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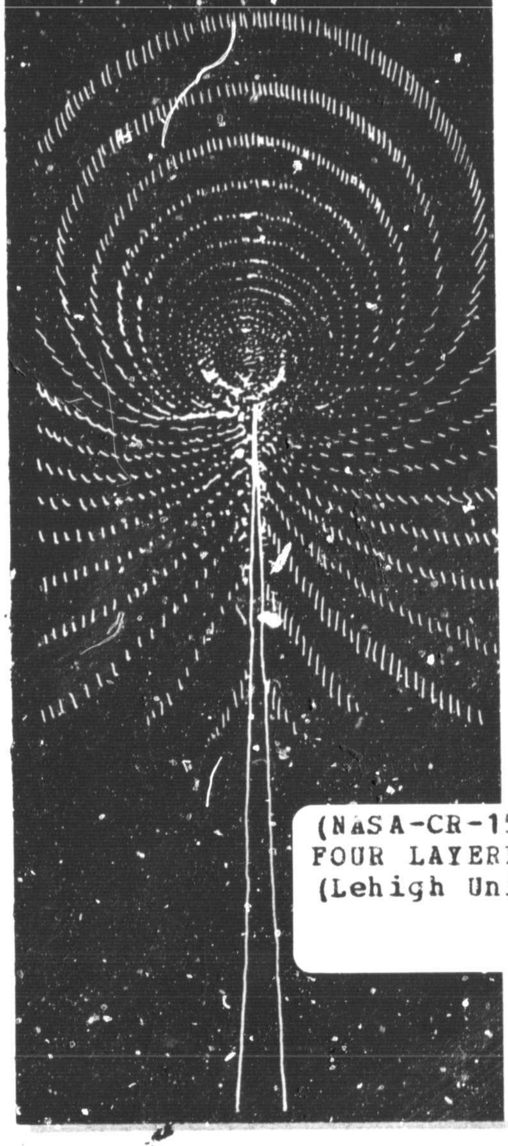

LEHIGH UNIVERSITY

SUDDEN STRETCHING OF A FOUR LAYERED-COMPOSITE PLATE

BY

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MARCH 1980
NASA CR 259870

MATERIALS AND STRUCTURES DIVISION
NASA-LEWIS RESEARCH CENTER
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(NASA-CR-159870) SUDDEN STRETCHING OF A FOUR LAYERED COMPOSITE PLATE Interim Report (Lehigh Univ.) 42 p HC A03/MF A01 CSCL 11D

N80-25383

Unclas
G3/24 23605

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR159870		

4. Title and Subtitle	5. Report Date
Sudden Stretching of a Four-Layered Composite Plate	March 1980
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
G. C. Sih and E. P. Chen	
	10. Work Unit No.
9. Performing Organization Name and Address Lehigh University Institute of Fracture and Solid Mechanics Bethlehem, PA 18015	11. Contract or Grant No. NSG3179
	13. Type of Report and Period Covered Interim Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington DC 20546	14. Sponsoring Agency Code

15. Supplementary Notes
Project Manager: C. C. Chamis
Structures and Mechanical Technologies Division
NASA-Lewis Research Center
21000 Brookpark Road, Mail Stop 49-3 Cleveland, OH 44135

16. Abstract

An approximate theory of laminated plates is developed by assuming that the extensional and thickness mode of vibration are coupled. The mixed boundary value crack problem of a four-layered composite plate is solved. Dynamic stress intensity factors for a crack subjected to suddenly applied stress are found to vary as a function of time and depend on the material properties of the laminate. Stress intensification in the region near the crack front can be reduced by having the shear modulus of the inner layers to be larger than that of the outer layers.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Elastodynamics, stress intensity, layered plates, stress analysis, cracks, computer program	Unclassified, unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	35	A03

* For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA-C-168 (Rev. 10-75)

FOREWORD

The results in this report on the sudden stretching of a four-layered composite plate were obtained during the course of research supported by the NASA-Lewis Research Center in Cleveland, Ohio, for the period February 13, 1979 through February 12, 1980 under Grant NSG 3179 with the Institute of Fracture and Solid Mechanics at Lehigh University. The Principal Investigator of the Project is Professor George C. Sih who wishes to acknowledge Dr. Christos C. Chamis, the NASA Project Manager, for the encouragement he provided during this Project.

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LIST OF SYMBOLS

a	- half crack length
$A,B,\ldots,D$	- unknowns in integrals, functions of (s,p)
Br	- Bromwich contour in complex p-plane
c_{21}	- shear wave speed for material 1
F, G	- known functions of (s,p)
h	- laminate thickness
H	- potential function
H^*	- Laplace transform of H
$H(t)$	- Heaviside unit step function
J_o	- Bessel function of order zero
$k_1(t)$	- dynamic stress intensity factor
$k_1^*(p)$	- Laplace transform of $k_1(t)$
$L(\xi,\eta,p)$	- kernel in Fredholm integral equation
N_o	- constant stress resultant
$N_x,N_y,\ldots,N_{xy}$	- stress resultants
p	- Laplace transform variable
R_x, R_y	- transverse shear forces
r, θ	- crack tip polar coordinates
s	- variable of integration
s_j	- parameter defined in equation (26) with $j = 1,2$
t	- time
u_x,v_y,w_z	- displacement components in the (x,y,z) coordinate system
v_x,v_y,v_z	- displacement functions of x and y
x,y,z	- rectangular coordinates

$\alpha_o, \gamma_o, \delta_o, \rho_o, \omega_o$	- material parameters
β, γ, δ	- material constants
Δ_j	- defined in equation (28) with $j = 1,2$
$\epsilon_x, \epsilon_y, \ldots, \gamma_{yz}$	- strain components
κ	- correction factor $\pi/\sqrt{12}$ in plate theory
μ_j	- shear moduli with $j = 1,2$
ν_j	- Poisson's ratio with $j = 1,2$
ξ, η	- variables of integration
ρ_j	- mass density for material j
λ_j, μ_j	- Lamé coefficients with $j = 1,2$
$(\Lambda_x)_j, (\Lambda_y)_j, \ldots, (\Lambda_{xy})_j$	- strain resultants
$\sigma_x, \sigma_y, \ldots, \tau_{yz}$	- stress components
ϕ^*	- potential function in Laplace transform plane
$\Phi^*(\xi, p)$	- unknown in Fredholm integral equation

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ABSTRACT

A research effort primarily concerned with the understanding of laminated composite plates with cracks subjected to time-dependent extensional loads is reported here. When loads are applied suddenly to a laminate, waves are reflected and refracted through the laminae and give rise to stresses and strains throughout the composite system. The process is three-dimensional in character and presents a formidable problem in the theory of elastodynamics, particularly in the presence of crack-like imperfections.

An approximate theory of laminated plates is developed by assuming that the extensional and thickness mode of vibration are coupled. The mixed boundary value crack problem of a four-layered composite plate is solved. Dynamic stress intensity factors for a crack subjected to suddenly applied stress are found to vary as a function of time and depend on the material properties of the laminate. Stress intensification in the region near the crack front can be reduced by having the shear modulus of the inner layers to be larger than that of the outer layers.

INTRODUCTION

The current interest in laminates for structural application is associated with the high strength-to-weight ratio which can be developed in laminates. These laminates are generally composed of layers which have been reinforced by embedding unidirectional fibers. The layers are adhered to each other such that the fiber direction varies from one layer to the next in a previously determined manner. The freedom of choice for fiber orientation in the layers of the composite system enables the development of laminates with special preferential directional properties for particular applications. Because of this characteristic of fibrous composites, the employment of these systems rather than equivalent homogeneous members will be clearly advantageous in many applications.

Because of the complicated internal structure of composite systems, stress analysis is much more difficult than for equivalent single-phase material. One fact which emerges very clearly from laminate studies is that the stress field in composite systems is truly three-dimensional in character. Thus, even the stress field in a symmetric laminate subjected to in-plane loading cannot be accurately modeled by standard two-dimensional methods of analysis. The previous work in this area further indicates that relatively little effort has been made to formulate laminate plate theories that can effectively solve for the redistribution of stresses and strains due to the presence of mechanical imperfections such as cracks.

One possible means of simplifying the three-dimensional equations of elasticity is to invoke the concept adopted in the formulation of plate theory. Approximate stress and strain dependence on the plate thickness coordinate are assumed such that the governing differential equations possess only two independent space variables. In addition, special attention must be given to the state of affairs near the crack when formulating plate theories for analyzing crack problems. With this in mind, Hartranft and Sih [1] developed an approximate three-dimensional theory for a single material plate containing a through crack. The condition of plane strain was preserved ahead of the crack as suggested by Sih [2]. This theory was later extended to laminates by Badaliance, Sih and Chen [3] to solve the problem of a through crack in a laminar plate subjected to in-plane loading. The through crack configuration represents a preliminary effort to model the damage of composite plates. Additional complications arise when the load is time dependent. These considerations will be taken into account in the development of a new dynamic theory of laminated composite plates subjected to extensional loads.

