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STEADY, OSCILLATORY, AND UNSTEADY SUBSONIC AND SUPERSONIC AERODYNAMICS - PRODUCTION VERSION (SOUSSA-P 1.1) - VOLUME I - THEORETICAL MANUAL

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STEADY, OSCILLATORY, AND UNSTEADY SUBSONIC AND SUPERSONIC AERODYNAMICS - PRODUCTION VERSION (SOUSSA-P 1.1)

Vol. I - Theoretical Manual

Final Report

by

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PREFACE

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ABSTRACT

A review and summary of recent developments of the Green's function method and the computer program SOUSSA (Steady, Oscillatory, and Unsteady Subsonic and Supersonic Aerodynamics) are presented. Applying the Green's function method to the fully unsteady (transient) potential equation yields an integro-differential-delay equation. With spatial discretization by the finite-element method, this equation is approximated by a set of differential-delay equations in time. Time solution by Laplace transform yields a matrix relating the velocity potential to the normal wash. Premultiplying and postmultiplying by the matrices relating generalized forces to the potential and the normal wash to the generalized coordinates one obtains the matrix of the generalized aerodynamic forces. The frequency and mode-shape dependence of this matrix makes the program SOUSSA very useful for multiple frequency and repeated mode-shape evaluations. The program SOUSSA is general, flexible, easy to use, and accurate. Applications to aerodynamic design are also discussed. The user/programmer manual for SOUSSA-P 1.1 is presented in Volume 2 of this report (NASA CR-159131).

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

a_∞	Speed of sound
AR	Aspect ratio
b	Span
B_h	See Equation (3-7)
B_{jh}	See Equation (3-10)
B'_h	See Equation (3-16)
B'_{hj}	See Equation (3-19)
c	Chord
C_p	Pressure coefficient
C_h	See Equation (3-7)
C_{jh}	See Equation (3-10)
D_h	See Equation (3-7)
D_{jh}	See Equation (3-10)
$D^{\pm}_h$	See Equation (3-16)
D'_{hj}	See Equation (3-19)
E(P)	See Equations (2-1), (2-55), (A-27) and (A-47)
$\underline{\hat{E}}$	See Equation (6-21)
$\tilde{E}^{(1)}_{hm}$	See Equation (4-29)
$\tilde{E}^{(3)}_{kh}$	See Equation (5-34)

$\hat{E}_{kh}^{(3)}$	See Equation (5-14)
$E_{nh}^{(4)}$	See Equation (5-41)
$E_{kh}^{(TE)}$	See Equation (5-30)
$\tilde{\underline{E}}_1$	See Equation (6-4)
$\tilde{\underline{E}}_1^{(0)}$	See Equation (7-1)
$\tilde{\underline{E}}_1^{(1)}$	See Equation (7-1)
$\tilde{\underline{E}}_2$	See Equation (6-17)
$\tilde{\underline{E}}_3$	See Equation (6-9)
$\tilde{\underline{E}}_3^{(0)}$	See Equation (7-2)
$\tilde{\underline{E}}_3^{(1)}$	See Equation (7-2)
$\underline{E}_4$	See Equation (6-10)
F_n	See Equation (3-7)
F_{jn}	See Equation (3-10)
F_n'	See Equation (3-16)
F_{hn}'	See Equation (3-19)
G_n	See Equation (3-7)
G_{jn}	See Equation (3-10)

$G_n^{\pm}$	See Equation (3-16)
G'_{hn}	See Equation (3-19)
H	Number of nodes on body
$H(\bar{P})$	See Equation (2-61)
k	Reduced frequency, $\omega\ell/U_\infty$
ℓ	Reference length
$L_n(\bar{P})$	Shape functions for $\Delta\Phi$, see Equation (3-2)
M	Mach number, U_∞/a_∞
$\bar{n}$	Normal to σ_B
$\bar{n}_1$	Normal to σ_W
N	Number of nodes on wake
$\bar{N}$	Normal to Σ_B
$\bar{N}_1$	Normal to Σ_W
$N_h(\bar{P})$	Shape functions for Ψ and Φ, see Equation (3-1)
p	Complex reduced frequency, $s\ell/U_\infty$
$\bar{p}$	Point having coordinates (x, y, z)
$\bar{P}$	Point having coordinates (X, Y, Z)
$\bar{P}_*$	Control point, (X_*, Y_*, Z_*)
q_n	Lagrangian generalized coordinates, see Equation (4-23)
r_β	See Equation (2-18)

r'_β	See Equation (2-57)
R	See Equation (2-28)
R'	See Equation (2-66)
s	Complex frequency (Laplace parameter)
S	See Equation (1-6)
S_B	See Equation (1-10)
S_U	See Equation (1-10)
S_V	See Equation (2-2)
S_W	See Equation (2-3)
S_{nh}	See Equation (3-5)
t	Time
T	Nondimensional time, see Equation (2-26)
$\tilde{u}$	Displacement, see Equation (4-10)
U_∞	Velocity of undisturbed flow
x, y, z	Space coordinates
$\tilde{x}$	Point of the unsteady surface σ, see Equation (4-10)
X, Y, Z	Nondimensional space coordinates, see Equations (2-26) and (2-62)
$\tilde{Y}_{jh}$	See Equations (3-13) and (3-21)
$\tilde{Z}_{jh}$	See Equations (3-14) and (3-21)
β	$= (1 - M^2)^{1/2}$
β'	$= (M^2 - 1)^{1/2}$

δ_{jh}	Kronecker delta, 1 for $j = h$, 0 for $j \neq h$
$\Delta\Phi$	Discontinuity of Φ across the wake, $\Phi_u - \Phi_\ell$
$\Delta\Phi_n$	Nodal values of $\Delta\Phi$
θ	Time for a disturbance to propagate from $\bar{P}$ to $\bar{P}_*$, see Equation (2-21)
$\theta^{\pm}$	See Equation (2-60)
Θ	See Equation (2-31)
$\hat{\Theta}$	See Equation (2-32)
Θ_h	Θ for $\bar{P} = \bar{P}_h$
Θ_{jh}	See Equation (3-10)
$\Theta^{\pm}$	See Equation (2-69)
$\hat{\Theta}^{\pm}$	See Equation (2-70)
$\Theta_h^{\pm}$	$\Theta^{\pm}$ for $\bar{P} = \bar{P}_h$
$\Theta_{hj}^{\pm}$	See Equation (3-19)
Π	Convection time of wake vortices, see Equation (2-53)
Π_n	Value of Π for $\bar{P} = \bar{P}_n$, see Equation (3-3)
ρ	Fluid density
σ	Unsteady surface at body
σ_B	Surface of body (x, y, z space)
σ_W	Surface of wake (x, y, z space)

Σ_B	Surface of body (X, Y, Z space)
Σ_W	Surface of wake (X, Y, Z space)
τ	See Equations (1-18) and (1-19)
φ	Perturbation velocity potential
ϕ	Velocity potential, $U_\infty x + \varphi$
Φ	Nondimensional perturbation velocity potential, $\varphi/U_\infty \ell$
Φ_i	Nodal values of Φ_U
$\tilde{\Phi}_i$	Laplace transform Φ_i
Φ_U	Unsteady component of Φ
ψ	Normal wash in x, y, z space
Ψ	Normal wash in X, Y, Z space
Ψ_i	Normal value of Ψ_U
$\tilde{\Psi}_i$	Laplace transform of Ψ_i
Ψ'	Conormal wash in X, Y, Z space, see Equation (2-65)
Ψ'_i	Nodal value of Ψ'_U
$\tilde{\Psi}'_i$	Laplace transform of Ψ'_i
Ψ_U	Unsteady component of Ψ
ω	Frequency
Ω	Solid angle
∇	Del operator in x, y, z space
∇_o	Del operator in X, Y, Z space

SUBSCRIPTS AND SUPERSCRIPTS

B	Body
I	Imaginary part of complex number
ℓ	Lower
M	Modified; see Equation (4-1) and Appendix C
R	Real part of complex number
TE	Trailing edge
u	Upper
U	Unsteady
W	Wake
∞	Free stream condition

SPECIAL NOTATATIONS

$(\tilde{\ })$	Laplace transform of ( )
$(\bar{\ })$	Vector
$(\underline{\ })$	Matrix
$(\dot{\ })$	Derivative with respect to T
{ }	Column matrix
[ ]	Matrix
$[\ \ ]^{\theta}$	Evaluation at time $t_* - \theta$; see Equation (2-25)
$\Delta(\ )$	Discontinuity of ( ) across wake, $(\ )_u - (\ )_\ell$

SECTION 1

INTRODUCTION

Presented here is a general formulation for steady, oscillatory, and unsteady subsonic and supersonic, potential aerodynamics for complex aircraft configurations. This formulation is the basis for the computer program SOUSSA-P, which may be used for a variety of aerodynamic computations. These include, for example, the following types of aerodynamic analysis:

- Unsteady State Applications
  - a) Flutter or Gust Analysis.
  - b) Flutter or Gust Analysis with multiple sets of frequencies.
  - c) Flutter or Gust Analysis with multiple sets of boundary-condition modes and/or generalized-force modes.
  - d) Flutter or Gust Analysis with multiple Mach numbers.
- Steady or Quasi-Steady State Applications
  - a) Steady-State Pressure Distributions.
  - b) Structural Design Loads.
  - c) Aerodynamic Coefficients.
  - d) Stability Derivatives.
  - e) Static Aeroelastic Analysis.

