\varepsilon(T) = \zeta T^{-3} e^{-Q/T} \equiv \zeta T^{\nu(T)} \quad (3)$$

where ζ and Q are constants. At the temperatures and densities of interest the opacity is due to Thomson scattering and is constant (in contrast to the model of [10]). In this simplified model the radii are assumed time-independent, so that the K_i become constants. The buffer zone is energetically inert and can only act as a reservoir of heat fed by radiation conduction. The effect of the core appears through its temperature T_c, assumed constant, and similarly the atmosphere is held at a constant temperature T_a. (It turns out that the model is highly insensitive to the values of T_c and T_a.)

We now analyze the conditions under which these equations (1) allow a <u>limit cycle</u> (LC) oscillation. The equilibrium curves (EC), $F_1\ (T_1,T_2) = 0$ for the shell (EC1) and $F_2\ (T_1,T_2) = 0$ for the buffer zone (EC2) are exhibited in the figure. If the EC1 has an S-shape as shown, it is clear that the fold points A and B are points of exchange of stability; if the function F_1 is positive to the right of EC1, as it turns out to be, the lower and upper branches are both stable, whereas the arc AB is unstable. The criterion for the existence of a fold point (A) is (similarly to [1]) that

$$(T_1^4 - T_2^4) / T_1^4 > 4/\nu(T_1) \tag{4}$$

while a second fold point (B) exists provided that $\nu(T_1)$ drops below 4 at higher T_1. These two conditions then guarantee the existence of an S-shaped EC. The EC for the second zone is monotonic. The hydrostatic adjustment factor is vital in making it unstable (F_2 is positive to the right of EC2). We assume now that EC1 and EC2 intersect at a point P located on the unstable branch AB of EC1, which point P is therefore an *unstable equilibrium* point.

Turning to a discussion of the existence of a stable oscillation, we first assume for the sake of argument that the ratio χ of the order of magnitudes of F_1 and F_2 is very large. A stable LC then occurs when the following two conditions are satisfied: First, both F_1 and F_2 are positive to the right of their respective EC, and P is on the unstable arc AB; *only* for this one among four sign combinations can there exist a *stable* LC; secondly, there is no other equilibrium point (like Q on the figure) inside the curve AB'BA'. This curve then represents the resulting *relaxation* oscillation. For arbitrary values of χ, no simple criterion for the existence of a stable LC can be formulated, but it is obvious that a LC still exists under condition (1) provided the other possible equilibrium points (like Q) are sufficiently far from the fold points.

It is interesting to note that a *linear* stability analysis in the neighbourhood of the equilibrium point P yields purely *real* eigen-values

$$\lambda_1 = \frac{4K_1}{c_{p1}\Delta m_1} \varsigma \, T_1^3 \quad + O(\chi^{-1}) \tag{5a}$$

$$\lambda_2 = \frac{4K_1}{c_{p2}\Delta m_2} \quad T_2^3 \, \frac{T_2}{T_{h2}} \quad (1 + \gamma + \frac{1}{\zeta}) \; + O(\chi^{-1})$$

where $\zeta = \frac{\nu}{4} \frac{T_1^4 - T_2^4}{T_1^4} - 1$, (5b)

and where we have neglected the small contributions of K_c and T_a.

Equations (5) of course also show that P is an unstable node point (λ_1 and $\lambda_2 > 0$). As the parameter $\gamma \equiv K_2/K_1$ is lowered, the point P moves south beyond A, becoming first a saddle point ($\lambda_1 < 0$, $\lambda_2 > 0$) and soon afterwards a stable node (λ_1 and $\lambda_2 < 0$).

The existence of an oscillation is in this model clearly the result of non-linearity. This can nicely be exhibited by an inspection of the "velocity" field $\underline{v} \equiv (\frac{dT_1}{dt}, \frac{dT_2}{dt}) = (F_1, F_2)$ in the two-dimensional "phase-plane" (T_1, T_2).

In the figure we show the trajectory of the oscillation in the temperature "phase-plane". The physical parameters have been chosen to be: $Q = 26$ (hence all temperatures in units of 1.66×10^8K), $T_c = 6$, $T_a = 0.5$, $K_c/K_1 = 10^{-4}$, $\Delta m_2 c_{p2} T_{h2} = 1$, $\zeta \Delta m_1/K_1 = 10^7$, $\gamma = 4$ and $\chi \equiv K_1/(c_{p1}\Delta m_1) =$ 50 and 500. The point P falls on the unstable branch AB and oscillations exist for values of γ in the range (1,8). The innermost cycle (AA'BB') corresponds to a relaxation oscillation ($\chi >> 1$). The middle and outer cycles correspond to $\chi = 500$ and $\chi = 50$ respectively. For smaller values of χ, e.g. $\chi = 5$ already, this limit cycle disappears because of the influence of the saddle point located at Q. For large χ, $\lambda_2 << \lambda_1$ and the order of magnitude of the period of the relaxation oscillation is given by $1/\lambda_2$. As χ is lowered the excursion into the nonlinear regime increases and the period lengthens. The profiles of the temperatures as a function of time are very similar to those found by Rose

Schwarzschild and Harm [2] have attributed the thermal oscillations to the interaction between the shell and the underlying zone; they reach this conclusion on the basis of a linear stability analysis in which the entropy eigen-functions exhibit a 90° phase difference between the shell and the zones below. Their argument, however, does not hold in general even for linear oscillations (except e.g. for mechanical oscillators when represented by the velocity and position). This criticism does not exclude, though, the possibility of such an interplay between shell and underlying zones; we, however, have not succeeded in reproducing such oscillations with a two-zone model unless we include additional physical effects, like neutrino losses, in the bottom zone and the model becomes rather contrived.

