Second Moments of Optical Degradation

Due to a Thin Turbulent Layer

by

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Abstract

The effect of thin turbulent layers, including boundary layers and shear layers, on light propagation is examined from a theoretical point of view. In particular, a mathematical model is developed to describe the interaction between the aerodynamic or, more precisely, the density fluctuations and the electromagnetic field. It is assumed that the turbulence induces a normally distributed phase aberration which is a homogeneous random function in the plane of the aperture. Hypotheses concerning the density fluctuations in the layer sufficient to guarantee such a phase aberration are exhibited.

The optical degradation is described in terms of the optical transfer function (OTF) and the Strehl ratio (I/I_o) which are random. Expressions for the first and second moments of these two parameters are developed from the definitions. Asymptotic ("large" aperture) approximations to these expressions are derived and discussed. Finally, the exact and approximate results are compared for several "typical" values of the ratios of aperture diameter to scale of density fluctuations and rms phase aberration to wave length respectively.

1. Introduction

The effect of turbulent layers on the propagation of electromagnetic waves has drawn much attention over the past three decades. In 1950, Booker and Gordon [1] developed a theory which explained the scattering of radio waves in the troposphere. In 1956, Stine and Winovich [2] conducted an experimental investigation of light diffusion through turbulent boundary layers which tended to validate the applicability of the Booker-Gordon theory in this new context. Then, in 1971, Sutton [3] analyzed the Stine and Winovich data further and concluded that, for imaging apertures much larger than the turbulence scale size, scattering decreases contrast but not resolution.

While the above investigators were examining the influence of turbulence on resolution by studying the induced scattering, others looked at the influence on the optical (or modulation) transfer function. Hufnagel and Stanley [4] developed an expression for the average or expected value of the MTF. In particular, they showed that the average transfer function can be written as the product of the diffraction limited transfer function and an attenuating factor which incorporates properties of the turbulence. Fried elaborated on this theme in a series of papers [5,6,7,8] in the late 1960's. Much of the recent theoretical development connecting optical and turbulent parameters has been summarized by Wolters [9].

In the present work, we will attempt to present the fundamental theory relating turbulence statistics and mean values of the optical parameters, describing in some detail the relevant statistical assumptions. Also, we will discuss the second moments of the optical parameters when the aperture is much larger than the scale of turbulence.

2. Mathematical Model of Aero-Optic Interaction

The model described below does not originate with the author, but appears, at least implicitly, in much of the current literature on aero-optic interaction. Perhaps the most complete discussion has been given by Wolters [9]. However, because the assumptions and hypotheses aren't always explicit, and because the development is not easily accessible in the literature, it is believed that the somewhat more thorough discussion here is warranted.

We begin by assuming that the turbulent layer of thickness L induces a phase aberration but no amplitude degradation of the optical wave front (monochromatic of wave length λ). Hence the wave field u on the aperture of the receiving optics* can be described as follows (see Figure 1):

where

$$u(x,y) = \exp\left[ik\Delta(x,y)\right], \qquad (2.1)$$

$$\Delta(x,y) = \int_0^L n(x,y,z)\,dz$$

$$= \int_0^L [1+n_1(x,y,z)]\,dz$$

$$= L + \int_0^L n_1(x,y,z)\,dz$$

$$= L + \Delta_1(x,y).$$

In the above, k is the wave number $(2\pi/\lambda)$, n is the local refractive index, and n_1 is the deviation of the refractive index from its value

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*Even though we shall be discussing a passive receiving system, it should be noted that an active or propagating system (e.g., a high energy laser beam) would experience similar degradation.

in a vacuum. The expression (2.1) is the first order geometrical optics approximation, where it has been assumed that $|n_1| << 1$ [10]. The phase shift Δ is sometimes called the optical length of the ray's path through the turbulent layer. (See, for example, [11,p115].)

Without making some assumptions regarding the statistical nature of the random phase Δ_1, further progress would be difficult if not impossible. In particular, we make two assumptions:

(i) The random variable $\Delta_1(x,y)$ is normally distributed for any x and y,

and

(ii) $\Delta_1(x,y)$ is a weakly homogeneous random process; that is,

$$<\Delta_1(x,y)> = \text{constant} \qquad (2.2)$$

$$<(\Delta_1(x_1,y_1) - <\Delta_1>)(\Delta_1(x_2,y_2) - <\Delta_1>)> = R(x_2-x_1,y_2-y_1),$$

where $<\ >$ denotes ensemble average or mathematical expectation.

Assumption (i) can be justified by considering the relationship between the optical parameter Δ_1 and the air density ρ. By the Gladstone-Dale law we have

$$n_1(x,y,z) = G\,\rho(x,y,z), \qquad (2.3)$$

where $G = 0.000223$ m^3/kg is the so-called Gladstone-Dale constant. Hence, substitution of (2.3) into (2.1) yields:

$$\Delta_1(x,y) = G \int_0^L \rho(x,y,z)\,dz. \qquad (2.4)$$

Assume that $L \gg \ell_z$, where ℓ_z is the integral scale of the density fluctuations in the direction of wave propagation. It follows that we can partition the random layer into strata of thickness h, with h (and hence N) chosen so that

$$\ell_z < h = L/N \ll L \ .$$

We then have that

$$\Delta_1(x,y) = G \sum_{j=1}^N \left[\int_{(j-1)h}^{jh} \rho(x,y,z)\,dz \right]. \qquad (2.5)$$

But the random variables determined by the integrals in the sum (2.5) are approximately mutually independent (since $h > \ell_z$); then, since $N \gg 1$, the Central Limit Theorem [12, p.266] gives the desired result. It should be observed that the assumption $L \gg \ell_z$ is often satisfied; indeed, in practice,

$$10 < L/\ell_z < 40.$$

(See, for example, [13 , Tables 1 and 2].)

The first of conditions (2.2) follows from (2.4) if we assume that $< \rho(x,y,z) >$ depends solely on z and not on the aperture coordinates (x,y). The second condition, translation invariance of R, follows from the assumption that the covariance of the density

fluctuations, R_ρ, satisfies

$$R_\rho(x_1,y_1,z_1;x_2,y_2,z_2) = R_\rho(x_2-x_1,y_2-y_1; z_1,z_2) \ . \qquad (2.6)$$

Then

$$R(x_2-x_1, y_2-y_1) = G^2 \int_0^L \int_0^L R_\rho(x_2-x_1, y_2-y_1; z_1,z_2)\,dz_1\,dz_2. \qquad (2.7)$$

The assumption (2.6) is consistent with experimental evidence which suggests that the rms density and the turbulence scale length in any direction (i.e., ℓ_x, ℓ_y, or ℓ_z) vary with z [13].