Reviews of other methods in this field are given in Refs. 1 and 2 and, therefore, are not presented here. The method is based upon the Green's function method (to transform the velocity-potential differential equation into an integral-differential-delay equation) and the finite element method (to transform the equation into a set of differential-delay equations in time).

A review of the development of the method is presented in the following subsection. The integral equation used in SOUSSA-P is obtained in Section 2, whereas the numerical formulation for the solution of the integral equation is presented in Section 3. The boundary conditions are considered in Section 4, whereas Section 5 deals with the evaluation of pressure coefficients from the potential and the evaluation of the generalized forces from the pressure coefficient. Section 6 discusses the matrix relating the generalized aerodynamic forces to the generalized coordinates describing the rigid-body motion and/or deformation of the aircraft. The use of the method in aerodynamic design (find the shape given the pressure distribution) is also discussed in Section 6. Concluding remarks are presented in Section 7.

Subtle points in the formulation are dealt with in the Appendices. In particular, the value of the function E for a point on the surface is obtained in Appendix A. A discussion of Laplace transform and the truncation of the wake is given in Appendix B. The effect of the motion of the surface of the body is discussed in Appendix C. Appendix D deals with the boundary conditions for a body-axis formulation. In Appendix E the closed-wake phenomenon is examined.

1.1 REVIEW OF DEVELOPMENT OF METHOD

The method is based upon a formulation developed by Morino (Refs. 3 and 4) in which the Green's function method is applied to the equation for the velocity potential. Use of the infinite-space Green's function method yields a representation of the potential φ at any point, $\bar{p}$, in the flow field (control point) in terms of the values of the potential and its normal derivatives on the surface, Σ, surrounding the body and its wake. The integral equation is obtained by imposing the requirement that the value of the potential at $\bar{p}$ approaches the value of φ on the surface if $\bar{p}$ approaches a point on the surface. The wake is a natural byproduct of the method and is treated as a layer of doublets. It may be noted that the integral equation is different from other ones currently used in aerodynamics in that it does not require that the boundary condition on the normal wash be satisfied, but rather makes use of the continuity of the potential as the control point approaches the surface, σ. The two boundary conditions are mathematical equivalents, however. The tangency boundary conditions are automatically satisfied by the type of representation obtained with the Green's theorem. A feasibility analysis by Morino and Kuo (Refs. 5 and 6) indicates that the method is fast and accurate. A general formulation for steady and oscillatory subsonic and supersonic flow is given in Ref. 7. In that formulation the surface of the aircraft and its wake was divided into small quadrilateral elements. Each element was replaced by a portion of a hyperboloidal paraboloid (hyperboloidal element) defined by the four corner points of the element. In this process, the continuity of the surface was maintained, although discontinuities in the slopes were introduced. The unknown potential was assumed to be constant within each element (zeroth-order element) and therefore, the integral equation was approximated with a system of algebraic equations.

An extension to the fully unsteady formulation, both in the time and complex-frequency (Laplace) domain, is presented in Ref. 8 for an arbitrary finite-element representation. Evaluation of the matrix of the generalized aerodynamic forces is presented in Ref. 9. Modification of the theory for coplanar surfaces is discussed in Ref. 10. Numerical results for subsonic and supersonic flows have been obtained for wing-body-tail configurations in steady oscillatory and full unsteady (transient) subsonic and supersonic flows (Refs. 9 and 10).

More recently the computer time for the evaluation of the aerodynamic influence coefficients has been reduced considerably by using numerical quadrature to evaluate the source and doublet integrals for "distant" elements (Ref. 11). Additional work includes the wake roll-up (Ref. 12). An overall review of the method is given in Ref. 13. An assessment of the method is given in Ref. 14.

1.2 THE COMPUTER PROGRAM SOUSSA-P 1.1

The method reviewed in Section 1.1 was originally imbedded in the general-purpose computer program SOUSSA-I (Steady, Oscillatory, and Unsteady Subsonic and Supersonic Aerodynamics - Interim Version), which was first presented in Referece 9. SOUSSA-I was at the time the most general code available for potential aerodynamic analysis. However, SOUSSA-I was developed in an academic environment, for research and proof-of-concept purposes. A more general and efficient version (SOUSSA-P) was planned for "production" applications. The SOUSSA-P program possesses the following qualitites:

- User Orientation - To enable its use without extensive specialized training.
- Generality - The program has been structured to facilitate the analysis of aerodynamic problems involving a wide range of flight speeds, arbitrary aircraft geometry, multiple sets of vibration and deformation modes, and multiple sets of frequencies. Furthermore, SOUSSA-P was designed to be compatible with most currently available geometry preprocessors.
- Computational Efficiency - A premium has been placed on the conservation of central processor time as well as central memory so that the program may be a useful tool for application to complicated configurations such as complete aircraft.
- Simplicity of Method - It provides a commonality between subsonic and supersonic, steady and unsteady flows. Also, the expressions for the coefficients are very simple due to the vector formulation of the problem. Furthermore, the use of quadrilateral hyperboloidal elements, described in terms of their corner points, allows for application to arbitrarily complex configurations.
- Accuracy - The method is accurate and fast, as indicated by the results obtained thus far.
- Modularity - To facilitate incorporating new or improved features of the method or the computational algorithms.

SOUSSA-P employs the sophisticated data handling capabilities of the SPAR Finite Element Structural Analysis, System Level II, computer program (Ref. 15), which is in part responsible for realizing the aforementioned qualities.

1.3 FORMULATION OF PROBLEM

The integral equation used in SOUSSA-P is derived in Ref. 3. For the sake of clarity, a simpler derivation is presented in this report, assuming from the beginning that the motion of the surface is infinitesimal.*

In this report, the isentropic inviscid flow of a perfect gas, initially irrotational, is considered. Under this hypothesis, the flow can be described by the velocity potential, ϕ, such that the velocity $\bar{V}_F$ of the fluid is given by

$$\bar{V}_F = \nabla \phi \tag{1-1}$$

where ∇ is the del operator (nabla). Consider a frame of reference $\bar{i}$, $\bar{j}$, $\bar{k}$ such that the undisturbed flow has velocity U_∞ in the direction of the $\bar{i}$ axis. Then the small-perturbation, linearized equation of the unsteady aerodynamic potential is

$$a_\infty^2 \quad \nabla^2 \phi = \frac{d^2 \phi}{dt^2} \tag{1-2}$$

where a_∞ is the speed of sound of the undisturbed flow, ∇^2 is the Laplacian operator, and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x} \tag{1-3}$$

is the linear substantial derivative. Then, it is convenient to introduce the perturbation potential φ, such that

$$\phi = U_\infty x + \varphi \tag{1-4}$$

Note that

$$\varphi = 0 \tag{1-5}$$

in the undisturbed flow.

*This point is discussed further in Appendix C.

