Theories of White Dwarf Oscillations

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Abstract

The current status of theoretical understanding of the oscillations observed in the ZZ Ceti stars and cataclysmic variables is briefly reviewed. Non-radial g-mode oscillations appear to provide a satisfactory explanation for the low-amplitude variables such as R548, with periods in the range ~ 200-300 seconds, but for the longer-periods (800-1000 second) oscillators, the situation is still unclear. Rotation may play an important role in this problem, and the effects of both slow and fast rotation upon the mode structure are discussed. In the cataclysmic variables, both accretion and thermonuclear burning may act to excite oscillations of the white dwarf, and recent work on this problem is summarized also.

I. Introduction

The purpose of this review is to provide a summary of some of the recent developments in the theory of white dwarf oscillations. It is beyond the scope of this paper to provide a comprehensive review of all of the work in this area since the last LASL/GSFC pulsation conference. Instead I shall restrict myself to those subjects on which there has been recent activity or which appear to me to be particularly important, and I shall concentrate almost exclusively on theoretical research carried out within the past two or three years. Earlier work on white dwarf oscillations has already been summarized in a number of review papers to which reference is made below.

The general theory of non-radial oscillations of stars has recently been reviewed by Ledoux (1974) and by Cox (1976), both of whom include brief summaries of the work on white dwarf oscillations. A review devoted exclusively to white dwarfs has been presented by Van Horn (1976). The observational basis for the theoretical investigations are provided by the periodicities that have been detected in the ZZ Ceti stars (single, variable white dwarfs) and in the cataclysmic variables. The observed data on the ZZ Ceti stars were reviewed by Robinson and McGraw (1976) at the last LASL/GSFC pulsation conference, and an excellent, comprehensive discussion of the observed pulsational properties of these stars has recently been provided by McGraw (1977). The properties of the cataclysmic variable oscillators have been reviewed by Warner (1976a) and by Robinson (1976), and a comprehensive and detailed discussion of these systems has been given by Warner (1976b). The.most recent survey of the observations of white dwarf oscillations has been provided by Robinson (1978) at this conference.

The plan of this paper is as follows. In § II some of the general

problems associated with non-radial oscillations, but which have a special

significance for white dwarfs, are reviewed. In § III research concerned

primarily with oscillations of single white dwarfs is discussed, except

for effects of rotation. Rotational modifications of the spectra of non-

radial oscillations are considered separately in § IV, and this is followed

in § V by a discussion of recent work on pulsational instabilities of high-

luminosity degenerate stars, including those produced by thermonuclear

burning. In § VI we conclude with recommendations concerning some problems

for future study.

II. General Problems of Non-Radial Oscillations

a. $\ell=1$ Modes

In the study of non-radial stellar oscillations, modes corresponding

to spherical harmonics of degree $\ell=1$ have generally been ignored. This is

a result of the mistaken impression that such modes correspond to displace-

ments of the center of mass, which of course cannot occur in isolated stars.

For "stars" composed of homogeneous, incompressible fluids, this claim is

true; however, real stars are neither homogeneous nor incompressible, and

in such cases non-zero displacements at the center of the star need not

correspond to a displacement of the center of mass. This was explicitly

shown by Smeyers (1966) for adiabatic oscillations, and a more general proof

has recently been given by Christensen-Dalsgaard (1975). Thus modes

corresponding to spherical harmonics of degree $\ell=1$ are physically realizable in stars. This is particularly important for the white dwarfs, because the $\ell=1$ modes have the longest periods of any of the g-modes, and because one of the most vexing problems connected with white dwarf oscillations has been the persistent inability of theory to identify low-order modes that have periods as long as those observed.

b. Connection between theory and observation

In contrast to radial oscillations, in which the relations between displacement, radial velocity, and luminosity variation are relatively straightforward, the connections of the radial and transverse displacements with the observed quantities are not simple in the case of non-radial oscillations. A pure eigenmode corresponding to the spherical harmonic $Y_{\ell m}(\theta,\phi)$, for example, has ℓ nodes in the colatitude interval $0 \leq \theta \leq \pi$, and the real part of the eigenfunction has m nodes equally spaced in azimuth between $0 \leq \phi \leq \pi$ (see Fig. 1). The observed light from a non-radially oscillating star will thus be a weighted integral over these angular functions. It is evident that the observed amplitude of a mode with ℓ or m even moderately large will be considerably reduced by cancellation. In addition, there will in general be mode-dependent phase shifts between the times of maximum radial velocity and maximum light. Dziembowski (1977a) has presented the general theory of the radial velocity and luminosity variations of non-radially oscillating stars. He has also evaluated the various integrals relating these quantities to the theoretical parameters for the case of an Eddington limb-darkening law, and he finds that the observed radial velocity amplitude is reduced to less than one-tenth of the maximum for $\ell \geq 4$. In a separate publication (Dziembowski 1977b) he has applied the theory specifically to evaluate the observed parameters
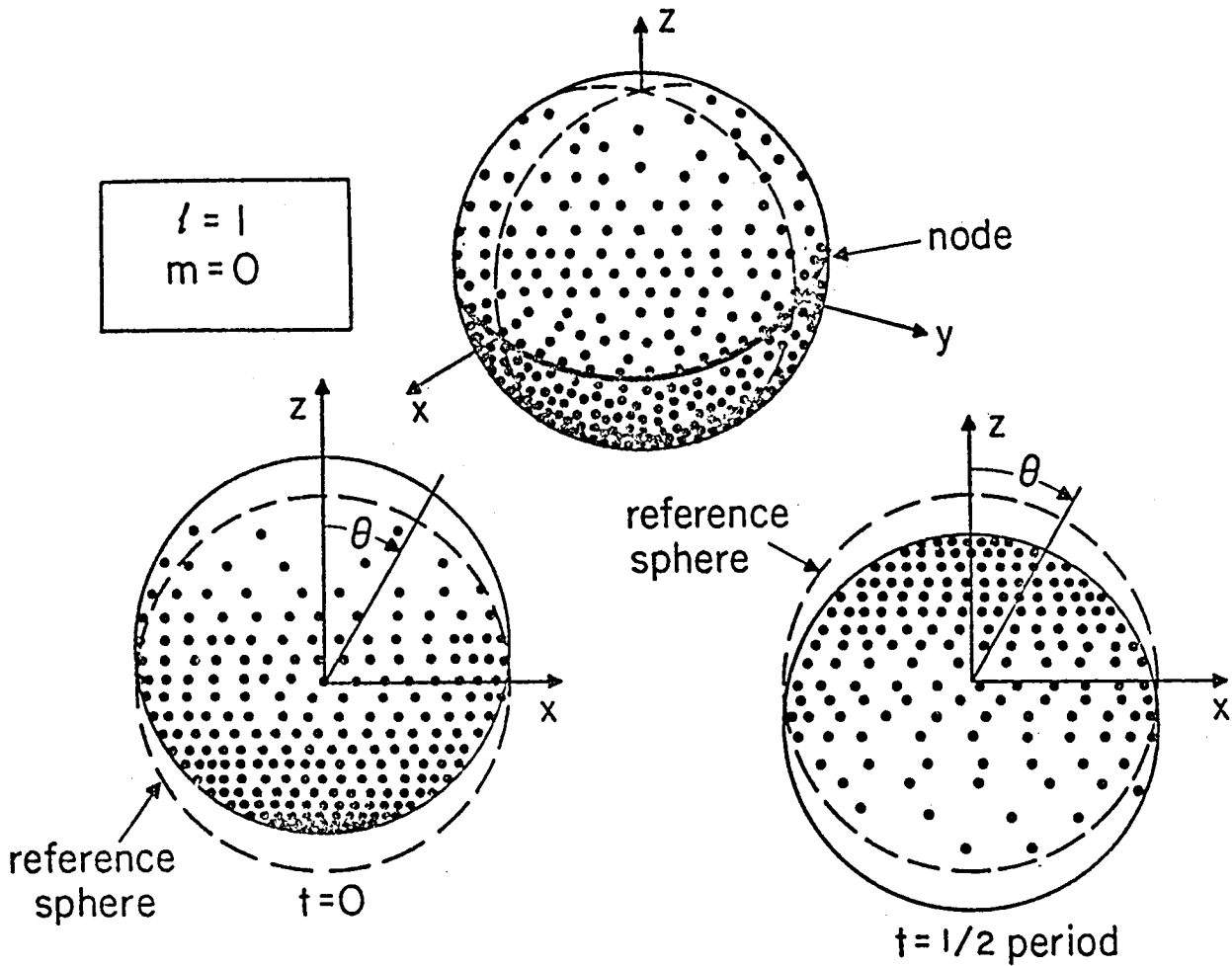
Fig. 1a — A pictorial representation of the luminosity variation corresponding to the spherical harmonic $Y_{1,0}(\theta,\phi)$. The upper figure gives a perspective view of the luminosity distribution projected onto a spherical surface. The figure at the lower left is a polar projection of the luminosity amplitude in the (x,z)-plane ($\phi=0$) at the time of peak luminosity at $\theta=0$. The distribution of luminosity amplitude in θ is given by the difference between the solid outline and the dashed curve representing the reference sphere. The figure at the lower left represents the luminosity distribution half a period later, when peak luminosity has shifted to $\theta=180°$. This mode of oscillation corresponds to the alternate brightening and darkening of the upper ($\theta<90°$) and lower ($\theta>90°$) hemispheres of the star.
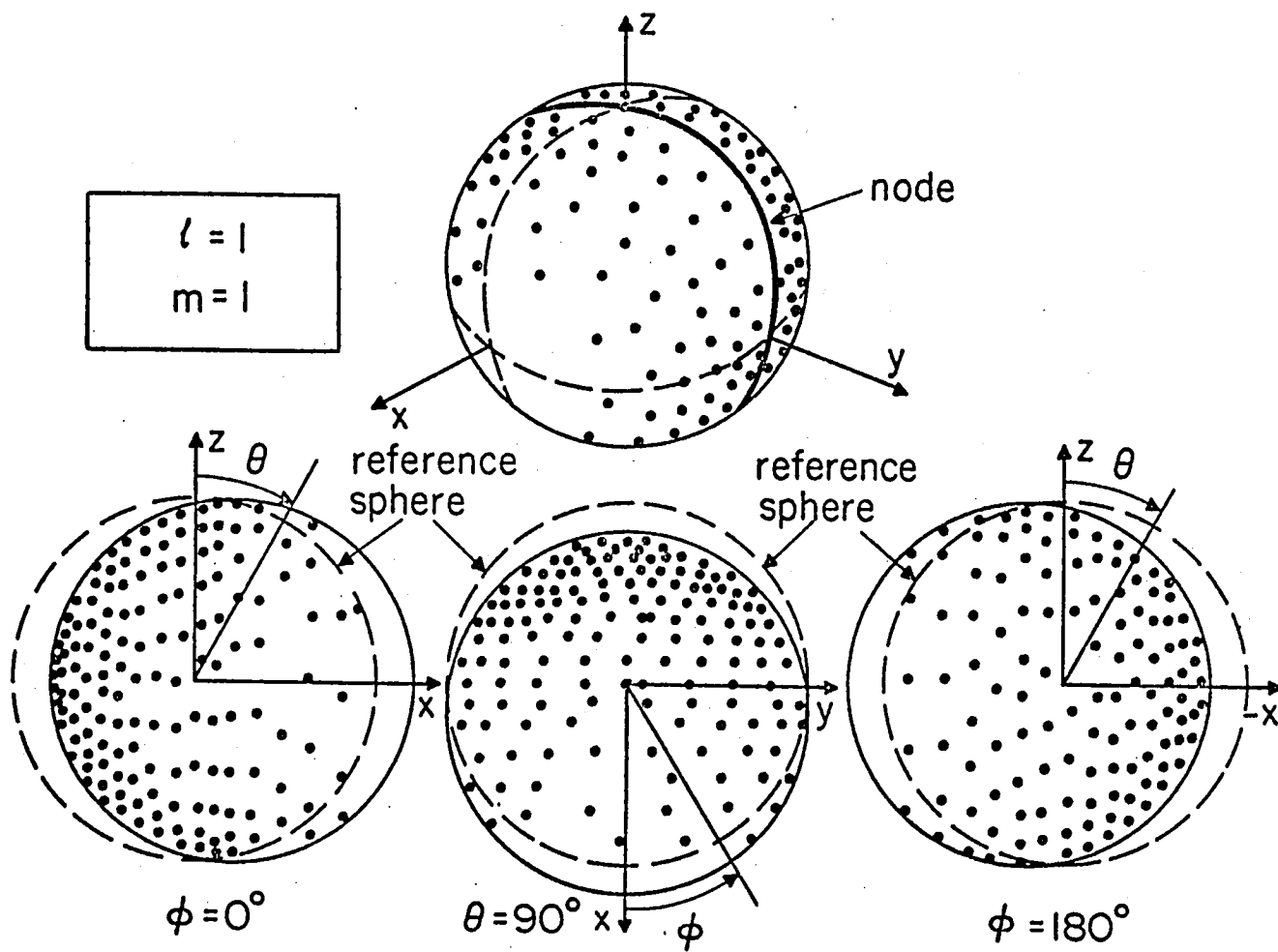
Fig. 1b – Similar to Fig. 1a, except for a different ℓ=1 mode.
The actual mode shown is the standing wave corresponding to the
combination $Y_{1,1}(\theta,\phi) + Y_{1,-1}(\theta,\phi)$. This mode represents the
alternate brightening and darkening of the forward ($|\phi|<90°$)
and backward ($|\phi|>90°$) hemispheres. The individual $m=\pm1$ modes
correspond to travelling waves in which the bright spot rotates
in the direction of $\mp\phi$, making one revolution in
the period of the oscillation. The figures at the lower left
and right are polar projections of the luminosity variation at
$\phi=0°$ and $\phi=180°$ (corresponding to the positions of maximum and
minimum luminosity at $\theta=90°$). The central figure gives the
luminosity distribution as a function of ϕ in the equatorial
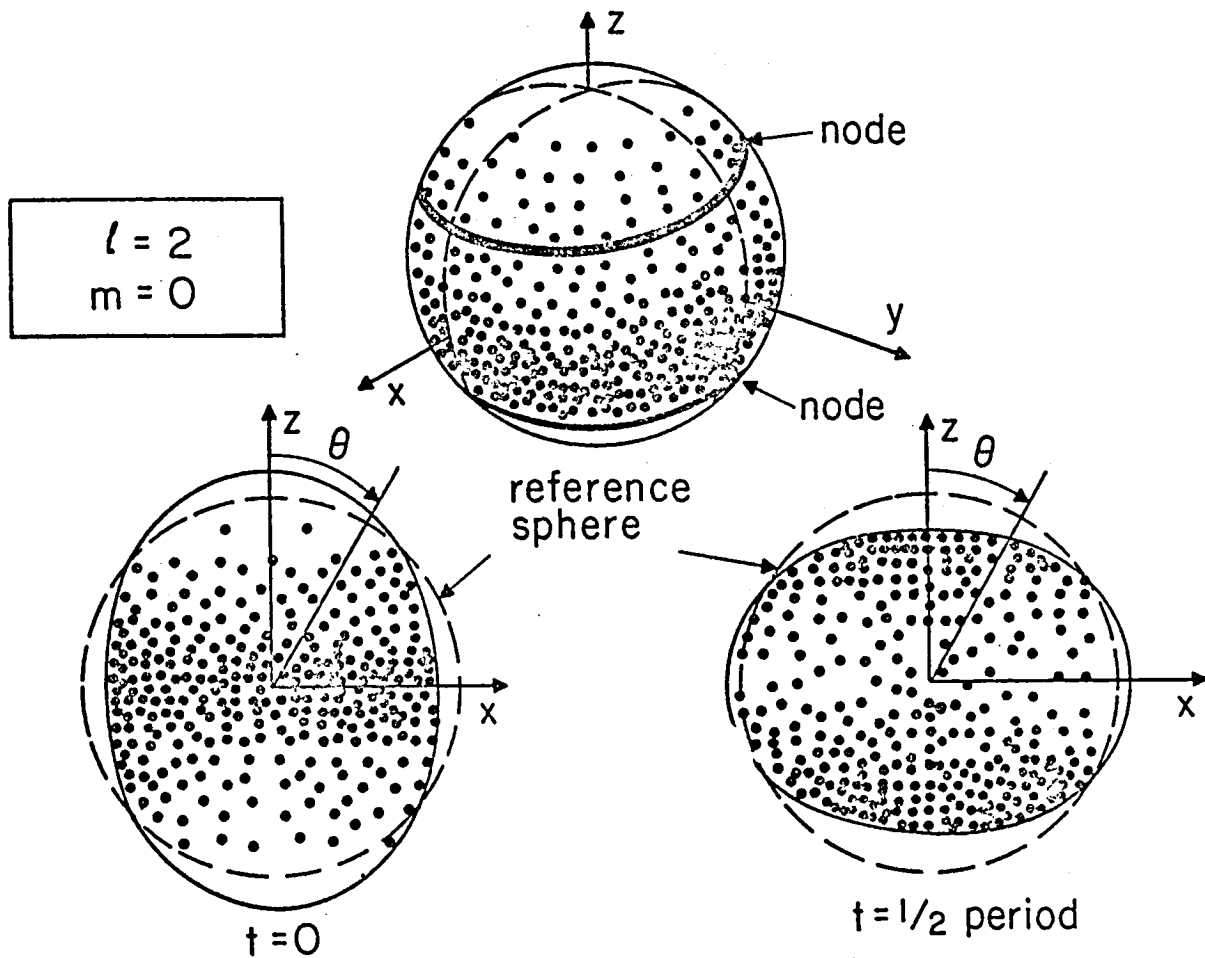plane ($\theta=90°$).