In practice, it has been found that R_ρ can, with good accuracy, be represented as follows [14, Figure 20]:

$$R_\rho(x_2-x_1,y_2-y_1; z_1,z_2) = R_\rho(x_2-x_1,y_2-y_1; u,v) \qquad (2.8)$$

$$= \sigma_\rho^2(u)\exp\left\{-\sqrt{\left(\frac{x_2-x_1}{\ell_x(u)}\right)^2 +\left(\frac{y_2-y_1}{\ell_y(u)}\right)^2 +\left(\frac{v}{\ell_z(u)}\right)^2}\right\}$$

where

$$u = (z_1+z_2)/2$$

$$\qquad (2.9)$$

$$v = z_2-z_1 \ .$$

But, making the change of variables (2.9) in (2.7) we find

$$R(x_2-x_1,y_2-y_1) = G^2 \left\{ \int_0^{L/2} \int_{-2u}^{2u} R_\rho(x_2-x_1,y_2-y_1; u,v) \, dv \, du \right.$$

$$\left. + \int_{L/2}^{L} \int_{-2(L-u)}^{2(L-u)} R_\rho(x_2-x_1,y_2-y_1; u,v) \, dv \, du \right\}. \qquad (2.10)$$

The relations (2.8) and (2.10) allow calculation of R once the z variations of $\sigma_\rho^2, \ell_x, \ell_y,$ and ℓ_z have been determined. Finally, from (2.8) and (2.10) it can be shown that

$$\sigma^2 = R(0,0) \sim 2G^2 \int_0^L \sigma_\rho^2(z) \, \ell_z(z) \, dz , \qquad (2.11)$$

for $L \gg \ell_z$.

3. Second Moments of Optical Parameters

We proceed now to develop expressions for the second order statistics of the optical transfer function τ. We begin with the definition:

$$\tau(x,y) = (4/\pi) \iint_{A_{x,y}} u(\xi + \frac{x}{2}, \eta + \frac{y}{2}) \, u^*(\xi - \frac{x}{2}, \eta - \frac{y}{2}) \, d\xi d\eta, \qquad (3.1)$$

where $A_{x,y}$ is the area common to two identical apertures (of unit diameter) displaced relative to each other a distance x and y along the ξ and η axes respectively. Note that $\tau(x,y) \equiv 0$ if $x^2 + y^2 \geq 1$.[*] Clearly, since u is random, then τ is itself random. Hence, taking the expectation of (3.1) we obtain:

$$\begin{aligned}
<\tau(x,y)> &= (4/\pi) \iint_{A_{x,y}} <u(\xi + \frac{x}{2}, \eta + \frac{y}{2}) \, u^*(\xi - \frac{x}{2}, \eta - \frac{y}{2})> \, d\xi \, d\eta \\
&= (4/\pi) \iint_{A_{x,y}} < \exp\left(ik [\Delta_1(\xi + \frac{x}{2}, \eta + \frac{y}{2}) - \Delta_1(\xi - \frac{x}{2}, \eta - \frac{y}{2})] \right) > d\xi \, d\eta \\
&= (4/\pi) \iint_{A_{x,y}} \exp\left(-k^2 [R(0,0) - R(x,y)] \right) d\xi \, d\eta \\
&= \exp\left(-k^2 \sigma^2 [1 - r(x,y)] \right) \tau_0(x,y), \qquad (3.2)
\end{aligned}$$

where $\sigma^2 = R(0,0)$ is the variance and $r(x,y)$ is the correlation coefficient of the phase aberration Δ_1 and $\tau_0(x,y)$ is the diffraction

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[*]The aperture coordinates $\xi, \eta, x,$ and y have been normalized with respect to the aperture diameter, D. Then (x,y) correspond to spatial frequencies $(f_x, f_y) = (Dx/\lambda R, Dy/\lambda R)$, where R is the focal length of the optical system.

limited optical transfer function associated with the receiving

optics. We have invoked both assumptions (i) and (ii) of Section 2

in obtaining (3.2). Hence, the average optical transfer function is

just the diffraction limited optical transfer function τ_o attenuated

by the factor

$$\tau_{att}(x,y) = \exp\left(-\ (2\pi\sigma/\lambda)^2[1 - r(x,y)]\right). \tag{3.3}$$

In general, to describe completely the random process $\tau(x,y)$,

we must obtain correlations of all orders in addition to the

expectation given by (3.2). This has in fact been accomplished by

Barakat [15] . We will be concerned here however only with the second

order correlation or auto-covariance function. But by following

the procedure outlined in the development of (3.2), it is a

straightforward task to verify that the auto-covariance function for

τ is given by:

$$R_\tau(x,y,x',y') = \tau_{att}(x,y)\tau_{att}(x',y')(4/\pi)^2 \iint_{A_{x,y}} \iint_{A_{x',y'}} \left[F(u,v)-1\right] d\xi\,d\eta\,d\xi'\,d\eta', \tag{3.4}$$

where

$$u = \xi'- \xi$$

$$v = \eta'- \eta$$

and

$$F(u,v) = \exp\left(-(2\pi\sigma/\lambda)^2\left[r(u+\tfrac{x'+x}{2},\ v+\tfrac{y'+y}{2})\right.\right.$$

$$-r(u+\tfrac{x'-x}{2},\ v+\tfrac{y'-y}{2})\ -r(u-\tfrac{x'-x}{2},\ v-\tfrac{y'-y}{2})$$

$$\left.\left.+r(u-\tfrac{x'+x}{2},\ v-\tfrac{y'+y}{2})\right]\right)\ .$$

It should be observed that since the covariance function R_τ depends explicitly on x,y,x', and y' rather than on the differences $x'-x$ and $y'-y$, the optical transfer function τ is not a homogenous random process. Indeed, since we are considering only phase aberrations, $|\tau(x,y)| \leq \tau_0(x,y) \leq 1$. Hence τ is clearly not even normally distributed. A fuller discussion of this matter can be found in [15]. Before proceeding, note that the variance of $\tau(x,y)$ is given by

$$\sigma_\tau^2(x,y) = R_\tau(x,y;x,y)\ . \tag{3.5}$$

We turn now to the so-called Strehl ratio, I/I_0, the ratio of the maximum intensity in the image plane with aberrations, I, to the maximum intensity without aberrations, I_0, both in response to a point source. We will use the following definition (See, for example, [16 , p 88]) :

$$I/I_0 = \iint\limits_{-\infty}^{\infty} \tau(x,y)\ dx\ dy\ /\ \iint\limits_{-\infty}^{\infty} \tau_0(x,y)\ dx\ dy\ . \tag{3.6}$$

Now since $\tau_0(x,y)$ is defined as the normallized convolution of the unaberrated pupil function, it can easily be shown (see, for example, [17 , p 166]) that

$$\int\int_{-\infty}^{\infty} \tau_0(x,y) \; dx \; dy = \pi/4 \; .$$

Hence (3.6) can be rewritten as:

$$I/I_0 = (4/\pi) \int\int_{-\infty}^{\infty} \tau(x,y) \; dx \; dy \; . \tag{3.7}$$

It follows easily from (3.7) that the first and second moments of the Strehl ratio are given by:

$$< I/I_0 > = (4/\pi) \int\int_{-\infty}^{\infty} <\tau(x,y)> \; dx \; dy \tag{3.8}$$

and

$$\sigma^2_{I/I_0} = (4/\pi)^2 \int\int\int\int_{-\infty}^{\infty} R_\tau(x,y,x',y') \; dx \; dy \; dx' \; dy' \tag{3.9}$$

respectively. It is interesting to observe that if the fluctuations in τ at different spatial frequencies (x,y) are perfectly correlated, then $R_\tau(x,y,x',y') = \sigma_\tau(x,y)\sigma_\tau(x',y')$ and hence, from (3.9),

$$\sigma_{I/I_0} = (4/\pi) \int\int_{-\infty}^{\infty} \sigma_\tau(x,y) \; dx \; dy \tag{3.10}$$

In this case, the rms Strehl ratio is determined simply by the volume under the rms optical transfer function. However, in general, the optical transfer function does not fluctuate uniformly over its width and so (3.10) cannot be expected to give accurate results.