A very general approach is considered in Ref. 3 by assuming that the body immersed in this flow has arbitrary shape and is moving with arbitrary motion, including deformation. Thus, the surface of the body is represented in the general form

$$S(x, y, z, t) = 0 \tag{1-6}$$

The no-flow-through boundary condition on the body is given by (see Subsection 4-1)

$$\frac{\partial S}{\partial t} + \nabla\phi \cdot \nabla S = 0 \qquad \text{on } S = 0 \tag{1-7}$$

By using Equations (1-3) and (1-4), Equation (1-7) reduces to*

$$\nabla\varphi \cdot \nabla S = -\left(\frac{\partial S}{\partial t} + U_\infty \frac{\partial S}{\partial x}\right) \tag{1-8}$$

or

$$\psi \equiv \frac{\partial\varphi}{\partial n} = -\frac{1}{|\nabla S|} \frac{dS}{dt} \tag{1-9}$$

As mentioned previously, a simplified approach is used here, in which the surface is assumed to move only infinitesimally with respect to the frame of reference traveling at velocity U_∞ (in direction of the negative x-axis) with respect to the undisturbed flow. Mathematically, it is assumed that the surface of the body is represented in the form

$$S(x, y, z, t) = S_B(x, y, z) + \epsilon S_U(x, y, z, t) = 0 \tag{1-10}$$

*This is the exact boundary condition: no small-perturbation assumption is used here. A nonzero flow-through boundary condition can be easily included in the formulation by adding the flow-through term in Equation (1-8).

with $\varepsilon << 1$. In Equation (1-10), S_B represents the steady-state geometry while S_U gives the unsteady perturbation.

Next consider the pressure distribution which can be evaluated from the exact nonlinear Bernoulli Theorem for barotropic fluids

$$\int_{p_\infty}^{p} \frac{1}{\rho} \, dp = -\left(\frac{\partial \varphi}{\partial t} + U_\infty \frac{\partial \varphi}{\partial x} + \frac{1}{2} \nabla \varphi \cdot \nabla \varphi \right) \tag{1-11}$$

or if nonlinear terms are neglected,

$$C_p = \frac{p - p_\infty}{\frac{1}{2} \rho U_\infty^2} = -2 \frac{1}{U_\infty^2} \left(\frac{\partial \varphi}{\partial t} + U_\infty \frac{\partial \varphi}{\partial x} \right) = -\frac{2}{U_\infty^2} \frac{d\varphi}{dt} \tag{1-12}$$

which is the form used in SOUSSA-P 1.1.

Then the nondimensional generalized aerodynamic force acting on the non-dimensional mode $\bar{M}_h$ is given by

$$e_h = \frac{-1}{\ell^2} \oiint_{\sigma_B} C_p \, \bar{n} \cdot \bar{M}_h \, d\sigma_B \tag{1-13}$$

where σ_B is the surface of the body and ℓ is a reference length.

Finally, consider the boundary condition on the wake, i.e., that the pressure discontinuity across the wake be equal to zero. Using Equation (1-11) one obtains*

$$\left[\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x} + \frac{1}{2} \left(\nabla \varphi_u + \nabla \varphi_\ell \right) \cdot \nabla \right] \left(\varphi_u - \varphi_\ell \right) = 0 \tag{1-14}$$

*The subscripts u and ℓ indicate the upper and lower sides of the wake. They are used even when the wake is vertical or rolled-up.

or

$$\frac{D}{Dt} \Delta\varphi = 0 \tag{1-15}$$

where $\Delta\varphi = \varphi_u - \varphi_\ell$ and D/Dt is the substantial derivative, i.e., the time derivative evaluated by following a wake particle which by definition has velocity

$$\bar{V} = U_\infty \bar{i} + \frac{1}{2} (\nabla\varphi_u + \nabla\varphi_\ell) \tag{1-16}$$

Note that for steady state Equation (1-15) becomes

$$\frac{d}{ds_W} \Delta\varphi = 0 \tag{1-17}$$

where s_W is the arc length along a wake streamline. Equations (1-15) and (1-17) are exact, i.e., involve no small-perturbation assumption. Equation (1-15) may be integrated to yield

$$\Delta\varphi(\bar{p}, t) = \Delta\varphi(\bar{p}_{TE}, t - \tau) \tag{1-18}$$

where τ is the time necessary for a particle in the wake to travel from the point, $\bar{p}_{TE}$ (origin of the vortex line at the trailing edge), to the point $\bar{p}$. Small perturbation hypothesis is used in SOUSSA-P 1.1 and hence τ is approximated by

$$\tau = (x - x_{TE})/U_\infty \tag{1-19}$$

This is consistent with using the linearized Bernoulli's theorem for the evaluation of the pressure since in this case Equation (1-12) yields

$$\left(\frac{\partial}{\partial t} + U_{\infty} \frac{\partial}{\partial x} \right) \Delta\varphi = 0 \tag{1-20}$$

or

$$\Delta\varphi = \Delta\varphi(x - U_{\infty} t) \tag{1-21}$$

An additional boundary condition is that the normal component of the velocity is zero on the wake, i.e.,

$$\frac{\partial \phi}{\partial n} = 0 \tag{1-22}$$

This condition may be used to obtain the geometry of the wake.

SECTION 2

SUBSONIC AND SUPERSONIC INTEGRAL EQUATIONS

Considered in this section is the integral equation for unsteady subsonic and supersonic potential aerodynamics for an aircraft having arbitrary shape. The rigid-body motion and/or deformation of the aircraft is assumed to consist of small perturbations (starting at $t = 0$) with respect to the constant-speed motion. The objective of this formulation is to describe the functional relationship between aerodynamic potential and its normal derivative (normal wash, $\psi = \partial\varphi/\partial n$) on the fluid boundary.

The analysis presented in this section is based upon an integral formulation presented in Refs. 3 and 4, which includes completely arbitrary motion. Note that in order to perform a linear-system analysis of the aircraft, it is convenient to use a general aerodynamic formulation, i.e., fully transient response for time-domain analysis, and the aerodynamic transfer function (Laplace transform of the fully unsteady operator) for complex-frequency-domain analysis. Consistent with this type of analysis, the unsteady contribution is assumed to start at time $t = 0$, so that for time $t < 0$ the flow is in steady state. Furthermore, the unsteadiness of the aircraft is assumed to consist of small (infinitesimal) perturbations around the steady-state configuration.

In this section the subsonic formulation is presented in detail; the extension to supersonic flows is also outlined.

2.1 GREEN'S THEOREM FOR POTENTIAL AERODYNAMICS

The purpose of this analysis is to obtain a representation of the potential in terms of its value (and the values of its derivatives) on the surface of the body and the wake using Green's Theorem.

Note that the equation of the aerodynamic potential given by Equation (1-2) is not valid on the wake, where discontinuities on φ exist. Thus, consider the volume V in which Equation (1-2) is valid. At any instant of time, this volume is given by the whole physical space except the volume, V_B, occupied by the body and the infinitesimally thin layer, V_W, representing the wake. As mentioned above, the volume V is assumed to be time-independent. Define the function E (see Figure 2-1).

$$\begin{aligned} E(x, y, z) &= 1 \qquad \text{in } V \\ &= 0 \qquad \text{in } V_B + V_W \end{aligned} \tag{2-1}$$

This function represents the domain of validity of the equation of the potential and will be called "domain function." Consider the surface of discontinuity of the function E, that is, the surface, σ, surrounding the volume $V_B + V_W$. Let

$$S_V(x, y, z) = 0 \tag{2-2}$$

be the equation of the surface σ.

Note that the surface σ is composed of two branches. The first, σ_B, is the surface of the body given by Equation (1-10) (with $\varepsilon = 0$ except for the boundary conditions*). The second is the surface, σ_W, of the wake

$$S_W(x, y, z) = 0 \tag{2-3}$$

Note that this surface σ_W is considered twice, since σ is a closed surface. In other words, the two sides of the wake are considered to be two independent surfaces having the same equation (but opposite outwardly-directed normals).

*In other words, the boundary conditions take into account the motion of the surface σ, but are applied at the mean (steady) position.

