

# THE g-MODES OF WHITE DWARFS\*

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## SUMMARY

The neutral g-modes of a degenerate fluid at zero temperature are analyzed. The g-modes of a degenerate fluid at finite but small temperatures are then expanded in terms of those of the zero temperature fluid. For non-relativistic degenerate fluids it is found that (a) the g-eigenvalues are proportional to  $T \mu_e^6 \mu_i^{-1}$ , where  $T$  is the internal temperature of the fluid,  $\mu_e$  and  $\mu_i$  are the mean molecular weights of electrons and ions, respectively; (b) the ion pressure is solely responsible for driving the g-modes. For white dwarfs of about a solar mass, the periods of the g-oscillations are in the range of a few hundred of seconds.

## 1. INTRODUCTION

It has been suggested that the short period oscillations observed in the cataclysmic variables are connected with the g-modes of the white dwarf component of these objects (e.g. Warner and Robinson, 1972; Osaki and Hansen, 1973). The thrust of the argument behind this suggestion is that the observed periodicities, of the order of several tens of seconds, are too long to be attributed to the p-modes. In connection with the longer periodicities of DA white dwarfs, of the order of several hundreds of seconds, some authors have also called upon the

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g-modes. For example, Wolff (1977) has proposed that the interaction of a slow rotation with the g-modes is capable of accounting for such periodicities. Most of these propositions have, at most, had partial acceptability by investigators in the field. Modifications to the theories have been proposed (e.g., Papaloizou and Pringle, 1978), and alternative explanations, not requiring the intervention of g-modes, have been put forward (e.g., Bath 1973, and Bath et al., 1974).

In recent years, Baglin and Schatzman (1969) have calculated some g-frequencies of white dwarfs with finite temperatures, Harper and Rose (1970). Osaki and Hansen (1973), and Brickhill (1975), each have analyzed the problem in varying details. Practically all previous works on non-radial oscillations of white dwarfs are attempts to integrate the differential equations governing the small displacements of the fluid. Some information on the eigenfrequencies and, to a lesser extent, on the eigenfunctions has been accumulated over the years.

The authors wish to draw attention to an alternative approach to the problem: The operator generating small displacements of a fluid, including those of a degenerate structure, is self-adjoint (Chandrasekhar, 1964). The normal modes of the fluid belong to a Hilbert space. If there is access to a basis set for this space, then one can expand the actual eigendisplacements of the fluid in terms of this basis set, transform the differential equations of motions into a matrix equation, and obtain the expansion coefficients by variational calculations. This alternative route is followed in the present paper. It provides more information on the eigenvalues and eigenfunctions of the system, and gives deeper insight into their behavior and their dependence on the physical properties of the fluid.

(6) is an exact solution of Equations (1) corresponding to  $\epsilon = 0$ . This completes the demonstration of the existence of neutral modes in the zero temperature fluid, the neutral g-modes.

#### IV. THE BASIS-SET

The neutral state of the zero temperature fluid is infinitely degenerate. Any displacement vector  $\underline{\zeta}$  satisfying Equation (6), but otherwise arbitrary, is an eigenfunction of  $\epsilon = 0$ . Sobouti (paper 1) has used this arbitrariness to propose a set of basis-vectors for the space of the g-modes of the neutral fluid. Thus, let  $\underline{\zeta}^j$ ,  $j = 1, 2, \dots$ , have the following spherical harmonic expansion

$$\underline{\zeta}^j : \left( \frac{\psi^j(r)}{r^2} Y_\ell^m, \frac{1}{\ell(\ell+1)} \frac{\chi^{j'}(r)}{r} \frac{\partial Y_\ell^m}{\partial \theta}, \frac{1}{\ell(\ell+1)} \frac{\chi^{j'}(r)}{r} \frac{1}{\sin \theta} \frac{\partial Y_\ell^m}{\partial \phi} \right), \quad (11a)$$

where  $\chi' = d\chi / dr$ .

Substitution of Equation (11a) in Equation (6) gives

$$\chi^{j'} = \psi^{j'} + \frac{\rho'}{\rho} \psi^j. \quad (11b)$$

One of the two functions  $\psi^j$  and  $\chi^j$  could be chosen arbitrarily. Equation (11b) can then be used to obtain the other one. One now appreciates the immense simplification that Equation (6), or equivalently Equation (11b), brings into the problem. These equations reduce the task of specifying a vector  $\underline{\zeta}^j$  to the determination of only one scalar function,  $\psi^j$ , say. The following expression for  $\psi^j$  is used in the numerical calculations of this paper:

$$\psi^j = -\frac{3}{4\pi G} \rho \rho' r^{-2} r^{\ell+2j-2}, \quad j = 1, 2, \dots \quad (11c)$$