The equations for the moments of τ and I/I_o developed above
could in principle be used to obtain quantitative results for a
given turbulent layer by utilizing a high speed digital computer to
carry out the required integrations. This procedure can be quite time
consuming however, especially for the four dimensional integrations
of (3.4) and (3.9). However, by pursuing an asymptotic ($D \uparrow \infty$)
analysis, an approximation to these integrals can be achieved. This
approach will be pursued in Section 4 below.

Before proceeding, we should remark that it is of some interest
to consider the modulus of the optical transfer function, $|\tau|$, sometimes
referred to as the modulation transfer function or MTF. Since τ is
in general a complex-valued function, the MTF does not yield information
concerning the phase of τ . However, in recent experiments, Kelsall
has utilized the fast shearing interferometer [18] which measures the
MTF rather than the desired optical transfer function. Although the
two parameters are obviously not equivalent, there is a relationship
between the two. In particular, it is not difficult to show from
their respective definitions that:

$$< \tau > \; \le \; < MTF > \; \le \; \tau_o \tag{3.11}$$

and

$$\sigma^2_{MTF} \; \le \; \sigma^2_\tau \; \le \; \tau^2_o (1 - \tau^2_{att}) \; < \; \tau^2_o \; . \tag{3.12}$$

Clearly, from (3.11) and (3.12), the MTF suffers less degradation
than does the optical transfer function τ . Note that even though,
from (3.2), $< \tau >$ is real, this does not necessarily imply that
$< \tau > \; = \; < MTF >$.

4. Asymptotic (Large Aperture) Approximations

The equations developed in Section 3 allow one, in principle, to calculate the first and second moments of the optical transfer function, τ, and the Strehl ratio, I/I_0. Indeed the expression (3.2) for $<\tau>$ is explicit and needs no further comment. However, the calculation of σ_τ^2, R_τ, $<I/I_0>$, and σ_{I/I_0}^2 require multiple numerical integrations which can consume large amounts of computer time. Furthermore, the exact expressions tend to conceal the influence of variables like rms aberration, σ, and aperture size, D, on the parameter in question. Hence, in this section, we will develop approximations for the moment expressions derived in Section 3 for the case $D >> \ell_x$.

Consider first the average Strehl ratio; by (3.2) and (3.8), we have:

$$< I/I_0 > = (4/\pi) \int\!\!\int_{-\infty}^{\infty} \exp\left(-k^2\sigma^2[1-r(x,y)]\right) \tau_0(x,y)\,dx\,dy$$

$$= (4/\pi)\,\exp(-k^2\sigma^2) \int\!\!\int_{-\infty}^{\infty} \exp\left(k^2\sigma^2 r(x,y)\right) \tau_0(x,y)\,dx\,dy \quad (4.1)$$

Now expand the exponential part of the integrand in a Taylor series,

$$\exp\left(k^2\sigma^2 r(x,y)\right) = \sum_{p=0}^{\infty} \frac{(k^2\sigma^2)^p}{p!}\, r^p(x,y) \ .$$

Then, substitution into (4.1) yields:

$$\langle I/I_o \rangle = \exp(-k^2\sigma^2)\left\{1 + \frac{4}{\pi}\sum_{p=1}^{\infty}\frac{(k^2\sigma^2)^p}{p!}\int\int_{-\infty}^{\infty} r^p(x,y)\,\tau_o(x,y)\,dx\,dy\right\}. \tag{4.2}$$

The expression (4.2) is exact; no approximations have yet been made.

Now let's assume that r is of the form*

$$r(x,y) = \exp\left[-\sqrt{(D/\ell_x)^2 x^2 + (D/\ell_y)^2 y^2}\right]. \tag{4.3}$$

We will consider the limit $D/\ell_x \uparrow \infty$ while ℓ_x/ℓ_y remains constant. Then (4.3) can be rewritten

$$r(x,y) = \exp\left[-(D/\ell_x)\sqrt{x^2 + (\ell_x/\ell_y)^2 y^2}\right]. \tag{4.4}$$

Clearly, as D/ℓ_x grows larger, the graph of $r(x,y)$ becomes narrower. In fact, it is an easy matter to verify (keeping in mind that the volume under the delta function $\delta(x,y)$ is unity) that

$$r^p(x,y) \sim \frac{2\pi}{p^2(D/\ell_x)(D/\ell_y)}\,\delta(x,y) \tag{4.5}$$

as $D/\ell_x \uparrow \infty$, for any $p > 0$. The details of derivation of (4.5) can be found in the appendix.

Substitution of (4.5) into (4.2) gives

$$\langle I/I_o \rangle \sim \exp(-k^2\sigma^2)\left\{1 + \frac{4}{\pi}\sum_{p=1}^{\infty}\frac{(k^2\sigma^2)^p}{p!}\frac{2\pi}{p^2(D/\ell_x)(D/\ell_y)}\int\int_{-\infty}^{\infty}\delta(x,y)\,\tau_o(x,y)\right\}$$

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*This requires either that the scale lengths ℓ_x and ℓ_y be constant or that $\ell_x(u)$ and $\ell_y(u)$ be replaced by average values in (2.8).

$$= \exp{(-k^2\sigma^2)} \left\{ 1 + \frac{8}{(D/\ell_x)(D/\ell_y)} \sum_{p=1}^{\infty} \frac{(k^2\sigma^2)^p}{p^2 p!} \right\} \tag{4.6}$$

as $D/\ell_x \uparrow \infty$. Since the sum occurring in (4.6) will appear again, let us define

$$F(x) = \sum_{p=1}^{\infty} \frac{x^p}{p^2 p!} . \tag{4.7}$$

(Note that this sum is convergent for all x.) Then, from (4.6), we have

$$< I/I_o > \sim \exp{(-k^2\sigma^2)} \left\{ 1 + \frac{8}{(D/\ell_x)(D/\ell_y)} F(k^2\sigma^2) \right\} \tag{4.8}$$

as $D/\ell_x \uparrow \infty$. (For reference, the function F has been graphed over an interval of x sufficient for most conceivable aberrations; see Figure 2.) It might be observed that the second term of (4.8) can be viewed as the contribution from what Hogge and his colleagues called the incoherent beam [19]. They concluded that for a phase-aberrated beam, the far-field irradiance distribution can be written as the sum of two beams; one beam is the diffraction limited beam attenuated by the factor $\exp{(-k^2\sigma^2)}$ and the other beam is much wider and contributes an amount, on-axis, proportional to the second term of (4.8).

We turn our attention now to the variance of the optical transfer function. From (3.4) and (3.5), we have

$$\sigma_\tau^2(x,y) = \tau_{att}^2(x,y)(4/\pi)^2 \iint\limits_{A_{x,y}} \iint\limits_{A_{x,y}} \left(\exp\left[-k^2\sigma^2 \, g(\xi,\eta,\xi',\eta')\right] - 1 \right) d\xi \, d\eta \, d\xi' \, d\eta',$$

$$\tag{4.9}$$

where

$$g(\xi, \eta, \xi', \eta') = r(u+x, v+y) - 2r(u,v) + r(u-x, v-y)$$

$$u = \xi' - \xi$$

$$v = \eta' - \eta \quad .$$

Following the procedure adopted in the analysis of $<I/I_o>$, we expand
the exponential in the integrand in a Taylor series and (4.9) becomes:

$$\sigma_\tau^2(x,y) = \tau_{att}^2(x,y) (4/\pi)^2 \sum_{p=1}^{\infty} \frac{(-1)^p (k^2 \sigma^2)^p}{p!} \iint_{A_{x,y}} \iint_{A_{x,y}} g^p(\xi, \eta, \xi', \eta') \, d\xi \, d\eta \, d\xi' \, d\eta'.$$

$$(4.10)$$

The expression (4.10) is exact but not very useful as it stands. It
remains to estimate the 4-dimensional integral in the case
$D \gg \ell_x$.

