TIME DOMAIN PERIOD DETERMINATION TECHNIQUES

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Abstract

Two simple period determination schemes are discussed. They are well suited to problems involving non-sinusoidal periodic phenomena sampled at a few irregularly spaced points. Statistical properties are discussed. The techniques are applied to the double-mode Cepheids BK Cen and TU CAS as test cases.

Introduction

Fourier techniques offer the "best possible" methods of spectral analysis in a number of respects, but they are not well suited to cases with strongly non-sinusoidal variation, or cases in which only a few scattered data points are available. Unfortunately, this is exactly the situation in many physical problems, including variable stars. I will discuss two classical techniques that do treat these cases efficiently in cases involving several discrete frequencies.

When departing from Fourier methods, invariably some type of sacrifice must be made. In this case "ghosts" appear at subharmonic frequencies of a strong spectral line. An important point in the analysis is to show that the unwanted subharmonics can be minimized and can be distinguished from the real frequency. Once understood, these "ghosts" can be used to good advantage as indications of the presence or absence of high frequency components.

A more complete description of these techniques will appear in Stellingwerf (1978).

I. THE PDM METHOD

A discrete set of observations can be represented by two vectors, $\underset{\sim}{x}$, and the observation times $\underset{\sim}{t}$, where the i^{th} observation is given by (x_i,t_i) and there are N points in all (i = 1,N). Let σ^2 be the variance of $\underset{\sim}{x}$, given by

$$\sigma^2 = \frac{\Sigma(x_i-\bar{x})^2}{N-1} \qquad (1)$$

where $\bar{x}$ is the mean, $\bar{x} = \Sigma x_i/N$. For any subset of x_i's we define the sample variance, s^2, exactly as in equation (1). Suppose we have chosen M distinct samples, having variances s_j^2 (j = 1,M) and containing n_j data points. The overall variance for all the samples is then given by

$$s^2 = \frac{\Sigma(n_j-1)s_j^2}{\Sigma n_j - M} \qquad (2)$$

as a consequence of equation (1).

We wish to minimize the variance of the data with respect to the mean light curve. Let Π be a trial period, compute a phase vector $\underset{\sim}{\phi}$: $\phi_i = t_i/\Pi - [t_i/\Pi]$; brackets indicate integer part. Equivalently, $\underset{\sim}{\phi} = \underset{\sim}{t} \bmod (\Pi)$. We now pick M samples from $\underset{\sim}{x}$ using the criterion that all the members of sample j have similar ϕ_i. Usually the full phase interval (0,1) is divided into fixed bins, but the samples may be chosen in any way that satisfies the criterion. All points need not be picked, or, alternatively, a point can belong to many samples. The variance of these samples gives a measure of the scatter around the mean light curve defined by the means of the x_i in each sample, considered as a function of ϕ. We define the statistic

$$\theta = \frac{s^2}{\sigma^2} \qquad (3)$$

where s^2 is given by equation (2) and σ^2 is given by equation (1). If Π is not a true period, then $s^2 \approx \sigma^2$ and $\theta \approx 1$, whereas if Π is a correct period θ will reach a local minimum compared with neighboring periods, hopefully near zero.

Since this technique seeks to minimize the dispersion of the data at constant phase, we will refer to it as "phase dispersion minimization", or PDM for short. Mathematically, this is a least-squares fitting technique, but rather than fitting to a given curve (such as a Fourier component), the fit is relative to the mean curve as defined by the means of each bin. We simultaneously obtain the best least-squares light curve and the best period. The PDM technique is thus a "Fouriergram" method [as discussed by Faulkner (1977)] of infinite order, since all harmonics are included in the fitted function. The Fourier series technique, a least squares fit to a truncated series with variable amplitudes and phase, often requires additional constraints and rather high orders for nonsinusoidal variations (Lucy, 1976).