This work is concerned with the formulation of a dynamic theory of laminated plates and reduces to that of Kane and Mindlin [4] for the single material plate. The idealized condition of stress and displacement continuity across the interface is replaced by assigning certain conditions of material nonhomogeneity in the thickness direction of the laminated plate as if it were a single layered nonhomogeneous plate. The nonhomogeneity is made equivalent to a symmetric laminate balanced with reference to its mid-plane. A through crack is assumed to exist in a four-layered laminate. Dynamic stress intensity factors are obtained for the case of a suddenly applied uniform in-plane loading and shown to vary as a function of time. Discussed are also the influence of material properties of the layers on the local stresses.

BASIC FORMULATION

The elastodynamic equations of generalized plane stress are adequate only if the frequency of vibration is lower than that of the first thickness mode and the wave length is large in comparison with the plate thickness. In other words, the coupling between extensional and thickness mode of vibration can be neglected. When laminated composite plates are stressed dynamically, loads are transmitted through the laminae by the reflection of thickness refraction of stress waves. The mode of vibration cannot be ignored, particularly in the vicinity of a crack-like imperfection where the stress state acquires a three-dimensional character.

A dynamic laminate plate theory will be developed to solve the problem of a four-layered composite plate with a through crack subjected to a suddenly applied uniform extensional load. The theory is a generalization to that given by Kane and Mindlin [4] for a single layer plate in which the extensional and thickness mode of vibration are assumed to be coupled. Accounted for is the lowest thickness-stretch mode such that the displacement is normal to the plate surface. Mindlin and Medick [5] have also considered a formulation in which the thickness-shear mode of vibration with displacement parallel to the plate surface is also included. The mid-plane of the plate is taken as the nodal plane of vibration.

Consider a four-layered composite plate of thickness h as shown in Figure 1 where each layer has the same thickness h/4. The two outer layers have material properties (μ_2,ν_2) or (λ_2,μ_2) while the two inner layers have material properties (μ_1,ν_1) or (λ_1,μ_1). The Lamé coefficients are denoted by λ_j and μ_j $(j = 1,2)$. The layers are stacked such that symmetry prevails across the mid-plane of the laminate composite. The crack of width 2a cuts through the entire thickness of the laminate.

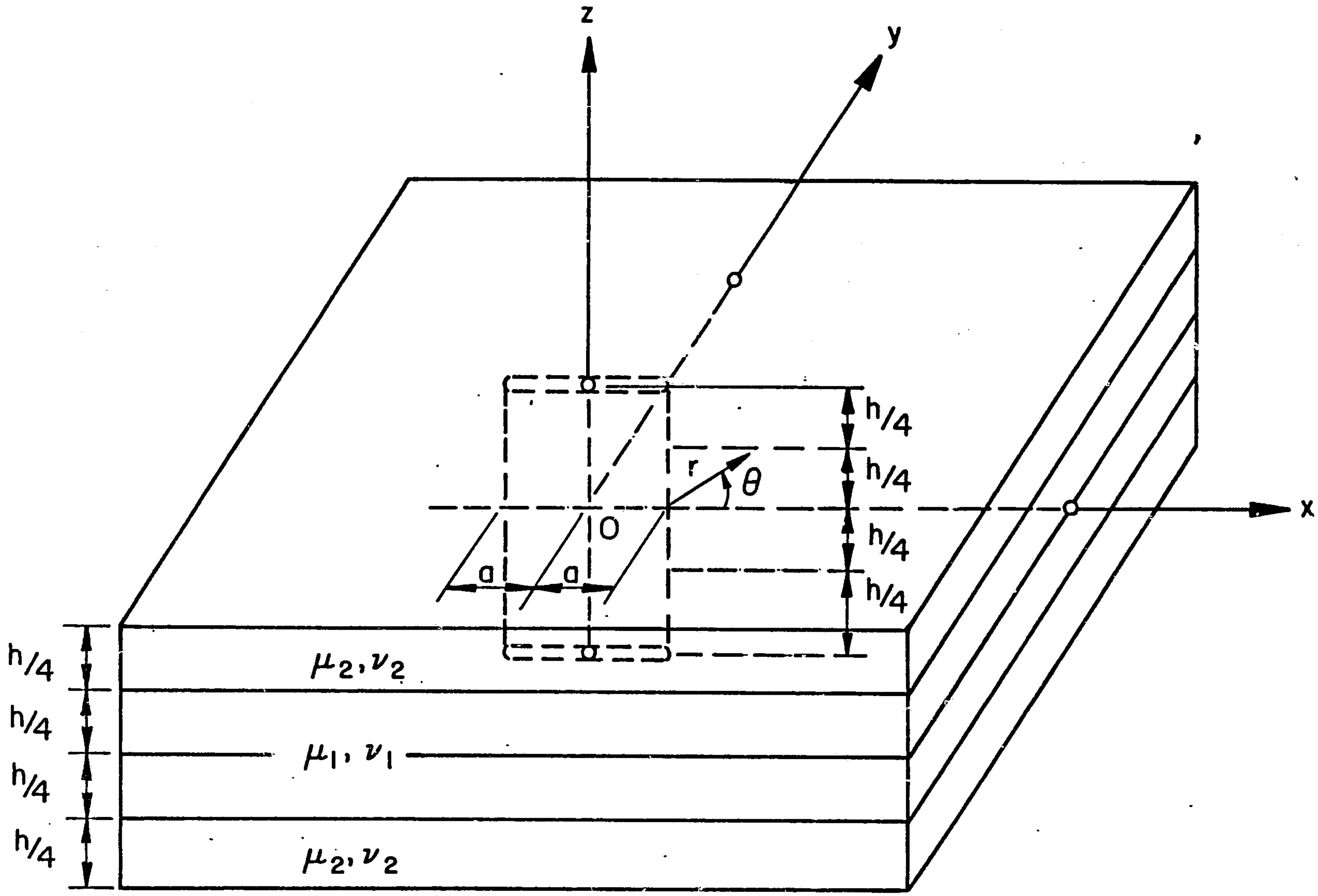

Figure 1 - A four-layered composite plate with a crack

The time-dependent displacement field is assumed to be

$$
\begin{aligned}
u_x &= v_x(x,y,t) \\
v_y &= v_y(x,y,t) \\
w_z &= \frac{2z}{h} v_z(x,y,t)
\end{aligned}
\tag{1}
$$

It follows that the strain components can be written as

$$
\begin{aligned}
\varepsilon_x(x,y,t) &= \frac{\partial v_x}{\partial x} \\
\varepsilon_y(x,y,t) &= \frac{\partial v_y}{\partial y} \\
\varepsilon_z(x,y,t) &= \frac{2}{h} v_z \\
\gamma_{xy}(x,y,t) &= \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \\
\gamma_{yz}(x,y,t) &= \frac{2z}{h} \frac{\partial v_z}{\partial y} \\
\gamma_{zx}(x,y,t) &= \frac{2z}{h} \frac{\partial v_z}{\partial x}
\end{aligned}
\tag{2}
$$

in which the transverse normal and shear strains are assumed to be linear in the thickness coordinate z. If each layer of the laminated composite plate is isotropic, then the following stress-strain relationships may be used:

$$
\begin{aligned}
\sigma_x &= (\lambda+2\mu)\varepsilon_x + \lambda(\varepsilon_y+\kappa\varepsilon_z) \\
\sigma_y &= (\lambda+2\mu)\varepsilon_y + \lambda(\varepsilon_x+\kappa\varepsilon_z) \\
\sigma_z &= (\lambda+2\mu)\kappa^2\varepsilon_z + \lambda\kappa(\varepsilon_x+\varepsilon_y)
\end{aligned}
$$

$$\tau_{yz} = \mu\gamma_{yz}$$

$$\tau_{zx} = \mu\gamma_{zx}$$

$$\tau_{xy} = \mu\gamma_{xy} \tag{3}$$

The constant

$$\kappa = \pi/\sqrt{12} \tag{4}$$

accounts for the coupling between the extensional and thickness mode of vibration. It is determined from the three-dimensional equations of elasticity. As in the development of plate theories, the resultant strain quantities $(\Lambda_x)_j$, $(\Lambda_y)_j$, ..., (Λ_{xy}) $(j = 1,2)$ will be defined:

$$[(\Lambda_x)_1, (\Lambda_y)_1, (\Lambda_z)_1, (\Lambda_{xy})_1] = \frac{2}{h}\int_{-h/4}^{h/4} [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}]dz$$

$$[(\Lambda_x)_2, (\Lambda_y)_2, (\Lambda_z)_2, (\Lambda_{xy})_2] = \frac{2}{h}\{\int_{h/4}^{h/2} [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}]dz$$

$$+ \int_{-h/2}^{-h/4} [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}]dz\} \tag{5}$$

$$[(\Lambda_{xz})_1, (\Lambda_{yz})_1] = \frac{96}{h^3}\int_{-h/4}^{h/4} [\gamma_{xz}, \gamma_{yz}]zdz$$

$$[(\Lambda_{xz})_2, (\Lambda_{yz})_2] = \frac{96}{7h^3}\int_{-h/2}^{-h/4} [\gamma_{xz}, \gamma_{yz}]zdz + \int_{h/4}^{h/2} [\gamma_{xz}, \gamma_{yz}]zdz \quad .$$