Again we assume that the correlation function r is of the form
(4.3) and note that it approximates a delta function as $D/\ell_x \uparrow \infty$.
Specifically, we have the asymptotic approximation given by (4.5).
Hence, each of the three terms in g (defined by (4.9) approach a
delta function in shape. But, if $(x,y) \neq (0,0)$, each delta function
is centered at a different point in the u-v plane. In this case,
cross terms in the product g^p can be neglected and we have

$$g^p(\xi, \eta, \xi', \eta') \approx r^p(u+x, v+y) + (-1)^p 2^p r^p(u,v) + r^p(u-x, v-y)$$

$$\sim \frac{2\pi}{p^2 (D/\ell_x)(D/\ell_y)} \left[\delta(u+x,v+y) + (-1)^p 2^p \, \delta(u,v) + \delta(u-x,v-y) \right],$$

$$(4.11)$$

as $D/\ell_x \uparrow \infty$. Note that this approximation is least accurate near $(x,y) = (0,0)$. Substitution of (4.11) into (4.10) gives

$$\sigma_\tau^2(x,y) \sim \tau_{att}^2(x,y) \left(\frac{4}{\pi}\right)^2 \frac{2\pi}{(D/\ell_x)(D/\ell_y)} \sum_{p=1}^{\infty} \left\{ \frac{(-1)^p (k^2 \sigma^2)^p}{p^2 p!} \right.$$

$$\iiint\limits_{A_{x,y}} \iint\limits_{A_{x,y}} \left[\delta(u+x,v+y) + (-1)^p 2^p \, \delta(u,v) + \delta(u-x,v-y) \right] d\xi d\eta d\xi' d\eta' \left. \vphantom{\sum} \right\} . \quad (4.12)$$

It remains to evaluate the integral.

We proceed now to analyze the first integral of (4.12). The remaining two will follow directly. We have

$$\iiint\limits_{A_{x,y}} \iint\limits_{A_{x,y}} \delta(u+x,v+y) \, d\xi d\eta d\xi' d\eta'$$

$$= \iiint\limits_{-\infty}^{\infty} G_o\left(\xi+\tfrac{x}{2},\eta+\tfrac{y}{2}\right) G_o\left(\xi-\tfrac{x}{2},\eta-\tfrac{y}{2}\right) G_o\left(\xi'+\tfrac{x}{2},\eta'+\tfrac{y}{2}\right) G_o\left(\xi'-\tfrac{x}{2},\eta'-\tfrac{y}{2}\right)$$
$$\delta(u+x,v+y) \, d\xi d\eta d\xi' d\eta'$$

$$= \iint\limits_{-\infty}^{\infty} G_o\left(\xi+\tfrac{x}{2},\eta+\tfrac{y}{2}\right) G_o\left(\xi-\tfrac{x}{2},\eta-\tfrac{y}{2}\right) G_o\left(\xi-\tfrac{3x}{2},\eta-\tfrac{3y}{2}\right) d\xi d\eta \quad , \quad (4.13)$$

where G_o is the unaberrated pupil function; i.e.

$$G_o(x,y) = \begin{cases} 1, & \sqrt{x^2+y^2} \le 1/2 \\ \\ 0, & \sqrt{x^2+y^2} > 1/2 \end{cases}.$$

From Figure 3, it can be seen that (4.13) is just the area of intersection of three circles of unit diameter. In fact, the middle circle contributes nothing, and we can rewrite (4.13) as follows:

$$\iint_{A_{x,y}} \iint_{A_{x,y}} \delta(u+x,\ v+y)\,d\xi\,d\eta\,d\xi'\,d\eta' = \iint_{-\infty}^{\infty} G_o\left(\xi+\frac{x}{2},\eta+\frac{y}{2}\right) G_o\left(\xi-\frac{3x}{2},\eta-\frac{3y}{2}\right)\,d\xi\,d\eta$$

$$= \frac{\pi}{4}\,\tau_o(2x,2y) \ . \tag{4.14}$$

The remaining integrals in (4.12) can be evaluated similarly and we find:

$$\sigma_\tau^2(x,y) \sim \frac{8}{(D/\ell_x)\,(D/\ell_y)}\,\tau_{att}^2(x,y)\left[F(2k^2\sigma^2)\,\tau_o(x,y)+2F(-k^2\sigma^2)\,\tau_o(2x,2y)\right], \tag{4.15}$$

as $D/\ell_x \uparrow \infty$, where F is given by (4.7).

One could conceivably carry out analogous arguments to estimate R_τ and hence σ^2_{I/I_o} . However this was not attempted.

5. Conclusions

In order to judge the accuracy of the asymptotic approximations derived in Section 4, those expressions and the corresponding exact results from Section 3 have been calculated for several values of phase aberration σ/λ and aperture size D/ℓ_x. These calculations were carried out on a Burroughs 6800 digital computer.

Figure 4 illustrates the variation of average Strehl ratio with D/ℓ_x for three phase aberrations. The exact Strehl ratio was calculated from (3.8) and (3.2) using, successively, Simpson's rule and the four-point Gaussian quadrature formula to evaluate the double integral. Note that if $D \ll \ell_x$, the optics are essentially insensitive to the turbulent layer. Then, as D/ℓ_x grows larger, the average Strehl ratio decreases to an asymptote determined by the aberration σ/λ. It is clear from Figure 4 and (4.8) that the so-called "infinite aperture" (or zeroth order) approximation

$$< I/I_o > \sim \exp(-k^2\sigma^2)$$

is reasonably accurate for $D > 6\ell_x$. The first order correction (given by the second term of (4.8)) varies from about 3% for $\sigma/\lambda = 0.08$ to 22% for $\sigma/\lambda = 0.2$ when $D = 6\ell_x$.

It is appropriate here to relate the expression for $< I/I_o >$ given by (4.8) to the work of Hogge, Butts, and Burlakoff [19]. In obtaining (4.8) from (4.1), the exponential term $\exp[k^2\sigma^2 \, r(x,y)]$ was expanded in a power series. In [19], only the first two terms of this series were retained, thus limiting the validity of those results

to σ/λ no larger than about 0.1. No such limitation applies to the results derived here. On the other hand, Hogge, et al, reached important conclusions regarding the shape of the focal plane irradiance distribution (discussed in Section 4), whereas the spatial distribution of irradiance has not been considered here.

Figures 5, 6, and 7 illustrate the correspondence between the exact (Equation (3.5)) and the approximate (Equation (4.15)) values of σ_τ for successively larger D/ℓ_x (3,6, and 10) with $\sigma/\lambda = 0.2$. The exact expression (3.5) was evaluated using a Monte Carlo technique to approximate the four-dimensional integration. As expected, agreement is best for $D/\ell_x = 10$. In fact, for this case, the accuracy of the Monte Carlo method employed to evaluate (3.5) is questionable and, hence, given the inherent computational errors, the exact and approximate values of σ_τ can be said to agree. It is clear, especially from Figure 5, that

$$\sigma_\tau \text{ (approx.) } \uparrow \infty \text{ as } x/D \downarrow 0.$$

This anomaly is the result of the assumption that (x,y) is not near $(0,0)$ which leads to (4.11).