Although the individual samples may be chosen in many ways, it is convenient to define a standard bin structure. We divide the unit interval into N_b bins of length $1/N_b$, and take N_c "covers" of N_b bins, each cover offset in phase by $1/(N_bN_c)$ from the previous cover, using

periodic conditions on the unit interval to obtain a uniform covering. We thus obtain $M = N_b N_c$ bins, each of length $1/N_b$, and whose midpoints are uniformly spaced along the unit interval at distance of $1/(N_b N_c)$. Clearly, each data point will fall in exactly N_c bins. Denote a given bin structure by (N_b, N_c).

To compute Θ, we take the ratio of variances of the two subsets of $\underset{\sim}{X}$, that of the actual observations, $\underset{\sim}{x}$, and that of the bins. Θ therefore has a probability density given by an F distribution with $\Sigma n_j - M$ and $N-1$ degrees of freedom. It is convenient to define F as a number greater than unity, so $F \equiv \Theta^{-1}$. The probability, P, that a given value of Θ is due to random fluctuations (also called the "significance") is twice the area of the F distribution above Θ^{-1} (two-sided test). This probability approaches unity as $\Theta \to 1$. Thus, for significance P, we compute

$$F_{(P/2, N_{1f}, N_{2f})} = 1/\Theta, \quad N_{1f} = N-1, \quad N_{2f} = \Sigma n_j - M. \tag{4}$$

P may then be obtained by reference to an F table, or using an approximation to $P(F, N_{1f}, N_{2f})$, see Abramowitz and Segun (1965), § 26.

If N is large (> 100), we may take $\sigma^2(\underset{\sim}{x}) \approx \sigma^2(\underset{\sim}{X})$. In this case we may use the somewhat simpler χ^2 test,

$$\chi^2_{(P/2, N_f)} = N_f \Theta , \quad N_f = \Sigma n_j - M. \tag{5}$$

Here P is twice the area of the χ^2 distribution below $N_f\Theta$.

II. THE WR METHOD

An interesting related method is discussed by Whittaker and Robinson (1926, "WR") in which one seeks the maximum variance of the bin means (as opposed to the mean of the bin variances). If s_m^2 is the variance of the bin means, define $\Theta_{WR} = 1 - s_m^2/\sigma^2$ for comparison purposes. In general Θ_{WR} will vary between 0 and $1 - \langle 1/n_j \rangle$, where $\langle n_j \rangle$ is the mean number of points per bin. At a true period $s_m^2 \approx \sigma^2$ and $\Theta_{WR} = 0$. Here we seek periods at which the amplitude of the mean curve is a maximum, which in most cases will correspond to minimum phase dispersion. The calculation of σ_m^2 is easier than s^2 (equation 2), but the number of degrees of freedom is much lower, suggesting less sensitivity (see § III). We will show below that in most cases this supposition is true, although for one range of parameters the WR technique may be preferrable. The WR method is discussed in detail by Chapman and Bartels (1940) under the title "persistence analysis".

A statistical criterion may also be derived for the WR technique. The distribution of bin means is normal if M is large with $\sigma_m^2 = \sigma^2/n_j$. Since M is generally not large, however, a t distribution with M-1 degrees of freedom will actually be obtained, with $\sigma_m^2 \approx (M-1)/(M-3)\ \sigma^2/n_j$. We may always select bins with $n_j = N/N_b$, for all j. Note that normal constant-phase bins will produce binomially distributed n_j and increase σ_m^2 since a harmonic mean of n_j appears in σ_m^2. We wish to test whether the observed s_m^2 is significantly bigger than σ_m^2. Evidently s_m^2/σ_m^2 follows an F distribution with M-1 and N-1 degrees of freedom. Using the definition of Θ_{WR}, we have

$$F_{(M-1,N-1)} \approx \frac{(M-3)}{(M-1)} \frac{N}{N_b} (1-\Theta_{WR}). \qquad (6)$$

III. APPLICATIONS

Here I will briefly describe the practical aspects of these methods; for details see Stellingwerf (1978).