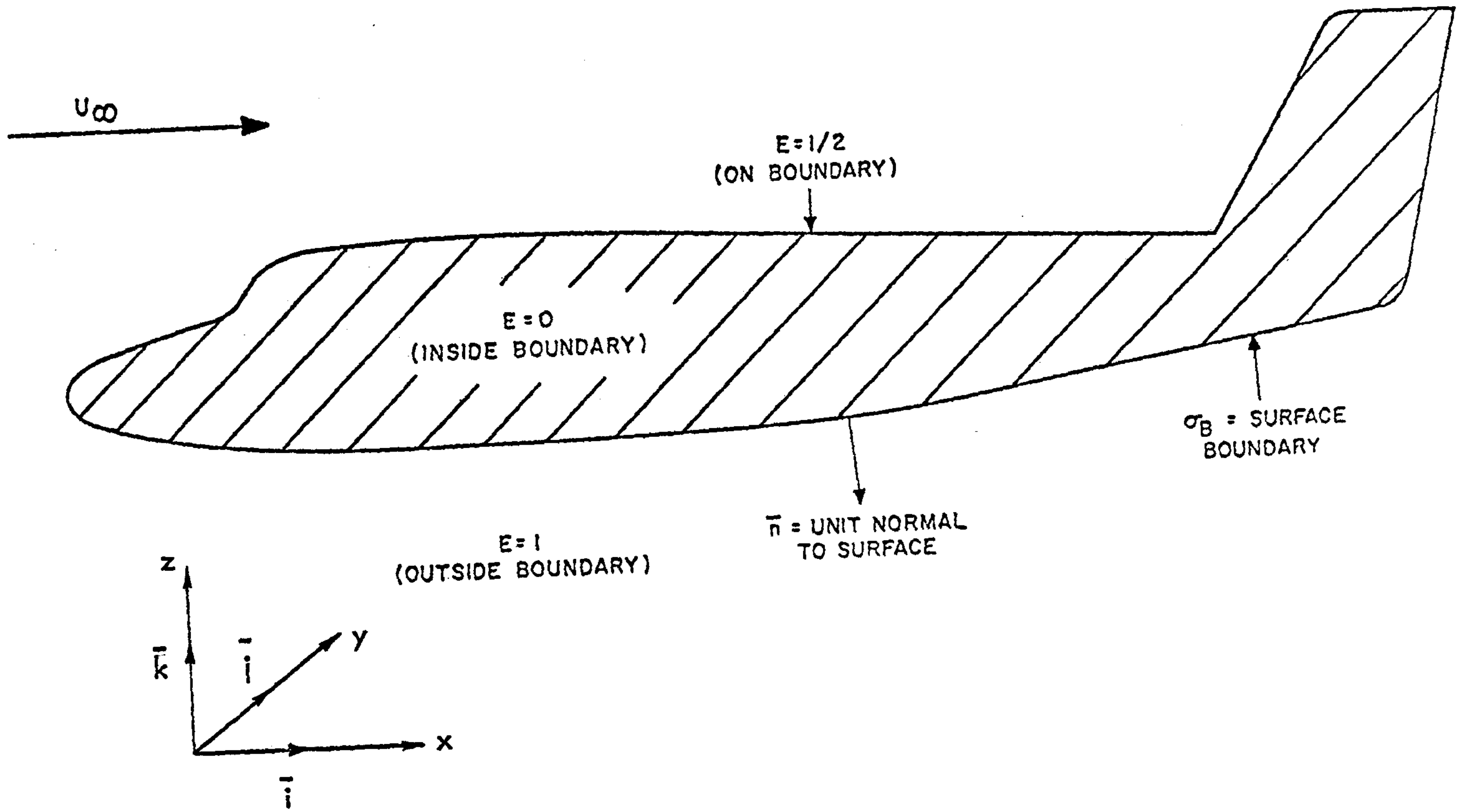

Figure 2-1. Aerodynamic Nomenclature for Potential Flow.

As mentioned previously, the present formulation is based upon the Green's function method. The infinite-space Green's function for the equation of potential is defined by

$$\nabla^2 G - \frac{1}{a_\infty^2}\left(\frac{d^2 G}{dt^2}\right) = \delta\,(x-x_*,\ y-y_*,\ z-z_*,\ t-t_*) \qquad (2\text{-}4)$$

(where δ is the Dirac delta function) with $G \equiv 0$ at infinity.

Multiplying the equation of the aerodynamic potential, Equation (1-2), by the Green's function G and subtracting Equation (2-4) multiplied by φ, yields

$$G\left(\nabla^2\varphi - \frac{1}{a_\infty^2}\,\frac{d^2\varphi}{dt^2}\right) - \varphi\left(\nabla^2 G - \frac{1}{a_\infty^2}\,\frac{d^2 G}{dt^2}\right) = -\varphi\,\delta \qquad (2\text{-}5)$$

Making use of the identities

$$\nabla\cdot(a\nabla b) \equiv \nabla a\cdot\nabla b + a\nabla^2 b \qquad (2\text{-}6)$$

and

$$\frac{d}{dt}\left(a\,\frac{db}{dt}\right) = \frac{da}{dt}\,\frac{db}{dt} + a\,\frac{d^2 b}{dt^2} \qquad (2\text{-}7)$$

Equation (2-5) reduces to

$$\nabla\cdot(G\nabla\varphi - \varphi\nabla G) - \frac{1}{a_\infty^2}\,\frac{d}{dt}\left(G\,\frac{d\varphi}{dt} - \varphi\,\frac{dG}{dt}\right) = -\varphi\delta \qquad (2\text{-}8)$$

Multiplying Equation (2-8) by the domain function E, defined by Equation (2-1), integrating over the whole four-dimensional space-time and noting that, for any function g, (see Equation (2-1))

$$\iiint_{-\infty}^{\infty} E\,g\,dV = \iiint_V g\,dV \qquad (2\text{-}9)$$

one obtains

$$\int_{-\infty}^{\infty} dt \iiint_V \left[\nabla \cdot (G\nabla\varphi - \varphi\nabla G) - \frac{1}{a_\infty^2} \frac{d}{dt} \left(G \frac{d\varphi}{dt} - \varphi \frac{dG}{dt} \right) \right] dV$$

$$= - \iiiint_{-\infty}^{\infty} E \, \varphi \, \delta \, dV \, dt \tag{2-10}$$

Using the relations

$$\int_{-\infty}^{\infty} \frac{\partial g}{\partial t} \, dt = g(\infty) - g(-\infty) = 0 \tag{2-11}$$

for $g(\pm\infty) = 0$ and

$$\iiint_V \frac{\partial g}{\partial x} \, dV = \oiint_\sigma g \bar{n} \cdot \bar{i} \, d\sigma = \oiint_\sigma g \frac{\partial S_V}{\partial x} \, |\nabla S_V|^{-1} \, d\sigma \tag{2-12}$$

and similarly

$$\iiint_V \nabla \cdot \bar{a} \, dV = \oiint_\sigma \bar{a} \cdot \bar{n} \, d\sigma = \oiint_\sigma \bar{a} \cdot \nabla S_V |\nabla S_V|^{-1} \, d\sigma \tag{2-13}$$

where $\bar{a}$ is an arbitrary three-dimensional vector and $\bar{n}$ is the outwardly-directed* normal to the surface

$$\bar{n} = \frac{\nabla S_V}{|\nabla S_V|} \tag{2-14}$$

---

*"Outwardly" is understood as "going from the body into the fluid", that is, from the region $E = 0$ into the region $E = 1$.

Equation (2-10) reduces to

$$\int_{-\infty}^{\infty} \oint\!\!\oint_{\sigma} \Bigg[\nabla S_V \cdot (G\nabla\varphi - \varphi\nabla G)$$

$$- \frac{1}{a_\infty^2} U_\infty \frac{\partial S_V}{\partial x} \left(G \frac{d\varphi}{dt} - \varphi \frac{dG}{dt} \right) \Bigg] |\nabla S_V|^{-1} \; d\sigma \; dt$$

$$= \iiiint_{-\infty}^{\infty} E \, \varphi \, \delta \, dV \, dt \tag{2-15}$$

Finally, making use of the definition of the Dirac delta function yields

$$E(\bar{p}_*) \varphi \; (\bar{p}_*, t_*) = \int_{-\infty}^{\infty} \oint\!\!\oint_{\sigma} \Bigg[\nabla S_V \cdot (G\nabla\varphi - \varphi\nabla G)$$

$$- \frac{1}{a_\infty^2} U_\infty \frac{\partial S_V}{\partial x} \left(G \frac{d\varphi}{dt} - \varphi \frac{dG}{dt} \right) \Bigg] |\nabla S_V|^{-1} \; d\sigma \; dt \tag{2-16}$$

Equation (2-16) is the desired Green's Theorem for potential subsonic and supersonic small-perturbation aerodynamics. (Compare Equation (3-19) of Ref. 3.)