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Fig. 1c – Similar to Fig. 1a, except for the mode $Y_{2,0}(\theta,\phi)$. In this mode both caps brighten and darken together, in anti-phase with the equatorial belt ($|\cos\theta|<1/\sqrt{3}$). The maximum luminosity amplitude at the equator is half that at the poles, as shown in the lower figure.

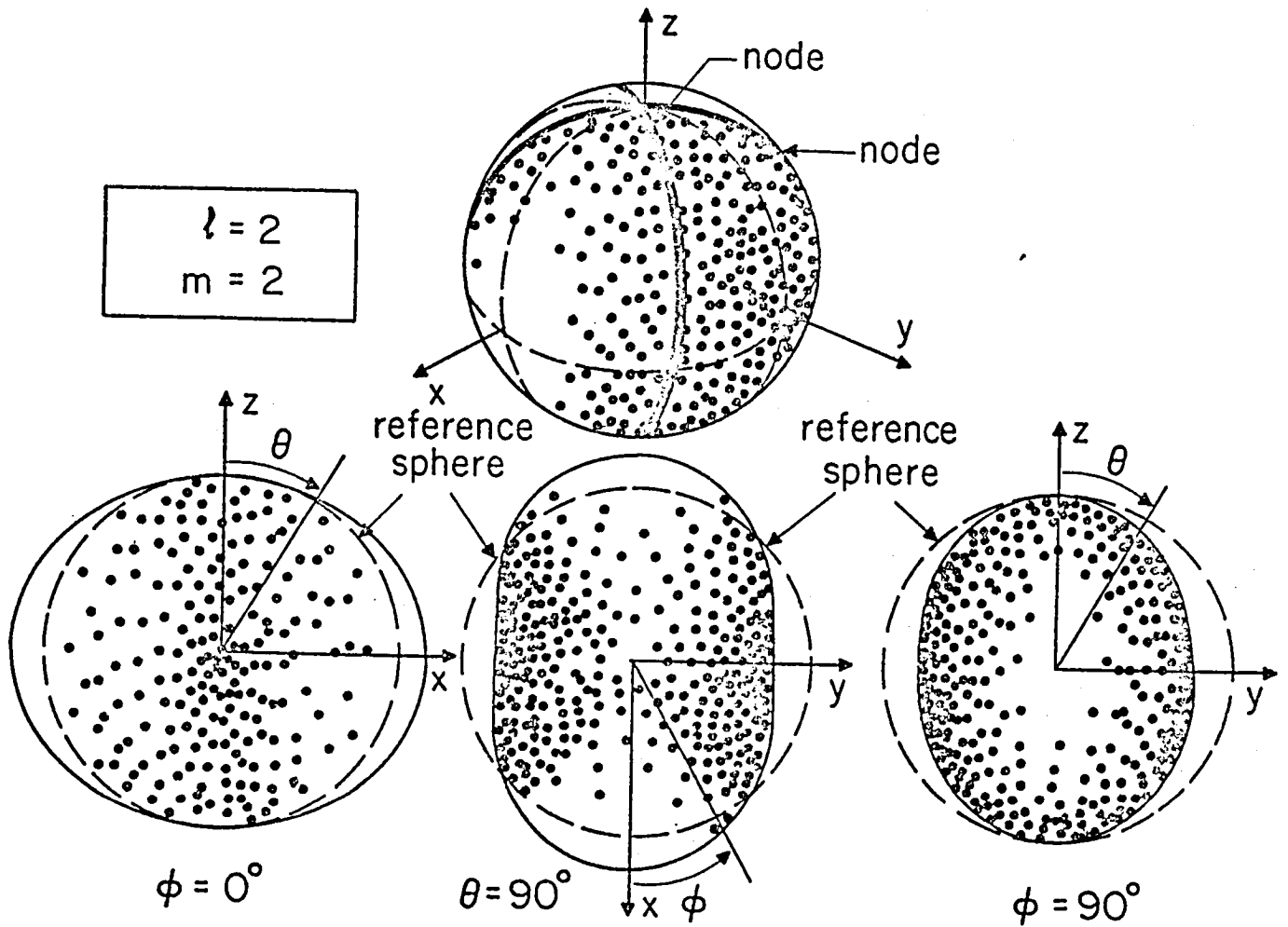Fig. 1d – Similar to Fig. 1b, except for the standing wave $Y_{2,2}(\theta,\phi) + Y_{2,-2}(\theta,\phi)$. The individual $m = \pm 2$ modes correspond to traveling waves rotating in the $\mp \phi$ directions with the period of the oscillation.

corresponding to a variety of g-mode oscillations of $0.6M_\odot$ white dwarf

models with $T_{eff} \approx 11,000°K$ and $12,000°K$. As expected, the observed

luminosity amplitude decreases rapidly with ℓ; for the fundamental g-modes,

he finds $\Delta M_{bol}(\ell=2)/\Delta M_{bol}(\ell=1) = 0.186$ at $11,000°K$ and 0.205 at $12,000°K$.