In conclusion one has a fair amount of confidence that our model, because of both its insensitivity to most of the parameters involved and its physical simplicity, is representative of the thermal oscillations as reported in helium burning shells; in addition, the physical parameters of the model, needed to obtain the oscillations, take on quite reasonable values. Our geometric approach of representing the oscillators in a phase-portrait is quite general and one hopes that it may prove useful in the search for similar oscillatory phases in different burning stages.

We gratefully acknowledge the support of the National Science Foundation (grant numbers AST 75-05012 and AST 77-17572). One of us (JRB) also thanks the Government of Luxembourg for a grant and the other of us (JP) the Belgian FNRS for travel support.

References

1) Schwarzschild, M. and Härm, R., 1965, Ap. J. 142, 855; 1967, Ap. J. 150, 961.

2) Härm, R. and Schwarzschild, M., 1972, Ap. J. 172, 403.

3) Rose, W. K., 1966, Ap. J. 146, 838.

4) Weigert, A., 1966, Zs. 7. Ap., 64, 395.

5) Sweigart, A. V., 1971, Ap. J. 168, 79; 1973, Ad Ap 24, 459.

6) Faulkner, D. J. and Wood, P. R., 1972, Ap. J. 178, 207.

7) Christy-Sackmann, I. J. and Paczynski, B. E., 1975, Mém. Soc. Roy. Sci. Liége, 6e sér., 8, 335.

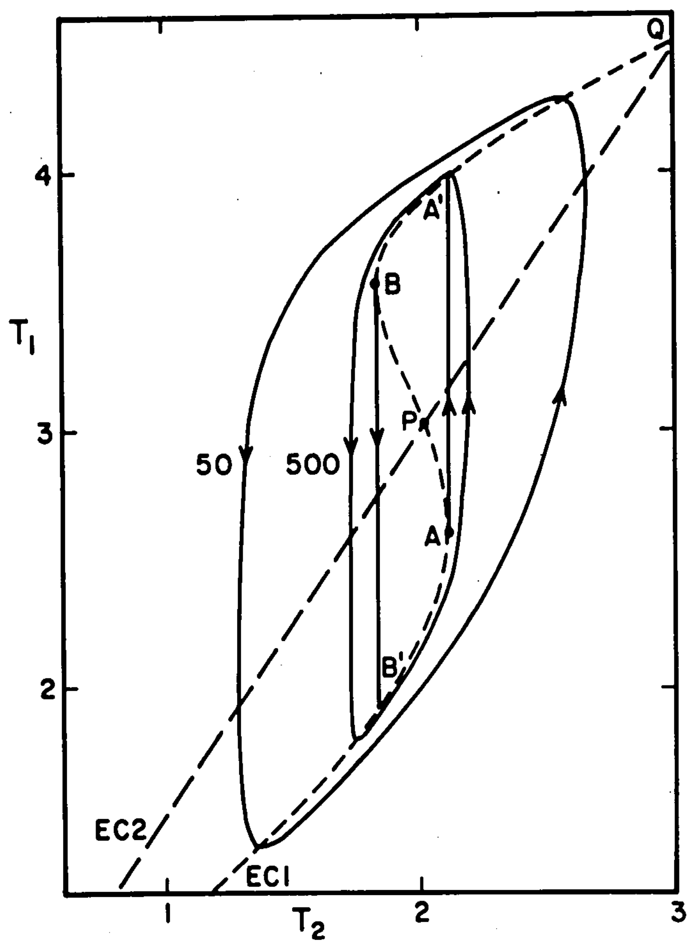
8) Iben, I., Jr., 1975, Ap. J. 196, 525, and 549; 1976, Ap. J. 208, 165.

9) Sugimoto, D. and Nomoto, K., 1975, Publ. Astron. Soc. Jap. 27, 197.

10) Defouw, R. J., 1973, Ap. J. 183, 215.

11) Buchler, J. R. and Perdang, J., 1978, "Thermal Relaxation Oscillations in a Two Zone Model", Ap. J. (submitted).

12) Henyey, L. G. and Ulrich, R. K., 1972, Ap. J. 173, 109.



Equilibrium curves (dashed lines) for the two zones in the temperature phase plane and trajectories of the oscillation for various values of χ: inner curve, relaxation oscillation, $\chi \gg 1$, middle curve, χ=500 and outer curve, χ=50.

Discussion

Adams: Can you relate the time between the pulses to the shell parameters?

Buchler: The numerical quantities we used were all scaled. It is difficult to relate them to actual physical numbers unless one determines what masses and radii to put in. It is uncertain what to use for the buffer zone mass.

Sweigart: Have you made any attempt to look at the stability of the hydrogen-burning shell in the subgiant branch phase of globular cluster stars?

Buchler: We have not applied it to anything like that.

Baker: Do you find the oscillations to be strictly periodic for all values of the parameters which you used?

Buchler: The abundances don't change, so it has to be periodic.

Baker: Could it be an aperiodic oscillator?

Buchler: We have not found any evidence of that. The relaxation oscillation is certainly strictly periodic. The three we have considered here are periodic. Rose, who only considered the helium burning shell, found very periodic oscillations. Schwarzschild and Harm found that the oscillation decayed, disappeared, and returned later. We're speculating on what may happen. Some of the parameters we kept constant may vary, so we need a three- or four-zone model because it is really the coupling of two different types of oscillations. So a zone may expand and relax, which is a regime of oscillation.