2.2 SUBSONIC INTEGRAL EQUATION

Equation (2-16) is valid for both subsonic and supersonic flow. In this subsection the flow is assumed to be subsonic, i.e., $M = U_\infty / a_\infty < 1$. In this case the Green's function is given by (Ref. 3)

$$G = - \frac{1}{4 \pi r_\beta} \; \delta_\theta \tag{2-17}$$

where

$$r_\beta = \left\{ (x-x_*)^2 + \beta^2 \left[(y-y_*)^2 + (z-z_*)^2 \right] \right\}^{1/2} \tag{2-18}$$

with

$$\beta = \sqrt{1 - M^2} \tag{2-19}$$

while

$$\delta_\theta = \delta\ (t - t_* + \theta) \tag{2-20}$$

with

$$\theta = \frac{1}{a_\infty \beta^2} \left[r_\beta + M\ (x-x_*) \right] \tag{2-21}$$

2.2.1 SUBSONIC GREEN'S THEOREM

Combining Equations (2-16) and (2-17) one obtains

$$\begin{aligned}
4\pi\ E(\bar{p}_*)\ \varphi(\bar{p}_*, t_*) = &- \int_{-\infty}^{\infty} \oiint_\sigma \left[\nabla S_V \cdot \nabla \varphi - \frac{U_\infty}{a_\infty^2} \frac{\partial S_V}{\partial x} \frac{d\varphi}{dt} \right] \frac{1}{r_\beta} \delta_\theta \left| \nabla S_V \right|^{-1} d\sigma dt \\
&+ \int_{-\infty}^{\infty} \oiint_\sigma \left[\nabla S_V \cdot \nabla \left(\frac{1}{r_\beta} \right) - \frac{U_\infty^2}{a_\infty^2} \frac{\partial S_V}{\partial x} \frac{\partial}{\partial x} \left(\frac{1}{r_\beta} \right) \right] \varphi\, \delta_\theta \left| \nabla S_V \right|^{-1} d\sigma dt \\
&+ \int_{-\infty}^{\infty} \oiint_\sigma \left[\nabla S_V \cdot \nabla \delta_\theta - \frac{U_\infty}{a_\infty^2} \frac{\partial S_V}{\partial x} \frac{d\delta_\theta}{dt} \right] \varphi \frac{1}{r_\beta} \left| \nabla S_V \right|^{-1} d\sigma dt
\end{aligned} \tag{2-22}$$

Next note that

$$\nabla \delta_\theta = \frac{\partial \delta_\theta}{\partial t} \nabla \theta$$

$$\frac{d\delta_\theta}{dt} = \left(\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x}\right) \delta_\theta = \frac{\partial \delta_\theta}{\partial t} \left(1 + U_\infty \frac{\partial \theta}{\partial x}\right) \tag{2-23}$$

Combining Equations (2-22) and (2-23) yields

$$4\pi E(\bar{p}_*)\, \varphi(\bar{p}_*, t_*) =$$

$$- \int_{-\infty}^{\infty} \oiint_{\sigma} \left[\nabla S_V \cdot \nabla \varphi - \frac{U_\infty}{a_\infty^2} \frac{\partial S_V}{\partial x} \frac{d\varphi}{dt}\right] \frac{1}{r_\beta}\, \delta_\theta\, |\nabla S_V|^{-1}\, d\sigma\, dt$$

$$+ \int_{-\infty}^{\infty} \oiint_{\sigma} \left[\nabla S_V \cdot \nabla \left(\frac{1}{r_\beta}\right) - \frac{U_\infty^2}{a_\infty^2} \frac{\partial S_V}{\partial x} \frac{\partial}{\partial x} \left(\frac{1}{r_\beta}\right)\right] \varphi \delta_\theta\, |\nabla S_V|^{-1}\, d\sigma\, dt$$

$$+ \int_{-\infty}^{\infty} \oiint_{\sigma} \left[\nabla S_V \cdot \nabla \theta - \frac{U_\infty}{a_\infty^2} \frac{\partial S_V}{\partial x} \left(1 + U_\infty \frac{\partial \theta}{\partial x}\right)\right] \frac{\varphi}{r_\beta} \frac{\partial \delta_\theta}{\partial t}\, |\nabla S_V|^{-1}\, d\sigma\, dt \tag{2-24}$$

Performing the integration with respect to time, one obtains*

$$4\pi E(\bar{p}_*)\,\varphi(\bar{p}_*, t_*) =$$

$$- \oint\!\!\oint_\sigma \left\{ \nabla S_V \cdot [\nabla\varphi]^\theta - \frac{U_\infty}{a_\infty^2} \frac{\partial S_V}{\partial x} \left[\frac{d\varphi}{dt}\right]^\theta \right\} \frac{1}{r_\beta} \frac{1}{|\nabla S_V|}\, d\sigma$$

$$+ \oint\!\!\oint_\sigma \left\{ \nabla S_V \cdot \nabla\left(\frac{1}{r_\beta}\right) - \frac{U_\infty^2}{a_\infty^2} \frac{\partial S_V}{\partial x} \frac{\partial}{\partial x}\left(\frac{1}{r_\beta}\right) \right\} [\varphi]^\theta \frac{1}{|\nabla S_V|}\, d\sigma$$

$$- \oint\!\!\oint_\sigma \left\{ \nabla S_V \cdot \nabla\theta - \frac{U_\infty}{a_\infty^2} \frac{\partial S_V}{\partial x} \left(1 + U_\infty \frac{\partial\theta}{\partial x}\right) \right\} \left[\frac{\partial\varphi}{\partial t}\right]^\theta \frac{1}{r_\beta} \frac{1}{|\nabla S_V|}\, d\sigma \tag{2-25}$$

Equation (2-25) is the desired subsonic Green's theorem.

2.2.2 GENERALIZED PRANDTL-GLAUERT TRANSFORMATION

In order to obtain a formally simpler expression for Equation (2-25), consider the generalized Prandtl-Glauert transformation and nondimensionalization

$$\begin{aligned} \Phi &= \varphi/U_\infty \ell \\ X &= x/\beta\, \ell \\ Y &= y/\ell \\ Z &= z/\ell \\ T &= U_\infty t/\ell \end{aligned} \tag{2-26}$$

where ℓ is a reference length.

---

* In Equation (2-25), the symbol $[\ ]^\theta$ indicates evaluation at time $= t_* - \theta$.

$$[\ ]^\theta \equiv [\ ]\,\big|_{t = t_* - \theta}$$

Note that

$$\frac{1}{r_\beta}\frac{d\sigma}{|\nabla S_V|}=\frac{1}{r_\beta}\frac{dy\,dz}{|\partial S_V/\partial x|}=\frac{\ell^2}{R}\frac{dY\,dZ}{|\partial S_V/\partial X|}=\frac{\ell^2}{R}\frac{d\Sigma}{|\nabla_o S_V|} \qquad (2\text{-}27)$$

where ∇_o is the gradient operator in X, Y, Z variables, and Σ is the transformed σ in the space X, Y, Z, and R is given by

$$R=[(X-X_*)^2+(Y-Y_*)^2+(Z-Z_*)^2]^{1/2} \qquad (2\text{-}28)$$

Using Equations (2-26) and (2-27), Equation (2-25) reduces to

$$4\pi E(\bar{P}_*)\,\Phi\,(\bar{P}_*,T_*)=-\oiint_\Sigma\left[\nabla_o S_V\cdot\nabla_o\Phi\right]^{\Theta}\frac{1}{R}\frac{d\Sigma}{|\nabla_o S_V|}$$

$$+\oiint_\Sigma \nabla_o S_V\cdot\nabla_o\left(\frac{1}{R}\right)[\Phi]^{\Theta}\frac{d\Sigma}{|\nabla_o S_V|}$$

$$-\oiint_\Sigma \nabla_o S_V\cdot\nabla_o\hat{\Theta}\left[\frac{\partial\Phi}{\partial T}\right]^{\Theta}\frac{1}{R}\frac{d\Sigma}{|\nabla_o S_V|} \qquad (2\text{-}29)$$

where

$$[\ \ ]^{\Theta}\equiv[\ \ ]\Big|_{T=T_*-\Theta} \qquad (2\text{-}30)$$

with

$$\Theta=U_\infty\theta/\ell=[M(X-X_*)+R]\,M/\beta \qquad (2\text{-}31)$$

and

$$\hat{\Theta}=[M(X_*-X)+R]\,M/\beta \qquad (2\text{-}32)$$

Introducing the normal derivative

$$\frac{\partial}{\partial N} = \bar{N} \cdot \nabla_o \equiv \frac{1}{|\nabla_o S_V|} \nabla_o S_V \cdot \nabla_o \tag{2-33}$$