Somewhat surprisingly, however, Dziembowski also finds that the observed

luminosity amplitude increases with the radial wavenumber k (k is the

number of radial nodes in the eigenfunction); for modes with $\ell=1$ or 2,

$\Delta M_{bol}(k=10)/\Delta M_{bol}(k=1) \sim 20$-$30$. Thus if high radial overtones can be

excited with amplitude $\delta r/r$ at the surface comparable to that of the

lower k-modes, the high overtones will dominate the observed light variation.

For the same reason, the observed oscillations must correspond to modes of

moderately low values of ℓ and m.

c.   Thermal imbalance

The problem of determining the pulsational stability of stars in

thermal imbalance (i.e., evolving stars, in which $T\partial s/\partial t \neq 0$) has by now

been studied quite extensively. It has been a subject of some controversy,

and although some of the questions raised have now been resolved, it is

not yet entirely clear (at least to this writer) that a complete under-

standing has been achieved. For the white dwarf stars this problem is of

some importance, because, except perhaps in the cataclysmic variables, they

have no thermonuclear energy sources; instead the luminosities of these

stars are supplied by the cooling of the hot interiors, thus necessitating

$T\partial s/\partial t \neq 0$.

A brief summary of recent work on this problem has been given by Cox

(1976). In particular, Aizenman and Cox (1975), Demaret (1975), Simon (1977),

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and Buchler (1978) have shown that the imaginary part of the complex pulsation eigenfrequency σ which gives exponential growth or decay of small-amplitude oscillations is necessarily different for the oscillation energy and the oscillation amplitude. In the linear, quasi-adiabatic case, they find

$$(\text{Im}\sigma)_E = (\text{Im}\sigma)_a - \frac{\dot{\omega}}{\omega} \, , \tag{1}$$

where $\dot{\omega}$ is the time rate of change of the oscillation frequency $\omega = \text{Re}\sigma$. This raises the interesting possibility that amplitude growth [$(\text{Im}\sigma)_a > 0$] may in some cases be associated with energy decay [$(\text{Im}\sigma)_E < 0$] and vice versa, and leads directly to the question of which complex eigenfrequency (if either!) is the more fundamental. This has been further discussed by Simon (1977), Buchler (1978), and Demaret and Predang (1978). An interesting example of amplitude growth coupled with energy decay has recently been found in a hydrogen shell-burning pre-nova model by Vemury (1978).

For our present purposes, it is sufficient to note that the thermal imbalance effects are appreciable only when the pulsation damping time $(\text{Im}\sigma)^{-1}$ is comparable to or greater than the timescale of evolution of the unperturbed star, $\omega/\dot{\omega}$. For a cooling white dwarf, the latter is in the range 10^7 to 10^{10} years (cf. Lamb and Van Horn 1975; Sweeney 1976; Shaviv and Kovetz 1976). In the absence of gravitational radiation damping (Osaki and Hansen 1973), the damping times for the f- and p-modes are generally of this same order, indicating that thermal imbalance effects may be appreciable. However, for all of the g-modes, and —with gravitational radiation damping included— for the f- and p-modes as well, the damping times are sufficiently short that thermal imbalance is not significant.

III. Non-radial Oscillations of Single White Dwarfs

a. What are the modes of oscillation?

It has been pointed out repeatedly that the observed oscillation periods of the single white dwarfs (ZZ Ceti stars) are much too long to correspond to p- or f-modes, and must perforce belong to the class of g-mode oscillations. However, there still remains a multiple infinity of g-modes. The system of differential equations governing the non-radial oscillations of a spherical star contain the spherical harmonic order ℓ explicitly (cf. Ledoux and Walraven 1958, Cox 1974, Ledoux 1974, Van Horn 1976), so that the oscillation periods must depend on ℓ. For each ℓ-value, there are $2\ell+1$ independent angular modes of oscillation, corresponding to the different spherical harmonics $Y_{\ell m}(\theta,\phi)$, with $m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell$. In the absence of rotation or magnetic fields, the oscillation periods corresponding to different m-values are all degenerate. (We shall defer discussion of the consequences of rotation to the following section). In addition, for given ℓ, it is possible to find eigensolutions of the set of differential equations plus boundary conditions which have $k \geq 1$ nodes in the radial eigenfunction. It is well-known that the periods of the high overtones increase without limit as $k \to \infty$ (Ledoux and Walraven 1958, Cox 1974, Ledoux 1974). These oscillatory modes are denoted as g_k^+ modes. If the star contains a convection zone (as is the case for all white dwarfs except DA stars with $T_{eff} \gtrsim 14,000°K$) an additional set of modes, which have a real exponential time dependence, becomes possible. These are termed g_k^- modes. We shall be mainly concerned with the oscillatory modes, however, and we shall therefore take "g-modes" to mean "g^+-modes" unless otherwise specified.

To investigate the nature of the oscillation modes observed in the ZZ Ceti stars, one may compare the observed periods with those computed for various theoretical models. This is shown in Fig. 2. The effective temperatures and the periods of the principal modes of the ZZ Ceti stars are taken from McGraw (1977), and theoretical values are taken from computations by Brickhill (1975) and Dziembowski (1977b), with the mode of oscillation identified. Several conclusions can be drawn from this immediately; most of them have been pointed out previously, especially by McGraw (1977) and references therein.

1. The observed region of instability for the white dwarfs lies in the range of effective temperatures $10,000°K \lesssim T_{eff} \lesssim 14,000°K$. Interestingly, this lies in precisely the region where the extrapolation of the Cepheid instability strip meets the white dwarfs (Fig. 3). This has prompted the suggestion by McGraw and Robinson (1976) that the same κ- and γ-mechanisms that operate in the hydrogen and helium ionization zones of the Cepheids are also responsible for exciting the oscillations in the ZZ Ceti stars. We shall return to this point again below.

2. The oscillations of the shortest period ZZ Ceti stars can be understood in terms of low overtones of $\ell=1$ or 2 g-modes. This point has been made previously by Brickhill (1975), by Robinson and McGraw (1976), and by Robinson, Nather and McGraw (1976), for the special case of R548.

3. The shortest period of oscillation changes abruptly from $\Pi \sim 200$ sec for stars with $\log T_{eff} \gtrsim 4.1$ to $\Pi \sim 800-900$ sec for those with $\log T_{eff} \lesssim 4.1$. The reason for this (if it is a real effect) is at present unknown.
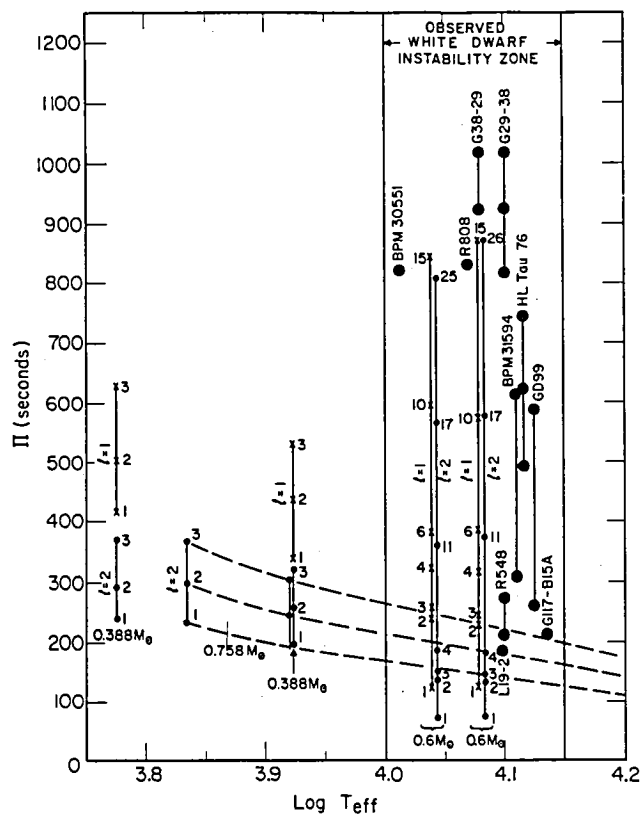
Figure 2. Observed periods of the ZZ Ceti stars from McGraw (1977) compared with theoretical g-mode periods calculated for different white dwarf models. For each star the various principal periods detected are plotted as large dots connected together and labelled with the star name. The effective temperatures are those tabulated by McGraw. The calculated periods for the 0.388M$_\odot$ and 0.758M$_\odot$ models are by Brickhill (1975); those for the 0.6M$_\odot$ models are by Dziembowski (1977). The lowest few obertones of the ℓ=1 and ℓ=2 modes are shown (labelled by the values of k and ℓ) for the theoretical calculations. The shortest observed periods of the hotter ZZ Ceti stars (log T$_{eff}$ $\gtrsim$ 4.1) are in the range of the low-order g-mode periods. The periods of the cooler ZZ Ceti stars (log T$_{eff}$ $\lesssim$ 4.1) require rather high radial overtones (k $\gtrsim$ 15-25) if these oscillations correspond to conventional g-modes.

