Recent Theoretical Work on Cepheids and Other Types of Variables

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ABSTRACT

Some important problems in the theory of Cepheids and of other types of variables are pointed out. Three of these are: (1) large-amplitude mode behavior, (2) convection, and (3) Cepheid masses, which must essentially always be inferred indirectly. Of the several types of indirect mass which can be defined, the inferred masses of the "beat (or double-mode) Cepheids," seem to be smaller than one expects, for this period range, by factors of 2-3. For the nonbeat Cepheids, the indirect masses also appear to be low, as compared with conventional stellar evolution theory, but by a smaller amount, say some 20-40 percent. Some conceivable ways of explaining these mass discrepancies are discussed. In particular, if the beat Cepheid masses, as usually inferred, are spuriously low (as appears likely), then two kinds of resolution appear possible: (1) accept pulsation theory, but question the assumed stellar envelope structure; (2) accept the conventional envelope structure, but question pulsation theory. In line with (2), we ask: is it possible that the apparently predominantly radial pulsations of the beat Cepheids could be contaminated with a small admixture of nonradial pulsations, so that the use of purely radial pulsation theory may not be applicable to the beat Cepheids? Some other conjectures which may bear on ordinary or beat Cepheids are offered.

I. INTRODUCTION

Despite the title, this paper will be a review of recent theoretical work on, primarily, classical Cepheids, although some things that are said may apply to related types of variables, such as RR Lyrae variables. We shall not discuss in detail the "line profile variable B stars" (Smith 1977), for which the obvious unsolved problem is the cause of the observed variations; the variable DA white dwarfs (the "ZZ Ceti stars," e.g., McGraw, 1977; Robinson, Nather, and McGraw 1976); nor the oscillations observed at some phases in the cataclysmic variables (e.g., Patterson, Robinson, and Nather 1977).

The observational properties of classical Cepheids have been well summarized in the past, and so will not be dealt with in detail here (see, e.g., Cox 1974; Pel 1978; and the excellent summary paper in this conference by Pel). Suffice it to say here that they are relatively cool (effective temperatures ~5000-7000°K), luminous (a few hundred to 10^4-10^5 solar luminosities) stars whose light output varies periodically with periods ~1^d to 50^d or 100^d.

The classical Cepheids are believed to be predominantly radial pulsators. The mechanism of excitation of the pulsations is now also reasonably well understood as due to "envelope ionization mechanisms" (especially He^+). These stars are thought to be in the core helium burning phase, on a "blue loop" just after core helium ignition in the red giant phase. This "blue loop" has intersected the "Cepheid instability strip," so that these stars will be Cepheids as long as they are in the instability strip. The blue edge of the instability strip is well defined theoretically, and agrees reasonably well with observations (e.g., Iben and Tuggle 1975). The red edge, being determined, most likely, by convection, is more of a problem theoretically, and will be discussed in more detail in §II.

In recent years a few (eleven, according to Stobie 1977) "beat Cepheids," with two or more periods (the longer period lying between about 1 and 7 days) superposed, have been discovered. Since the longer period is only 1 to 7 days, the beat Cepheids would occupy the lower portions of the instability strip. As Stobie (1977), Petersen (1973, 1978), and others have pointed out, these variables provide us with a new way of determining Cepheid masses. These mass values as determined in this way present us with further problems which will be discussed in §III.

As seen by this reviewer, there are at least three remaining outstanding theoretical problems of Cepheids. These problems are: (1) the large-amplitude mode behavior, (2) convective transfer, and (3) Cepheid masses.

These problems, as well as perhaps others, will be discussed in §II. In §III we shall review what is perhaps the most controversial and most talked-about difficulty, that of Cepheid masses. We shall present some conjectures in §IV. The purpose of these conjectures is to offer possible clues to the eventual solution or alleviation of some of the above problems.

II. SOME IMPORTANT PROBLEMS IN CEPHEID THEORY

The three main problems in Cepheid theory (in the opinion of this reviewer) referred to in §I, i.e., (1) the large-amplitude mode behavior, (2) convection, and (3) Cepheid masses, will be discussed in turn in the following subsections.

A. Large-Amplitude Mode Behavior

It would be desirable to know the answer to the following question: Suppose a star or stellar model is found by means of a linear theory to be unstable in two or more pulsation modes. Then, at large amplitudes, where nonlinear effects are important, which one (or ones) of the above unstable modes will be present? Thus, for example, in the case of the beat Cepheids, are the two periods that are observed (representing, presumably, two pulsation modes) a permanent feature of these stars, or are they in the process of switching modes?

Opinions seem to be divided on this point. Thus, according to Stobie (1977), there has been no evidence for a change in amplitude observed for the beat Cepheids over at least a 50-year time span. Theoretical evidence derived by Stellingwerf (1975a) indicates a switching time of some 80 years. In fact, Faulkner (1977a) suggests that any amplitude changes, if they exist, should be detectable within a few more years. On the other hand, Takeuti (1973) has suggested that the beat Cepheids may be in the process of mode switching (a view also espoused by Stobie 1970 and Rodgers 1970; see also Cox, Hodson, and Davey 1976).

In the early days of nonlinear calculations of pulsating stars (e.g., Christy 1964, 1966a,b; Cox, Brownlee, and Eilers 1966; Cox, Cox, Olsen, King, and Eilers 1966) the first question posed above was answered (hopefully!) by the "brute force" method of long computer runs, in which the model is followed in time from given initial values through hundreds or thousands of pulsation periods. This method is obviously inelegant, and it is also possibly subject to accumulation of rounding errors and other numerical problems.

Attempts to improve on the above method were initiated by Baker and collaborators (e.g., Baker 1970, 1973; Baker and von Sengbusch 1969). They sought periodic solutions of the nonlinear equations of hydrodynamics and heat flow. Their method was reasonably successful, and also yielded information on the modal stability, i.e., on the stability of a given nonlinear mode against switching to some other mode.

A new method was originated by Stellingwerf (1974a,b), in which the above initial-value techniques and the seeking of periodic solutions of the nonlinear equations were combined. This method also appeared to be successful, and was applied to the problems of the modal behavior of RR Lyrae variables, with applications to beat Cepheids (Stellingwerf 1975a,b). The method was subsequently adapted to and applied to another hydrodynamic pulsation code by Cox, Hodson, and Davey (1976). Again, modal stability information was obtained.