Substituting equations (2) into (5), it is found that

$$(\Lambda_x)_1 = (\Lambda_x)_2 = \frac{\partial v_x}{\partial x}$$

$$(\Lambda_y)_1 = (\Lambda_y)_2 = \frac{\partial v_x}{\partial y}$$

$$(\Lambda_z)_1 = (\Lambda_z)_2 = \frac{2}{h} v_z$$

$$(\Lambda_{xy})_1 = (\Lambda_{xy})_2 = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \tag{6}$$

$$(\Lambda_{xz})_1 = (\Lambda_{xz})_2 = \frac{2}{h} \frac{\partial v_z}{\partial x}$$

$$(\Lambda_{yz})_1 = (\Lambda_{yz})_2 = \frac{2}{h} \frac{\partial v_z}{\partial y}$$

The laminate plate theory can be most conveniently formulated in terms of the stress resultants

$$(N_x, N_y, N_z, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}) dz \tag{7}$$

and the transverse shears

$$(R_x, R_y) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) z dz \tag{8}$$

The stress-strain relations in equations (3) when enforced yield

$$N_x(x,y,t) = \frac{h}{2} [(\beta+2\gamma) \frac{\partial v_x}{\partial x} + \beta \frac{\partial v_y}{\partial y}] + \beta\kappa v_z$$

$$N_y(x,y,t) = \frac{h}{2} [(\beta+2\gamma) \frac{\partial v_y}{\partial y} + \beta \frac{\partial v_x}{\partial x}] + \beta\kappa v_z \tag{9}$$

$$N_z(x,y,t) = (\beta+2\gamma)\kappa^2 v_z + \frac{1}{2} \beta\kappa h \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)$$

$$N_{xy}(x,y,t) = \frac{1}{2} \gamma h \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

and

$$R_x(x,y,t) = \frac{1}{48}\,\delta h^2\,\frac{\partial v_z}{\partial x}$$

$$R_y(x,y,t) = \frac{1}{48}\,\delta h^2\,\frac{\partial v_z}{\partial y} \tag{10}$$

In equations (9) and (10), β, γ and δ stand for

$$\beta = \lambda_1+\lambda_2,\ \gamma = \mu_1+\mu_2,\ \delta = \mu_1+7\mu_2 \tag{11}$$

Denoting ρ_1 and ρ_2 as the mass density of the inner and outer layers of the laminate, the three equations of motion governing N_x, N_y,..., R_y become

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \frac{1}{2}\,h(\rho_1+\rho_2)\,\frac{\partial^2 v_x}{\partial t^2}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \frac{1}{2}\,h(\rho_1+\rho_2)\,\frac{\partial^2 v_y}{\partial t^2} \tag{12}$$

$$\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} - N_z = \frac{1}{48}\,h^2(\rho_1+7\rho_2)\,\frac{\partial^2 v_z}{\partial t^2}$$

The result of substituting equations (9) and (10) into (12) renders

$$\gamma\nabla^2 v_x + (\beta+\gamma)\,\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) + \frac{2\beta\kappa}{h}\,\frac{\partial v_z}{\partial x} = (\rho_1+\rho_2)\,\frac{\partial^2 v_x}{\partial t^2}$$

$$\gamma\nabla^2 v_y + (\beta+\gamma)\,\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) + \frac{2\beta\kappa}{h}\,\frac{\partial v_z}{\partial y} = (\rho_1+\rho_2)\,\frac{\partial^2 v_y}{\partial t^2} \tag{13}$$

$$\delta\nabla^2 v_z - \frac{48}{h^2}\,(\beta+2\gamma)\kappa^2 v_z - \frac{24\beta\kappa}{h}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) = (\rho_1+7\rho_2)\,\frac{\partial^2 v_z}{\partial t^2}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator in two dimensions.

METHOD OF SOLUTION

Equations (13) will be solved by introducing two potential functions $\phi(x,y,t)$ and $H(x,y,t)$ as

$$v_x(x,y,t) = \frac{\partial \phi}{\partial x} + \frac{\partial H}{\partial y}$$
$$v_y(x,y,t) = \frac{\partial \phi}{\partial y} - \frac{\partial H}{\partial x} \qquad (14)$$

Making the appropriate algebraic manipulations, the governing equations for the potential functions can be derived by enforcing equations (13):

$$\gamma\nabla^2 H = (\rho_1+\rho_2)\frac{\partial^2 H}{\partial t^2} - \delta(\beta+2\gamma)\nabla^4\phi + \frac{192\ \kappa^2\gamma(\beta+2\gamma)}{h^2}\nabla^2\phi$$
$$= \{(\rho_1+\rho_2)[(\rho_1+7\rho_2)\frac{\partial^4\phi}{\partial t^4} + \frac{48}{h^2}(\beta+2\gamma)\frac{\partial^2\phi}{\partial t^2}]$$
$$- [\delta(\rho_1+\rho_2) + (\beta+2\gamma)(\rho_1+7\rho_2)\frac{\partial^2}{\partial t^2}(\nabla^2\phi)]\} \qquad (15)$$

Once $\phi(x,y,t)$ and $H(x,y,t)$ are known, v_x and v_y can be obtained from equations (14) and

$$v_z(x,y,t) = \frac{h}{2\beta\kappa}[(\rho_1+\rho_2)\frac{\partial^2\phi}{\partial t^2} - (\beta+2\kappa)\nabla^2\phi] \qquad (16)$$

Suppose that a uniform stress resultant N_o is applied suddenly to the crack surfaces and the resulting deformation is symmetrical with respect to the x-axis, then the following conditions are to be specified:

$$N_y(x,o,t) = -N_o H(t), \; x<a$$
$$v_y(x,o,t) = 0, \; x \geq a \tag{17}$$

where H(t) is the Heaviside unit step function. The condition of symmetry further requires that

$$N_{xy}(x,o,t) = R_y(x,o,t) = 0, \text{ for all } x \tag{18}$$

Use will now be made of the Laplace transform. Let $\phi^*(x,y,p)$, $H^*(x,y,p)$, etc., denote the Laplace transforms of the functions $\phi(x,y,t)$, $H(x,y,t)$, etc. Equation (15) when expressed in the Laplace transform domain become

$$(\nabla^2-\omega_1^2)\phi_1^*(x,y,p) = 0$$
$$(\nabla^2-\omega_2^2)\phi_2^*(x,y,p) = 0 \tag{19}$$
$$(\nabla^2-\omega_3^2)H^*(x,y,p) = 0$$

where the potential $\phi(x,y,t)$ has been separated into two parts:

$$\phi(x,y,t) = \phi_1(x,y,t) + \phi_2(x,y,t) \tag{20}$$

in terms of time t or

$$\phi^*(x,y,p) = \phi_1^*(x,y,p) + \phi_2^*(x,y,p) \tag{21}$$

in terms of the Laplace transform variable p. The parameters ω_j (j = 1,2,3) in equations (19) are defined as

$$\omega_{1,2}^2 = \frac{6\kappa^2}{h\delta_0}\,[(\alpha_0+\delta_0)(\frac{p}{\omega_0})^2 + \rho_0 \pm \{[(\alpha_0+\delta_0)(\frac{p}{\omega_0})^2 + \rho_0]^2$$

$$- 4\alpha_0\delta_0\,(\frac{p}{\omega_0})^2\,[(\frac{p}{\omega_0})^2 + \rho_0]\}^{1/2}] \tag{22}$$

$$\omega_3^2 = (\rho_1+\rho_2)p^2\gamma^{-1}$$

in which the newly defined quantities are

$$\alpha_0 = \frac{\beta+2\gamma}{(\rho_1+\rho_2)\gamma_0^2},\quad \delta_0 = \frac{\delta}{(\rho_1+7\rho_2)\gamma_0^2}$$

$$\rho_0 = \frac{4(\rho_1+\rho_2)}{\rho_1+7\rho_2},\quad \omega_0^2 = \frac{12\,\kappa^2(\beta+2\gamma)}{h^2(\rho_1+\rho_2)} \tag{23}$$

and γ_0 takes the form

$$\gamma_0^2 = \frac{4\gamma(\beta+\gamma)}{(\rho_1+\rho_2)(\beta+2\gamma)} \tag{24}$$

Equations (19) then give

$$\phi_1^*(x,y,p) = \frac{2}{\pi}\int_0^\infty A(s,p)\,\cos(sx)\,\exp(-s_1y)ds$$

$$\phi_2^*(x,y,p) = \frac{2}{\pi}\int_0^\infty B(s,p)\,\cos(sx)\,\exp(-s_2y)ds \tag{25}$$

$$H^*(x,y,p) = \frac{2}{\pi}\int_0^\infty C(s,p)\,\sin(sx)\,\exp(-s_3y)ds$$

with s_j being given by

$$s_j = \sqrt{s^2+\omega_j^2},\; j = 1,2,3 \tag{26}$$

The dynamic problem has now been reduced to finding the three unknown functions $A(s,p)$, $B(s,p)$ and $C(s,p)$.