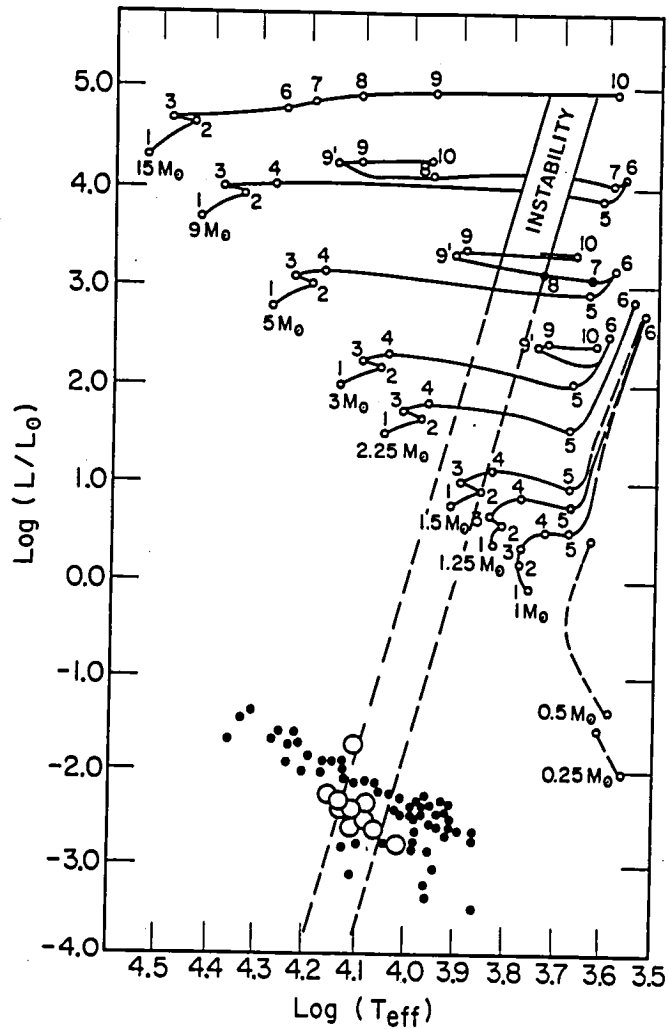Figure 3. The instability strip in the H-R diagram. The upper portion of the figure showing the theoretical evolutionary tracks and the location of the Cepeheids is adapted from Figure 1 of Henden and Cox (1976). The instability strip has been extrapolated linearly along the dashed lines into the region occupied by the white dwarfs. The locations of the variable white dwarfs are shown by the large open circles.

4.   In order to interpret the oscillations of the long-period ZZ Ceti stars in terms of modes which are presently understood, it is necessary to identify the observed oscillations as rather high overtones ($k \gtrsim 15$ for $\ell = 1$; $k \gtrsim 25$ for $\ell = 2$). Note that the high-order spectra belonging to various ℓ-values overlap, so that periods alone are insufficient to provide unique mode identifications in this range.

There are a few additional indications from the data which reinforce the interpretation of the long-period modes as high radial overtones. As pointed out by Robinson and McGraw (1976), there is a rough correlation of oscillation period with observed oscillation amplitude. This is shown in Fig. 4, where McGraw's (1977) periods and amplitudes are plotted. Note that the practice of assigning a single amplitude to the star is not really adequate for this purpose, as McGraw points out; a more detailed comparison with the theory can be made by using the observed amplitudes of the individual oscillation modes. Also shown in this figure are McGraw's estimates of the stability of the oscillations.

For a comparison we have also plotted in Fig. 4 Dziembowski's (1977b) bolometric magnitude variations, reduced by a factor of 10^{-3}, for the various radial overtones of the $\ell=1$ and 2 g-modes computed for his $0.6M_{o}$, $T_{eff} \approx$ 11,000°K model; almost identical results are obtained for the $T_{eff} \approx$ 12,000°K model. This suggests the following conclusions:

1.   The general trends of the observational data are broadly consistent with the theoretical curves. This is at least not in disagreement with the hypothesis that the long-period modes are high radial overtones.
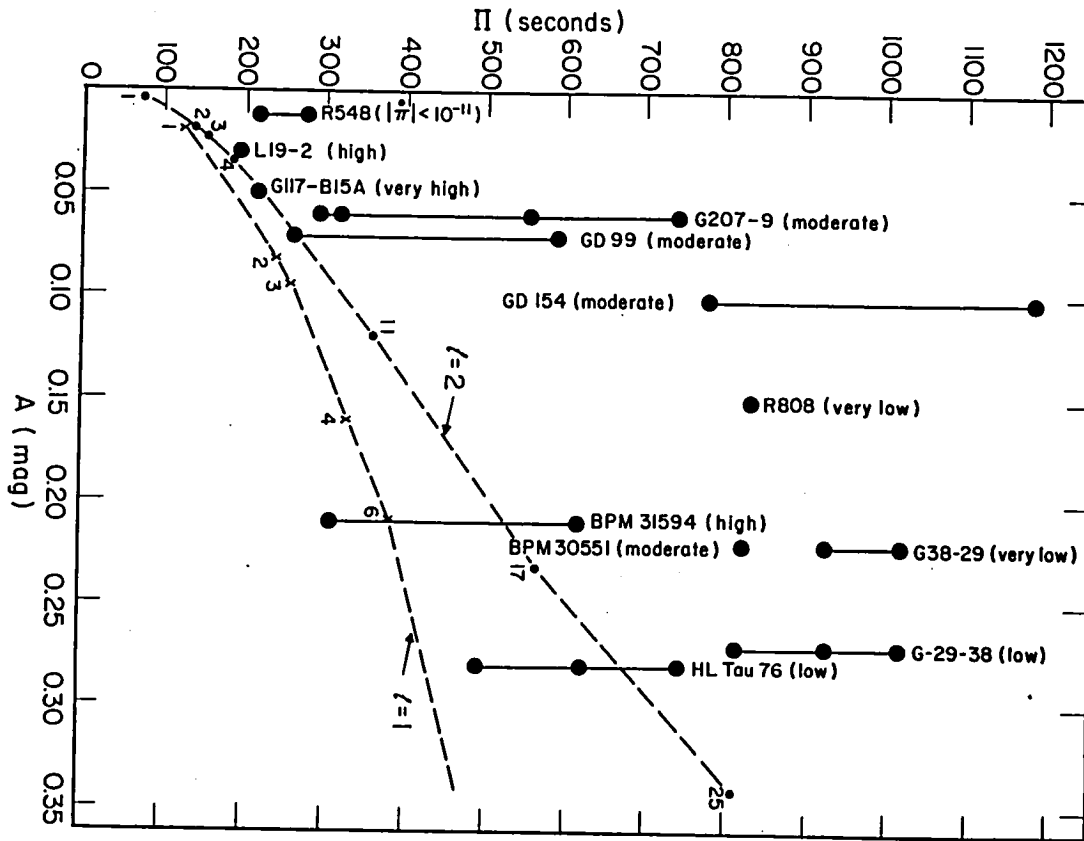
Figure 4. The correlation between period and amplitude of the oscillation for the ZZ Ceti stars. The observational data are taken from McGraw (1977), and his estimates of the stability of the oscillation modes are indicated in parentheses after the star name. Note that the "amplitude" plotted here is a "typical value" for each star; a more informative comparison is possible by making use of the amplitudes of the individual oscillation modes as determined from the power spectra. Also shown for comparison are the predicted bolometric magnitude variations for the various radial overtones for the $\ell=1$ and 2 g-modes as calculated by Dziembowski (1977). The plotted points are labelled by the k and ℓ values of the modes. Dziembowski's tabulated values of ΔM_{bol} have been reduced by a factor of 10 for this graph, suggesting that the amplitudes of the radial oscillations are $\delta r/r \lesssim 10^{-3}$ for these stars.

2. The stability of the oscillations displays an interesting correlation with the periods. The shortest period oscillators display a very stable mode structure, while the longest period oscillators tend to be the least stable. This is also consistent with the suggested identification of the short periods with low-k modes and the long periods with high-k modes. Because the density of modes increases very rapidly with k in frequency space, the possibility of mode-coupling is enormously greater for the high-k modes than at low k. This may lead to beating or mode-switching that may be observed as apparent lack of stability of the oscillation. Precisely this type of interaction has been shown by Robinson, Nather, and McGraw (1976) to occur even in the most stable ZZ Ceti star, R548; when this is taken into account, the underlying mode structure of the star is found to have $|\dot{\Pi}| < 10^{-11}$.

3. The oscillation amplitudes as determined from the ratio of Dziembowski's calculations of ΔM_{bol} with the observed amplitude of variation A are quite small; $\delta r/r \lesssim 10^{-3}$ at the stellar surface.

As the previous discussion has been intended to show, great progress has been made in the past several years in understanding the nature of the oscillations in the ZZ Ceti stars. Despite this, the mode identifications, except perhaps in R548, are still·unsettled. One reason for this is that theoretical models have yet to demonstrate pulsational instability in the modes suggested by the observations. A second reason is that additional oscillation modes may exist in white dwarfs which have not yet been studied adequately or that rotational modifications of existing g-modes may be important. We therefore turn next to considerations of these questions.

b. Excitation and non-linear mode-coupling

Only a few of the recent non-radial oscillation calculations have investigated the problem of excitation of the modes, and this has usually been restricted to the linear, quasi-adiabatic approximation. In their classic paper, Osaki and Hansen (1973) studied radiative, neutrino, and gravitational-radiation damping of non-radial oscillations. However, the pure ^{56}Fe white dwarf models they employed did not include a treatment of the ionization zones, and thus no pulsational instability was found. More recently, Hansen, Cox, and Van Horn (1977; see also Van Horn 1976) have studied the low-k, ℓ=1 and 2 non-radial oscillation modes of detailed models of pure ^{12}C white dwarfs. These models did include a careful treatment of the ionization/convection zone, and a hint of instability in the ℓ=2 g_1^+ and g_2^+ modes was found for a model with $T_{eff} \approx$ 58,000°K. It is not clear that this is a real instability, however, because the result depends upon the assumed location of the "transition zone" (cf. Cox and Giuli 1967, ch. 27) where the quasi-adiabatic approximation breaks down. In any case, this "instability" is clearly irrelevant to the ZZ Ceti stars.

The only other theoretical study of the excitation mechanism in models for the ZZ Ceti stars of which this writer is aware is that by Dziembowski (1977b). In this important work, Dziembowski investigated the oscillations and stability of two 0.6M$_\odot$ white dwarf models with $T_{eff} \approx$ 11,000°K and 12,000°K. These models, with element abundance distributions taken from the last model of Paczynski's (1971) planetary nebula sequence, did include hydrogen/helium envelopes with ionization zones. The results of Dziembowski's fully non-adiabatic calculations were: i) g-modes with k $\leq$ 15-25 were found not to be self-excited, and ii) modes corresponding to very high orders ($\ell \sim$ 100-400, k $\sim$ 15-20) were driven violently unstable,

primarily by the HeII ionization zone, with growth timescales of days.

However, the periods of these modes are much too short ($\Pi \sim$ 5-20 sec),

and the ℓ-value much too large for these modes to correspond to the

observations. (For example, Dziembowski finds $\Delta M_{bol} \sim 10^{-2} - 10^{-5}$ for

these high-ℓ modes as opposed to $\Delta M_{bol} \sim 200$ for the $\ell = 2$ modes of

comparable k-order).