Typical transforms are illustrated by Figure 1, which shows the computed Θ-transforms for a sine wave [sin(2πft)], panel A, and a

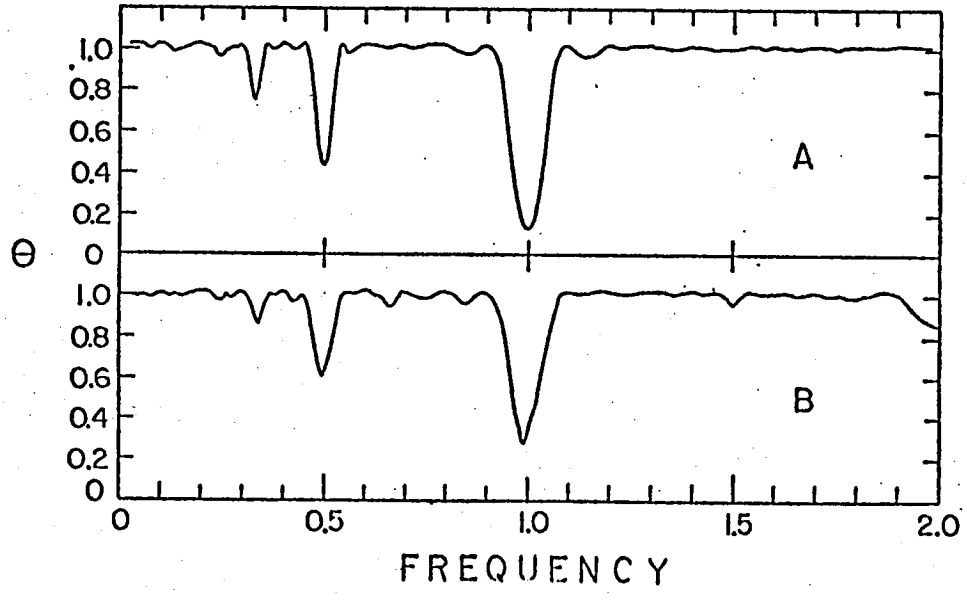

Fig. 1: Θ statistic versus frequency for the two test cases described in the text. Panel A: sine-wave transform; Panel B: sawtooth function transform.

"saw-tooth" function (fractional part of ft), panel B. These functions have been selected because they represent the two extremes found in variable star light curves. In each case x consisted of 201 data points evenly distributed over ten periods with f = 1, T = 10. A bin structure of $(N_b,N_c) = (5,2)$ was used. The main line at f = 1 is virtually identical to the Fourier power spectrum line shape $[(\sin(x)/x)^2]$. Two subharmonics are present (the third, just visible at f = 1/4 is below the cutoff $n = N_b \approx 3$, and is not significantly different from 1). The great similarity of the two curves shown in Figure 1 is, of course, the strong point of the PDM technique: highly nonsinusoidal variations are handled just as efficiently as a sine wave. The presence of subharmonic response could be a disadvantage if oscillations with widely spaced frequencies are present. In practice, however, subharmonics can be distinguished in at least three ways: 1) light curve shape, 2) narrow line widths, 3) reduced significance with increasing bin size. If initial scans of the full frequency range use broad bin sizes [(5,2), say], subharmonics should pose no problems.

It is important to know the number of observations required to achieve a given significance with these methods. This quantity ("N") is plotted in Figures 2 and 3 as a function of the signal-to-noise ratio, $\varepsilon = \sigma/\sigma_{Noise}$. A variety of bin structures are shown. It is seen that coarse bins and multiple covers are needed for small N and ε. Clearly, the PDM technique is superior at small N, while the WR technique seems preferable in noisy cases ($\varepsilon < 0$) if N is large enough. Accuracy is lost if bins are too wide, and multiple covers will not help if many bins contain the same points. In general (5,2) structure works very well. To obtain mean curves for prewhitening, bins should be made as fine as possible.