DUAL COUPLED INTEGRAL EQUATIONS

Before the boundary and symmetry conditions can be enforced, it is necessary to obtain $v_x^*(s,y,p)$, $v_y^*(x,y,p)$, etc., in terms of the unknowns in equations (25). With the help of equations (14) and (16), it can be shown that

$$v_x^*(x,y,p) = -\frac{2}{\pi}\int_0^\infty [sA(s,p)\exp(-s_1 y) + sB(s,p)\exp(-s_2 y) + s_3 C(s,p)\exp(-s_3 y)]\sin(sx)ds$$

$$v_y^*(x,y,p) = -\frac{2}{\pi}\int_0^\infty [s_1 A(s,p)\exp(-s_1 y) + s_2 B(s,p)\exp(-s_2 y) + sC(s,p)\exp(-s_3 y)]\cos(sx)ds \quad (27)$$

$$v_z^*(x,y,p) = \frac{2}{\pi}\int_0^\infty [\Delta_1 A(s,p)\exp(-s_1 y) + \Delta_2 B(s,p)\exp(-s_2 y)]\cos(sx)ds$$

The quantities Δ_j $(j = 1,2)$ are given by

$$\Delta_j = \frac{h(\beta+2\gamma)}{2\beta\kappa}\left[\frac{(\rho_1+\rho_2)p^2}{\beta+2\gamma} - \omega_j^2\right],\ j = 1,2 \quad (28)$$

Similarly, the Laplace transform of $N_x^*(x,y,p)$, $N_y^*(x,y,p)$ become

$$N_x^*(x,y,p) = \frac{2}{\pi}\gamma h\int_0^\infty \left\{\left[\frac{(\rho_1+\rho_2)p^2}{2\gamma} - s_1^2\right]A(s,p)\exp(-s_1 y) + \left[\frac{(\rho_1+\rho_2)p^2}{2\gamma} - s_2^2\right]B(s,p)\exp(-s_2 y) - ss_3 C(s,p)\exp(-s_3 y)\right\}\cos(sx)ds$$

$$N_y^*(x,y,p) = \frac{2}{\pi}\gamma h \int_0^\infty \{[s^2 + \frac{(\rho_1+\rho_2)p^2}{2\gamma}][A(s,p)\exp(-s_1y) + B(s,p)\exp(-s_2y)]$$

$$+ ss_3C(s,p)\exp(-s_3y)\}\cos(sx)ds$$

$$N_{xy}^*(x,y,p) = \frac{2}{\pi}\gamma h \int_0^\infty \{ss_1A(s,p)\exp(-s_1y) + ss_2B(s,p)\exp(-s_2y)$$

$$+ \frac{1}{2}(s^2+s_3^2)\,C(s,p)\exp(-s_3y)\}\sin(sx)ds \qquad (29)$$

while $R_x^*(x,y,p)$ and $R_y^*(x,y,p)$ take the forms

$$R_x^*(x,y,p) = -\frac{\delta h^2}{24\pi}\int_0^\infty s[\Delta_1A(s,p)\exp(-s_1y) + \Delta_2B(s,p)\exp(-s_2y)]\sin(sx)ds$$

$$R_y^*(x,y,p) = -\frac{\delta h^2}{24\pi}\int_0^\infty [s_1\Delta_1A(s,p)\exp(-s_1y) + s_2\Delta_2B(s,p)\exp(-s_2y)]\cos(sx)ds \qquad (30)$$

The symmetry conditions in equations (18) when applied show that $A(s,p)$, $B(s,p)$ and $C(s,p)$ can be expressed in terms of a single unknown $D(s,p)$:

$$A(s,p) = \frac{s^2+s_3^2}{s_1}D(s,p)$$

$$B(s,p) = -s_1\left[\frac{(\rho_1+\rho_2)p^2}{\beta+2\gamma} - \omega_1^2\right]/\left\{s_2\left[\frac{(\rho_1+\rho_2)p^2}{\beta+2\gamma} - \omega_2^2\right]\right\}A(s,p) \qquad (31)$$

$$C(s,p) = -\frac{2ss_1(\omega_1^2-\omega_2^2)}{(s^2+s_3^2)\left[\frac{(\rho_1+\rho_2)p^2}{\beta+2\gamma} - \omega_2^2\right]}A(s,p)$$

Application of the mixed boundary conditions in equations (17) leads to a system of dual integral equations

$$\int_0^\infty D(s,p)\cos(sx)ds = 0, \quad x \geq a$$

$$\int_0^\infty sF(s,p)\,D(s,p)\cos(sx)ds = -\frac{\pi N_o}{2\gamma hp}, \quad x<a \tag{32}$$

The function F(s,p) is known:

$$F(s,p) = \frac{s^2+s_3^2}{ss_1}\left[\left[s^2 + \frac{(\rho_1+\rho_2)p^2}{2\gamma}\right]\left\{1 - \frac{s_1\left[\frac{(\rho_1+\rho_2)p^2}{\beta+2\gamma} - \omega^2\right]}{s_2\left[\frac{(\rho_1+\rho_2)p^2}{\beta+2\gamma} - \omega_2^2\right]}\right\} - \frac{2s^2s_1s_3}{s^2+s_3^2}\left[\frac{\omega_1^2-\omega_2^2}{(\rho_1+\rho_2)p^2/(\beta+2\gamma)-\omega_2^2}\right]\right] \tag{33}$$

The standard procedure by Copson [6] may be applied to solve equations (32) and the result is

$$D(s,p) = -\frac{\pi N_o a^2}{2\gamma hp}\left\{\frac{[(\rho_1+\rho_2)p^2/(\beta+2\gamma)]-\omega_2^2}{\frac{(\rho_1+\rho_2)p^2}{\gamma}\,(1 - \frac{\gamma}{\beta+2\gamma})(\omega_1^2-\omega_2^2)}\right\} \times \int_0^1 \sqrt{\xi}\,\Phi^*(\xi,p)\,J_o(sa\xi)d\xi \tag{34}$$

in which $\Phi^*(\xi,p)$ can be computed from a Fredholm integral equation of the second kind:

$$\Phi^*(\xi,p) + \int_0^1 \Phi^*(\eta,p)\,L(\xi,\eta,p)d\eta = \sqrt{\xi} \tag{35}$$

The kernel $L(\xi,\eta,p)$ is

$$L(\xi,\eta,p) = \sqrt{\xi\eta}\int_0^\infty s\left[G(\tfrac{s}{a}, p) - 1\right] J_o(s\xi)\,J_o(s\eta)ds \tag{36}$$

while the function G(s,p) is related to F(s,p) in equation (33):

$$G(s,p) = \frac{\{[(\rho_1+\rho_2)p^2/(\beta+2\gamma)]-\omega_2^2\}\gamma}{p^2[1-\gamma/(\beta+2\gamma)](\omega_1^2-\omega_2^2)} F(s,p) \tag{37}$$

DYNAMIC STRESS INTENSITY FACTORS

Of interest is the intensification of the dynamic stresses ahead of the crack. Hence, the integrals in equations (29) and (30) must be evaluated for large values of s which corresponds to distances near the crack edge $x = \pm a$ and $y=0$. In terms of the polar coordinates r and θ in Figure 1, the Laplace transform of the stress resultants for small r are found:

$$\begin{aligned}
N_x^*(r,\theta,p) &= \frac{k_1^*(p)}{\sqrt{2r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + O(r^0) \\
N_y^*(r,\theta,p) &= \frac{k_1^*(p)}{\sqrt{2r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + O(r^0) \\
N_{xy}^*(r,\theta,p) &= \frac{k_1^*(p)}{\sqrt{2r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + O(r^0) \\
R_x^*(r,\theta,p) &= R_y^*(r,\theta,p) = O(r^0)
\end{aligned} \tag{38}$$

in which $k_1^*(p)$ is the Laplace transform of $k_1(t)$:

$$k^*(p) = \frac{\Phi^*(1,p)}{p} N_0\sqrt{a} \tag{39}$$

The Laplace inversion theorem may now be applied to give

$$N_x(r,\theta,t) = \frac{k_1(t)}{\sqrt{2r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + O(r^0)$$

$$N_y(r,\theta,t) = \frac{k_1(t)}{\sqrt{2r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + O(r^0) \qquad (40)$$

$$N_{xy}(r,\theta,t) = \frac{k_1(t)}{\sqrt{2r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + O(r^0)$$

$$R_x(r,\theta,t) = R_y(r,\theta,t) = O(r^0)$$

Equations (40) reveal that dynamic loading does not affect the functional relationship of r and θ. The stress intensity factor, however, is a function of time:

$$k_1(t) = \frac{N_o\sqrt{a}}{2\pi i}\int_{Br} \frac{\Phi^*(1,p)}{p}\exp(pt)dp \qquad (41)$$

where Br denotes the Bromwich path of integration. Once $\Phi^*(\xi,p)$ is calculated from equation (35) and evaluated at $\xi=1$, equation (41) may be solved numerically.