Thus none of the theoretical calculations has yet succeeded in

discovering pulsational instabilities in those modes which the observations

indicate to be excited in the ZZ Ceti stars.

One possibility for resolving this problem was suggested by

Dziembowski (1977b). He pointed out that non-linear interactions among

the very high-ℓ modes which he finds to be excited may provide resonant

excitation of the lower-order modes in the observed range of frequencies.

(Vandakurov (1977) has subsequently considered non-linear driving of

radial pulsations by these same unstable non-radial modes).

This would be expected to lead to variability of the observed mode ampli-

tudes, as is found in the long-period (800-1000 second) oscillators. It

is difficult to accept this explanation for the short-period ($\sim$200-second)

oscillators, however. In particular, the great stability of the oscillations

in R458 strongly suggests that these modes are indeed correctly identified

as low-ℓ, low-k oscillations that are self-excited. If this is correct,

then there is an essential aspect of the non-radial excitation mechanism

that we have yet understood.

Another, related, problem that may be important in the white dwarfs

is the interaction between oscillations and convection. There are two

aspects of this: the effect of convective flux variations upon the

pulsational stability and the direct non-linear coupling between pulsational

and convective motions. Work on the general problem of convection in

pulsation theory has been briefly reviewed by Cox (1976). In the context

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of white dwarfs, there has been no work on this problem, and time-variations in the convective flux have been entirely ignored in the pulsational stability analysis. From research on the effects of convection on other types of stars, however, it appears likely that this is inadequate. For example, in a series of papers concerned with the full hydrodynamic treatment of convection in Cepheids and RR Lyrae stars, Deupree (1976, pp. 222, 229, and references therein) has shown that convection becomes important at the red edge of the instability strip, where it dominates the damping of the oscillations. Is it too great an extrapolation to suppose that convection may play a similar role in the ZZ Ceti stars, perhaps even in determining the transition from stable 200-second oscillations to fluctuating 800-1000 second oscillations?

Unfortunately, Deupree's detailed numerical approach is ill-adapted for use in a survey of stability among non-radial modes, especially those of moderately high order. For this purpose one would prefer a simpler approximation that could be employed with linear theory. Two recent groups of papers are of interest in this regard. First, Gough (1977) has recently reviewed the time-dependent generalizations of mixing-length theory and has presented them in a form suitable for use in studies of radial pulsation. Second, Goldreich and Keeley (1977a,b) have presented a careful analysis of the interaction between convection and pulsation in connection with low-amplitude oscillations of the sun. They find (1977a) that turbulent dissipation renders unstable radial modes marginally stable. In their second paper (1977b) the treatment of convection/pulsation interactions is generalized to the case of non-radial oscillations, and they find that non-linear interactions lead to a tendency for equipartition of

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pulsation modes which are close to resonance with convective eddies.
In the white dwarfs, it has been pointed out before (Van Horn 1976)
that convective timescales become comparable with g-mode periods for
the cooler He-envelope white dwarfs with M $\sim$ 0.4-0.6M$_\odot$. Clearly this
problem needs further attention from the theorists.

c. New modes?

The difficulty of identifying the long-period oscillations of the
ZZ Ceti stars with modes of low order has suggested to some the possibility
of finding additional oscillation modes of the white dwarfs beyond the
conventional p-,f-, and g-modes. Motivated in part by this, Van Horn and
Savedoff (1976; see also Denis 1975) undertook a preliminary investigation
of the effects of a solid core upon the oscillation spectrum of a white
dwarf. Although their analysis has not yet been carried through for a
complete stellar model, they were able to show that the ability of the
solid core to sustain shearing motions permitted torsional oscillations of
the core, just as in the case of the solid Earth. In addition, they found
that the non-vanishing shear modulus produced modifications in the p- and
g-mode oscillation periods; for the p-modes the change was only a few
percent, while for the g-modes, the period ranged from the normal value
in the case of small shear moduli, to that appropriate to the torsional
oscillations. Since the torsional oscillation periods Π_t were estimated
to be no more than 3 to 10 times longer than the p-mode periods, or $\Pi_t \sim$
30-100 seconds, the effect of the solid core does not appear to be relevant
to the problem of the long-period oscillations. In addition, while core
crystallization begins near $T_{eff} \approx$ 13,000°K in a 1M$_\odot$, ^{12}C white, it will
not occur until considerably lower temperatures in stars of lower masses.

From McGraw's (1977) values of log g for the ZZ Ceti stars, the masses are expected to lie in the range of 0.5-$0.7M_\odot$; the values of T_{eff} at crystallization for these masses are so low that core crystallization will not begin until these stars have cooled well below the observed instability strip.

Another, as yet unexplored, possibility exists for introducing new modes into the white dwarfs, however. This has to do with the effect of composition discontinuities upon the mode structure of the star[1]. It is well-known that white dwarfs have undergone thermonuclear processing. They must initially have been more massive in order to have evolved off the main sequence and to have become white dwarfs; thus some of the hydrogen has been burnt into helium. In addition, the masses are larger than the minimum for helium burning; thus some of the helium has also been processed into carbon and oxygen. It is unlikely that the white dwarfs have undergone further nuclear processing, although this cannot yet be rigorously established. It is also well-known that the gravitational fields in white dwarfs are sufficiently high so that gravitational settling of the elements will have proceeded to its limit (cf. Schatzman 1958). Thus the compositional structure of a white dwarf is expected to consist of layers of virtually pure elements; hydrogen overlying helium overlying carbon and oxygen. Vauclair and Reisse (1977) and Koester (1976) describe the structure of the outer layers of such a star.

In cases with such layered structures, it is anticipated that additional modes of oscillation associated with the density discontinuities will appear. For the case of the heterogeneous incompressible sphere this has been

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[1] I am indebted to M.P. Savedoff for drawing my attention to this point.

confirmed within the past five years (cf. Ledoux 1974). The eigenfunctions associated with the additional modes peak near the density discontinuities; thus these modes can be regarded as "surface waves" associated with the discontinuities. The effect of the thickness of the various composition layers upon the oscillation spectrum of a white dwarf, however, has not yet been investigated. It is tempting to speculate that the location of the H/He composition boundary relative to the location of the HeII ionization zone may even be responsible for the existence of non-variables within the instability strip, just as Baglin (1976) has suggested in the region of the main sequence A stars.

IV. Effects of Rotation on Non-Radial Oscillations

a. Slow rotation

Up to this point we have ignored the effects of rotation upon the mode structure of a non-radially oscillating star. However, as we shall discuss in this section, rotation exerts a profound influence upon the periods of such oscillations. It is convenient to begin this discussion with the case of very slow rotation (rotation frequency $\Omega \ll \omega = 2\pi/\Pi$), both because it can be treated as a perturbation on the non-rotation case, and because it may be especially relevant to the ZZ Ceti stars we have been considering.

The theory of the leading (i.e., linear) corrections to the frequencies of non-radial oscillations due to uniform rotation has been well-established for some time now (cf. Ledoux 1951, Ledoux and Walraven 1958) and yields, in the inertial observer's frame,

$$\sigma_{k\ell m} = \sigma_{k\ell} - m\Omega(1 - C_{k\ell}). \tag{2}$$

Here $\sigma_{k\ell m}$ is the (complex) eigenfrequency, $\sigma_{k\ell}$ is the value of $\sigma_{k\ell m}$ in the absence of rotation, m is the azimuthal spherical harmonic index ($m = -\ell$, $-\ell+1$, ...,ℓ), and $C_{k\ell}$ is defined by

$$C_{k\ell} = \frac{\int \rho r^2 dr(2ab+b^2)}{\int \rho r^2 dr[a^2+\ell(\ell+1)b^2]}. \tag{3}$$

In (3), the quantities a and b are, respectively, the amplitudes of the radial and tangential displacement eigenfunction, defined by

$$\underset{\sim}{\xi}_{k\ell m} = (\xi_r,\xi_\theta,\xi_\phi)=(\delta r,r\delta\theta,r\sin\theta\delta\phi)=[a(r)Y_{\ell m},b(r)\frac{\partial Y_{\ell m}}{\partial\theta}, \frac{b(r)}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial\phi}]. \tag{4}$$

For moderately large k-values, $|a|$ is generally much smaller than $|b|$, and (4) reduces to

$$C_{k\ell} \approx \frac{1}{\ell(\ell+1)}. \tag{4a}$$

Brickhill (1975) comments that (4a) is a good approximation for $k \geq 4$, but he gives no quantitiative details beyond this.

Recently, Wolff (1977) has applied this formalism to interpret the multiperiodicities observed in the ZZ Ceti stars as beats produced by non-linear mode-coupling of rotationally-split g-mode oscillations. He argues (see Wolff 1974 for details) that g-modes with spherical harmonic index m and -m should be excited to comparable levels, and hence that the corresponding retrograde (m > 0) and prograde (m < 0) modes will combine to yield a not-quite-standing mode with azimuth and time dependence of the form (in the non-rotating inertial frame):

$$\cos\{|m|[\phi + \Omega(1-C_{k\ell})t]\}e^{i\sigma_{k\ell}t}. \tag{5}$$

The rate of azimuthal drift of this wave pattern is thus given by

$$-\dot{\phi} = \Omega(1-C_{k\ell}) \equiv \Omega_\ell . \tag{6}$$

In the case of uniform rotation and sufficiently large k, $C_{k\ell}$ assumes the simple form given by (4a), and the drift rate of the wave pattern then depends only upon ℓ and Ω, but not upon k or m.

Wolff then argues that it is the "slow" relative drift of groups of modes with comparable ℓ-values that dominates the observed variations. This leads him to consider the pattern frequencies given by (6) and (4a) together with simple differences of these frequencies as defining an oscillation spectrum depending only upon the stellar rotation rate Ω. A schematic illustration of this concept and its application to four of the ZZ Ceti variables are shown in Fig. 5, adapted from Wolff's paper. The coincidence of the theoretical spectrum with the observed power spectra is rather striking, despite some obvious differences. This, together with the fact that it yields potentially testable predictions of the rotation rates of the oscillating white dwarfs, makes Wolff's model of interest for further study.