We note the following: (a) On the surface of star  $\psi$  vanishes, while  $\chi'$ , the non-radial component of the displacement, remains finite. (b) The exponent of  $r$  is chosen such that  $\nabla \cdot \underline{\zeta}$  behaves as  $r^\ell$  near the origin and is an even or odd polynomial depending on whether  $\ell$  is even or odd; these properties are shown to be required by Hurley, Roberts and Wright (1966). (c) The polynomials in  $r$  are helpful in achieving the completeness of the proposed vectors of

## 11. THE MODEL AND THE EQUATIONS OF MOTION

A fluid consisting of non-relativistic degenerate electrons and non-degenerate ions will be considered. It will be assumed that the temperature is constant throughout the fluid, except in a thin envelope. In this envelope, the temperature will decrease to zero in a manner determined by whether the envelope is in convective or in radiative equilibrium.

i) Equations of Motion: Adiabatic Lagrangian displacements of the fluid,  $\underline{\xi}(\underline{r}) \exp(i \epsilon^{1/2} t)$ , are governed by the following equations:

$$\mathcal{W}\underline{\xi} = \epsilon\rho\underline{\xi}, \quad (1a)$$

where

$$\mathcal{W}\underline{\xi} = \underline{\nabla}\delta\rho - \frac{1}{\rho} \delta\rho\underline{\nabla}\rho - \rho\underline{\nabla}\delta\Omega, \quad (1b)$$

$$\delta\rho = -\rho\underline{\nabla}\cdot\underline{\xi} - \underline{\nabla}\rho\cdot\underline{\xi}, \quad (2a)$$

$$\delta p = -\left(\frac{\partial p}{\partial \rho}\right)_{\text{ad}} \rho\underline{\nabla}\cdot\underline{\xi} - \underline{\nabla}\rho\cdot\underline{\xi}, \quad (2b)$$

$$\nabla^2 \delta\Omega = -4\pi G\delta\rho, \quad (2c)$$

and  $\rho$ ,  $p$  and  $\Omega$  are the density, pressure and gravitational potential of the fluid in hydrostatic equilibrium, respectively.

ii) The Equation of State: The pressure of the fluid is the sum of the partial pressures of the degenerate free electrons and the non-degenerate ions.

Thus,

$$p(\rho, T) = p_e(\rho, T) + p_i(\rho, T). \quad (3a)$$

No correction for charge separation is allowed in Equation (3a). The number density of the free electrons and the positive ions is assumed to be proportional to each other, so as to maintain macroscopic charge neutrality. Therefore, both  $p_e$  and  $p_i$  are given in terms of the same density  $\rho$ . At temperatures

$T$ , much below the Fermi temperature  $T_F$  of electrons, temperature corrections in  $p_e$  are of the order of  $(T/T_F)^2$ . In contrast, the ions treated as a perfect gas have their pressure  $p_i$  proportional to the temperature. Thus, up to and including the first order terms in  $T/T_F$ , Equation (3a) gives

$$p(\rho, T) = p_e(\rho) + \frac{k}{\mu_i H} \rho T = \frac{k}{\mu_e H} \rho T_F \left[ \frac{2}{5} + \frac{\mu_e}{\mu_i} \frac{T}{T_F} \right], \quad T_F \propto \rho^{2/3}, \quad (3b)$$

where the first term on the right hand side is the electron pressure at zero temperature,  $\mu_e$  and  $\mu_i$  are the mean molecular weights of electrons and ions,  $k$  and  $H$  are the Boltzmann constant and the mass of hydrogen atom, respectively. As a simplifying assumption,  $\mu_e$  and  $\mu_i$  will be considered to be constant throughout the star and also in the course of any thermodynamic process which may take place in it.

iii) The Adiabatic Processes: The entropy of the fluid is the sum of the electronic and ionic contributions,  $s_e$  and  $s_i$ , respectively. Thus, the change in the entropy is

$$ds = ds_e + ds_i. \quad (4a)$$

For the unit mass of ions, treated as a perfect gas, one has

$$ds_i = \frac{k}{\mu_i H} \left( \frac{3}{2} \frac{dT}{T} - \frac{d\rho}{\rho} \right). \quad (4b)$$

The entropy per unit mass of non-relativistic degenerate electrons is

$s_e = (\frac{1}{2} \pi^2 k / \mu_e H) T / T_F$  (see for example, Landau and Lifshitz, 1959; Morse, 1965). From the last expression, and considering the fact that  $T_F$  is proportional to  $\rho^{2/3}$ , one obtains

$$ds_e = \frac{1}{2} \pi^2 \frac{k}{\mu_e H} \frac{T}{T_F} \left[ \frac{dT}{T} - \frac{2}{3} \frac{d\rho}{\rho} \right] = \frac{\pi^2}{3} \frac{\mu_i}{\mu_e} \frac{T}{T_F} ds_i. \quad (4c)$$