Figure 2 gives a plot of $\Phi^*(1,p)$ as a function of c_{21}/pa where $c_{21} = (\mu_1/\rho_1)^{1/2}$ is the shear wave velocity referred to the material in the inner layers. For $\rho_1 = \rho_2$, $\nu_1 = \nu_2 = 0.3$ and $\rho_1 = \rho_2$, $\Phi^*(1,p)$ is seen to increase monotonically with c_{21}/pa. Three different ratios of $\mu_2/\mu_1 = 0.2$, 1.0 and 5.0 are considered. Making use of the results in Figure 2, $k_1(t)$ in equation (41) may be computed. Refer to Figure 3 for a display of $k_1(t)/N_o\sqrt{a}$ versus $c_{21}t/a$. The resultant stress intensity factors are observed to vary as a function of time. Their amplitude rise quickly reaching a peak and then declines. The solution for a homogeneous plate corresponds to $\mu_2/\mu_1 = 1.0$ as the Poisson's ratio and mass density

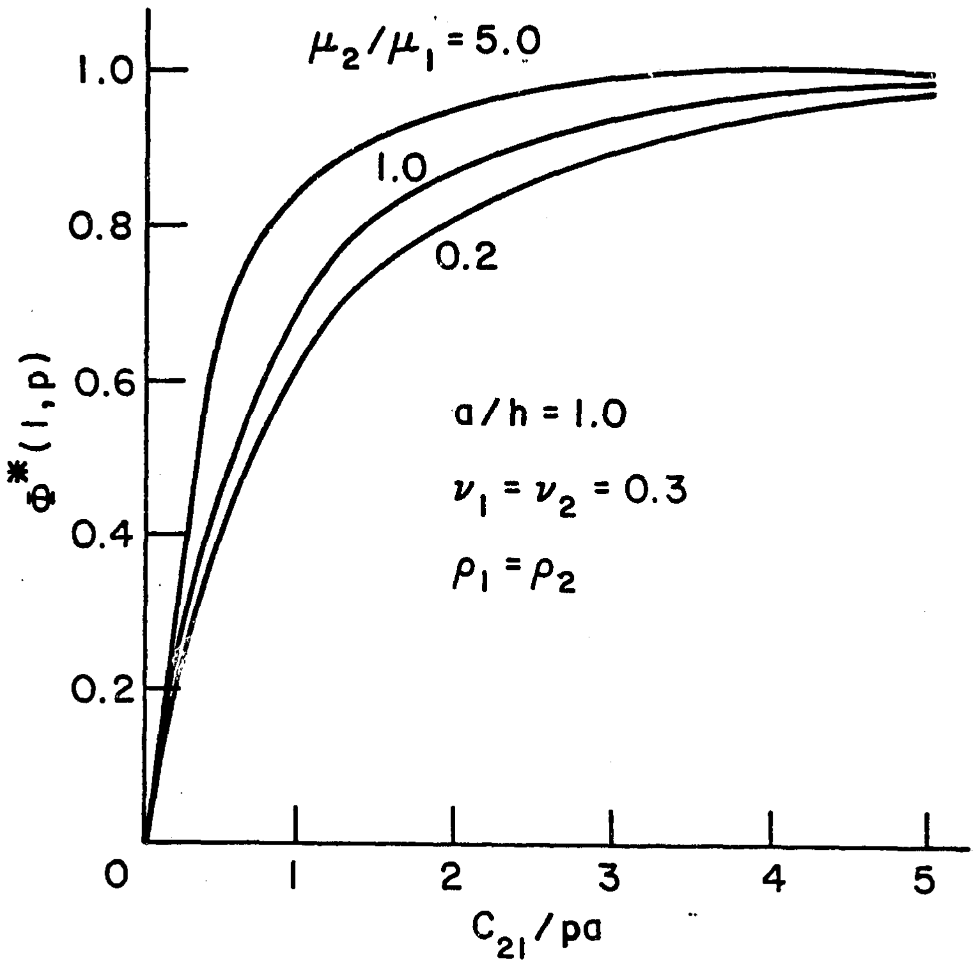

Figure 2 - Numerical values of $\phi^*(1,p)$ as a function of $c_{2\bar{1}}/pa$ for a/h = 1.0

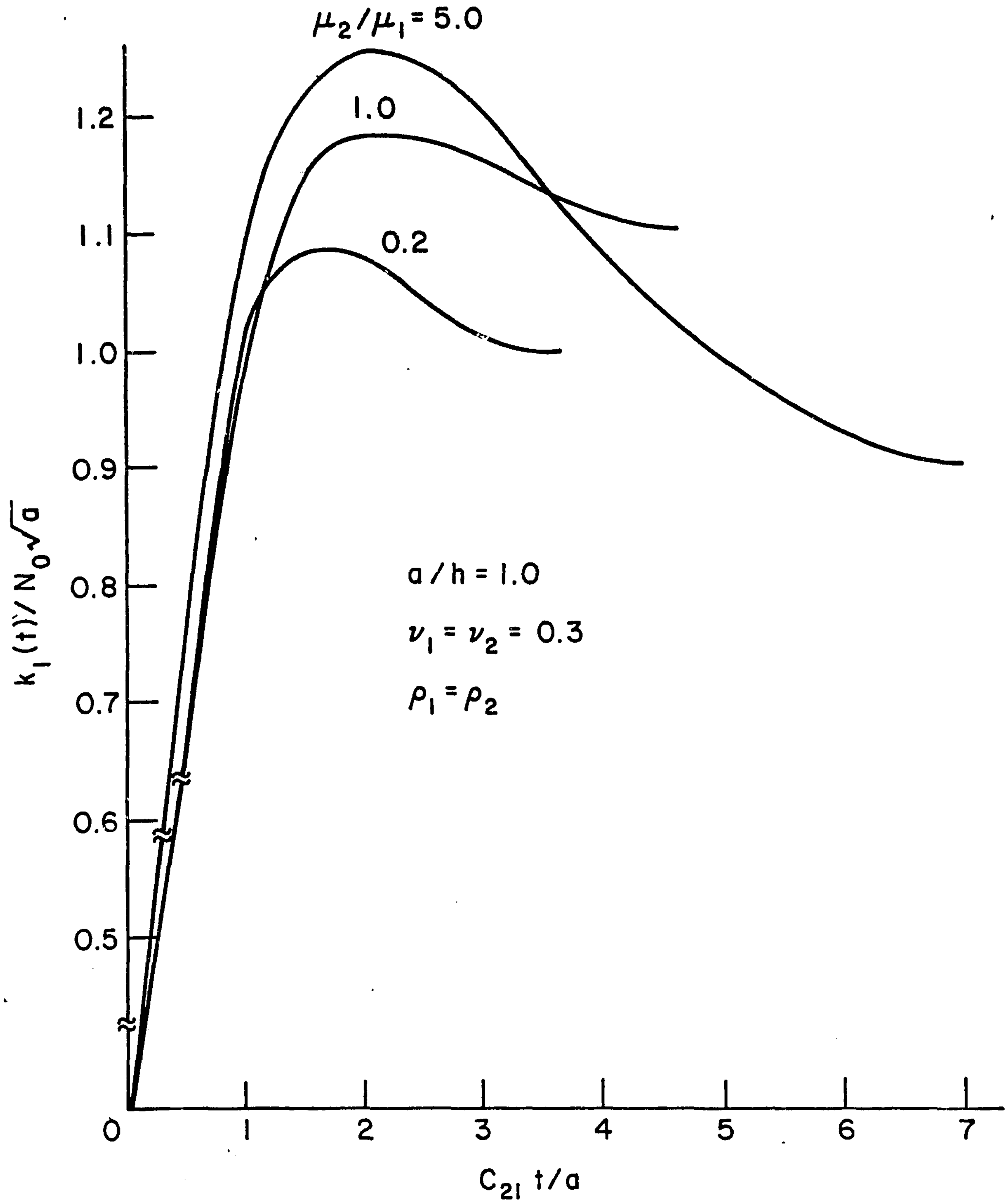

Figure 3 - Normalized resultant stress intensity factor versus $c_{21}t/a$ for a/h = 1.0

for the inner and outer layers are assumed to be equal. The peak value of $k_1(t)$ is greater than that of the homogeneous plate solution for $\mu_2>\mu_1$ while the opposite is found for $\mu_2<\mu_1$. Hence, the intensity of the crack border stress field can be reduced by having the shear modulus of the outer layers to be smaller than that of the inner layers.