However, the modal stability information as obtained above does not seem to agree, at least in the several cases that have been investigated, with the results of extended nonlinear, initial-value calculations (e.g., King, Cox, Eilers, and Cox 1975; Cox and Cox 1976; Hodson and Cox 1976). For example, one model studied by Stellingwerf (1975a) (his model 2.6) showed, according to the modal stability analysis, instability of the fundamental mode to a switch-over to the first overtone, and at the same time instability of the first overtone mode to a switch-over to the fundamental mode. (However, in a subsequent calculation different stability results were obtained; see Stellingwerf 1976). Double-mode behavior, i.e., a fairly permanent mixture of these two pulsation modes, seems to be implied by these results. Yet this same model (or at least a slightly different numerical version of it) showed only pure fundamental or pure first overtone mode behavior when examined by initial-value techniques (Cox, Hodson, and King 1978), in apparent disagreement with Stellingwerf (1975a).

As another example, consider the $6 \cdot M_\Theta$ (M_Θ = solar mass) Cepheid model which was computed using the initial-value techniques by King, Cox, Eilers, and Davey (1973). This model was found by the above authors to be slowly switching modes from the fundamental to the first overtone. This model was further investigated by Cox, Hodson, and King (1978), who found, also using initial-value techniques, that it eventually actually switched to the first overtone. Yet the modal stability analysis, when applied to this model, showed the fundamental mode to be stable against switch-over to the first overtone mode! The above stability analysis also yielded the periodic fundamental mode, whose amplitude was considerably larger than the amplitude of the corresponding mode in the initial-value calculations of King, Cox, Eilers, and Davey (1973). This apparent discrepancy between the results of the modal stability analysis technique and those of the initial-value technique were, in fact, attributed by Cox, Hodson, and King (1978) to this difference in amplitude. Presumably, a larger-amplitude fundamental mode would have been found, using the initial-value approach, to remain more or less permanently in this mode in this model.

Why does this discrepancy exist? More importantly, why has it not been possible to get multiple mode behavior in initial-value type pulsation calculations, in spite of the fact that this kind of behavior may occur in real stars? It appears to this reviewer that these questions have not yet been satisfactorily answered, at least in the published literature.

B. Convection

Here we shall once again repeat the oft-made statement that what is desperately needed for the theoretical study of pulsating stars, is a reliable and (hopefully!) usable time-dependent theory of convective transfer. Considerable efforts have been made, and some tangible results obtained, about stellar convection in general, by Toomre, Spiegel, and collaborators (Latour, Spiegel, Toomre, and Zahn 1976; Toomre, Zahn, Latour, and Spiegel 1976).

An attempt to come to grips with the problems of convective transfer in pulsating stars has recently been made by Deupree in a series of papers (Deupree 1975a,b; 1976a,b,c; 1977a,b,c,d). This work represents perhaps the most ambitious attempt so far in this direction, and has given us, for the first time, perhaps some idea as to how pulsation is "throttled" by convection at the red edge of the instability strip (see below). Although these calculations are almost certainly not definitive, and although questions may legimately be raised about some of Deupree's assumptions, and also about how well his results actually mimic convection in real stars, still it must be admitted that his results are suggestive, and perhaps even, essentially, physically correct. They are certainly the most far-reaching obtained so far.

Deupree actually solved the equations of mass, momentum, and energy conservation in time and in two spatial dimensions in a convectively unstable region. Because of computer time limitations, he could follow only the motion of the largest "eddies." He treated the break-up of these large eddies into smaller ones and the subsequent turbulent cascade which eventually shows up as heat, by the introduction of an "eddy viscosity" coefficient. Perhaps the most serious of Deupree's assumptions is that convection can be adequately treated with two (as compared to three in nature) spatial dimensions. While his results may nevertheless be qualitatively correct, this may still be a serious assumption.

However, if we accept Deupree's results at face value, we have a theoretical calculation of the red edge of the instability strip, at least for RR Lyrae variables, and presumably also for Cepheids, not based on any kind of phenomenological theory such as a mixing-length theory of convection (however, Baker and Gough 1967 computed a red edge on the basis of the quasiadiabatic approximation in linear theory and of a variant of mixing-length theory). Moreover, Deupree's calculated red edge is in reasonable agreement with observations. The dependence of this red edge on various factors, such as luminosity, composition, and overtone pulsations, has been examined by Deupree (1977a,b,c).

From the above work the following physical picture emerges as to how convection may "throttle" instability at the red edge of the instability strip. Deupree finds that the convective heat flux is greatest at about the instant of maximum compression (near minimum stellar radius) of the convective regions. Thus, convection causes the energy which has been "dammed up" by the operation of the kappa and gamma mechanisms in the He^+ ionization zone, to "leak out" at this phase. Thus, the driving effects of the ionization zone(s) are essentially "undone" by the convection. According to Deupree, only a small amount of convection in the relevant regions (say with the convective flux amounting to only a few percent of the total flux) is needed effectively to "throttle" pulsations on the red side of the instability strip.

This picture certainly seems plausible, and, as Deupree (1977a) points out essentially on grounds of overall reasonableness, the picture is not likely to be fundamentally incorrect in any important respect. Nevertheless, one must bear in mind that the picture is based on a much simplified, two-dimensional treatment of convection, although the computational complications are still horrendous. Whether this picture will be modified appreciably by a more elaborate treatment of convection remains an open question, to be answered (presumably) in the future.

C. Cepheid Masses

As this subject will be discussed in some detail in §III, we simply refer the interested reader to that section.

III. CEPHEID MASSES

It is well known that no Cepheid is a member of a binary system whose stars are close enough together to admit of a reliable mass determination (the relevant empirical information has been summarized by Latyshev 1969 and Abt 1959). Therefore, one is forced to resort to indirect methods of mass determination for the Cepheids. (However, it is pointed out by Pel 1978 that perhaps a fourth of all Cepheids are members of binaries, and that the companions are usually unseen. He correctly states that this frequent binary membership might have a significant effect on some of these indirect methods of mass determination.) These we shall discuss in this section.

These indirect methods of Cepheid mass determination have been discussed very thoroughly in the paper at this conference by A. N. Cox (see also Cox 1978b). We shall therefore only summarize some of this information, together with some brief discussion, in §IIIA. Then, we shall discuss these results further in §IIIB.