One observes that (a)  $ds_e$  is smaller than  $ds_i$  by a factor of  $T/T_F$ , and (b) in the course of an adiabatic process (in which by definition  $ds = 0$ ) both  $ds_e$  and

$ds_i$  vanish simultaneously. These results simplify the forthcoming calculations considerably. Thus, one obtains

$$\left(\frac{\partial p}{\partial \rho}\right)_{ad} = \left(\frac{\partial p_e}{\partial \rho}\right)_{s_e} + \left(\frac{\partial p_i}{\partial \rho}\right)_{s_i} = \frac{5}{3} \frac{p_e}{\rho} + \gamma \frac{p_i}{\rho}, \quad (5)$$

where  $\gamma = 5/3$  is ratio of the specific heats of the ion gas.

### III. THE NEUTRAL MODES OF THE ZERO TEMPERATURE FLUID

At zero temperature the g-modes of a degenerate fluid are neutral, that is, the g-eigenvalues become zero (e.g., Harper and Rose, 1970). We intend to analyze these neutral modes in some detail and subsequently use them as a basis set to expand the modes of the finite temperature model. The formalism is the same as that of Sobouti (1977a, henceforth referred to as paper I) for the g- and p-modes of ordinary fluids.

Let us investigate if there is a displacement  $\underline{\zeta}(\underline{r}) \neq 0$  which is a solution of Equation (1) and for which the corresponding  $\delta p$  is zero. This last condition, by Equation (2a), requires that

$$\delta p = -\rho \nabla \cdot \underline{\zeta} - \underline{\nabla} \rho \cdot \underline{\zeta} = 0. \quad (6)$$

Substitution of Equation (6) in Equation (2b) gives

$$\delta p = -\left[ \left(\frac{\partial p}{\partial \rho}\right)_{ad} - \left(\frac{\partial p}{\partial \rho}\right)_{st} \right] \rho \nabla \cdot \underline{\zeta}, \quad (7)$$

where we have utilized the fact that in hydrostatic equilibrium  $p$  is a function of  $\rho$  alone and  $\underline{\nabla} p = (\partial p / \partial \rho)_{st} \underline{\nabla} \rho$ . The expression  $(\partial p / \partial \rho)_{st}$  is the derivative of  $p$  with respect to  $\rho$  as prevailing in the equilibrium structure. The subscript "st" is inserted to distinguish this derivative from the adiabatic derivative  $(\partial p / \partial \rho)_{ad}$ . From Equation (3b) one has

$$\left(\frac{\partial p}{\partial \rho}\right)_{st} = \left(\frac{\partial p_e}{\partial \rho}\right)_{st} + \frac{k}{\mu_i H} T + \frac{k}{\mu_i H} \rho \left(\frac{\partial T}{\partial \rho}\right)_{st}. \quad (8a)$$

The first derivative on the right hand side of Equation (8a) will be calculated for the zero-temperature structure and is equal to  $\frac{5}{3} p_e / \rho$ . With regards to  $(\partial T / \partial \rho)_{st}$ , let us define  $\gamma'$  as follows:

$$\left(\frac{\partial T}{\partial \rho}\right)_{st} = (\gamma' - 1) \frac{T}{\rho}. \quad (8b)$$

We observe that in the isothermal core,  $\gamma' = 1$ . In the envelope, where the temperature decreases outwards, one has the following information: (a) In any region in convective equilibrium,  $\gamma' > \gamma$ . (b) In any convectively neutral region,  $\gamma' = \gamma$ . (c) In any region in radiative equilibrium,  $\gamma' < \gamma$ . From the theory of g-modes in ordinary fluids one knows that in case (a) some unstable g-motions (or g-modes, if they develop into standing patterns) arise. These are, however, confined to the convective layers. In case (c) stable g-modes develop and again are confined to the radiative region. We shall assume that the envelope is convectively neutral, case (b). By this assumption the envelope will not contribute to the g-modes of the system and the role of the degenerate core will be singled out. In view of these considerations, Equation (8a) becomes

$$\left(\frac{\partial p}{\partial \rho}\right)_{st} = \frac{5}{3} \frac{p_e}{\rho} + \frac{p_i}{\rho}, \text{ in the core,} \quad (8c)$$

$$= \frac{5}{3} \frac{p_e}{\rho} + \gamma' \frac{p_i}{\rho}, \text{ in the envelope.} \quad (8d)$$