CONCLUDING REMARKS

A dynamic laminate plate theory has been developed for solving crack boundary value problems. The complexity of the problem owing to material nonhomogeneity and dynamic stress analysis necessitates certain simplifying assumptions so that effective analytical solutions can be obtained. It is shown that the dynamic stresses near a mechanical imperfection such as a crack are intensified depending on the stacking sequence of the laminae. In general, this intensity tends to increase quickly for small time reaching a peak and then decreases to the static solution for sufficiently long time. When the modulus of the outer layers are smaller than that of the inner layers, the crack border stress intensity reaches a maximum quicker than the homogeneous solution but with a smaller magnitude. The opposite holds for the case when the outer layers are stiffer than the inner layers. Information of this type is useful for evaluating the resistance of laminate plates to impact loadings.

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```
        PROGRAM FLAP(INPUT.OUTPUT)
        REAL NON(4),F(4,4,2),G(4,4),D(4),PT(4)
        REAL B(4),C(4)
        REAL LP(19),DTA(19)
        EQUIVALENCE (NON,B)
         COMMON K1,K2,K3,K4
        COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
        LP(1)=0.0
        DTA(1)=0.0
        READ 2,K1,K2,K3,K4
     2  FORMAT(I2)
*    K1 = ORDER OF SYSTEM OF EQUATIONS
*    K2 = NO. OF DISTINCT KERNELS
*    K3 = NO. OF DATA POINTS
*    K4 = NO. OF DATA SETS TO BE EVALUATED
*    SET UP DATA POINTS
        AK=K3
        DO 5 N=1,K3
        AN=N
     5  PT(N)=AN/AK
*    SET UP INTEGRATION MATRIX
        M=K3-2
        N=K3-1
        A=K3
        A=1./(3.*A)
        DO 10 K=2,M,2
    10  D(K)=2.*A
        DO 15 K=1,N,2
    15  D(K)=4.*A
        D(K3)=A
*    CALCULATE NONHOMOGENEOUS TERMS
        RHS=1.0
        DO 22 I=1,K2
        PRINT 9
     9  FORMAT(1H1)
        DO 999 II=1,K4
        DO 35 N=1,K3
    35  NON(N)=RHS*SQRT(PT(N))
        CALL CONST(I)
*    CALCULATE KERNEL MATRICES
        DO 20 N=1,K3
        DO 20 M=1,K3
        F(M,N,I)=FU(I,PT(M),PT(N))
    20  CONTINUE
        CALL CHANGE(F,G,D,I)
        CALL LINEQ(G,B,C,    K3)
         DO 40 L=1,K3
        PRINT 6,PT(L),NON(L)
     6  FORMAT(5X,F8.4,F15.6)
    40  CONTINUE
        LP(II+1)=NON(K3)
        DTA(II+1)=P
   999  CONTINUE
        CALL LAPINV(DTA,LP)
    22  CONTINUE
        END
```

```
      FUNCTION SIMP(I,A,B)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      MXYZ=2**15
      DEL=0.25*(B-A)
      IF(DEL)40,45,50
   45 SIMP=0.0
      RETURN
   50 CONTINUE
      SA=Z(I,A)+Z(I,B)
      SB=Z(I,A+2.*DEL)
      SC=Z(I,A+DEL)+Z(I,A+3.*DEL)
      S1=(DEL/3.)*(SA+2.*SB+4.*SC)
      IF(S1.EQ.0.0) GO TO 45
      K=8
   35 SB=SB+SC
      DEL=0.5*DEL
      SC=Z(I,A+DEL)
      J=K-1
      DO 5 N=3,J,2
      AN=N
    5 SC=SC+Z(I,A+AN*DEL)
      S2=(DEL/3.)*(SA+2.*SB+4.*SC)
      DIF=ABS((S2-S1)/S1)
      ER=0.01
      IF(DIF-ER)30,25,25
   30 SIMP=S2
      RETURN
   25 K=2*K
      S1=S2
      IF(K-MXYZ)35,35,40
   40 PRINT 42,I,A,B
   42 FORMAT(5X,* INT. DOES NOT CONVERGE *,I3,2F9.4)
      PRINT 60,X,Y
   60 FORMAT(2F10.5)
      DO 70 J=1,10
      DIP=J
      DIP=DIP/10.
      W=Z(I,DIP)
      PRINT 60,W
   70 CONTINUE
      CALL EXIT
      END
```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
4 SIMP

VARIABLES		SN	TYPE	RELOCATION			
0	A		REAL	F.P.	260	AN	REAL
0	B		REAL	F.P.	4	BMU	REAL
250	DEL		REAL		262	DIF	REAL
264	DIP		REAL		263	ER	REAL

```
                     SUBROUTINE CHANGE(F,G,D,I)
                     COMMON K1,K2,K3,K4
                     REAL F(4,4,2),G(4,4),D(4)
                     DO 10 N=1,K3
                     DO 10 M=1,K3
                     G(M,N)   =F(M,N,I)*D(N)
                  10 CONTINUE
                     DO 20 N=1,K3
                 20  G(N,N)=G(N,N)+1.0
                     RETURN
                     END
```

```
      SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
    3  CHANGE

VARIABLES      SN   TYPE              RELOCATION
    0  D           REAL        ARRAY      F.P.            0  F           REAL
    0  G           REAL        ARRAY      F.P.            0  I           INTEGER
    0  K1          INTEGER               / /              1  K2          INTEGER
    2  K3          INTEGER               / /              3  K4          INTEGER
   53  M           INTEGER                               52  N           INTEGER

STATEMENT LABELS
    0  10                                       0   20

LOOPS  LABEL     INDEX       FROM-TO     LENGTH    PROPERTIES
   17  10      * N            4 7         17B                NOT INNER
   30  10        M            5 7          3B      INSTACK
   43  20        N            8 9          4B      INSTACK

COMMON BLOCKS   LENGTH
      / /          4

STATISTICS
  PROGRAM LENGTH                     65B      53
  SCM BLANK COMMON LENGTH             4B       4
        47000B SCM USED
```

```
      SUBROUTINE LINEQ(A,B,T,N)
      REAL A(N,N),B(N),T(N)
      DO 5 I=2,N
    5 A(I,1)=A(I,1)/A(1,1)
      DO 10 K=2,N
      M=K-1
      DO 15 I=1,N
   15 T(I)=A(I,K)
      DO 20 J=1,M
      A(J,K)=T(J)
      J1=J+1
      DO 20 I=J1,N
      T(I)=T(I)-A(I,J)*A(J,K)
   20 CONTINUE
      A(K,K)=T(K)
      IF(K.EQ.N) GO TO 10
      M=K+1
      DO 25 I=M,N
   25 A(I,K)=T(I)/A(K,K)
   10 CONTINUE
*  BACK SUBSTITUTE
      DO 31 I=1,N
      T(I)=B(I)
      M=I+1
      IF(M.GT.N) GO TO 31
      DO 30 J=M,N
      B(J)=B(J)-A(J,I)*T(I)
   30 CONTINUE
   31 CONTINUE
      DO 35 I=1,N
      K=N+1-I
      B(K)=T(K)/A(K,K)
      K1=K-1
      IF(K1.EQ.0) GO TO 35
      DO 36 J1=1,K1
      J=K-J1
      T(J)=T(J)-A(J,K)*B(K)
   36 CONTINUE
   35 CONTINUE
      RETURN
      END
```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 LINEQ

VARIABLES		SN	TYPE		RELOCATION			
0	A		REAL	ARRAY	F.P.	0	B	REAL
172	I		INTEGER			175	J	INTEGER
176	J1		INTEGER			173	K	INTEGER
177	K1		INTEGER			174	M	INTEGER
0	N		INTEGER		F.P.	0	T	REAL

```
      FUNCTION FU(I,A,B)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      X=A
      Y=B
      IF(A*B)5,10,5
   10 FU=0.0
      RETURN
    5 SUM=SIMP(I,0.0,5.0)
      ER=0.01
      DEL =5.0
   20 UP=DEL+5.0
      ADDL=SIMP(I,DEL,UP)
      DEL =UP
      TEST=ABS(ADDL/SUM)
      SUM=SUM+ADDL
      IF(TEST-ER)15,20,20
 15   FU=SQRT(X*Y)*SUM
      RETURN
      END
```