Equation (2-29) may be rewritten simply as

$$4\pi E(\bar{P}_*)\,\Phi(\bar{P}_*, T_*) = -\oiint_\Sigma [\Psi]^{\Theta} \frac{1}{R}\, d\Sigma + \oiint_\Sigma \frac{\partial}{\partial N}\left(\frac{1}{R}\right)[\Phi]^{\Theta} d\Sigma$$

$$- \oiint_\Sigma \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N} \left[\frac{\partial \Phi}{\partial T}\right]^{\Theta} d\Sigma \tag{2-34}$$

where

$$\Psi = \frac{\partial \Phi}{\partial N} \tag{2-35}$$

indicates the component of the nondimensional velocity in the direction of the normal $\bar{N}$ to the surface Σ of the X, Y, Z space (not in the direction of the normal, $\bar{n}$, to the surface σ of the physical space) and is known from the boundary conditions (see Section 4). The relationship between Ψ and ψ is given by

$$\Psi = \bar{N} \cdot \nabla \Phi = \frac{1}{|\nabla_o S|}\left(\frac{\partial S_V}{\partial X}\frac{\partial \Phi}{\partial X} + \frac{\partial S_V}{\partial Y}\frac{\partial \Phi}{\partial Y} + \frac{\partial S_V}{\partial Z}\frac{\partial \Phi}{\partial Z}\right)$$

$$= \frac{\ell/U_\infty}{|\nabla_o S|}\left[(1-M^2)\frac{\partial S_V}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial S_V}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial S_V}{\partial z}\frac{\partial \varphi}{\partial z}\right]$$

$$= \frac{\ell}{U_\infty}\frac{|\nabla S_V|}{|\nabla_o S_V|}\nabla\varphi\cdot(\bar{n} - M^2 n_x \bar{i}) = \frac{\ell}{U_\infty}\frac{|\nabla S_V|}{|\nabla_o S_V|}\left(\psi - M^2 n_x \frac{\partial \varphi}{\partial x}\right) \tag{2-36}$$

Equation (2-36) may be rewritten in a more interesting form by noting that $\bar{i}$ can be decomposed into two vectors; the first one, $\bar{i}_n = n_x \bar{n}$, normal to Σ and the second one, $\bar{i}_t$, tangent to Σ, in the $(\bar{i}, \bar{n})$ plane

$$\bar{i} = \bar{i}_n + \bar{i}_t = n_x \bar{n} + \bar{i}_t \tag{2-37}$$

Note that

$$\bar{i}_t = \bar{i} - n_x \bar{n} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} - n_x \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 1 - n_x^{\,2} \\ n_x n_y \\ n_x n_z \end{Bmatrix} = i_t \bar{t} \tag{2-38}$$

where

$$\bar{t} = \bar{i}_t / |\bar{i}_t| \tag{2-39}$$

is a unit vector tangent to Σ and

$$i_t = |\bar{i}_t| = \left[(1 - n_x^{\,2})^2 + n_x^{\,2} n_y^{\,2} + n_x^{\,2} n_z^{\,2} \right]^{1/2}$$

$$= \left[1 - 2n_x^{\,2} + n_x^{\,2} (n_x^{\,2} + n_y^{\,2} + n_z^{\,2}) \right]^{1/2} = (1 - n_x^{\,2})^{1/2} \tag{2-40}$$

Note also that

$$\frac{|\nabla_o S_V|}{|\nabla S_V|} = \ell \left[\frac{(1 - M^2)\left(\dfrac{\partial S_V}{\partial x}\right)^2 + \left(\dfrac{\partial S_V}{\partial y}\right)^2 + \left(\dfrac{\partial S_V}{\partial z}\right)^2}{\left(\dfrac{\partial S_V}{\partial x}\right)^2 + \left(\dfrac{\partial S_V}{\partial y}\right)^2 + \left(\dfrac{\partial S_V}{\partial z}\right)^2} \right]^{1/2}$$

$$= \ell \, (1 - M^2 n_x^2)^{1/2} \tag{2-41}$$

Combining Equations (2-36) to (2-41) one obtains

$$\Psi = \frac{1}{U_\infty}\left(1-M^2 n_x^2\right)^{-1/2} \nabla\varphi \cdot \left\{ \bar{n} - M^2 n_x \left[n_x \bar{n} + \left(1-n_x^2\right)^{1/2} \bar{t} \right] \right\}$$

$$= \frac{1}{U_\infty}\left(1-M^2 n_x^2\right)^{-1/2} \nabla\varphi \cdot \left[\left(1-M^2 n_x^2\right)\bar{n} - M^2 n_x \left(1-n_x^2\right)^{1/2} \bar{t} \right]$$

$$= \frac{1}{U_\infty}\left[\left(1-M^2 n_x^2\right)^{1/2} \psi - \left(\frac{1-n_x^2}{1-M^2 n_x^2} \right)^{1/2} M^2 n_x \frac{d\varphi}{ds_t} \right] \qquad (2\text{-}42)$$

where s_t is the arc length along the direction of the unit tangent $\bar{t}$. Note that if $n_x = 1$, then $\Psi = \beta\psi/U_\infty$, whereas if

$$M n_x \cong 0 \qquad (2\text{-}43)$$

then

$$\Psi = \frac{1}{U_\infty} \psi \qquad (2\text{-}44)$$

The approximation introduced in Equation (2-43) implies small-perturbation from free-stream velocity for $M > 0$ and is consistent with the use of the linearized differential equation for the perturbation velocity potential of Equation (1-2).

It should be noted that, in Equation (2-34) the surface Σ is assumed to be fixed with respect to the frame of reference. However, the effect of the motion of the surface is retained in the boundary conditions, which are considered in Section 4.* Also note that (consistent with the hypothesis of small perturbation with respect to the steady-state configuration) the surface of the wake is assumed to be the one of the steady-state case. It may be noted that since Σ is fixed, Equation (2-34) represents a

*This point is discussed further in Appendix C.

linear operator. Hence, flow can be decomposed into steady and unsteady contributions, which are decoupled and can be evaluated separately. As mentioned previously, the flow is assumed to be steady-state for $T < 0$. Therefore, the unsteady parts of the potential and the normal wash, Φ_U and Ψ_U, respectively, are identically equal to zero for $T < 0$:

$$\Phi_U \equiv 0, \quad \Psi_U \equiv 0 \quad (T < 0) \tag{2-45}$$

These conditions are introduced to permit the use of Laplace transform. This point is discussed further in Appendix B.4.

2.2.3 CONTRIBUTION OF WAKE

In order to understand the nature of the aerodynamic operator, Equation (2-34), it is convenient to isolate the contribution of the wake.

Note that, as mentioned in Section 2.1, the surface Σ is composed of two branches. The first is the (closed) surface of the body, Σ_B. The second is the (open) surface of the wake, Σ_W*: note that this surface is considered twice since Σ is a closed surface. In other words upper and lower sides of the wake are considered to be two independent surfaces, having the same equation but opposite outwardly-directed normals, $\bar{N}_u$ and $\bar{N}_\ell$, respectively. This may be expressed as

$$\Sigma = \Sigma_B + \Sigma_u + \Sigma_\ell \tag{2-46}$$

$$\bar{N}_u = -\bar{N}_\ell \tag{2-47}$$

*Note: In inviscid flows the wake has zero thickness. However, the method may be used for finite-thickness wake with appropriate changes in the wake boundary conditions. For a fuselage with a truncated base, inappropriate modeling of the wake as a closed wake yields a nonunique solution to the integral equation. This phenomenon is analyzed in detail in Appendix E.