Substitution of Equations (5), (8c) and (8d) in Equation (7) gives

$$\delta p = -(\gamma - 1) p_i \nabla \cdot \underline{\xi} = -(\gamma - 1) \frac{k}{\mu_1 H} T \rho \nabla \cdot \underline{\xi}, \text{ in the core,} \quad (9a)$$

$$= 0, \text{ in the envelope } (\gamma = \gamma'). \quad (9b)$$

This completes the reduction of  $\delta p$ . From Equations (2c) and (6) one has

$$\delta \Omega = 0. \quad (10)$$

We now observe that at zero temperature,  $\delta p$ ,  $\delta \rho$ , and  $\delta \Omega$ , generated by the displacements prescribed by Equation (6), vanish identically. Thus  $\underline{\xi}$  of Equation

Equations (11) in the g-subspace of the normal modes of the neutral-fluid. The completeness of the proposed vectors, though not yet established theoretically, has, however, been born out by numerical computations of Sobouti (1977b) and of this paper. Variational calculations with the basis vectors of Equations (11) enable one to isolate the g-modes of a fluid systematically and with satisfactory accuracies.

We have only discussed the g-modes of the neutral fluid. A second basis set for the p-modes can be generated from the requirement that the p-set is orthogonal to the g-set. These two basis sets may then be used to expand the eigendisplacement vectors of any other fluid at finite temperatures (see paper I for the p-basis set and for further details). Silverman and Sobouti (1978), and Sobouti and Silverman (1978) have carried out such an expansion for ordinary fluids. Their analysis shows that in the limit of small departures from the neutral state (the limit of small  $T$  in the present problem), the g-modes, as given in terms of the vectors of Equations (11), are independent of the p-basis vectors. This property stems from the fact that the g-states at  $T = 0$  are degenerate. In the language of linear vector spaces, to a degenerate state there corresponds a subspace of the normal modes of the system. The effect of a small perturbation is to specify the principal directions of this subspace but it leaves the subspace unaltered. Thus, up to the order  $T/T_F$ , the g-eigendisplacements can be expanded in terms of the basis set of Equation (11) alone. No intervention of the p-basis set will be necessary.

#### V. THE EQUATION GOVERNING THE g-MODES:

Let  $\xi^j$ ,  $j = 1, 2, \dots$  be a sequence of the g-eigendisplacements of the finite temperature fluid. The Eulerian variations  $\delta\rho$ ,  $\delta p$  and  $\delta\Omega$ , generated by these vectors are given by Equations (6), (9) and (10), respectively.

Substitution of these in Equations (1) gives



$$\omega_{\underline{\xi}}^j = \underline{\nabla} \delta^j p = \epsilon^j \rho \underline{\xi}^j. \quad (12)$$

The density  $\rho$  in Equation (12), and in Equation (9) for  $\delta^j p$ , is the density of the zero temperature fluid. This does not imply, however, that the density and the pressure are approximated by their values at  $T = 0$ . We have been able to show that Equation (12) is correct up to the terms of first order in  $T/T_F$ . The terms arising from  $\delta p$ ,  $\delta \rho$  and  $\delta \Omega$  in Equations (1), on account of the fact that the density distribution at  $T \neq 0$  is different from that at  $T = 0$ , cancel out each other.

Let  $\underline{\xi}^j$  have the following expansion in terms of the basis set of Equations (11):

$$\underline{\xi}^j = \sum \underline{\zeta}^k z^{kj}; \quad k, j = 1, 2, \dots \quad (13)$$

The expansion coefficients  $z^{kj}$  can be considered as the elements of a matrix  $Z$ . Thus,

$$Z = [z^{kj}]; \quad k, j = 1, 2, \dots \quad (14a)$$

Each column of  $Z$  is an eigenvector of the system. Let  $E$  be the diagonal matrix of the eigenvalues  $\epsilon^j$ ,

$$E = [\epsilon^j] \text{ diagonal}; \quad j = 1, 2, \dots \quad (14b)$$

In connection with Equation (13), let us introduce the following matrices:

$$W^{kj} = \int \underline{\zeta}^{k*} \cdot \omega_{\underline{\zeta}}^j dv, \quad (15a)$$

$$S^{kj} = \int \rho \underline{\zeta}^{k*} \cdot \underline{\zeta}^j dv. \quad (15b)$$

We are now ready to convert the differential Equation (12) into an equivalent matrix equation. Let us substitute Equation (13) in Equation (12), premultiply the resulting equation by  $\zeta^{i*}$  (say), and integrate over the volume of the fluid. We obtain the  $(i,j)$ th element of the following matrix Equation:

$$WZ = SZE. \quad (16a)$$

The matrix  $Z$  simultaneously diagonalizes  $W$  and  $S$ . A normalization condition on the eigendisplacements  $\xi^j$ , or on the eigenvectors  $Z$ , could be imposed such that  $S$  diagonalizes to a unit matrix  $I$  (see Silverman and Sobouti, 1978). Thus

$$Z^\dagger S Z = I. \quad (16b)$$

Given the matrices  $W$  and  $S$ , the eigenvalue matrix  $E$  and the eigenvector matrix  $Z$  can be solved from Equations (16).