A. Types of Masses

As stated by A. Cox (see also Cox, Deupree, King, and Hodson 1977), there are (at least) some six kinds of these indirect masses, denoted by M_{evol}, M_T, M_Q, M_W, M_{bump}, and M_{beat}. These are defined as follows.

M_{evol}: This is the mass, for given mean or equilibrium luminosity L, while the star is in the Cepheid region of the Hertzsprung-Russell diagram, as computed from conventional stellar evolution theory. One generally assumes that the observed Cepheids are undergoing the second crossing of the instability strip after having left the main sequence, on a "blue loop." This second crossing is normally considerably slower than any other crossing, because the star derives most of its energy in this phase from core helium burning (but there is also a hydrogen-burning shell in this phase; see, e.g., Iben 1967), and is thus nearly in thermal equilibrium. The relation between mass M and L during this core-helium burning phase is given as ($L_\odot$ and $M_\odot$ denote solar luminosity and mass, respectively)

$$\log(L/L_\odot) = 3.48 \log(M/M_\odot) + 0.68 \tag{1}$$

(King, Hansen, Ross, and Cox 1975) or as

$$\log(L/L_\odot) = 3.1 + 4[\log(M/M_\odot) - 0.7] - 4(X - 0.7) - 12(Z - 0.02) \tag{2}$$

(Iben 1974), where X and Z are, respectively, the mass fractions of hydrogen and "metals." Equation (1) is purportedly accurate to within a factor of about two in $L/L_\odot$ (King, Cox, Eilers, and Davey 1973). This uncertainty includes uncertainties in chemical composition, which crossing of the instability strip is involved, etc. For $X = 0.7$ and $Z = 0.02$, equations (1) and (2) agree in $(L/L_\odot)$ to within better than a factor of two for $2 \lesssim (M/M_\odot) \lesssim 15$. The formula given by Becker, Iben, and Tuggle (1977) agrees with equation (2) to about the same accuracy for X and Z close to the above values.

Despite the above formulae, one should bear in mind that these "evolutionary masses," for given luminosity, are themselves rather uncertain, perhaps by some 20-30 percent. The main reason is that they depend on the rather poorly known composition (mostly helium). Also, one should realize that these stars are in a rather advanced (beyond the main sequence) evolutionary phase, and that the uncertainties in computations of stellar evolution increase with increasing evolution away from the main sequence. In particular, the exact causes of the occurrence of the loops are not really known. These loops have been discussed at some length in the literature (e.g., Hofmeister 1967; Schlesinger 1969; Hallgren and Cox 1970; Paczyński 1970; Robertson 1971a,b, 1972; Lauterborn, Refsdal, and Roth 1971; Lauterborn, Refsdal, and Weigert 1971; Lauterborn, Refsdel, and Stabell 1972; Fricke and Strittmatter 1972; Lauterborn and Siquig 1974a,b,c, 1976; Flower 1976; Durand, Eoll, and Schlesinger 1976; Schlesinger 1977), and they are generally thought to show some correlation with the details of the hydrogen abundance profile in the shell-burning region. These precautionary remarks are perhaps even more worth bearing in mind, since it is not so certain that even the sun is well understood -- witness the "solar neutrino problem" (e.g., Bahcall 1977 and references therein). Also, one should beware of certain complicating factors, such as secular instabilities during these relatively late evolutionary phases (see Hansen 1978 and references therein), and possibly also differential rotation (Wiita 1978).

M_T: This mass was defined by Cox, Deupree, King, and Hodson (1977), and is obtained as follows. One assumes an L-M relation, such as in equations (1) or (2), based on conventional stellar evolution theory; a period-mean density relation which is a relation involving mostly Π (period), M, and R (radius), for a given assumed mode of pulsation; and the defining relation connecting L, R, and T_e (effective temperature). With five quantitites, i.e., L, M, R, Π, and T_e; and three relations; specification of some two will then determine the rest. Cox, Deupree, King, and Hodson (1977) use an empirical value of Π for a given star, and assume a value of T_e such that the star is in the instability strip. Thus, the remaining parameters are determined. Masses obtained in this way are called "theoretical masses," and are denoted by M_T. This mass should be close to M_{evol}, because the underlying assumptions are essentially the same; the near equality of these two kinds of mass has, in fact, been shown by Cox (1978b).

A mass somewhat similar to M_T may be inferred from Cogan's (1978) work. He assumes the above relations, plus an additional one, namely the relation among, say, L, M, and R along the center of the instability strip or its blue edge. With one additional relation among the same five quantities, only one quantity, for example, the observed period Π, need be specified. Thus, the mass introduced by Cogan (1978) differs from M_T only in that the T_e as used in Cogan's mass is such that the star is guaranteed to be in the instability strip, whereas in M_T the value of T_e is chosen in such a way that the star should be in the instability strip.

In the following we shall not distinguish between M_{evol} and M_T, and shall only speak of M_{evol}.

M_Q: This mass, sometimes called the "pulsation mass," is obtained as follows. One somehow obtains the average absolute visual magnitude, say $\langle V \rangle$, and the average color, say $\langle B \rangle - \langle V \rangle$, for the star, where both quantities are assumed to have been corrected for reddening and extinction. For example, for the thirteen Cepheids in galactic clusters (see, e.g., Cogan 1970), one gets these quantities fairly directly. Otherwise, say for "field" Cepheids, one may use a "period-luminosity-color" (PLC) relation (e.g., Sandage and Tamman 1969, 1971). At any rate, given $\langle V \rangle$ and $\langle B \rangle - \langle V \rangle$, one may then apply a bolometric correction to get the mean luminosity L, and an adopted color-effective temperature (T_e) relation to get a corresponding mean value of T_e. The mean radius R then follows from the defining relation connecting L, R, and T_e ($L \propto R^2 T_e^4$).

One then assumes a certain (radial) pulsation mode. As shown by Cogan (1970) and by Cox, King, and Stellingwerf (1972), the pulsation period Π of a star in a given mode is determined mostly by M and R. Hence, given Π from observations as determined, for example, above, one can now compute, for a given assumed pulsation mode, M, the "pulsation mass" (denoted above as M_Q). Results as found by Cogan (1970) yielded $M_Q/M_{evol} \approx 0.7 - 0.8$ (similar results were reported by Stobie 1974).