```
        SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
    4  FU

VARIABLES      SN  TYPE            RELOCATION
    0  A           REAL                  F.P.        62  ADDL      REAL
    0  B           REAL                  F.P.         4  BMU       REAL
   60  DEL         REAL                              57  ER        REAL
   55  FU          REAL                               0  H         REAL
    0  I           INTEGER               F.P.         1  P         REAL
    2  PK1         REAL                 AUX           3  PK2       REAL
   56  SUM         REAL                              63  TEST      REAL
   61  UP          REAL                               5  X         REAL
    6  Y           REAL                 AUX

EXTERNALS          TYPE    ARGS
      SIMP      REAL          3                         SQRT      REAL

INLINE FUNCTIONS   TYPE    ARGS
      ABS       REAL          1  INTRIN

STATEMENT LABELS
   14  5                                    0  10           INACTIVE
   22  20

COMMON BLOCKS   LENGTH
      AUX            7

STATISTICS
  PROGRAM LENGTH                    64B        52
  SCM LABELED COMMON LENGTH          7B         7
        47000B SCM USED
```

```
      FUNCTION BESJO(A)
      IF(A-3.)5,5,10
    5 B=A*A/9.
      W=1.-2.2499997*B
      Z=B*B
      W=W+1.2656208*Z
      Z=Z*B
      W=W-.3163866*Z
      Z=Z*B
      W=W+.0444479*Z
      Z=Z*B
      W=W-.0039444*Z
      Z=Z*B
      BESJO=W+.00021*Z
      RETURN
   10 B=3./A
      W=.79788456-.00000077*B
      V=A-.78539816-.04166397*B
      Z=B*B
      W=W-.0055274*Z
      V=V-.00003954*Z
      Z=Z*B
      W=W-.00009512*Z
      V=V+.00262573*Z
      Z=Z*B
      W=W+.00137237*Z
      V=V-.00054125*Z
      Z=Z*B
      W=W-.00072805*Z
      V=V-.00029333*Z
      Z=Z*B
      W=W+.00014476*Z
      V=V+.00013558*Z
      BESJO=W/SQRT(A)*COS(V)
      RETURN
      END
```

```
        SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
    4   BESJO

VARIABLES       SN  TYPE            RELOCATION
    0   A           REAL                 F.P.         114  B          REAL
  113   BESJO       REAL                              117  V          REAL
  115   W           REAL                              116  Z          REAL

EXTERNALS            TYPE    ARGS
        COS         REAL      1 LIBRARY                   SQRT        REAL

STATEMENT LABELS
    0   5            INACTIVE               26   10
```

```
      SUBROUTINE CONST(I)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      PK1=0.3
      PK2=0.3
      BMU=50.0
      H=1.0
      READ 2,P
    2 FORMAT(F10.5)
      HH=1./H
      PRINT 1,BMU,PK1,PK2,HH,P
    1 FORMAT(/////5X,*  MU2/MU1 =*F6.2,* NU1 =*F4.2,* NU2 =
     1A/H =*F4.2,* C21/PA =*F4.2//)
      RETURN
      END
```

```
        SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
     3  CONST

VARIABLES      SN  TYPE           RELOCATION
     4  BMU        REAL              AUX              0  H          REAL
    55  HH         REAL                               0  I          INTEGER
     1  P          REAL              AUX              2  PK1        REAL
     3  PK2        REAL              AUX              5  X          REAL
     6  Y          REAL              AUX

FILE NAMES         MODE
        INPUT      FMT               OUTPUT     FMT

STATEMENT LABELS
    37  1        FMT                         25  2        FMT

COMMON BLOCKS    LENGTH
        AUX            7

STATISTICS
  PROGRAM LENGTH                    56B       46
  SCM LABELED COMMON LENGTH          7B        7
        47000B SCM USED
```

```
      FUNCTION Z(I,S)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      COMPLEX DA,DL1,DL2,SA,SB,SC,SD
      COMPLEX GA,GB,CA,CB,CC,F,G
      PI=3.1415926
      PP=P*P
      PG=2./PP/(1.+BMU)
      AA=2.*(1.-PK1)/(1.-2.*PK1)
      AB=2.*(1.-PK2)/(1.-2.*PK2)
      PA=2./PP/(AA+BMU*AB)
      PO=2.*H*H/PI/PI/(AA+BMU*AB)/PP
      BA=(1.+BMU)/(AA+BMU*AB)
      BB=1.-BA
      BC=(1.+7.*BMU)/4./(1.+BMU)
      BD=PI*PI/2./H/H
      ALP=1./4./BA/BB
      DLP=BC/4./BB
      DD=((ALP+DLP)*PO+1.)**2-4.*ALP*DLP*PO*(PO+1.)
      G=CMPLX(DD,0.0)
      DA=CSQRT(G)
      DL1=BD/DLP*((ALP+DLP)*PO+1.+DA)
      DL2=BD/DLP*((ALP+DLP)*PO+1.-DA)
      SC=S*S+DL1
      SD=S*S+DL2
      GA=CSQRT(SC)
      GB=CSQRT(SD)
      GC=SQRT(S*S+PG)
      SA=(PA-DL2)/(DL1-DL2)
      SB=(PA-DL1)/(PA-DL2)
      CA=SA/PG/BB
      CB=2.*(S*S+PG/2.)**2/GA*(1.-GA/GB*SB)
      CC=2.*S*S*GC/SA
      F=CA*(CB-CC)
      Q=REAL(F)
      QA=AIMAG(F)
      IF(QA-0.0)5,10,5
   10 Z=(Q-S)*BESJO(S*X)*BESJO(S*Y)
      RETURN
    5 PRINT 9,P,S,F
    9 FORMAT(4F10.5)
      CALL EXIT
      END
```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
4 Z

VARIABLES		SN	TYPE	RELOCATION			
276	AA		REAL		277	AB	REAL
306	ALP		REAL		302	BA	REAL
303	BB		REAL		304	BC	REAL
305	BD		REAL		4	BMU	REAL

```
      SUBROUTINE LAPINV(GLAM,PHI)
C     THIS PROGRAM EVALUATES THE COEFFICIENTS FOR SERIES
C     OF JACOBI POLYNOMIALS WHICH REPRESENTS A LAPLACE
C     INVERSION INTEGRAL
      REAL MUL
      DIMENSION A(50),GLAM(50),PHI(50),C(4,50)
      DIMENSION BK(101),TT(101)
      COMMON/2/TI,TF,DT,MN,BK,TT
      READ 1,NN,MN,MM
    1 FORMAT(3I2)
      READ 2,TI,TF,DT
    2 FORMAT(3F10.5)
      PRINT 99
   99 FORMAT(1H1)
      CALL SPLICE(GLAM,PHI,MM,C)
      PRINT 101
  101 FORMAT(/////5X,*   GLAM             PHI    *)
      PRINT 102,(GLAM(I),PHI(I),I=1,MM)
  102 FORMAT(5X,F10.5,5X,F10.5)
      M11=MM-1
      PRINT 99
      DO 10 I=1,NN
      READ 3,BET,DEL
    3 FORMAT(2F10.5)
      PRINT 98,BET,DEL
   98 FORMAT(/////5X,*BETA =*F5.3,* DELTA =*F5.3)
      DO 11 L=1,MN
      AL=L
      S=1./(AL+BET)/DEL
      CALL SPLINE(GLAM,PHI,MM,C,S,G)
      F=G*S
      IF(AL-2.)81,82,83
   81 A(1)=(1.+BET)*DEL*F
      GO TO 11
   82 A(2)=((2.+BET)*DEL*F-A(1))*(3.+BET)
      GO TO 11
   83 CONTINUE
      TOP=1.
      L1=L-1
      AL1=L1
      DO 12 J=1,L1
      AJ=J
      TOP=AJ*TOP
   12 CONTINUE
      L2=2*L-1
      BOT=1.
      DO 13 J=L,L2
      AJ=J
      BOT=(AJ+BET)*BOT
   13 CONTINUE
      MUL=BOT/TOP
      SUM=0.0
      DO 14 N=1,L1
      AN=N
      IF(AN-2.)85,86,87
   85 TOD=1.
      GO TO 88
```