The  $W$ -and  $S$ -Matrices: From Equations (15a), (12), and (9), after an integration by parts, one obtains

$$W^{kj} = \int (\gamma - \gamma') \rho_i \nabla \cdot \xi^{k*} \nabla \cdot \xi^j dv, \quad (17a)$$

where  $\gamma'$  is defined by Equation (8b). The value of  $\gamma'$  is equal to unity in an isothermal core; and is equal to  $\gamma$ , the ratio of specific heats of ions, in a convectively neutral envelope. Substitution of Equation (11) in Equation (17a) and an integration over the solid angle gives

$$W^{kj} = (\gamma - 1) \frac{kT}{\mu_i H} \int_0^R \frac{\rho^2}{\rho} \psi^k \psi^j r^{-2} dr, \quad (17b)$$

where  $R$  is the radius of the star. The assumption of a convectively neutral envelope confines the domain of integration to the core of the star. (i) The thickness of the envelope, however, is proportional to  $T$  and to  $R/M$ .

The latter is, in turn, proportional to  $T_{Fc}^{-1}$ . Therefore, the ratio of the thickness of the envelope to the radius of the star is of the order

of  $T/T_{Fc}$ . (ii) The density, temperature, and pressure tend to zero at the surface. These two factors make the contribution of the envelope to the  $W$ -matrix of the second order in  $T/T_{Fc}$ . Therefore, extending the domain of integration over the whole star does not introduce first order errors. This completes the reduction of the  $W$ -matrix. For the  $S$ -matrix, after an integration over the angles, one gets

$$s^{kj} = \int_0^R \rho \left[ \frac{\psi^k \psi^j}{r^2} + \frac{1}{\ell(\ell+1)} x^{k'} x^{j'} \right] dr. \quad (18)$$

The symmetry and positive definiteness of the W- and S-matrices are manifest from Equations (17) and (18).

## VI. SOME PROPERTIES OF THE EIGENVALUES AND EIGENVECTORS

i) The Sign and the Asymptotic Behavior of the Eigenvalues: From Equations (12) and (9) one can readily write down the following integral expression for  $\epsilon^j$ :

$$\epsilon^j = (\gamma - 1) \frac{kT}{\mu_i H} \int \rho \underline{\nabla} \cdot \underline{\xi}^{j*} \underline{\nabla} \cdot \underline{\xi}^j dv / \int \rho \underline{\xi}^{j*} \cdot \underline{\xi}^j dv. \quad (19)$$

Either from Equation (19) or from Equations (17) and (18) for the W- and S-matrices, one can draw the following conclusions:

a) The numerator and the denominator in Equation (19) are both positive for any  $\underline{\xi}^j$ , or, the W- and S-matrices are symmetric and positive definite. Therefore, all eigenvalues  $\epsilon^j$  are positive. That is, the g-modes of a degenerate fluid are all stable. This, of course, is not surprising. The g-modes are prototypes of convective motions. An isothermal core is highly subadiabatic and is stable against convective motions.

b) In Equation (19), as the mode-number  $j$  increases, the numerator becomes progressively smaller than the denominator. This can be seen from Equations (17b) and (18), where the integrations over the solid angles have been carried out. This asymptotic behavior is well established for the g-modes of ordinary fluids (e.g., Ledoux and Walraven 1958).

ii) The Unit and the Order of Magnitude of the Eigenvalues: In Equation (19), let us use a dimensionless radius variable  $x = r/R$ . Thus, one obtains

$$\epsilon^j = (\gamma - 1) \frac{k}{\mu_i H} \frac{T_{Fc}}{R^2} \frac{T}{T_{Fc}} \epsilon^j, \quad (20a)$$

$$\epsilon^j = \int \rho \nabla_x \cdot \xi^{j*} \nabla_x \cdot \xi^j d^3x / \int \rho \xi^{j*} \cdot \xi^j d^3x, \quad (20b)$$

where  $T_{Fc}$  is the Fermi temperature at the center. In Equation (20a) the factors  $(T/T_{Fc})$  and  $\epsilon^j$  are dimensionless. The first of these factors depends on the model under study. The second factor is calculated for a zero temperature star and is given in Tables 1, 2, and 3. The remaining factor in Equation (20a) has physical dimensions. On expressing  $T_{Fc}$  and  $R$  in terms of the central density  $\rho_c$  (see Chandrasekhar, 1939) one gets

$$\epsilon^j = \frac{4\pi G \rho_c}{\eta_1^2} (\gamma - 1) \frac{\mu_e}{\mu_i} \frac{T}{T_{Fc}} \epsilon^j, \quad (21a)$$