In particular, the rotation rates Wolff predicts are in the range $2\pi/\Omega = 200$ to 500 seconds. This would be consistent with the contraction of a star having approximately solar angular momentum from the main sequence
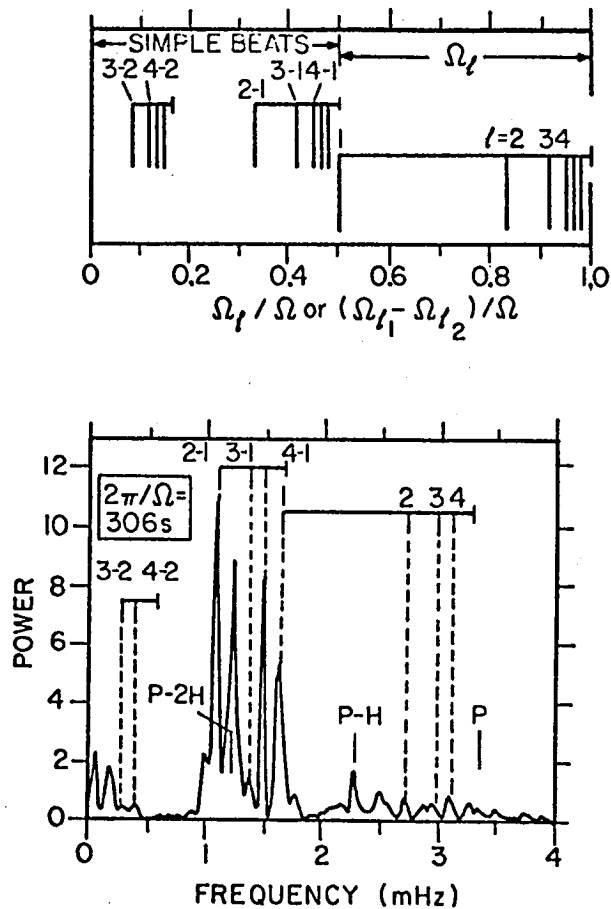
Fig. 5 - Wolff's theory for the oscillations of the ZZ Ceti stars. This figure is adapted from Wolff (1977). The right half of the upper panel shows schematically the azimuthal drift rates Ω_ℓ given by (6) for modes with $\ell = 1,2,3,4...$ in units of the stellar rotation frequency Ω. The left half of the same panel shows the beat frequencies $\Omega_{\ell=2} - \Omega_{\ell=1}$, ..., $\Omega_{\ell=4} - \Omega_{\ell=2}$, etc., also in units of Ω. The lower panel shows one example of a comparison of this theoretical spectrum with the observed oscillation spectrum of the ZZ Ceti star G29-38. The assumed stellar rotation period $2\pi/\Omega = 306$ seconds has been chosen to provide the best match with this data. The frequencies labelled P, P-H, and P-2H have no theoretical foundation, but are introduced to correspond to other large peaks in the power spectrum. The frequency P is supposedly analogous to the so-called "prograde mode" identified in the sun while H is "the frequency of the highest maximum in the observed spectrum" (Wolff 1977).

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to white dwarf dimensions without loss of angular momentum, but it is _much_ faster than the rotation rates found so far in any other single white dwarf ($>10^3$-10^4 seconds in 14 DA white dwarfs: Greenstein and Peterson 1973, Greenstein et al. 1977; ~ 2.2 hours in Feige 7: Leibert et al. 1977; and 1.3 days in G195-19: Angel et al. 1972). A rotation period of ~ 200 seconds corresponds to an equatorial velocity of ~ 300 km s^{-1} and to a rotational Doppler broadening of ~ 4.5Å at H_γ. This is amply large enough to be measured, even in the presence of the very large pressure broadening in white dwarfs, and the results of such measurements are of very considerable interest. If such large rotation broadening is found, it will favor Wolff's theory and provide a new puzzle: why do the variable white dwarfs rotate so much faster than non-variables? If rapid rotation is not observed, the measurement will at least place useful limits on the rotation periods of the ZZ Ceti stars and reaffirm an existing puzzle: why and how are high-order g-modes excited?

Apart from its virtue of potential for observational test, there are a number of shortcomings of Wolff's model from the point of view of theory. Instead of developing his model from first principles, Wolff has introduced a number of ad hoc assumptions that should be checked. For example, he adopts the large-k limit for $C_{k\ell}$ for all of his modes; is this adequate? In part to address this question Hansen, Cox, and Van Horn (1977) have computed the rotational splitting of $\ell=2$ g-modes in ^{56}Fe and ^{12}C white dwarf models in or near the observed white dwarf instability strip. For uniform rotation they find values of $C_{k\ell}$ ranging from -0.020 to 0.165, depending upon the stellar mass, T_{eff}, and the k-value of the mode; the result given by (4a) is $1/\ell(\ell+1) = 1/6 = 0.166$.

Hansen, Cox, and Van Horn have also carried out preliminary calculations of the effects of _differential_ rotation in white dwarfs, assuming a "rotation law" of the form adopted by Ostriker and Bodenheimer (1968) in their study of massive differentially-rotating white dwarfs. For $1M_\odot$, ^{12}C white dwarf models, the calculations indicate a considerably different (and T_{eff}-sensitive) splitting of the $\ell=2$ g_1- and g_2-modes, although this result is rather model-dependent. (There is almost _no_ difference in the g_1- and g_2-mode splittings for the much cruder ^{56}Fe white dwarf models in the T_{eff}-range of interest). This result is of interest in connection with the extremely careful and detailed analysis of the oscillations of R548 by Robinson, Nather, and McGraw (1976). They found that the power spectrum of this star consisted of two main peaks at periods of about 213s and 274s, and that these are each split into close pairs with _very_ stable periods ($|\dot{\Pi}| < 10^{-11}$). The two close pairs, presumably split by rotation, each beat together to yield difference frequencies corresponding to periods of $\sim 1\overset{d}{.}44$ (for the 213s oscillations) and $\sim 1\overset{d}{.}66$ (for the 274s oscillations). From (2), these differences correspond to the quantity $2|m|\Omega(1-C_{k\ell})$, and the different splitting frequencies for the two modes thus require different values of $C_{k\ell}$. As the calculations of Hansen _et al._ show, this is not at all surprising for the low-order g-modes thought to be present in R548. Although it is premature to identify the precise mode and rotation period of R548 (other than that it appears to be of the order of a day or two) — much less to associate the observed splitting with differential rotation — the prospects for the future seem promising.

b. Rapid rotation and disk accretion

In contrast to the single white dwarfs, the white dwarfs in cataclysmic variables may exhibit rapid rotation as a byproduct of accretion.

In these systems, accretion onto the white dwarf must ultimately take place from the inner edge of the accretion disk, which rotates with the Keplerian circular velocity $v_K \sim (GM/R_*)^{1/2}$. For a white dwarf with mass $M = 1M_\odot$ and radius $R_* \sim 10^9$ cm, this velocity is $v_K \sim 3000$ km s^{-1} which corresponds to an orbital period ~ 20 seconds. If such rapid rotation can be transferred efficiently to the white dwarf, the assumption of "slow" rotation, upon which the mode-splitting calculations are based, will be violated. For this reason, a number of recent papers have begun investigations of the effects of rapid rotation upon the oscillation spectra of stars.

A general formulation of the theory of non-radial oscillations of differentially rotating stars has been presented by Aizenman and Cox (1975; hereinafter denoted by AC). This approach was subsequently applied by Hansen, Aizenman, and Ross (1976) to a study of the non-radial oscillations of uniformly rotating isothermal cylinders. They found very peculiar behaviors of certain g-modes under rapid rotation and showed that some of the modes correspond to dynamically unstable spiral waves. This intriguing result stimulated Hansen and his collaborators to undertake further investigations of the effects of rotation upon the non-radial oscillation modes of stars. Hansen, Cox, and Carroll (1978) have accordingly adapted the theoretical formulation developed by AC to study this problem. In the limit of slow rotation (only correction terms to the eigenfunctions and eigenvalues that are linear in Ω are considered), they showed that the AC formalism recovers the conventional frequency-splitting constant C given by (3). However, Hansen et al. also extended their calculation to the quasi-adiabatic analysis of modal stability; they found the interesting

result that <u>retrograde (m>0) modes are slightly more stable than prograde ones</u>. Further, in a noteworthy appendix, Hansen, Cox, and Carroll studied the non-radial oscillations of rotating, cylindrical "white dwarfs" (the analogs of the rotating isothermal cylinders of Hansen, Aizenman, and Ross). In this work, which was <u>not</u> restricted to slow rotation, they found large effects on the g-mode periods even at rather moderate rotation rates ($\Pi \lesssim 500$ seconds for $\ell = 2$ g_1-modes). They also found substantial differences between the retrograde and prograde modes, as well as significant departures from the linear theory, even at periods as long as 1000 seconds. For this reason, they recommended a careful re-examination of Wolff's theory of the ZZ Ceti oscillations; if the mode splitting differs significantly from that given by (4a) with (2) and (6), the spectrum shown schematically in the top part of Fig. 5 will be modified, and it is not clear whether the degree of agreement with the observed oscillation spectra of the ZZ Ceti variables will be maintained.

An exciting new development in the theory of oscillations of rotating stars is contained in an important recent paper by Papaloizu and Pringle (1978). In this work, they pointed out the existence of a new class of modes which appear in rotating stars and which they have termed "r-modes" because of their similarity to Rossby waves. These modes have previously been missed by most workers because they belong to a completely different mode class (the <u>toroidal</u> modes) than do the <u>spheroidal</u> p-, f-, and g-modes, and because – for spherical stars – the toroidal modes are all degenerate at zero frequency.

The existence of the class of toroidal modes of stellar oscillation was noted in a group-theoretical paper by Perdang (1968; see also Chandrasekhar 1961). The nature of these modes was further clarified in an ex-

ceptionally careful study by Aizenmann and Smeyers (1977). They showed that the displacement fields $\rho\xi$ that correspond to oscillations of non-zero frequency have no toroidal component. In this case, they found that the displacement eigenfunction $\xi_{k\ell m}$ can be written in the familiar form (cf. equation [4])

$$\xi_{k\ell m} = (\xi_r, \xi_\theta, \xi_\phi) = [a(r)Y_{\ell m}, b(r)\frac{\partial Y_{\ell m}}{\partial\theta}, \frac{b(r)}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial\phi}] \ . \tag{7a}$$

This separation of variables yields the spheroidal oscillation modes, for which Cowling (1941) introduced the subclassifications of p-, f-, and g-modes. Aizenman and Smeyers went on to show, however, that the modes which are degenerate at zero frequency in a spherical star consist of the f-mode belonging to $\ell=1$ and a new class of modes which have no radial component of the displacement. For the latter modes, the displacement eigenfunction can be written in the form (see also Van Horn and Savedoff 1976)

$$\xi_{k\ell m} = [0, \frac{c(r)}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial\phi}, -c(r)\frac{\partial Y_{\ell m}}{\partial\theta}) \ , \tag{7b}$$

where $c(r)$ is a function of radius only. This separation of variables yields the toroidal oscillation modes.