IV. MULTIPLE PERIODS OF BK CEN

To illustrate the use of these methods, we briefly present an analysis of the light variation of the double-mode Cepheid BK Centauri. For this purpose we will use only the 49 photoelectric measurements made by C. J. van Houten in 1965 for which Leotta-Janin (1967) has published the ΔV values. This author comments that "their number is too small to allow a reliable determination of the beat period" and so uses 25 years of plate estimates as well as knowledge of other Cepheid period ratios to obtain $\Pi_0 = 3^d.17389$, $\Pi_1 = 2^d.2366$. We have found that the photoelectric points alone unambiguously confirm these results, and provide information on mode interaction as well.

The 49 measures span 137 days. Line widths will therefore be about 0.007 in frequency, so the frequency step size was taken to be .0015 for moderate resolution. Twelve nights have two observations, providing minimal alias discrimination. Although the measurements are quite accurate, we nonetheless estimate $\varepsilon \approx 1.5$ due to the secondary oscillation. Figures 2 and 3 indicate that the PDM and the WR techniques should be about equally good for this case, provided $N_b<10$. We choose a (5,2) bin structure to maintain about 10 points per bin.

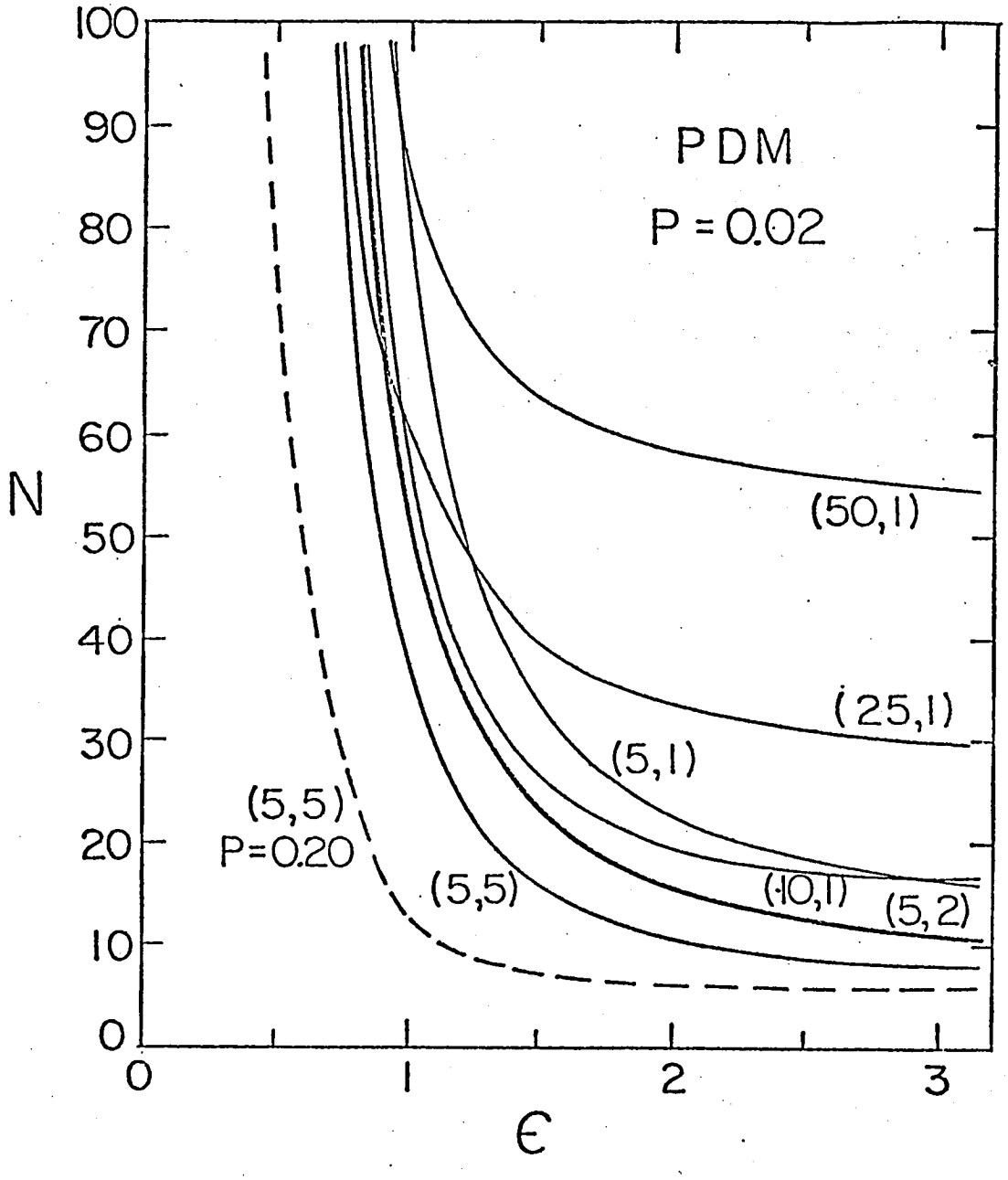

Fig. 2: Statistical characteristics of the PDM method. Notation as in text with various bin structures (N_b, N_c) as indicated. Solid lines refer to P = 0.02, dashed line is P = 0.20.