However, as pointed out by Cox, King, and Stellingwerf (1972) and others, M_Q is quite sensitive to R (roughly, $M_Q \propto R^3$). Thus, a 10 percent uncertainty in R, for example, will appear as approximately a 30 percent uncertainty in M_Q. The value of R is, in turn, fairly sensitive to the assumed color-effective temperature relation, the distance scale, and the reddenings adopted. As shown by King, Hansen, Ross, and Cox (1975), the color-effective temperature relation needed to make $M_Q \approx M_{evol}$ results in lower effective temperatures, for given color, by some 300°K to 600°K, than are given by the relation of Schmidt (1972). This needed relation is close to others that have been proposed (e.g., Böhm-Vitense 1972; van Paradijs 1973; Flower 1977). It is interesting to note that the solution favored by Iben and Tuggle (1972a,b) is based on an increase in the above value of R by virtue of an assumed slightly increased distance, and hence luminosity L (corresponding to a 0.3 magnitude increase in L) of the Cepheids (since, for given color $[T_e]$, $L \propto R^2$).

It has been pointed out by A. Cox (1978b) that the new Hanson (1975, 1977) Hyades distance scale (implying an increase in stellar luminosities by $\sim 0^m.27$) and the lower effective temperatures inferred from the data of Pel (1978) both conspire to yield larger radii than previously thought. A. Cox (1978b) points out that with these larger values of R, the ratio M_Q/M_{evol} is now sometimes larger, sometimes smaller, than unity and that this ratio differs from unity by only some 10-20 percent or less.

M_W: This mass, the "Wesselink mass," discussed extensively by A. Cox (1978b), is obtained as follows. One first obtains the "Wesselink radius," which we shall denote simply by R. As is well known, the method originally devised by Baade (1926) and Wesselink (1946, 1947) is based on the assumption that the total radiant power, say L_ν, emitted by a star in a given spectral region depends only on the color of the star and on its radius (in particular, $L_\nu \propto R^2$). Thus, measurements of L_ν at two phases of equal color give a measure of the <u>ratio</u> of the radii of the star at these two phases. Measurements of the radial velocity of the star at these two phases, and integration of the velocity curve, give one also the <u>difference</u> in radii at these two phases. From the <u>ratio</u> and the <u>difference</u> of the two radii, one can thus solve for the actual radius values at these two phases. Repeating this procedure around a pulsation cycle, and averaging over the cycle, gives one a value of R. Some of the difficulties in getting R in this manner have been discussed by, for example, Parsons (1972), Karp (1975), Cox and Davis (1975), and Evans (1976).

Once R is known, the procedure for obtaining M_W is exactly the same as in obtaining M_Q from R, as explained above. This procedure involves assuming a certain pulsation mode for the star. Also, just as in the case of M_Q, any uncertainty in R is magnified quite considerably in M_W. As pointed out by A. Cox (see also Cox 1978b), the results show a rather large scatter for M_W. Nevertheless, straight averages give M_W/M_{evol} close to 0.7 - 0.8. The possible significance of these results will be discussed in §IIIB below.

M_{bump}: As indicated originally by Christy (1966b) and Stobie (1969), M_{bump} is the mass required for the nonlinear pulsation calculations to yield a secondary bump on the descending portion of the velocity curve in the period range $\sim 7^d - 10^d$, where a corresponding secondary bump usually shows up on the descending portion of observed light curves of Cepheids (the "Hertzsprung relation"; see, e.g., Payne-Gaposchkin 1951, 1961; Payne-Gaposchkin and Gaposchkin 1966; Cox 1974). (It is usually assumed that the phasings of the secondary bump on the relatively easily observed light curve and on the relatively reliably computed velocity curve are the same. This assumption has been discussed by Stobie 1976, who presents empirical results for six Cepheids. He finds that the phasing of the secondary bump on the descending portion of the velocity curve is always about $1^d.3$ later than on the descending portion of the light curve.) Results found by Christy (1966b) and by Stobie (1969) gave

$M_{bump}/M_{evol} \sim 0.4 - 0.6$ (see also Stobie 1974). Similar results have been found by other workers (summarized in Fischel and Sparks 1974).

However, these results are based, for the most part, on a comparison of two different things -- observed light curves and computed velocity curves. It seems to this reviewer that only a comparison of similar things, for example, observed and computed light curves, will yield really reliable and trustworthy results regarding M_{bump}/M_{evol}.

Computations of Cepheid light curves are greatly facilitated by the new non-Lagrangian, moving-mesh kind of calculation originated by J. Castor and presently being pursued by Adams, Davison, Davis, and others (see, e.g., Castor, Davis, and Davison 1977; Davis and Davison 1978). Results of this kind of calculation should be forthcoming soon, and some of them may in fact be reported at this conference (see paper by Adams, Castor, and Davis). Preliminary results of Davis and Davison (1978) (also Adams 1978) suggest that $M_{bump} < M_{evol}$, and perhaps $M_{bump}/M_{evol} \approx 2/3$, at least for one Cepheid model (the "Goddard model").

We note that it has been suggested by Simon and Schmidt (1976) that the above bumps on the theoretical velocity curves of Christy (1966b) and Stobie (1969) correspond to stars for which a reasonance exists between the fundamental and the second overtone, such that the ratio of the second overtone period Π_2 to the fundamental period Π_0 is close to 0.50. If this interpretation is correct, it suggests that the value $M_{bump}/M_{evol} \approx 0.5$ may apply at least to radiative Cepheid models (see §III).

M_{beat}: As stated in §I, M_{beat} applies only to Cepheids for which (usually) two distinct periods may be derived. These stars have been discussed by Fitch (1970), Stobie (1970, 1972, 1976, 1977), Stobie and Harwardin (1972), Rodgers and Gingold (1973), Schmidt (1974), and others.