In a rotating star, the equations governing the fluid motions contain additional terms not present in the spherical case. These correspond to the effects of the centrifugal and Coriolis forces, and they affect both the equilibrium configurations and the small-amplitude perturbations about equilibrium. These additional terms are the ones responsible for producing the mode-splitting that breaks the $(2\ell+1)$-fold degeneracy of the g-modes of

order ℓ that has been discussed previously (cf. equation [2]). However, as Papaloizu and Pringle (1978) show, the existing analysis of g-mode splitting (except for the very general formulation given by Aizenman and Cox 1975) is based upon the assumption of slow rotation of the star, and they argue that this is probably invalid for the white dwarfs in cataclysmic variables. Furthermore, as Papaloizu and Pringle also show (see also Perdang 1968), rotation breaks the degeneracy of the zero-frequency modes, producing in addition to the rotationally-modified g-modes a spectrum of toroidal modes with frequencies approximately given by (for uniform rotation)

$$\sigma_{k\ell m} \approx -m\Omega[1-\frac{2}{\ell(\ell+1)}], \ \ell \neq 0. \tag{8}$$

These are the modes Papaloizu and Pringle have named r-modes. For the case $\ell=1$, (8) still yields zero frequency; in this case a slightly better approximation yields a non-zero result, which, however, is very small (cf. Papaloizu and Pringle 1978).

Papaloizu and Pringle go on to discuss some possible applications of their theory to the interpretation of observational data. In particular, they question whether the 20-30 second oscillations observed in some cataclysmic variables, usually in the post-maximum decline of the light curve, may be r-modes rather than g-modes as conventionally assumed (cf. Warner and Robinson 1972). There are three main points to their argument. First, they point out that although p- and f-modes generally have periods much shorter than 20-30 seconds in hot white dwarfs, the g-mode periods — although much shorter than the ~ 200 second periods found in models with T_{eff} ~

10,000°K — tend still to be longer than 20-30 seconds[2]. Secondly, they note that the timescales over which period changes are observed to occur in the cataclysmic variable stars (often $\leq$ 18 hours : cf. Warner 1976b) are enormously short compared to the characteristic e-folding (decay) timescales of the g-modes[2]. Third, they emphasize that the rotation expected in the cataclysmic variable white dwarfs is likely to be rapid enough to invalidate the slow-rotation approximation for g-mode splitting, and may be fast enough to permit the existence of r-modes with periods comparable to those observed.

In order to obtain decay timescales as short as the timescales of observed period changes in the cataclysmic variables, Papaloizu and Pringle argue that it is necessary to consider white dwarf models in which the mass involved in the oscillations is confined to the very outermost surface layers. To this end they have studied the properties of some $\ell=2$ g-modes for models with a luminosity source embedded in the outermost $\sim10^{-10}M_{\odot}$ of envelopes containing $\sim10^{-6}$ to $10^{-7}M_{\odot}$ of hydrogen. These models are intended to simulate accretional heating of the surface layers, and the results indicate decay times as short as $\sim 10^{4}$ seconds for these highly surface-concentrated modes. For the low-k g-modes the periods are still too long for these models, however.

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[2]However, see the discussion of accreting white dwarf models with nuclear burning in Section V below, especially the calculations by DeGregoria 1977 and by Sienkiewicz and Dziembowski 1978).

The interaction between the accretion disk and the surface layers of the white dwarf is further elaborated by Papaloizu and Pringle in Appendix 1 of their paper. Here they consider a simple model of the transition layer over which the angular velocity changes from the Keplerian value appropriate to the inner edge of the disk to the slower rotation rate of the star. They show that the perturbation equations for this flow lead to the analogs of Kelvin-Helmholtz instabilities, and they estimate conditions for the onset of instability. In the text of their paper they speculate that either this instability or the direct interaction of the accretion disk with the surface of the white dwarf may be responsible for the excitation of non-radial oscillations of the star, a concept already implicit in the work of Pringle (1977). This process, which is rather similar to the production of musical tones in a flute, represents an important new mechanism for the excitation of stellar oscillations and clearly merits considerably more detailed work. To date only a few papers have seriously considered the complex problems involved in this interaction region. In addition to those works already cited, we must add the fundamental paper of Lynden-Bell and Pringle (1974), the quasi-steady calculations of Durisen (1977), and the recent work by Kippenhahn and Thomas (1978). The latter authors in particular point out that the interface region in disk accretion differs drastically from that involved in spherical accretion flows.

V. Instabilities in Planetary Nucleus Stars and Nuclear-Burning White Dwarfs

To complete this survey of theoretical research on the oscillations of degenerate stars, mention must also be made of recent work on degenerate stars other than the ZZ Ceti variables and the cataclysmics. The two types of systems which have received attention are i) the central stars of planetary nebulae and ii) white dwarfs with H- or He-burning shells, usually thought of as resulting from accretion. To date, these theoretical models have only limited correspondence with the observational data, and for this reason my discussion of them will be brief.

a. Planetary nuclei

Research on the oscillations of planetary nuclei published within the past two or three years has been limited, to this writer's knowledge, to two short papers by Stothers (1977) and by Dziembowski (1978). Stothers' paper is another in his series of applications of T.R. Carson's new radiative opacities (cf. Carson 1976). In this newest work, Stothers finds that the fundamental model of radial pulsation (and in a few cases, the first overtone as well) is excited in very high luminosity degenerate stars by the κ-mechanism operating in the CNO ionization zone. This is a direct consequence of a large "bump" in the Carson CNO opacities that occurs over a wide temperature range aroung 10^6°K at low densities. The region of the H-R diagram in which pulsational instability is driven by this mechanism covers a range in luminosity given by $3.5 \lesssim \log L/L_\odot \lesssim 4$, with effective temperatures cooler than $\log T_{eff} \approx 5.0$, extending at least to the red of $\log T_{eff} \approx 4.65$ and possibly beyond. This is the same region occupied by the highest luminosity central stars of planetary nebulae, and Stothers suggests that this mechanism may perhaps explain some of the rapid vari-

ability that has been observed in some of the planetary nuclei. He quotes the observed timescales of variation as ranging from weeks to perhaps as short as seconds, while the theoretical models yield fundamental mode periods ranging from fractions of an hour to about a day.

Dziembowski's (1978) paper is concerned with short-period variability in FG Sge. Over the period from 1972 to 1975, during which time the spectral type changed from F6 to G2, observations quoted by Dziembowski indicate short-term variability with a period of 60 days in 1972 and 20 days in 1975. He has accordingly carried out calculations of the nonadiabatic pulsational instability of double (H- and He-) shell burning models in order to find the high-luminosity extension of the instability strip. For models with masses between $0.52M_\odot$ and $1M_\odot$ and with luminosities greater than about $10^3 L_\odot$, he finds that nonadiabatic effects are important, and by matching the models to the observed periods he is able to deduce the luminosity and mass of the star. He finds that a 60 day pulsation period can be fitted by a model with $L \approx 6300 \, L_\odot$ and $M \approx 0.63 \, M_\odot$, while a 20 day period requires $L \approx 1600 L_\odot$, which may be too low for the development of a He shell-flash.

b. Nuclear-burning white dwarfs

In a recent series of papers, Vila and Sion (1976), Sion and Vila (1976), Vila (1977), and DeGregoria (1977) have examined the pulsational and thermal stability of a number of static models of white dwarfs without accretion and with nuclear burning of a H/He envelope assumed to provide the entire luminosity of the star. Vila and Sion (1976) constructed static models with masses of 0.6 and $1.0M_\odot$ in which H shell-burning by pp and CNO reactions in an appropriately chosen envelope provided the only energy source. They found nuclear-energized instability of the fundamental radial (F) mode in the

luminosity interval $0 \leq \log L/L_\odot \leq 3$, with growth timescales of $\sim 10^6$–10^8 years. At $\log L/L_\odot$, the first radial overtone (H_1) mode was also excited, on a timescale of 10^8 years. In a closely related work, Sion and Vila (1976) found nuclear-energized pulsations of the F-mode alone in the range $0 \leq \log L/L_\odot \leq 3$ for models with He-burning shells near the surfaces. Vila (1977) subsequently undertook to examine the thermal stability of the H shell-burning models and came to the unexpected conclusion that the models were all thermally stable. A similar negative result was also found by Sion, Acierno, and Turnshek (1978), who discussed the thermal stability of models having masses $1.2 \leq M/M_\odot \leq 1.38$ and undergoing steady-state accretion with H-burning due to CNO reactions in the envelope as the only energy source. Unfortunately, from the brevity of the description given by these authors, it is unclear whether all of these calculations refer to models undergoing stationary accretion or whether that is only true of some of them. More seriously, it is not clear why the conclusions regarding the lack of thermal instabilities in these models differ from the conclusions for similar models based on the calculations of Sienkiewicz and Dziembowski, discussed below, and from the detailed, time-dependent shell-flash calculations carried out over the past several years, especially by Gallagher and Starrfield (1978), Sparks, Starrfield, and Truran (1977), and references therein. Until the differences are satisfactorily explained, these results must be used with caution.

Another recent paper dealing with nuclear-energized pulsations in white dwarfs is that of DeGregoria (1977), which follows up earlier, similar work by Cameron (1975). DeGregoria has investigated the radial and non-radial pulsational instability of static white dwarf models with masses between 0.6 and $1.4 M_\odot$ and in which the sole energy source is

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H-burning due to the CN reactions. His models have luminosities ranging from 10^{36} to 10^{38} ergs s^{-1}, and he finds instabilities in both radial and non-radial modes, the models with larger masses being consistently the more unstable. In most of DeGregoria's models, the fundamental radial (F) mode is unstable, with periods of a few seconds and growth times longer than about 10^4 years. In models with lower luminosities and higher masses, radial overtones as high as H_3 may be excited, and in cases where the first overtone (H_1) mode is excited, it is found to be far more unstable than the F-mode. In all cases, the $\ell=2$ g_1-mode (the only g-mode considered), with periods ~ 5-40 seconds was excited, with very short growth timescales, ranging from less than a week to some tens of years. The $\ell=2$ Kelvin- (f-) mode and the p-modes were found always to be stable, however. The results of these calculations were discussed in the context of pulsating X-ray sources.