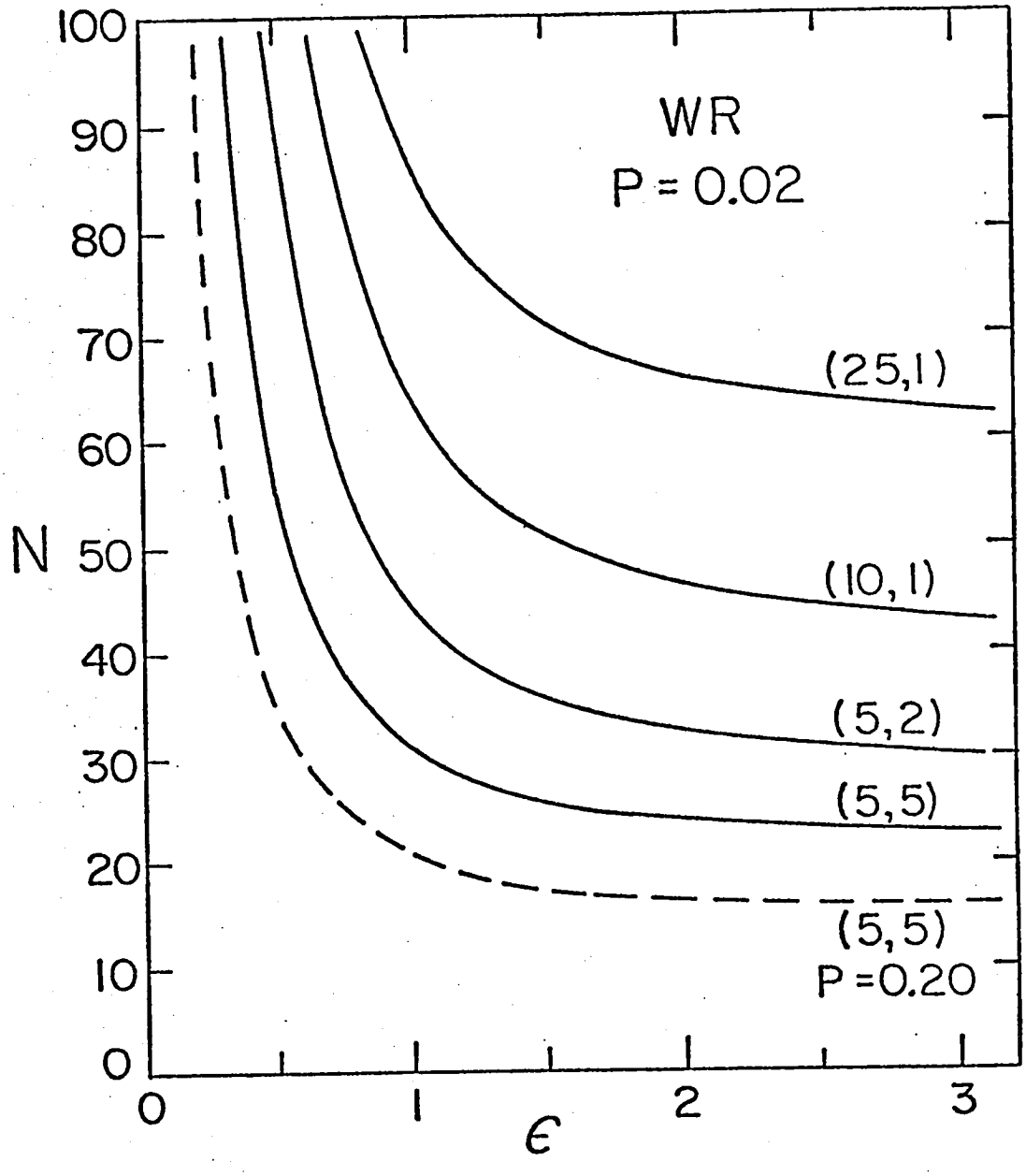

Fig. 3: Statistical characteristics of the WR method. Notation as in Fig. 2.

Figure 4, panel A shows the Θ transform of the raw data. The primary period is clearly evident (minimum e), with minima a and b forming the standard subharmonic sequence found in Figure 1. Minimum f is narrow and showed a two-cycle mean curve, it is therefore the

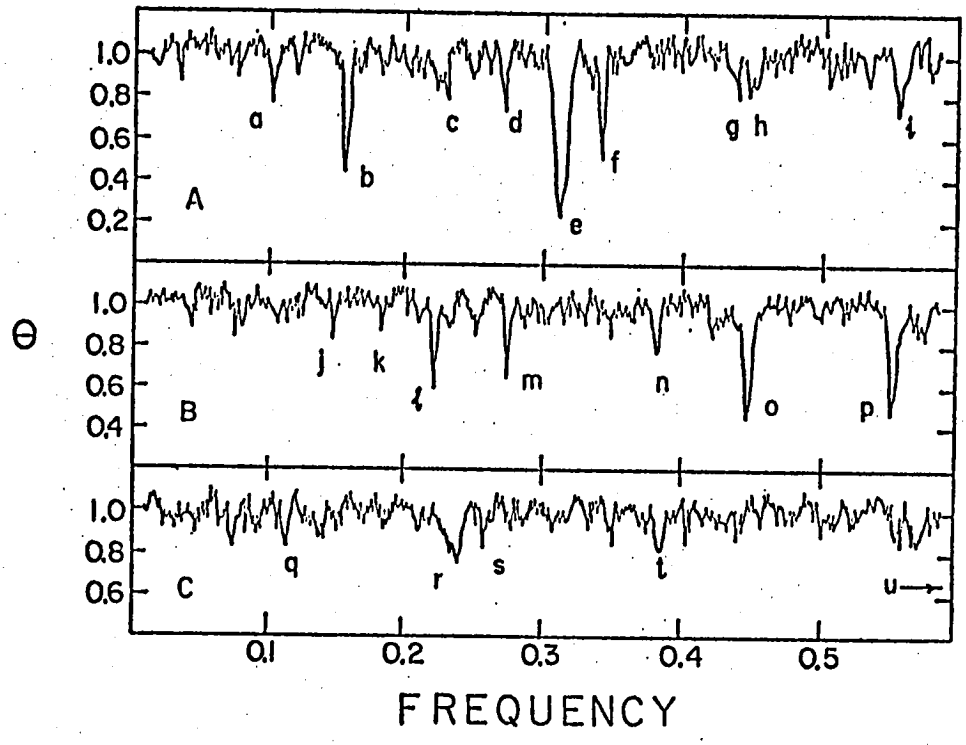

Figure 4: Θ transforms of the BK Cen data, plotted versus frequency in cycles/day. Panel A: transform of the raw data. Panel B: transform of the data after removal of the fundamental oscillation. Panel C: transform of the data after removal of the fundamental and first overtone oscillations. Labeled minima are identified in Table 1.

first subharmonic of a sizeable minimum off the scale, at $f \approx 0.7$. This is the main alias of e, occuring at $1-f_0$. Comparison of the subharmonics b and f indicates that e is the principal frequency. The identifications of the minima, together with their approximate frequencies and significances, are given in Table 1. Among the barely significant group, we identify g as the second subharmonic of the second alias of e, at $1+f_0$, while d, h and i are first overtone features.