It was originally suggested by Petersen (1973) that, since two periods (say radial fundamental and first overtone) were available for each such star, and in view of Cogan's (1970) result that the period is determined mostly by mass M and radius R, <u>both</u> M and R should be obtainable for each star. Such an analysis yielded $1 \lesssim M/M_\odot \lesssim 2$; the luminosities of these stars are, according to the available information, probably a few hundred solar luminosities. The mass determined by this kind of analysis is usually called M_{beat}. Note that $M_{beat}/M_{evol} \approx 1/3 - 1/2$. This discrepancy with respect to the evolutionary masses is perhaps even worse than for the other types of masses, and has generated a great deal of discussion in the literature (e.g., Petersen 1973, 1974, 1978; King, Hansen, Ross, and Cox 1975; Takeuti 1973; Stobie 1976, 1977; Cox, Hodson, and Davey 1976; King, Cox, and Hodson 1976; Cox and Cox 1976; Hodson and Cox 1976; Henden and Cox 1976; Cogan 1977; Cox, King, Hodson, and Henden 1977; Faulkner 1977a,b; Cox, Deupree, King, and Hodson 1977; Saio, Kobayashi, and Takeuti 1977; Cox 1978a,b; Cox, King, and Hodson 1978). Other investigators, making similar (and conventional) assumptions, have obtained results in general agreement with those of Petersen (1973) (however, see Cox, Deupree, King and Hodson 1977).

Some of the above results have been summarized in Table 1, and a general discussion will be given in §§IIIB and IV.

We note that Cogan (1978) has chosen to compare theory and observations in regard to pulsation theory and Cepheid variables <u>not</u> by a comparison of some inferred mass based on pulsation theory and the evolutionary mass, but as follows. As explained above in relation to M_T, the five quantities L, M, R, T_e, and Π may be related by the following four relations: a mass-luminosity relation, involving mostly L and M, based on

Table 1

Cepheid Masses

Type of Mass	M/M_{evol}
M_{evol}	-----
M_Q	0.7 - 1.3
M_W	0.7 - 0.8
M_{bump}	0.4 - 0.7 (?)
M_{beat}	1/2 - 1/3

conventional stellar evolution theory; a period-mean density relation (for a given assumed mode of pulsation), involving mostly Π, M, and R; a relation defining the location of the instability strip, involving mostly L, M, and T_e; and the defining relation among L, R, and T_e ($L \propto R^2 T_e^4$). Consequently, any one of these quantities may be expresed as a function of only one other of them, say the directly (and accurately!) observed period, Π, which is the independent parameter that Cogan (1978) has chosen. One of these expressions is a period-radius relation. Cogan (1978) has compared radii, for a given period, based on this expression, with radii inferred directly from the observations. He has obtained "empirical" radii for over 100 Cepheids, and refers to work by Balona (1977) in which the empirical radii of some 50 Cepheids have been obtained. He finds that different methods give somewhat different results (differing by up to, say, 20-30 percent) regarding radii of Cepheids. However, to this reviewer the overall agreement between theory and observation, when looked at this way, appears fairly satisfactory; i.e., there seem to be no glaring discrepancies between theory and observation.

B. Is There a Mass Discrepancy?

The above question may seem superfluous and unnecessary, in view of the above remarks. However, it appears to this reviewer that only in the case of M_{beat} is there incontrovertible evidence for a real mass discrepancy. Some conjectures regarding M_{beat} will be presented in §IV. We have tried to point out above some of the many uncertainties involved in obtaining the above indirect estimates of Cepheid masses. Except in the case of M_{beat}, the uncertainties in the above estimates are, for the most part, in the general range of 20-40 percent or less. As pointed out in the excellent and very thorough discussions of Fricke, Stobie, and Strittmatter (1971, 1972), it is impossible to exclude uncertainties of this order, due to various factors, both observational and theoretical.

Nevertheless, the weight of the evidence does seem to suggest that these estimates of Cepheid masses (again excluding M_{beat}) do seem to be somewhat less than conventional evolutionary masses. If this is a correct statement, then it appears to this reviewer that at least two general routes are open.

First, one may regard these conventional evolutionary masses as essentially correct, and then try to account for the discrepancy. The implication would then be that there is something wrong with pulsation theory, the way in which this theory is being applied to stellar envelopes, or the envelopes themselves. In line with the last two of these possibilities, it has been argued by Carson and Stothers (1976) that larger opacities can increase M_{bump}, and so remove at least part of the discrepancy. However, this explanation is not agreed upon by all workers (e.g., Cox, Deupree, King, and Hodson 1977).

Proceeding along the lines of seeking a modification of the envelope structure, Cox, Deupree, King, and Hodson (1977) have pointed out that a chemically inhomogeneous envelope, in particular one with a helium-enriched layer near the surface (having a helium mass fraction of at least 0.5 for temperatures less than 10^5°K), can increase M_Q (although this theory was originally conceived in relation to M_{beat}). Assuming that the Simon and Schmidt (1976) resonance-effect explanation of bumps on the nonlinear velocity curves is correct, the authors conclude that helium enrichment of the outer layers will also probably increase M_{bump}.

The existence of such a helium-enriched layer near the stellar surface raises new questions, which of course do not invalidate this possibility. Thus, it is known that such an "inverted μ-gradient" is unstable against slow mixing of the helium-enriched material down to the hydrogen-rich material below (e.g. Ulrich 1972; Kippenhahn 1974), as has been admitted by Cox, King, and Hodson (1978). Such a helium-enriched outer layer must therefore be continuously maintained, in order to offset this tendency to mix with the underlying material. It is suggested by A. Cox (1978a) that such a maintenance might be effected by radiation pressure acting on the helium atoms in the outer stellar layers, or perhaps by means of a "Cepheid wind" that is rich in hydrogen and that thus leaves the helium behind. (This last possibility has also been suggested by Cox, King, and Hodson 1978).

Second, one could assume that pulsation theory and the way it is being applied to stellar envelopes are essentially correct, and question the evolutionary calculations, and hence M_{evol}. This is perhaps a somewhat unpopular approach, but the precautionary remarks made earlier should be kept in mind, in particular as regards the relative lateness of those evolutionary phases relevant to the Cepheid question. For example, could mass loss have occurred before the Cepheid stage, perhaps while the star was a red giant? It is true that Lauterborn, Refsdal, and Weigert (1971) have shown that a loss of mass of more than about 10 percent or less than some 80 percent will prevent a Cepheid model from executing a loop. However, considering the many uncertainties involved in calculations of blue loops, and our apparent lack of understanding of what causes them, this reviewer wonders about the universality of such a conclusion. Thus, could such a conclusion conceivably be code-dependent? To the best of our knowledge, a similar conclusion, obtained with a different stellar evolution code by an entirely independent group, has not appeared in the published literature.[1] (Iben 1974 also acknowledges the possibility of mass loss prior to the Cepheid phase.) Such a conclusion would seem to be sufficiently important and far-reaching that it should be checked in as independent a way as possible.