Combining Equations (2-34), (2-46) and (2-47) yields

$$4\pi\, E(\bar{P}_*)\Phi(\bar{P}_*, T_*) = -\oiint_{\Sigma_B} [\Psi]^{\Theta} \frac{1}{R}\, d\Sigma_B +$$

$$\oiint_{\Sigma_B} [\Phi]^{\Theta} \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_B - \oiint_{\Sigma_B} \left[\frac{\partial \Phi}{\partial T}\right]^{\Theta} \frac{1}{R}\, \frac{\partial \hat{\Theta}}{\partial N}\, d\Sigma_B$$

$$- \iint_{\Sigma_W} [\Delta\Psi]^{\Theta} \frac{1}{R}\, d\Sigma_W + \iint_{\Sigma_W} [\Delta\Phi]^{\Theta} \frac{\partial}{\partial N_u}\left(\frac{1}{R}\right) d\Sigma_W$$

$$- \iint_{\Sigma_W} \left[\frac{\partial \Delta\Phi}{\partial T}\right]^{\Theta} \frac{1}{R}\, \frac{\partial \hat{\Theta}}{\partial N_u}\, d\Sigma_W \tag{2-48}$$

Note that, according to Equation (1-22), $\Delta\psi = 0$. However, in general $\Delta\Psi \neq 0$. For, using Equations (1-22) and (2-42), one obtains

$$\Delta\Psi = \frac{1}{U_\infty}\left(\frac{1 - n_x^{\,2}}{1 - M^2 n_x^{\,2}}\right)^{1/2} M^2 n_x \frac{d\Delta\varphi}{ds_t} \tag{2-49}$$

Thus, excluding the case $n_x = 1$, which violates the small perturbation assumption,

$$\Delta\Psi = 0 \tag{2-50}$$

only if

$$M^2 n_x \frac{d}{ds_t} \Delta\varphi = 0 \tag{2-51}$$

i.e., only for

(1) incompressible flow ($M = 0$)

(2) vortex-lines parallel to x-axis ($n_x = 0$)

(3) steady-state and small perturbations ($\frac{d}{ds_t}\Delta\varphi \simeq \frac{d}{ds_W}\Delta\varphi = 0$, see Equation (1-17)).

Note that the small perturbation assumption (required for the linearization of Equation (1-2)) implies that $n_x \simeq 0$. Therefore, $\Delta\Psi \simeq 0$ and hence Equation (2-48) may be approximated with

$$4\pi E(\bar{P}_*)\,\Phi\,(\bar{P}_*, T_*) = -\oiint_{\Sigma_B} [\Psi]^{\Theta}\,\frac{1}{R}\,d\Sigma_B$$
$$+\oiint_{\Sigma_B}\left\{[\Phi]^{\Theta}\,\frac{\partial}{\partial N}\left[\frac{1}{R}\right] - \left[\frac{\partial\Phi}{\partial T}\right]^{\Theta}\,\frac{1}{R}\,\frac{\partial\hat{\Theta}}{\partial N}\right\} d\Sigma_B$$
$$+\iint_{\Sigma_W}\left\{[\Delta\Phi]^{\Theta}\,\frac{\partial}{\partial N_u}\left(\frac{1}{R}\right) - \left[\frac{\partial\Delta\Phi}{\partial T}\right]^{\Theta}\,\frac{1}{R}\,\frac{\partial\hat{\Theta}}{\partial N_u}\right\} d\Sigma_W \qquad (2\text{-}52)$$

where Σ_B is the closed surface of the body, while Σ_W is the open (i.e., one side) surface of the wake, and $\Delta\Phi$ is the potential-discontinuity across the wake evaluated in the direction of the normal, $\bar{N}_u$ (i.e., $\Delta\Phi \equiv \Phi_u - \Phi_\ell$). It should be noted that the value of $\Delta\Phi$ is not an additional unknown, since Equation (1-18) may be rewritten, in nondimensional form, as

$$\Delta\Phi\,(\bar{P}, T) = \Delta\Phi\,(\bar{P}_{TE}, T - \Pi) \qquad (2\text{-}53)$$

where Π is the nondimensional time necessary for a particle in the wake to travel (along a vortex line, within the steady flow) from the point $\bar{P}_{TE}$ (origin of the vortex line at the trailing edge), to the point $\bar{P}$. Small perturbation hypothesis is used for the steady-state flow, and hence, Π is given by the nondimensional form of Equation (1-19), i.e.,

$$\Pi = \beta\,(X - X_{TE}) \qquad (2\text{-}54)$$

Equation (2-52) is the basis for SOUSSA-P 1.1.

2.2.4 BASIC EQUATIONS USED FOR SOUSSA-P 1.1

It may be noted that, as shown in Appendix A, Equation (2-52) is valid also on the surface Σ if the function E, defined by Equation (2-1), is generalized as

$$
\begin{aligned}
E &= 1 \quad \text{outside } \Sigma_B \\
&= 1/2 \quad \text{on } \Sigma_B \text{ (regular point)} \\
&= 0 \quad \text{inside } \Sigma_B
\end{aligned}
\tag{2-55}
$$

If the point $\bar{P}_* \equiv (X_*, Y_*, Z_*)$ is on the surface Σ, Equation (2-52) is an integral equation (with differential-delay dependence upon time) relating the values of Φ and $\frac{\partial \Phi}{\partial T}$ on the surface Σ to the values of the normal derivative $\frac{\partial \Phi}{\partial N}$ with $\Delta\varphi$ given by Equation (2-53). This is the equation used in SOUSSA-P 1.1. (On the other hand, if $\bar{P}_*$ is not on Σ, Equation (2-52) is not an integral equation but simply an integral representation of the potential Φ at any point of the flow field in terms of the values of Φ and $\frac{\partial \Phi}{\partial T}$ and the normal derivative $\frac{\partial \Phi}{\partial N}$ on the surface Σ.)

2.3 SUPERSONIC INTEGRAL EQUATION

Equation (2-16) was derived without assuming the flow to be subsonic; therefore, Equation (2-16) is valid for supersonic flow as well. The supersonic Green's function is also defined by Equation (2-4) which may be solved to obtain (see Ref. 3).

$$
G = -\frac{H}{4\pi r'_\beta} \left(\delta_{\theta^+} + \delta_{\theta^-} \right) \tag{2-56}
$$

where

$$
r'_\beta = \left\{ (x-x_*)^2 - \beta'^2 \left[(y-y_*)^2 + (z-z_*)^2 \right] \right\}^{1/2} \tag{2-57}
$$

with

$$\beta' = \sqrt{M^2 - 1} \tag{2-58}$$

while

$$\delta_{\theta^{\pm}} = \delta(t - t_* + \theta^{\pm}) \tag{2-59}$$

with

$$\theta^{\pm} = \frac{1}{a_{\infty}\beta'^2}\left[M\,(x_* - x) \pm r'_{\beta} \right] \tag{2-60}$$

and finally

$$\begin{aligned} H(\bar{P}) &= 1 \quad x_* - x > \beta' \sqrt{(y-y_*)^2 + (z-z_*)^2} \\ &= 0 \quad x_* - x \leq \beta' \sqrt{(y-y_*)^2 + (z-z_*)^2} \end{aligned} \tag{2-61}$$

Repeating the same procedure used to derive Equation (2-52) from Equation (2-16), using β' instead of β in Equation (2-26), i.e.,

$$X = x/\beta'\ell \qquad Y = y/\ell \qquad Z = z/\ell \qquad T = U_{\infty} t/\ell \tag{2-62}$$

one obtains

$$4\pi\, E(\bar{P}_*)\,\Phi\,(\bar{P}_*, T_*) = - \oint\!\!\oint_{\Sigma_B} \left([\Psi']^{\Theta^+} + [\Psi']^{\Theta^-} \right) \frac{H}{R'}\, d\Sigma_B$$

$$+ \oint\!\!\oint_{\Sigma_B} \left([\Phi]^{\Theta^+} + [\Phi]^{\Theta^-} \right) \frac{\partial}{\partial N^c} \left(\frac{H}{R'} \right) d\Sigma_B$$