The work described here suggests further research toward understanding the second-order statistics of the optical parameters I/I_o and τ, namely:

1. It is desirable to obtain approximations for R_τ and σ_{I/I_o}. Since the exact expression for these parameters involve complicated four-dimensional integrations, an approximate (closed form) expression would be especially useful. An

analysis similar to that described in this paper might prove fruitful.

2. Perhaps by obtaining higher order asymptotic ($D/\ell_x \uparrow \infty$) approximations to $< I/I_o >$ and σ_τ, better accuracy can be achieved for smaller values of D/ℓ_x. This would require a more accurate description of the aberration correlation $r(x,y)$ than the δ-function analysis provides.

3. Finally, the experimental data tends to substantiate the mathematical model employed here. For example, see Reference 13 for a discussion of the aperture scaling inferred by Figure 4. Also, a comparison can be made between the data and the expression for $< \tau >$ given by (3.2). However, there has been very limited effort expended to compute σ_{I/I_o} from measured data and virtually no attempt to compute σ_τ. Since these calculations can be accomplished routinely by modifying existing data reduction codes, it is strongly urged that this information be provided in future reports of experimental data.

The defining properties of the two-dimensional delta function, $\delta(x-x_o, y-y_o)$, are as follows:

$$\int\!\!\!\int_{-\infty}^{\infty} f(x,y)\, \delta(x-x_o, y-y_o)\, dx\, dy = f(x_o,y_o)$$

$$\delta(x-x_o, y-y_o) = 0, \quad (x,y) \neq (x_o,y_o).$$

Although no ordinary function can satisfy these requirements, it is possible to construct sequences of functions, $S_k(x-x_o, y-y_o)$, which approach the symbolic "function" $\delta(x-x_o, y-y_o)$ as $k \uparrow \infty$.
In particular, if

$$\int\!\!\!\int_{-\infty}^{\infty} S_k(x-x_o, y-y_o)\, dx\, dy = 1,$$

for every k, and

$$\lim_{k\uparrow\infty} S_k(x-x_o, y-y_o) = 0, \quad (x,y) \neq (x_o,y_o),$$

then we say that

$$\lim_{k\uparrow\infty} S_k(x-x_o, y-y_o) = \delta(x-x_o, y-y_o).$$

Now, consider the function

$$r^p(x,y) = \exp\left(-p\sqrt{(D/\ell_x)^2 x^2 + (D/\ell_y)^2 y^2} \right).$$

Then, after transforming from rectangular coordinates (x,y) to elliptic coordinates (r, θ), where

$$r^2 = (D/\ell_x)^2 x^2 + (D/\ell_y)^2 y^2,$$

we have

$$\int\!\!\!\int_{-\infty}^{\infty} r^p(x,y) \; dx \; dy = \int_0^{2\pi} \frac{d\theta}{(D/\ell_x)^2 \cos^2\theta + (D/\ell_y)^2 \sin^2\theta} \int_0^{\infty} e^{-pr} \; r \; dr$$

$$= \frac{2\pi}{p^2 (D/\ell_x)(D/\ell_y)} \quad .$$

Then, it follows that

$$\lim_{(D/\ell_x) \uparrow \infty} \frac{p^2 (D/\ell_x)(D/\ell_y)}{2\pi} \; r^p(x,y) = \delta(x,y) \quad .$$

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u(x,y) = 1   (PLANE WAVE)

z = 0

$\Delta(x,y)$ = constant

$u(x,y) = \exp\left[ik\ \Delta(x,y)\right]$

z = L
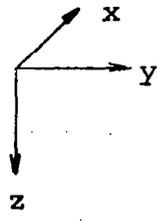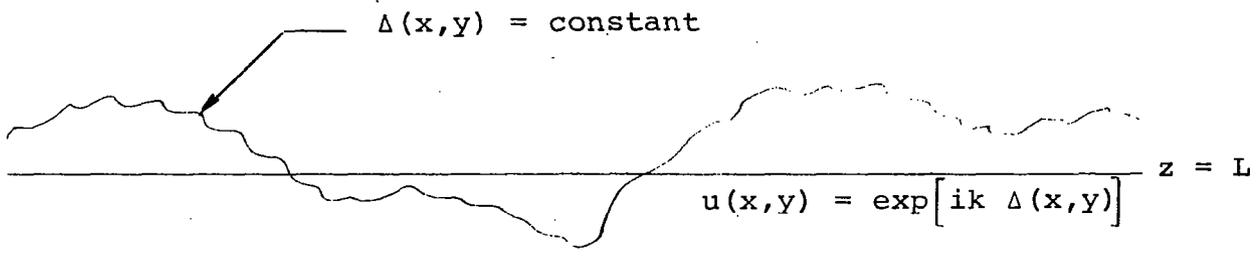
Figure 1. Phase-aberrated plane wave.

279
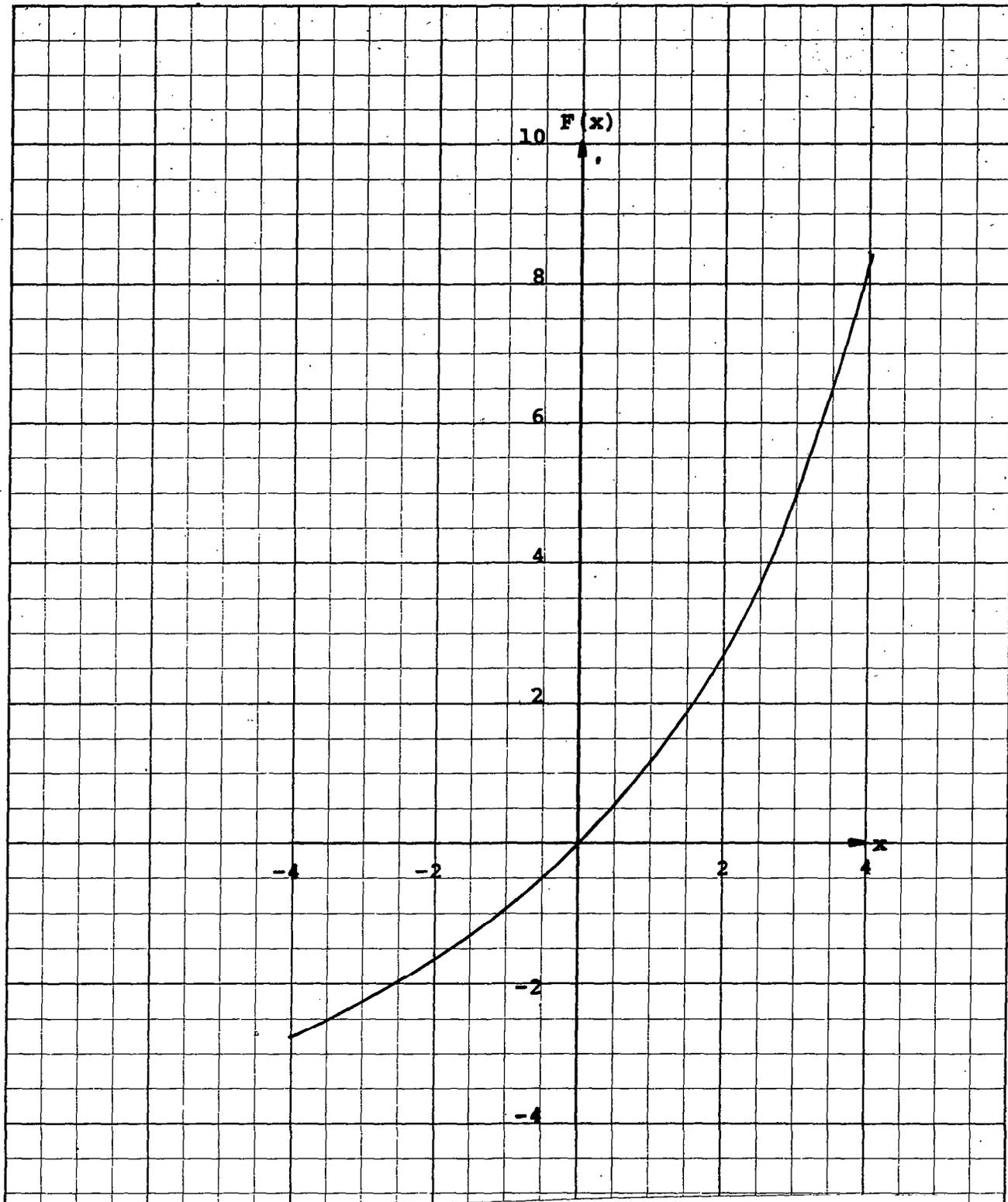
Figure 2. Graph of $F(x) = \displaystyle\sum_{p=1}^{\infty} \frac{x^p}{p^2 p!}$.