We are here only making a plea that we not be too complacent, and that we not put too much weight on conclusions that have been reached in the recent past. For example, in the course of this reviewer's lifetime the value of the Hubble constant has changed by about a factor of ten, despite statements occasionally made that the value of this constant is known at any given time to within 15 or 20 percent! Hopefully, Cepheid masses are not as difficult to obtain as the value of the Hubble constant, but the principle is the same. Also, it is a bit sobering to realize that an error of a factor of 4 in Cepheid luminosities persisted among astronomers for some 40 years prior to the early 1950's (the zero point of the Cepheid period-luminosity relation, the history of which has been described so beautifully by Fernie (1969)! These examples will, hopefully, encourage us to be very cautious in our acceptance of at least some quantitave results.

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[1]Recently, it has been shown by Sreenivaran and Wilson (1978) that a loss of 10% of the mass in the red giant phase of a 15 $M_{\odot}$ stellar model will suppress a blue loop.

IV. SOME CONJECTURES ABOUT BEAT CEPHEIDS

As we have pointed out before, it seems difficult to avoid the conclusion that the masses of the beat Cepheids (M_{beat} in §III), as usually computed, are significantly less (by a factor of 1/3 to 1/2) than conventional evolutionary masses, say (3-4) $M_{\odot}$ for Cepheids in this period range (1^d to 7^d for the longer period). This discrepancy apparently exists even if we do not accept a mass discrepancy for the other kinds of mass considered in §III.

Apparently, one can adopt the view that these stars really have the low masses found as described in §III, and that these stars thus represent a new kind of stellar object. This was the opinion expressed by King, Hansen, Ross, and Cox (1975). However, as pointed out by Stobie (1977), this view runs into the difficulty that the existence of such objects is hard to understand within the context of stellar evolution theory.

The alternative view is that the above small mass values are for some reason illusory, and that the beat Cepheids really have normal Cepheid masses for their periods. We have, according to this standpoint, for some reason been led astray as regards the masses of these stars. The present reviewer is now somewhat inclined to this viewpoint.

Assuming that there is a mass discrepancy, and that the masses as given by the usual kind of analysis are for some reason too low, we seem to have available at least two alternative sets of assumptions: (1) Pulsation theory (assumed purely radial) is correct, but the stellar envelope structure normally assumed in the calculation is incorrect. Or, (2) the conventional envelope structure is correct, but pulsation theory, as it is usually assumed and used, is incorrect.

In keeping with the set of assumptions (1), Cogan (1977) pointed out that the presence of convection in the envelope, with a rather large value of the ratio α of mixing length to pressure scale height, say $\alpha \approx 1.5$ or 2.0, could sufficiently modify the envelope structure so as to result in considerably larger masses than for purely radiative envelope models. This modification is, as might be expected, more and more pronounced, the lower is the effective temperature T_e. The way in which the presence of convection in the envelope works, is as follows: It is known (e.g., Table 27.3 of Cox and Giuli 1968) that lowering the relative mass concentration of a star lowers the value Π_1/Π_0 of the ratio of radial first overtone to fundamental periods. Now, it is known observationally (see, e.g., Cogan 1977) that the beat Cepheids have $\Pi_1/\Pi_0 = 0.70\text{-}0.71$ compared to the value $\Pi_1/\Pi_0 = 0.74\text{-}0.75$ found for purely radiative models having expected masses of, say, (3-4) $M_\odot$ (e.g., Henden and Cox 1976; Cox, King, Hodson, and Henden 1977). Essentially, the lower mass concentration increases Π_0 more than Π_1. With a smaller mass concentration produced by the presence of convection, Cogan (1977) found that the observed period ratio Π_1/Π_0 could be obtained with "normal" Cepheid masses for the relevant period range.

More specifically, Cogan (1977) assumed that the fundamental and first overtone periods were, respectively, $\Pi_0 = 2^{\mathrm{d}}.5684$ and $\Pi_1 = 1^{\mathrm{d}}.8248$ ($\Pi_1/\Pi_0 = 0.7105$) for U TrA. He then constructed a series of model envelopes, in which convection was included according to the mixing-length formalism with a given value α of the ratio of mixing length to pressure scale height. The equilibrium or average luminosity L was the parameter in this series. Since $L \propto R^2 T_e^4$ and since both Π_0 and Π_1 are functions of L, M (mass), and T_e, then specification of L, Π_0, and Π_1 leads to values of M, R, and T_e. For values of L corresponding to rather low values of T_e (near the red edge of the instability strip), and for $\alpha \gtrsim 1.5 - 2.0$, Cogan (1977) found that $M \approx 4\ M_\odot$, about the expected value, for this star.

However, Cogan's (1977) conclusion has been criticized by Henden and Cox (1976); Cox, Deupree, King, and Hodson (1977); and Deupree (1977c), who come to quantitatively different conclusions. It is found by these other investigators that the presence of convection, as usually modelled, has very little influence on derived masses of beat Cepheids.

In a recent study by Saio, Kobayashi, and Takeuti (1977), the influence of convection, again with use of the mixing-length formalism, on the periods of beat Cepheids was reexamined. Conclusions qualitatively similar to those of Cogan (1977), and not greatly different quantitatively, were reached by these investigators: namely, that with (mixing-length) convection included in the static model, if the beat Cepheids are fairly cool, then the observed periods can be obtained with "normal," or expected, masses. Saio, Kobayashi, and Takeuti (1977) attributed the (rather small) quantitative differences from the results of Cogan (1977) to differences in the formulation of mixing length theory, in particular to differences in convective efficiency.