```
86 TOD=AL1
   GO TO 88
87 CONTINUE
   TOD=1.
   ICH=L1-(N-2)
   DO 15 J=ICH,L1
   AJ=J
   TOD=AJ*TOD
15 CONTINUE
88 CONTINUE
   BOD=1.
   JA=L1+N
   DO 16 J=L,JA
   AJ=J
   BOD=BOD*(AJ+BET)
16 CONTINUE
   CO=TOD/BOD
   SUM=SUM+CO*A(N)
14 CONTINUE
   A(L)=MUL*(DEL*F-SUM)
11 CONTINUE
   CALL JACSER(DEL,A,BET)
10 CONTINUE
999 CONTINUE
   RETURN
   END
```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 LAPINV

VARIABLES

SN	VARIABLE	TYPE		RELOCATION
377	A	REAL	ARRAY	
354	AL	REAL		
371	AN	REAL		
4	BK	REAL	ARRAY	2
366	BOT	REAL		
376	CO	REAL		
2	DT	REAL		2
356	G	REAL		
347	I	INTEGER		
363	J	INTEGER		
353	L	INTEGER		
365	L2	INTEGER		
3	MN	INTEGER		2
350	M11	INTEGER		
345	NN	INTEGER		
355	S	REAL		
1	TF	REAL		2
372	TOD	REAL		
151	TT	REAL	ARRAY	2
364	AJ	REAL		
362	AL1	REAL		
351	BET	REAL		
374	BOD	REAL		
461	C	REAL		
352	DEL	REAL		
357	F	REAL		
0	GLAM	REAL		
373	ICH	INTEGER		
375	JA	INTEGER		
361	L1	INTEGER		
346	MM	INTEGER		
344	MUL	REAL		
370	N	INTEGER		
0	PHI	REAL		
367	SUM	REAL		
0	TI	REAL		
360	TOP	REAL		

```
      SUBROUTINE JACSER(D,C,B)
      DIMENSION C(50),SF(50),P(50)
      DIMENSION BK(101),TT(101)
      COMMON/2/TI,TF,DT,MN,BK,TT
      TT(1)=0.0
      BK(1)=0.0
      LM=1
      T=TI
   12 T=T+DT
      X=2.*EXP(-D*T)-1.
      CALL JACOBI(MN,X,B,P)
      SF(1)=C(1)*P(1)
      DO 10 L=2,MN
      L1=L-1
      AL=L
      SF(L)=SF(L1)+C(L)*P(L)
   10 CONTINUE
      LM=LM+1
      BK(LM)=SF(5)
      TT(LM)=T
      IF(T.LE.TF) GO TO 12
      PRINT 97
   97 FORMAT(/////5X,*   T          K            T           K
     1      T         K     *)
      DO 31 MY=1,25
      MA=MY+1
      MB=MA+25
      MC=MB+25
      MD=MC+25
      PRINT 96,TT(MA),BK(MA),TT(MB),BK(MB),TT(MC),BK(MC),T1
   96 FORMAT(5X,F5.2,3X,F7.5,3X,F5.2,3X,F7.5,3X,F5.2,3X,F7.
     1F7.5)
   31 CONTINUE
      RETURN
      END
```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 JACSER

VARIABLES		SN	TYPE	RELOCATION				
151	AL		REAL			0	B	REAL
4	BK		REAL	ARRAY	2	0	C	REAL
0	D		REAL		F.P.	2	DT	REAL
147	L		INTEGER			144	LM	INTEGER
150	L1		INTEGER			153	MA	INTEGER
154	MB		INTEGER			155	MC	INTEGER
156	MD		INTEGER			3	MN	INTEGER
152	MY		INTEGER			241	P	REAL
157	SF		REAL	ARRAY		145	T	REAL
1	TF		REAL		2	0	TI	REAL
151	TT		REAL	ARRAY	2	146	X	REAL

```
      SUBROUTINE JACOBI(N,X,B,PB)
C     THIS PROGRAM CALCULATES JACOBI POLYNOMIALS OF ORDER
C     K-1 WITH ARG X AND PARAMETER B GT -1
      DIMENSION PB(N)
      AN=N
      IF(AN-2.)1,2,3
    1 PB(1)=1.
      RETURN
    2 PB(1)=1.
      PB(2)=X-B*(1.-X)/2.
      RETURN
    3 BSQ=B*B
      BONE=B+1.
      PB(1)=1.
      PB(2)=X-B*(1.-X)/2.
      DO 4 K=3,N
      AK=K
      AK1=AK-1.
      AK2=AK-2.
      K1=K-1
      K2=K-2
      CO1=((2.*AK1)+B)*X
      CO1=((2.*AK2)+B)*CO1
      CO1=((2.*AK2)+BONE)*(CO1-BSQ)
      CO2=2.*AK2*(AK2+B)*((2.*AK1)+B)
      CO=2.*AK1*(AK1+B)*((2.*AK2)+B)
    4 PB(K)=(CO1*PB(K1)-CO2*PB(K2))/CO
      RETURN
      END
```

```
      SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
    3   JACOBI

VARIABLES        SN  TYPE              RELOCATION
  106   AK           REAL                                     107  AK1        REAL
  110   AK2          REAL                                     102  AN         REAL
    0   B            REAL                       F.P.          104  BONE       REAL
  103   BSQ          REAL                                     115  CO         REAL
  113   CO1          REAL                                     114  CO2        REAL
  105   K            INTEGER                                  111  K1         INTEGER
  112   K2           INTEGER                                    0  N          INTEGER
    0   PB           REAL          ARRAY        F.P.            0  X          REAL

STATEMENT LABELS
    0   1            INACTIVE                        24   2
    0   4

LOOPS  LABEL      INDEX       FROM-TO      LENGTH      PROPERTIES
  47   4          K              16 27        25B          OPT
```

```
      SUBROUTINE SPLINE(X,Y,M,C,XINT,YINT)
      DIMENSION X(50),Y(50),C(4,50)
      IF(XINT-X(1))1,10,11
   10 YINT=Y(1)
      RETURN
   11 CONTINUE
      IF(X(M)-XINT)1,12,13
   12 YINT=Y(M)
      RETURN
   13 CONTINUE
      K=M/2
      N=M
    2 CONTINUE
      IF(X(K)-XINT)3,14,5
   14 YINT=Y(K)
      RETURN
    3 CONTINUE
      IF(XINT-X(K+1))4,15,7
   15 YINT=Y(K+1)
      RETURN
    4 CONTINUE
      YINT=(X(K+1)-XINT)*(C(1,K)*(X(K+1)-XINT)**2+C(3,K))
      YINT=YINT+(XINT-X(K))*(C(2,K)*(XINT-X(K))**2+C(4,K))
      RETURN
    5 CONTINUE
      IF(X(K-1)-XINT)6,16,17
    6 K=K-1
      GO TO 4
   16 YINT=Y(K-1)
      RETURN
   17 N=K
      K=K/2
      GO TO 2
    7 LL=K
      K=(N+K)/2
    8 CONTINUE
      IF(X(K)-XINT)3,14,18
   18 CONTINUE
      IF(X(K-1)-XINT)6,16,19
   19 N=K
      K=(LL+K)/2
      GO TO 8
    1 PRINT 101
  101 FORMAT(* OUT OF RANGE FOR INTERPOLATION  *)
      STOP
      END
```

SYMBOLIC REFERENCE MAP (R=1)

```
ENTRY POINTS
    3  SPLINE
```

```
      SUBROUTINE SPLICE(X,Y,M,C)
      DIMENSION X(50),Y(50),D(50),P(50),E(50),C(4,50)
      DIMENSION A(50,3),B(50),Z(50)
      MM=M-1
      DO 2 K=1,MM
      D(K)=X(K+1)-X(K)
      P(K)=D(K)/6.
    2 E(K)=(Y(K+1)-Y(K))/D(K)
      DO 3 K=2,MM
    3 B(K)=E(K)-E(K-1)
      A(1,2)=-1.-D(1)/D(2)
      A(1,3)=D(1)/D(2)
      A(2,3)=P(2)-P(1)*A(1,3)
      A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)
      A(2,3)=A(2,3)/A(2,2)
      B(2)=B(2)/A(2,2)
      DO 4 K=3,MM
      A(K,2)=2.*(P(K-1)+P(K))-P(K-1)*A(K-1,3)
      B(K)=B(K)-P(K-1)*B(K-1)
      A(K,3)=P(K)/A(K,2)
    4 B(K)=B(K)/A(K,2)
      Q=D(M-2)/D(M-1)
      A(M,1)=1.+Q+A(M-2,3)
      A(M,2)=-Q-A(M,1)*A(M-1,3)
      B(M)=B(M-2)-A(M,1)*B(M-1)
      Z(M)=B(M)/A(M,2)
      MN=M-2
      DO 6 I=1,MN
      K=M-I
    6 Z(K)=B(K)-A(K,3)*Z(K+1)
      Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
      DO 7 K=1,MM
      Q=1./(6.*D(K))
      C(1,K)=Z(K)*Q
      C(2,K)=Z(K+1)*Q
      C(3,K)=Y(K)/D(K)-Z(K)*P(K)
    7 C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
      RETURN
      END
```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 SPLICE

VARIABLES		SN	TYPE	RELOCATION					
373	A		REAL	ARRAY		621	B	REAL	
0	C		REAL	ARRAY	F.P.	145	D	REAL	
311	E		REAL	ARRAY		144	I	INTEGER	
141	K		INTEGER			0	M	INTEGER	
140	MM		INTEGER			143	MN	INTEGER	
227	P		REAL	ARRAY		142	Q	REAL	
0	X		REAL	ARRAY	F.P.	0	Y	REAL	