280
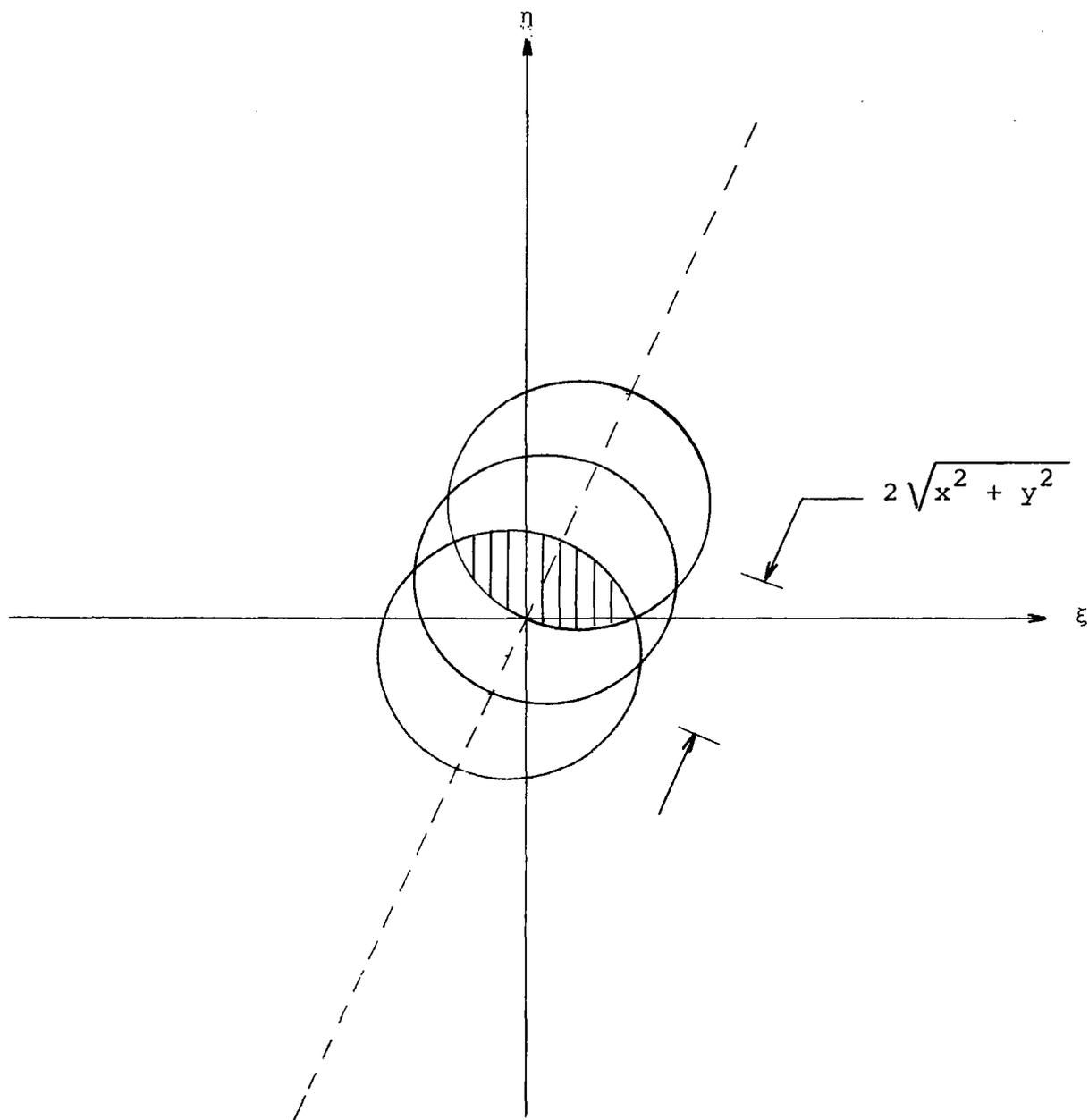
Figure 3. $\displaystyle\iint\limits_{-\infty}^{\infty} G_0(\xi+\tfrac{x}{2}, \eta+\tfrac{y}{2})\ G_0(\xi-\tfrac{x}{2}, \eta-\tfrac{y}{2})\ G_0(\xi-\tfrac{3x}{2}, \eta-\tfrac{3y}{2})\ d\xi\ d\eta$.
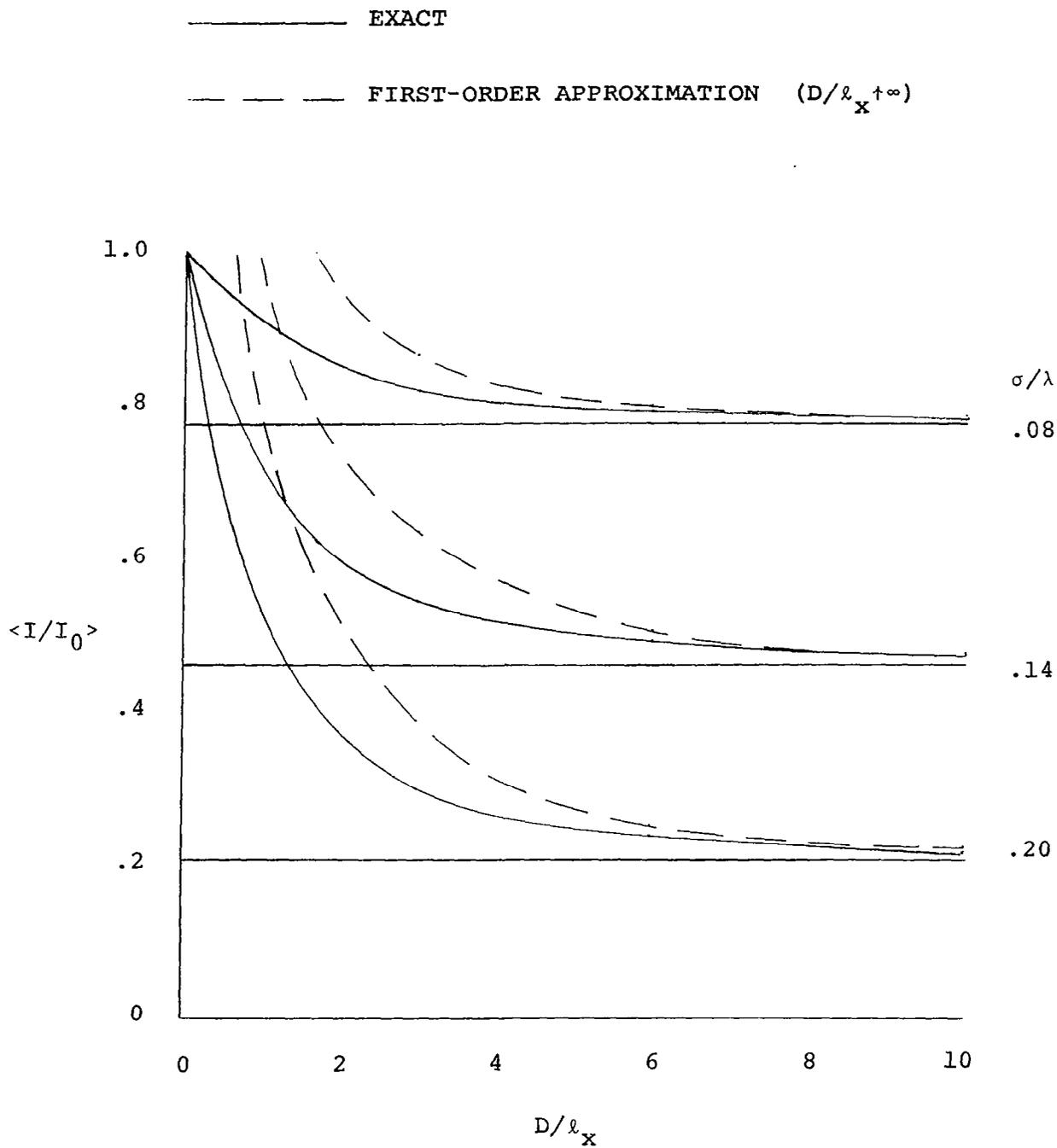
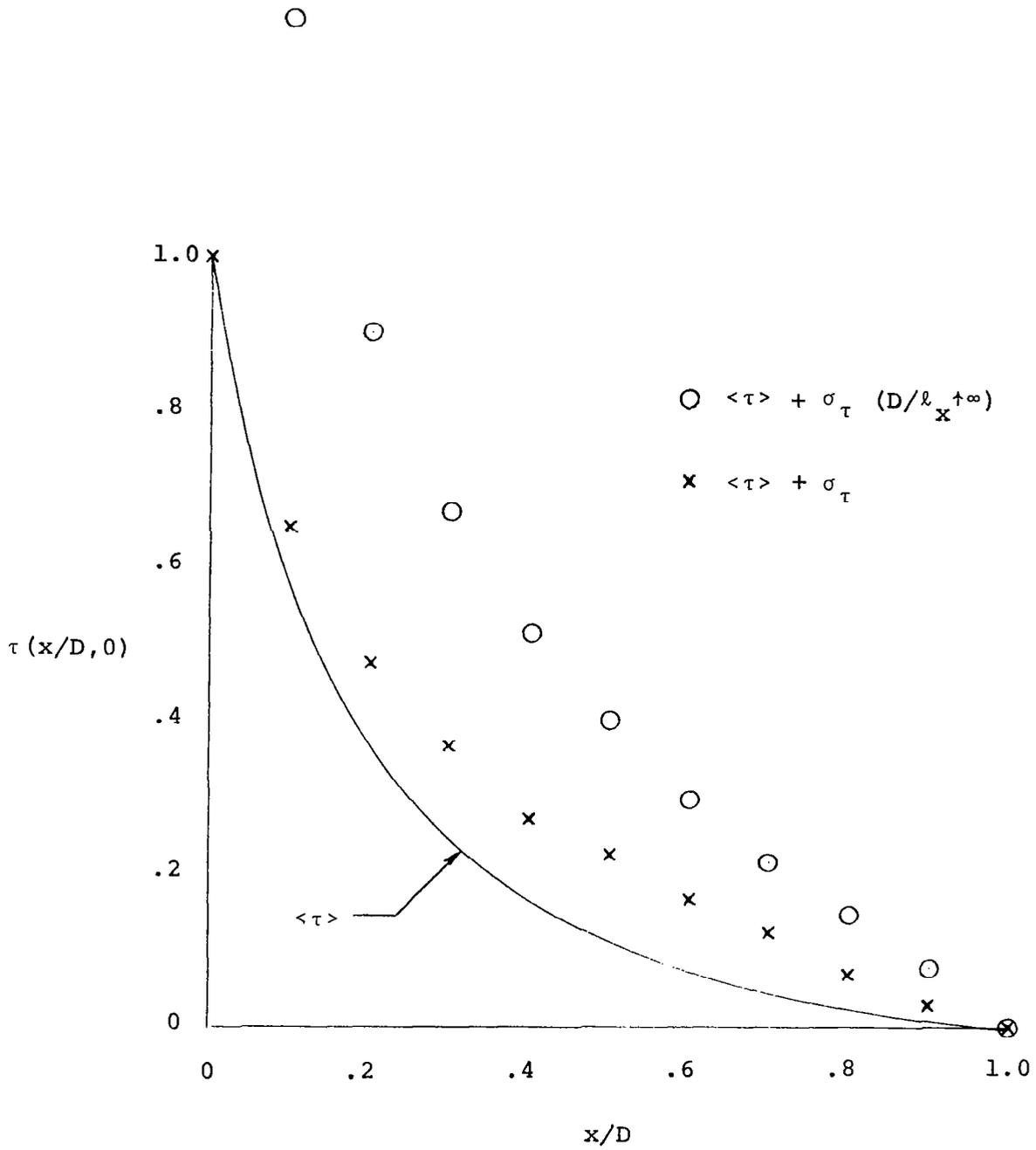Figure 4. Variation of average Strehl ratio with aperture size $(\ell_x/\ell_y = 2)$.