where  $\eta_1$  is the Emden radius of the polytrope 1.5 (we note that the structure of a non-relativistic degenerate fluid at zero temperature is that of the polytrope 1.5). The  $p$ -eigenvalues of the fluid are of the order of  $4\pi G \rho_c / \eta_1^2$ . Therefore, the  $g$ -eigenvalues of the white dwarfs are smaller than the  $p$ -eigenvalues by a factor of  $T/T_{Fc}$ . Upon expressing  $\rho_c$  in terms of the total mass of the star, one obtains an alternative expression for  $\epsilon^j$  (see Chandrasekhar, 1939),

$$\epsilon^j = \frac{128\pi^2}{125} \frac{G^4}{K^3} \eta_1^{-6} \left( \frac{d\theta}{dn} \right)_1^{-2} (\gamma - 1) \mu_e^5 \frac{\mu_e}{\mu_i} \left( \frac{M}{M_\odot} \right)^2 \frac{T}{T_{Fc}} \epsilon^j, \quad (21b)$$

where  $(d\theta/dn)_1$  is the surface value of the derivative of the Emden temperature, and  $k$  is defined by the relation  $p = K(\rho/\mu_e)^{5/3}$ . The oscillation period

$P^j = 2\pi / \sqrt{\epsilon^j}$ , in the cgs units, is given by

$$P^j \text{ (sec)} = 69.0 (\gamma - 1)^{-1/2} \mu_e^{-5/2} \left( \frac{\mu_i}{\mu_e} \right)^{1/2} \frac{M_\odot}{M} \left( \frac{T_{Fc}}{T} \right)^{1/2} (j)^{-1/2}. \quad (22a)$$

For  $\gamma = 5/3$ ,  $\mu_e = 2$ , and  $\mu_i = 4$ , one has

$$P^j \text{ (sec)} = 21.1 \frac{M_\odot}{M} \left( \frac{T_{Fc}}{T} \right)^{1/2} (\epsilon^j)^{-1/2} \quad (22b)$$

iii) Behavior of the Eigendisplacements at the Center and at the Surface: Once the eigenvector matrix  $Z$  is calculated from Equation (16), the

eigendisplacement vectors  $\xi^j$ , can be obtained by Equations (13) and (11). We note the following properties of  $\xi^j$ :

a) At the center  $\nabla \cdot \xi$  tends to zero as  $r^\ell$ . See the remarks following Equation (11c).

b) At the surface, the radial component of  $\xi$  tends to zero as  $r-R$ . The non-radial component and  $\nabla \cdot \xi$  remain finite. These requirements from the g-modes of any fluid are analyzed in paper I.

## VII. NUMERICAL RESULTS

Equation (16) is solved in a Rayleigh-Ritz approximation, that is, by approximating the infinite matrices of Equation (13)-(16) by finite matrices. The matrix size was varied from one by one to seven by seven. Values of  $\ell = 1, 2, 3, 4, 5$  and 6 were considered. For  $\ell = 1$  and 2 the dimensionless eigenvalues  $\epsilon^j$ , Equation (21b), and the eigenvectors  $Z$  at various Rayleigh-Ritz approximations are given in Tables 1 and 2. The eigenvalues are displayed in lines marked by an asterisk. The column below each eigenvalue is the corresponding eigenvector. The eigenvectors are normalized according to Equation (16b). For  $\ell = 3, 4, 5$ , and 6 the dimensionless eigenvalues are given in Table 3. The convergence of eigenvalues is satisfactory. The same ansatz, Equation (11c), for non-degenerate fluids, however, gives much faster convergence (Sobouti, 1977b). Motivated by this observation a search for a more suitable ansatz for degenerate fluids is being made.

## VIII. CONCLUDING REMARKS:

The salient features of our model can now be summarized. The white dwarf core is taken to be an isothermal fluid, composed of ions and non-relativistic degenerate electron gas. In this formalism, corrections up to first order in  $T/T_{FC}$  are included, where  $T$  is the temperature of the core and  $T_{FC}$  is the Fermi

Table 1. The eigenvalues and the eigenvectors of the g-modes of white dwarfs, corresponding to  $\ell = 1$ . The eigenvalues (Equation 20b) are displayed in rows marked by asterisks. The column below each eigenvalue is the corresponding eigenvector. Computations are in the Rayleigh-Ritz approximation, using from one to seven variational parameters. The last two digits in the entries represent the exponents of 10.