We see here a distinct advantage of the subharmonic response of this method: any other major frequency in the range .6<f<1.2 would show a subharmonic on Figure 1(A) comparable to b or f. Any minimum in the range 1.2<f<1.8 would show a second subharmonic comparable to

TABLE 1

Identification of Features in Figure 4

Feature	Identification	Frequency (d^{-1})	Significance
a	$f_0/3$	.105	0.13
b	$f_0/2$	.158	0.00
c	$(1-f_0)/3$	.226	0.39
d	$(1-f_1)/2$	.276	0.09
e	f_0	.316	0.00
f	$(1-f_0)/2$	.344	0.00
g	$(1+f_0)/3$	.439	0.24
h	f_1	.447	0.27
i	$1-f_1$	.553	0.05
j	$f_1/3$	.148	0.50
k	$(1-f_1)/3$	.183	0.55
l	$f_1/2$	.224	0.04
m	$(1-f_1)/2$	.276	0.08
n	$(f_0+f_1)/2$	.385	0.33
o	f_1	.447	0.00
p	$1-f_1$	.553	0.00
q	$(1-(f_0+f_1))/2$	.116	0.60
r	$1-(f_0+f_1)$	.235	0.27
s	$(f_0+f_1)/3$	.253	0.55
t	$(f_0+f_1)/2$	.383	0.43
u	f_0+f_1	.764	0.11

a or g. Since all features have been identified, no high frequency components exist. This "look-ahead" feature can be extended by increasing N_b, at the expense of further complicating the spectrum.

Having identified the principal frequency (f_0), this component was then removed from the data (using linear interpolation between bin means). The transform of the reduced data is shown in panel B of Figure 1. Here j, l, o are due to the first overtone, while k, m, p represent the aliases. Note that each alias feature is slightly less significant than the corresponding "real" feature. A mode interaction feature, n, has also appeared. We can again conclude that no more significant modes exist to f = 1.8, since no subharmonics appear.

Panel C shows the transform of the doubly-reduced data. No significant feature appears, but many marginal features are related to the mode interaction term f_0+f_1. An extension of the transform to higher frequencies showed that this mode did indeed appear at the P = 0.11 level. This is the only remaining significant mode. This particular interaction term (f_0+f_1) has also been found to be excited in several other studies of similar objects (Fitch 1976, Henden 1976, Faulkner 1977a,b, Fitch et.al. 1978). In particular, this term invariably dominates the beat frequency (f_1-f_0) component. At present the

meaning of this is not clear.

The periods and amplitudes of the three components were determined using a finer frequency scan, and are given in Table 2. Allowing for the bin size, the actual amplitudes should be about 20% larger, this is

TABLE 2

Summary of Principal Components

Mode	Frequency (d^{-1})	Period (d)	Amplitude	Adjusted Amplitude	Phase*
f_0	0.3153	3.172	$0^m.544$	$0^m.648$	0.40
f_1	0.4473	2.236	$0^m.222$	$0^m.264$	0.56
f_0+f_1	0.7662	1.302	$0^m.151$	$0^m.180$	0.16

*Phase = $(t_{max}-t_1)/\Pi_0$, t_1 = JD2,438,813.48

shown in the "corrected amplitudes". The sum of these estimates is $1^m.1$, in agreement with the range of the photoelectric measures. The standard deviation of the residuals (triply-reduced data) was $0^m.048$.

The frequency labeled f_0+f_1 in Table 2 is listed as derived and differs from the sum of the principal components. The discrepancy may not be real, since it amounts to about 1/T -- suggesting a side lobe problem.

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Discussion

A. Cox: What is the σ of the data after you've taken the periods out?

Stellingwerf: The residual σ following the triple reduction was $\sigma \approx 0.05$ magnitudes.