The final calculations to be discussed here are those of Sienkiewicz (1975) and Sienkiewicz and Dziembowski (1978). In the first of these papers, Sienkiewicz discussed the construction of white dwarf models of masses 1.0 and 1.39$M_\odot$, which are undergoing steady-state accretion with nuclear burning of the material at the same rate as it is accreted. Accretion rates between ~10^{-11} $M_\odot$ yr^{-1} and a few times 10^{-7} $M_\odot$ yr^{-1} were used, and the systematic behaviors of the H- and He- burning shells with accretion rate and stellar mass were studied. In the later paper, Sienkiewicz and Dziembowski (1978) investigated the thermal and vibrational stability of the 1 $M_\odot$ models from Sienkiewicz's accretion calculations. All of the models were found to be thermally unstable, in contrast to the conclusions of Vila (1977), with instability produced by the H-burning shell for $\dot{M} \lesssim 4 \times 10^{-8} M_\odot$ yr^{-1} and by the He-burning shell for $\dot{M} \gtrsim 3 \times 10^{-7} M_\odot$ yr^{-1}. Vibrational instability was

also found at all but the highest accretion rates, but the growth rate of
the oscillations was slower than the thermal instabilities, except in the
range $4 \times 10^{-8} M_\odot \, yr^{-1} \lesssim \dot{M} \lesssim 3 \times 10^{-7} M_\odot yr^{-1}$. (see Fig. 6). Both radial and
non-radial modes were found to be excited (similar multi-mode excitation
has also been found in non-accreting nova and pre-nova models by Sastri and
Simon 1973 and by Vemury (1978), although the growth rate of the most
rapidly-excited g-modes was more than three orders of magnitude faster than
that of the radial modes. The $\ell=1$ g_2-mode was found to be the most unstable,
with growth rates of a few months in the range of $\dot{M}$ where pulsations develop
more rapidly than thermal instabilities. The periods of these models are
about 30 seconds, although a broad spectrum of g-modes corresponding to
$\ell=1$-5 and with periods ranging from 10 to 50 seconds is excited simultaneously.
Because of the high rates of accretion and nuclear burning in the pul-
sationally unstable models, the luminosities of these cases are quite large
($L \gtrsim 10^3 L_\odot$, log $T_{eff} \gtrsim 5$), but because of the simultaneous excitation of
many different modes, the pulsations may not be easy to detect observationally
despite the high luminosities.

VI. Conclusions And Some Problems That Merit Further Work

Despite the real progress in clarifying the theoretical bases for
an understanding of the observed oscillations of white dwarfs, some major
problems remain to be resolved before a satisfactory comparison of theory
and observation can be achieved. In particular, no theoretical calculation
has yet succeeded in demonstrating pulsational instability in the oscillation
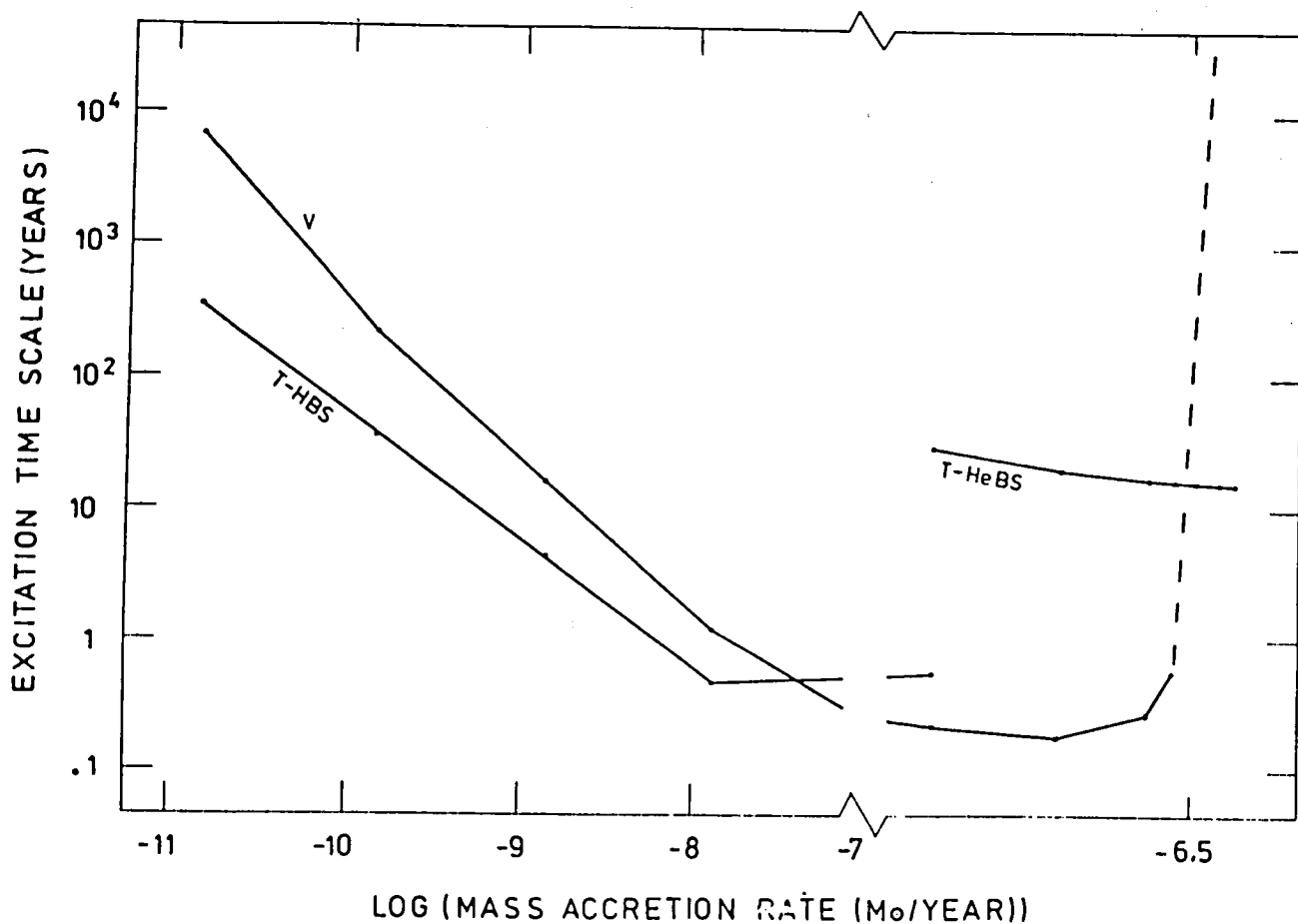modes which appear to be excited in these stars. On the positive side,

Fig. 6 – Characteristic timescales for the growth of thermal and vibrational instabilities, taken from Sienkiewicz and Dziembowski (1978) for white dwarfs undergoing steady-state accretion and nuclear burning. The curves labelled T-HBS and T-HeBS give the growth timescales for thermal instabilities in the H- and He- burning shells, respectively. The curve marked V gives the excitation timescale for vibrational instability in the most unstable mode; this is the fastest-growing instability for accretion rates in the range $4 \times 10^{-8} M_{\odot} yr^{-1} \lesssim \dot{M} \lesssim 3 \times 10^{-7} M_{\odot} yr^{-1}$.

theorists have begun to carry out considerably refined calculations for increasingly realistic stellar models, and some have begun to attack the extremely complex and important problems associated with the effects of rotation on non-radial stellar oscillations. Preliminary work has also been done in a few cases involving accretion and nuclear burning.

Among the many problems that need attention before we can claim an understanding of the oscillations of white dwarfs, the following appear to this writer to be some of the more important ones:

1. How does the layered compositional structure of a white dwarf affect the frequencies and excitation rates of non-radial oscillations? In particular, the location and extent of the He ionization zone is expected to depend rather sensitively upon the thickness of the overlying hydrogen layer; how does this affect the excitation rates? Does the depth of the hydrogen layer determine whether or not a white dwarf lying in the instability strip will be a variable? Also, do the H/He or He/C interfaces introduce additional g-modes ("surface waves") into the non-radial oscillation spectrum? If so, what are the periods of these modes; can they explain the 800-1000 second period oscillations?

2. What effect does the coupling between convection and oscillations have for the white dwarfs? For the purposes of a preliminary investigation of this problem, the use of a very simple form of time-dependent mixing-length theory may be sufficient; certainly this seems to be a logical and necessary first step.

3. From the standpoint of observations, can refined versions of the period-amplitude plot shown in Fig. 4 provide further clues to the nature of the ZZ Ceti oscillations? In addition, is it possible to achieve sufficient frequency resolution in the large-amplitude ZZ Ceti stars to determine

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whether the variability of the mode amplitudes observed here is due to beating between colsely-spaced high -k modes, as the period-amplitude correlations tend to suggest?

4. Also in regard to observations, is Wolff's model of beating between rotationally=split g-modes supported or rejected by observations of the rotation-broadening of the Balmer lines in the ZZ Ceti stars? If it is confirmed, the theoretical basis for the model requires further devleopment.

5. The toroidal r-modes introduced by Paploizu and Pringle need to be investigated carefully in the context of improved stellar models. What are the mode frequencies and excitation rates in such models? Do these modes play a role in the long-period ZZ Ceti stars as well as in the cataclysmic variables? Is there direct observational evidence of the required rotation in either type of system?

6. Can the interaction between the accretion disk and the white dwarf in cataclysmic variables drive oscillations of the star? How can this be calculated, and under what conditions (if any) can such excitation occur?

7. Finally, is there observational evidence for the existence of high-luminosity degenerate variables?

With the interest and activity on problems of white dwarf oscillations that has now been generated, perhaps it is not too much to hope that significant progress is answering these and related questions may be achieved before the next pulsation conference.

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Discussion

Sobouti: I would have to dispute your formula for the expansion of the

g-mode. There are certain criteria for any perturbation expansion, which in

the case of g-modes are not met. In perturbation expansions, you always

have an energy denominator which is the difference of two energies. If you

look at the spectra of g-modes, you can find an infinite number of pairs of

energy levels which are infinitely small and infinitely close to each other,

and if you insert these into the denominator the series won't converge. So

the g-modes cannot accept the perturbation expansion.

Van Horn: Thank you, I hope you will say more about this in your paper later

on. All I can say is, this is the classical expression that has been quoted

since Ledoux.

Sobouti: That formula is fine for p-modes, but not for g-modes, I am afraid.

Van Horn: Will you say something about this in your talk?

Sobouti: I don't know. I am only allowed twelve minutes. [Laughter]

Cahn: Would you like to continue the discussion into the planetary aspects?

Van Horn: I can tell you in one word what I was going to say, and that is

that Stothers has looked at some models for planetaries recently using the

new Carson opacities, and he finds that a mysterious bump which occurs in

those opacities drives instability at high luminosity for an effective

temperature range from 100,000 K down. He doesn't know where it terminates.
It is apparently a difference in the Los Alamos opacities and the Carson
opacities.

Wolff: I would like to make two further suggestions to the observers about
testing this model of mine, which Van Horn summarized so nicely. There may
be a tendency for the light curve to repeat itself after an interval of one
or two weeks. The precise interval is proportional to the rotation period,
and the theory says what this interval is for each star. So, do an auto-
correlation of the light curve, displaced by a certain amount, and measure
the tendency for repetition. The other suggestion that I would make is that
if you observe a few stars intensively, that would be a great deal of help
in testing these models.

Nather: I will exercise the Chairman's prerogative and make one comment
while the next speaker is coming up here. The one star that has been
observed most intensively is R548, and there we do find splitting which is
inconsistent with rapid rotation. It has to be slow.