						* 0.5184+00
						0.6600-02
					* 0.2654+00	0.6724+00
					-0.8996-02	0.1352-02
					0.3167-01	0.2471-01
					* 0.1684+00	0.3497+00
					0.1109-01	-0.2199-02
					-0.8377-01	-0.3596-01
					0.1183+00	0.1072+00
						0.7312+00
					* 0.1172+00	0.2166+00
					-0.1307-01	0.3303-02
					0.1607+00	0.4027-01
					-0.4936+00	-0.3102+00
					0.4151+00	0.3772+00
						0.2980+00
						0.7494+00
					* 0.8629-01	0.1461+00
					-0.1501-01	-0.4496-02
					0.2673+00	-0.3400-01
					-0.1340+01	0.6051+00
					0.2449+01	-0.1707+01
					-0.1452+01	0.1283+01
						0.2500+00
						0.4288+00
						0.7554+00
					* 0.6619-01	0.1043+00
					-0.1690-01	0.5793-02
					0.4082+00	0.1235-01
					-0.2958+01	-0.9435+00
					0.8779+01	0.4760+01
					-0.1124+02	-0.8043+01
					0.5136+01	0.4376+01
						0.1673+00
						0.2743+00
						0.4471+00
						0.7580+00
					* 0.5163-01	0.7871-01
					-0.1879-01	0.7072-02
					0.5878+00	-0.2476-01
					-0.5740+01	-0.1264+01
					0.2444+02	0.1042+02
					-0.5036+02	-0.2968+02
					0.4927+02	0.3544+02
					-0.1830+02	-0.1508+02
						0.1190+00
						0.1842+00
						0.2924+00
						0.4594+00
						0.7593+00
						-0.9869-02
						-0.3549-02
						-0.6866-01
						0.1195+01
						-0.6686+01
						0.1854+02
						-0.2444+02
						0.1185+02
						-0.3225+01
						0.2161+01
						0.5469-02
						-0.6653-01
						0.8952+00
						-0.5338+01
						0.1475+02
						-0.1953+02
						0.9812+01
						-0.1605-02
						-0.4936-01
						0.5724+00
						-0.3339+01
						0.9375+01
						-0.1237+02
						0.6348+01
						0.2360-02
						-0.1686-02
						0.1523+00
						-0.7685+00
						0.2229+01
						-0.2916+01
						0.1519+01

$g_7$

$g_6$

$g_5$

$g_4$

$g_3$

$g_2$

$g_1$

Table 2. The eigenvalues and the eigenvectors of the g-modes of white dwarfs, corresponding to  $\ell = 2$ . See the legend for Table. 1 for further details

						* 0.1199+01
						0.2148-01
					* 0.6630+00	0.1567+01
					-0.3548-01	-0.1502-02
					0.9286-01	0.7693-01
					* 0.4339+00	0.8694+00
					0.5113-01	-0.9241-03
					-0.2908+00	-0.1505+00
					0.3389+00	0.3096+00
						0.1733+01
						0.1076-01
						-0.4392-01
						0.1990+00
					* 0.3077+00	0.5522+00
					0.6907-01	0.5869-02
					-0.6427+00	0.2205+00
					0.1635+01	-0.1083+01
					-0.1200+01	0.1080+01
						0.1002+01
						0.1800+01
						-0.2000-01
						-0.2757-03
						0.8814-01
						-0.2668+00
						0.4554+00
					* 0.2299+00	0.3798+00
					0.8946-01	-0.1280-01
					-0.1208+01	-0.2673+00
					0.5007+01	0.2499+01
					-0.7972+01	-0.5668+01
					0.4257+01	0.3722+01
						0.6367+00
						0.1088+01
						0.1830+01
						0.8014-02
						-0.6347-01
						0.5345+00
						-0.1167+01
						0.1085+01
					* 0.1783+00	0.2765+00
					0.1124+00	0.2150-01
					-0.2053+01	0.2660+00
					0.1227+02	-0.4656+01
					-0.3160+02	0.1816+02
					0.3630+02	-0.2644+02
					-0.1523+02	0.1296+02
						0.4334+00
						0.6996+00
						0.1146+01
						0.1846+01
						0.8077-03
						-0.1678-01
						0.2506-03
						0.1108+00
						-0.7626+00
						0.2920+01
						-0.4595+01
						0.2855+01
					* 0.1408+00	0.2105+00
					0.1377+00	0.3290-01
					-0.3255+01	0.1637+00
					0.2617+02	-0.7301+01
					-0.9649+02	0.4443+02
					0.1780+03	-0.1078+03
					-0.1594+03	0.1152+03
					0.5511+02	-0.4508+02
						0.7055+01
						0.5960+01
						-0.2857+02
						0.6946+02
						-0.8137+02
						0.3571+02
						-0.5112-02
						-0.2624-01
						0.4246+00
						-0.4739+01
						0.2349+02
						-0.5700+02
						0.6736+02
						-0.3048+02
						0.3197-02
						-0.3363+00
						0.3396+01
						-0.1684+02
						0.4117+02
						-0.4870+02
						0.7393-02
						-0.9939-01
						0.1279+01
						-0.6090+01
						0.1507+02
						-0.1778+02
						0.8291+01
97	96	95	94	93	92	91

Table 3. The g-eigenvalues of white dwarfs (Equation 216), corresponding to  $\lambda=3,4,5$ , and 6. See the legend for Table 1 for further details.