Figure 5. Second-order statistics of the optical transfer function for $D/\ell_x = 3$ ($\ell_x/\ell_y = 2$, $\sigma/\lambda = .2$).
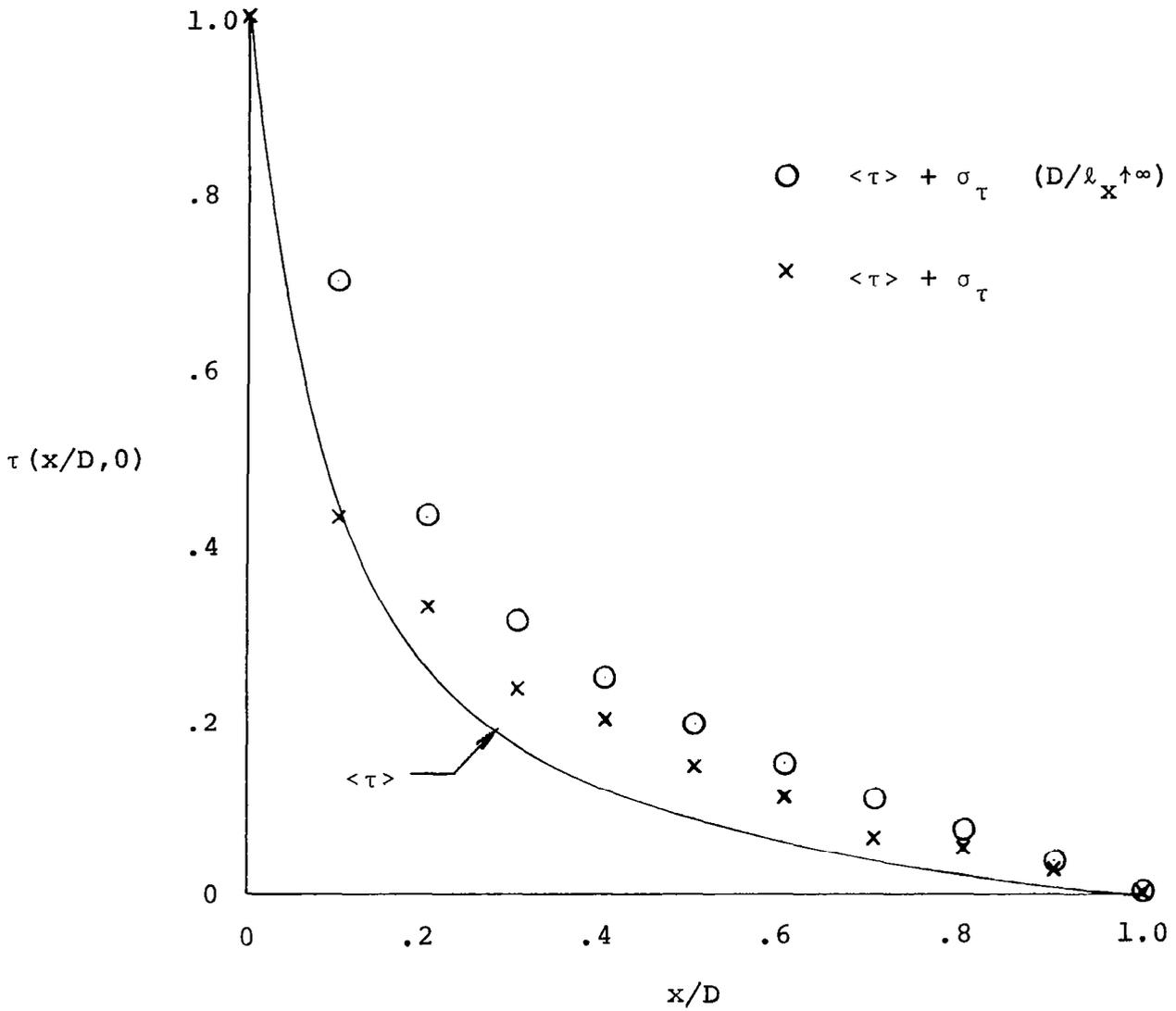
Figure 6. Second-order statistics of the optical transfer function for $D/\ell_x=6$ ($\ell_x/\ell_y=2$, $\sigma/\lambda=.2$).
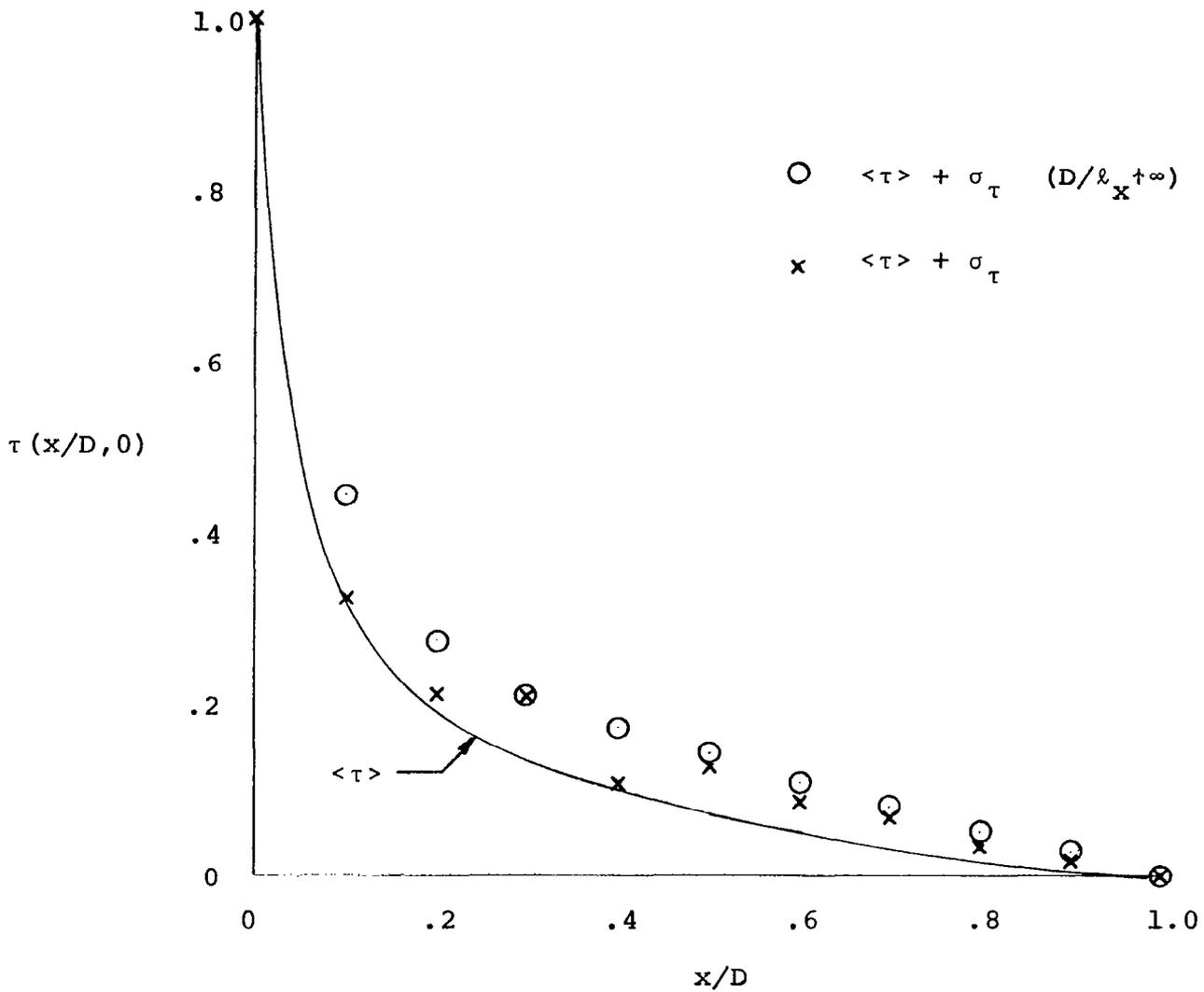
Figure 7. Second-order statistics of the optical transfer function for $D/\ell_x=10$ $(\ell_x/\ell_y=2, \sigma/\lambda=.2)$.