Saio, Kobayashi, and Takeuti (1977) also examined the effect of (mixing-length) convection with $\alpha = 1.5$ on the ratio Π_2/Π_0 of second-overtone period to fundamental period. They found that this ratio achieved the approximate value of 0.50 appropriate to the Simon-Schmidt (1976) resonance interpretation of the bumps on certain Cepheid velocity curves (see earlier) for a mass of $M/M_\odot = 4.7$ for the bump Cepheid S Nor. With the value $\log(L/L_\odot) = 3.518$ adopted by Saio, Kobayashi, and Takeuti (1977) and equations (1) or (2), we obtain $M_{evol}/M_{\Theta} = 6.5$. If the above value of $M = 4.7\ M_{\Theta}$ can be equated to M_{bump}, we obtain $M_{bump}/M_{evol} \approx 0.72$. This ratio is comparable to the value given above as obtained by Davis and Davison (1978), but larger than the value suggested by Christy (1966b), Stobie (1969), Fischel and Sparks (1974), and others. (The value given by Cox 1978b for S Nor is $\log[L/L_{\Theta}] = 3.7$; the corresponding value of M_{evol} is $M_{evol}/M_{\Theta} \approx 7.4$, and $M_{bump}/M_{evol} \approx 0.64$.)

Once again, this reviewer would like to urge exteme caution in our acceptance of at least certain kinds of quantitative information. After all, convection is one of the most poorly understood phenomena in astrophysics, and its effects may be much more important than our current thinking indicates. So the possibility suggested by Cogan (1977) should not, in this reviewer's opinion, be excluded out of hand.

The proposal made by Cox, Deupree, King, and Hodson (1977) regarding helium enrichment of the outer stellar layers, is also along the lines of the set of assumptions (1), i.e., keep (radial) pulsation theory but modify the envelope structure. The helium-rich outer layers serve to decrease the mass concentration of the star. Physically, the helium-enriched material compresses the outer stellar layers, and leaves the inner regions nearly unaffected. The net result is, just as in the case of a significant amount of convection in the envelope, that the observed first overtone-to-fundamental period ratio for the beat Cepheids can be obtained with "normal" masses of about (3-4) M_{Θ}.

However, as an alternative viewpoint, this reviewer has recently begun wondering if perhaps we ought not to look more carefully at the set of assumptions (2), i.e., accept currently assumed envelope structures but question pulsation theory as it is usually used. It is well known that the derived masses (and radii) of beat Cepheids are very sensitive to the details of the theory of pulsation, as has been emphasized by Stobie (1977). So far, all work on the question of beat Cepheid masses has assumed purely radial pulsations. Suppose that the actual pulsations were mostly radial, but that there was a small admixture of nonradial pulsations. Then, if the nonradial component were sufficiently small, the usual interpretation of single-mode variables as radial pulsators would not be appreciably affected. However, this small admixture of nonradial pulsations might conceivably alter the period ratios just enough to imply (assuming strictly radial pulsations) anomalously small masses for the beat Cepheids.

After all, it has recently been shown by Osaki (1977) and Dziembowski (1977) that stars in the Cepheid instability strip are unstable against not only radial pulsations, but also certain nonradial pulsations. These nonradial pulsations are excited by the same envelope-ionization mechanisms that excite radial pulsations, and the growth rates of these nonradial pulsations are found to be comparable to those for radial pulsations. In particular, nonradial acoustic modes having values of the index $\ell \gtrsim 4$ in the spherical harmonic $Y_\ell^m(\theta,\phi)$ are found to be excited. (The index ℓ in $Y_\ell^m[\theta,\phi]$ is equal to the number of "node lines" in the star.) Osaki (1977) finds that p_1 modes are excited for $\ell \geq 4$ in a $7\ M_\odot$ Cepheid model, whereas Dziembowski finds that p_1 modes are excited for $\ell \geq 6$ for approximately $0.6\ M_\odot$ RR Lyrae models.

An inspection of the properties of nonradial oscillations (e.g., Cox 1976) shows that the periods of these acoustic modes decrease as ℓ increases. Moreover, the periods of those p_1 modes found by Osaki to be overstable (i.e., those with $\ell \geq 4$) for his $7\ M_\odot$ model, when divided by the fundamental radial period, give values of the period ratio $\lesssim 0.6$. If the presence of a small amount of nonradial pulsation would have the effect of lowering the period ratio, then the presence of only a small amount of nonradial pulsation might therefore be sufficient to lower the period ratio to the observed value of 0.70-0.71.

It is also of some interest that, according to numerous investigations (e.g., Dziembowski 1977), the number of modes that may be excited increases as one descends the instability strip. And, as was pointed out earlier, the beat Cepheids occupy just the lower portions of the instability strip. Very low down on the instability strip are the Delta Scuti variables, in which, as Dziembowski (1975, 1977) has suggested, many different modes, both radial and nonradial, may be excited, thus possibly accounting for the complex observed behavior of these stars. (The Delta Scuti variables in the lower part of the instability strip.) Perhaps at the very bottom of the instability strip are the variable DA white dwarfs (the ZZ Ceti stars) which, as McGraw (1977) has suggested, may be pulsating exclusively in nonradial g modes.

The proposal that the pulsations of the beat Cepheids may involve a small amount of nonradial pulsation raises a number of questions, which will not be answered here. For example, would a mixture of predominantly radial pulsations with a small amount of nonradial pulsations produce a lowering of the observed period ratio? Would the relevant nonradial modes be excited in a star that was already executing radial pulsations? (The models assumed by Osaki 1977 and Dziembowski 1977 were static models.)

It is likely that this conjecture does not make life any simpler for the theorist in seeking an interpretation of the beat Cepheids. The combination of (mostly) radial and nonradial pulsations is a complicated phenomenon indeed. However, no one has ever guaranteed us that life would be simple in nature. The main point of this conjecture is that purely radial pulsation theory may not be strictly applicable to the beat Cepheids.

Below are a few other conjectures which may bear either specifically on the beat Cepheids or on the problem of Cepheid masses in general. We do not necessarily expect these conjectures to be taken seriously, but we hope that they will at least provide some "food for thought."

The possibility of some kind of mass loss prior to the Cepheid stage should not be rejected until or unless more work has been done. This statement applies even in spite of the work of Lauterborn, Refsdal, and Weigert (1971) which suggests that mass loss, except for a very little bit or a lot, will prevent a star from making a blue loop, and thus becoming a Cepheid. As stated in §III, our knowledge of the causes of the loops, and what they are affected by, is still extremely rudimentary.