$\lambda = 3$							0.2005+01
					0.1146+01	0.2645+01	
				0.7623+00	0.1496+01	0.2967+01	
			0.5477+00	0.9626+00	0.1733+01	0.3122+01	
		0.4135+00	0.6708+00	0.1109+01	0.1896+01	0.3204+01	
	0.3237+00	0.4941+00	0.7637+00	0.1221+01	0.2014+01	0.3254+01	
0.2612+00	0.3788+00	0.5557+00	0.8381+00	0.1310+01	0.2105+01	0.3289+01	
$\lambda = 4$							0.2942+01
					0.1694+01	0.3916+01	
				0.1136+01	0.2206+01	0.4444+01	
			0.8231+00	0.1427+01	0.2565+01	0.4724+01	
		0.6266+00	0.1003+01	0.1643+01	0.2823+01	0.4888+01	
	0.4947+00	0.7447+00	0.1139+01	0.1812+01	0.3017+01	0.4995+01	
0.3997+00	0.5785+00	0.8341+00	0.1250+01	0.1952+01	0.3173+01	0.5074+01	
$\lambda = 5$							0.4023+01
					0.2299+01	0.5395+01	
				0.1546+01	0.2993+01	0.6174+01	
			0.1126+01	0.1934+01	0.3492+01	0.6614+01	
		0.8620+00	0.1366+01	0.2229+01	0.3862+01	0.6887+01	
	0.6841+00	0.1021+01	0.1551+01	0.2463+01	0.4149+01	0.7074+01	
0.5445+00	0.7978+00	0.1151+01	0.1712+01	0.2669+01	0.4393+01	0.7221+01	
$\lambda = 6$							0.5260+01
					0.2960+01	0.7096+01	
				0.1986+01	0.3856+01	0.8166+01	
			0.1450+01	0.2481+01	0.4514+01	0.8798+01	
		0.1115+01	0.1755+01	0.2861+01	0.5014+01	0.9205+01	
	0.8802+00	0.1320+01	0.1998+01	0.3176+01	0.5417+01	0.9500+01	
0.2131+00	0.1089+01	0.1512+01	0.2174+01	0.3359+01	0.5615+01	0.9625+01	
$g_7$	$g_6$	$g_5$	$g_4$	$g_3$	$g_2$	$g_1$	



temperature at the center.

From Equations (21), it is seen that the eigenvalues for g-mode oscillations are proportional to the internal temperature  $T$ . We also note that the factor  $(\gamma - 1)$  appears in Equation (21a) because of the assumption of an isothermal core. If there is any decrease in the temperature outwards, this factor should be replaced by  $(\gamma - \gamma')$ , where  $\gamma'$  is defined by Equation (8b). This replacement will result in a decrease in the eigenvalues.

Again, from Equation (21b) we see that the eigenvalues are very sensitive functions of the mean molecular weight of the electrons, varying as  $\mu_e^6$ . Dependence on  $\mu_i$ , the mean molecular weight of the ions, is not so pronounced. These features should be compared with Brickhill's remark that the periods of gravity oscillations of white dwarfs do not depend critically on the composition of the stars.

The periods of g-modes are given by Equation (23), where the values of  $\epsilon^j$  are of the order of unity (see Tables 1, 2, and 3). Thus, the periods of oscillations come out to be of the order of a few hundred seconds. As the harmonic number  $\ell$  increases, the periods decrease. These conclusions, as well as the linear dependence of our eigenvalues on the temperature are in agreement with similar results obtained, through an entirely different approach, by Papaloizou and Pringle.

Let us also emphasise an important difference between the g-modes of a degenerate structure and those of an ordinary fluid. According to Cowling (1941), the pressure fluctuations associated with the g-modes of an ordinary fluid are less prominent than the corresponding density fluctuations. For degenerate fluids, the opposite appears to be the case. Equations (12) and (9a) show that the Eulerian changes in ionic pressure are mainly responsible for the g-modes. The effects of Eulerian changes in the density and in the electronic pressure remain unimportant (in the limit of non-relativistic

degeneracies). That the g-modes are primarily driven by the buoyancy forces remains valid for degenerate as well as for non-degenerate fluids.

These calculations have been carried out for non-relativistic fluids. There are strong indications that in a relativistic case, the oscillation periods will decrease.

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