The use of the linear theory may not really be adequate, as Dziembowski and Koslowski (1974) have pointed out. It is true that the work of Stellingwerf (1975b) and of Cox, Hodson, and King (1978) make it appear very likely that the linear theory can be relied upon to give a good interpretation of observed periods of Cepheids. In the case of single-mode pulsators, linear theory periods are almost certainly adequate. However, in beat Cepheids, as also was pointed out above, even a slight change in the periods can make a big difference in the inferred masses. In Cox, Hodson, and King (1978) the linear and nonlinear periods were compared, and found to be nearly identical, in the case of a decaying mode and a growing mode. However, in a beat Cepheid it may be that <u>both</u> modes are excited, and it is well known that an oscillator that is either excited or damped will have a slightly different period from that of a constant-amplitude oscillator. Perhaps the question of the strict applicability of a linear theory cannot really be answered until we can calculate true double- (or multiple-) mode behavior!

It is also possible that the existence of finite-amplitude pulsations in a star could affect its evolution, so that the conventional theory of evolution, applicable to nonpulsating stars, may not be appropriate for Cepheids. It has, in fact, been shown by Buchler (1978) that for large enough pulsation amplitudes in a star there can indeed be a feedback, so that the pulsations can affect the evolution of the star. Simon (1974) had suggested earlier that such a feedback may exist in a pulsating star.

V. SUMMARY AND CONCLUSIONS

We have reviewed most of the recent theoretical work on Cepheids and RR Lyrae variables, and have indicated what appear to this reviewer to be the main remaining problems. These are (1) the large-amplitude mode behavior, (2) convection, and (3) Cepheid masses.

It is pointed out, with respect to the large-amplitude mode behavior, that not only do two methods of calculating it seem to yield contradictory results, but also true double- (or multiple-) mode behavior in model stars has apparently not yet been calculated to everyone's satisfaction. With respect to convection, we are still apparently without a really workable and entirely satisfactory theory of time-dependent convection, in spite of the beautiful and pioneering work of Deupree on two-dimensional convection (see §II).

Cepheid masses are still a problem. Except in the case of the beat Cepheids, the inferred masses may be low by some 20-40 percent as compared to conventional evolutionary calculations. Whether this discrepancy is serious, is apparently a matter on which opinions differ. To this reviewer, this discrepancy, while it is somewhat disturbing, is not necessarily alarming, in view of the many uncertainties involved, both observational and theoretical.

It is only in the case of the beat Cepheids, in our opinion, that the very low inferred masses are cause for genuine and/or serious concern. As pointed out in §IV, one can retain conventional (strictly radial) pulsation theory, and attempt to resolve the beat Cepheid mass problem by assuming an alteration of the envelope structure. Evidently, a lowering of the mass concentration can resolve the mass problem. Attempts in this direction have called upon convection in the static model, as suggested by Cogan (1977) and Saio, Kobayashi, and Takeuti (1977); or the assumption of a helium-enriched outer layer, as suggested by Cox, Deupree, King, and Hodson (1977).

Alternatively, one can retain presently computed static envelope models, but question the applicability of strictly radial pulsation theory to the beat Cepheids. This reviewer wonders if a small admixture of nonradial pulsations would not be sufficient to invalidate the use of a theory of strictly radial pulsations for these stars. It is pointed out that Cepheid models have recently been shown by Osaki (1977) and Dziembowski (1977) to be unstable to certain nonradial modes, with growth rates comparable to those of the radial modes. Moreover, the simultaneous excitation of several pulsation modes is more likely in the lower parts of the instability strip, where the beat Cepheids are, than in the upper parts (Dziembowski 1975, 1977). It looks as if we have problems enough in attempting to understand Cepheids, to keep us busy for some time!

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Discussion

Hillendahl: On your last slide, you showed various ways of getting the mass, and all of them relied rather heavily on theoretical interpretations. But if you look at the models various people have made for classical Cepheids, one thing that seems to be rather universal is that the atmospheres appear to be in gravitational collapse between phases 0.5 and 0.7, so the photosphere stays essentially in the same mass zone. Now if you use only that idea, you can get an actual physical displacement in centimeters by measuring the displacement of the Fe II lines at those phases; and from the fact that the atmosphere is moving linearly with time, you can get the gravity. Thus you have g and R, and from them you can deduce values of M around 0.6 of the evolutionary mass, using very little theory.

J. Cox: Is that related to the method described by Art Cox this morning, involving gravities?

Hillendahl: I don't think so. I first tried this method using some data from a 1937 paper by A. H. Joy (Ap. J. 86, 363). He did a velocity survey, and I noticed that if you plotted the rate of change of velocity during that period, you got straight lines from which you would deduce g. You plot this versus period and at zero period you get a gravity of around 4. As you go to longer periods, the gravity becomes less - it's a very nice relationship. I think that if you could combine this with something that requires less theoretical interpretation, it might give you guidance as to which one of these masses might be nearer the truth.

A. Cox: What is the ℓ value for these high nonradial modes?

J. Cox: About 4 to 6.

A. Cox: Is that observable with so many bumps on the star?

J. Cox: I think that's a very good question - how do you detect this? I can't answer that, but maybe somebody else can.

Aizenman: I think the question of how you tell whether a higher mode is present has been addressed by Dziembowski in 1977, in a paper in Acta Astronomica. He described general observational characteristics of nonradial pulsation. If you take the ratio of the observed radial velocity, integrated over the disk of the star, to the change in the bolometric magnitude, this is independent of the angle of inclination. If you know the radius of the star and assume a limb darkening law, you can get a very strong criterion for the value of ℓ ($\ell = 0,1,2,...$).

Deupree: Would you expect these modes to have $m = 0$ or $m \neq 0$, and does it matter? Do these nonradial modes show the oscillations in the interior that were found in Dzienbowski's 1972 paper? As you may remember, there were oscillations in the eigenfunctions, and instead of going smoothly to zero, as you went into the deep interior, there was coupling of the g-modes.

J. Cox: The modes that are excited are the p_1-modes, and their amplitudes become rather small in the interior. So the answer to your second question is that the essential parts are not excited at large amplitude. With regard to your first question about the m value, it doesn't matter, as long as there are no perturbing influences. If you have rotation or a magnetic field, then the m value does matter. I must say that I don't have a very specific picture of these nonradial modes, except to say that that possibility might exist. It raises more questions than it answers. For example, if there were nonradial modes present, would the period ratio go in the right direction? Is a star that is already executing radial oscillations stable against nonradial oscillations? The models studied by Osaki and Dziembowski were static models -- now we are asking about nonstatic models. So there are many questions to be asked.