(Equation continued on next page)

$$
- \oint\!\!\oint_{\Sigma_B} \left(\left[\frac{\partial \Phi}{\partial T} \right]^{\Theta^+} \frac{\partial \hat{\Theta}^+}{\partial N^c} + \left[\frac{\partial \Phi}{\partial T} \right]^{\Theta^-} \frac{\partial \hat{\Theta}^-}{\partial N^c} \right) \frac{H}{R'} \, d\Sigma_B
$$

$$
+ \iint_{\Sigma_W} \left([\Delta \Phi]^{\Theta^+} + [\Delta \Phi]^{\Theta^-} \right) \frac{\partial}{\partial N_u^c} \left(\frac{H}{R'} \right) d\Sigma_W
$$

$$
- \iint_{\Sigma_W} \left(\left[\frac{\partial \Delta \Phi}{\partial T} \right]^{\Theta^+} \frac{\partial \hat{\Theta}^+}{\partial N_u^c} + \left[\frac{\partial \Delta \Phi}{\partial T} \right]^{\Theta^-} \frac{\partial \hat{\Theta}^-}{\partial N_u^c} \right) \frac{H}{R'} \, d\Sigma_W \qquad (2\text{-}63)
$$

where $E(\bar{P}_*)$ is given by Equation (2-55),

$$
\frac{\partial}{\partial N^c} = - N_x \frac{\partial}{\partial X} + N_y \frac{\partial}{\partial Y} + N_z \frac{\partial}{\partial Z} \qquad (2\text{-}64)
$$

is the conormal derivative,

$$
\Psi' = \frac{\partial \Phi}{\partial N^c} \qquad (2\text{-}65)
$$

is the conormal wash, whereas

$$
R' = \left[(X - X_*)^2 - (Y - Y_*)^2 - (Z - Z_*)^2 \right]^{1/2} \qquad (2\text{-}66)
$$

and

$$
\begin{aligned}
H &= 1 \qquad \text{for } X_* - X > \left[(Y - Y_*)^2 + (Z - Z_*)^2 \right]^{1/2} \\
&= 0 \qquad \text{for } X_* - X \leq \left[(Y - Y_*)^2 + (Z - Z_*)^2 \right]^{1/2}
\end{aligned} \qquad (2\text{-}67)
$$

Furthermore

$$\left[\quad\right]^{\Theta^{\pm}} \equiv \left[\quad\right]\Big|_{T_* - \Theta^{\pm}} \tag{2-68}$$

indicates evaluation at time $T = T_* - \Theta^{\pm}$ with

$$\Theta^{\pm} = [\,M\,(X_* - X) \pm R'\,]\;M/\beta' \tag{2-69}$$

and finally

$$\hat{\Theta}^{\pm} = [\,M\,(X - X_*) \pm R'\,]\;M/\beta' \tag{2-70}$$

Equation (2-63) is the basis for the supersonic SOUSSA and is presented here for the sake of completeness and for future reference. The supersonic option is not available in SOUSSA-P 1.1.

SECTION 3

NUMERICAL SOLUTION OF THE INTEGRAL EQUATION

Equations (2-52) and (2-63) fully describe the problem of linearized unsteady subsonic and supersonic potential aerodynamics around complex configurations. In order to solve this problem, it is necessary, in general, to obtain a numerical approximation for Equations (2-52) and (2-63). The numerical formulation used in SOUSSA-P is derived in this section. For the sake of completeness a general formulation for an arbitrary finite-element representation is presented first. Then the zeroth order formulation used in SOUSSA-P 1.1 is obtained.

3.1 SUBSONIC FORMULATION

Consider first the subsonic integral equation given by Equation (2-52). In order to discretize the space integral operator over the surface Σ_B it is convenient to use a finite-element representation for the normal wash Ψ and the potential Φ.*

3.1.1 SPACE DISCRETIZATION

Using a general finite-element representation, it is possible to write (Ref. 8),

$$\Psi(\bar{P}, T - \Theta) = \sum_{h=1}^{H} \Psi_h (T - \Theta_h) N_h(\bar{P})$$

$$\Phi(\bar{P}, T - \Theta) = \sum_{h=1}^{H} \Phi_h (T - \Theta_h) N_h(\bar{P}) \qquad (3\text{-}1)$$

*Finite-element representation is meant here in a very broad sense: actually any interpolation formula of the type given by Equations (3-1) and (3-2) is consistent with the formulation presented here. However, for arbitrarily complex configurations, only the finite-element interpolation (including splines over patches) is sufficiently general to be of interest here.

where $\Psi_h(T - \Theta_h)$ and $\Phi_h(T - \Theta_h)$ are time dependent values of Ψ and Φ at the point, $\bar{P}_h$, on Σ_B at the time $T - \Theta_h$ (where Θ_h is the disturbance-propagation time from $\bar{P}_h$ to $\bar{P}_*$); furthermore $N_h(\bar{P})$ are prescribed global shape functions, obtained by standard assembly of the element shape function (see for instance Ref. 16). The points $\bar{P}_h$ will be referred to as nodes*; H is the total number of nodes on the body. For simplicity the same shape functions are used for Φ and Ψ, although this is not essential to the method.

Next consider the integration over the wake. In order to facilitate the use of Equation (2-53), it is convenient to divide the wake into strips defined by (steady-state) vortex lines emanating from the nodes on the trailing edge. The strips are then divided into elements with nodes along the vortex lines. The potential discontinuity can then be expressed as

$$\Delta\Phi(\bar{P}, T - \Theta) = \sum_{n=1}^{N} \Delta\Phi_n(T - \Theta_n)\, L_n(\bar{P}) \tag{3-2}$$

where N is the number of nodes on the wake, $\Delta\Phi_n(T - \Theta_n)$ is the value of $\Delta\Phi$ at the nth node $\bar{P}_n^{(W)}$ on the wake at time $T - \Theta_n$ (where Θ_n is the propagation time from $\bar{P}_n^{(W)}$ to $\bar{P}_*$), and $L_n(\bar{P})$ is the global shape function relative to the nth node of the wake. Note that according to Equation (2-53)

$$\Delta\Phi_n(T) = \Delta\Phi_{m(n)}^{(TE)}(T - \Pi_n) \tag{3-3}$$

*These nodes used to define the functions Ψ and Φ do not necessarily coincide with the points used to define the geometry of the surface Σ_B.

where $m = m(n)$ identifies the trailing-edge node which is on the same vortex-line as the nth node $\bar{P}_n^{(W)}$. Furthermore, Π_n is the time necessary for the vortex-point to be convected from the trailing-edge node $\bar{P}_{m(n)}^{(TE)}$ to the wake node $\bar{P}_n^{(W)}$. It may be worth noting that $\Delta\Phi_m^{(TE)} = \Phi_{h_u} - \Phi_{h_\ell}$ where h_u and h_ℓ identify the trailing-edge nodes (upper and lower sides, respectively) on the body corresponding to the mth node on the trailing edge. Therefore, it is possible to write

$$\Delta\Phi_{m(n)}^{(TE)} = \sum_{h=1}^{H} S_{nh}\Phi_h \tag{3-4}$$

where $S_{nh} = 1$ ($S_{nh} = -1$), if h identifies the upper-side (lower-side) node $\bar{P}_h$ on the body corresponding to the nth node $\bar{P}_n^{(W)}$ on the wake (i.e., $\bar{P}_h$ coincides with the node $\bar{P}_{m(n)}^{(TE)}$ on the trailing edge), and $S_{nh} = 0$ otherwise. Thus,

$$\begin{aligned} S_{nh} &= +1 \qquad \text{if } \bar{P}_h \equiv \bar{P}_{m(n)}^{(TE)} \text{ is on the upper side of } \Sigma_B \\ &= -1 \qquad \text{if } \bar{P}_h \equiv \bar{P}_{m(n)}^{(TE)} \text{ is on the lower side of } \Sigma_B \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{3-5}$$

Combining Equations (2-52), (3-1) and (3-2) one obtains*

$$\begin{aligned} 2E(\bar{P}_*)\,\Phi(\bar{P}_*, T) &= \sum_h B_h \Psi_h (T - \Theta_h) \\ &+ \sum_h C_h \Phi_h(T - \Theta_h) + \sum_h D_h \dot{\Phi}_h(T - \Theta_h) \\ &+ \sum_n F_n \Delta\Phi_n (T - \Theta_n) + \sum_n G_n \Delta\dot{\Phi}_n(T - \Theta_n) \end{aligned} \tag{3-6}$$

*For notational simplicity T_* is replaced with T in this whole section.

where

$$B_h = -\frac{1}{2\pi} \oiint_{\Sigma_B} N_h(\bar{P}) \frac{1}{R} d\Sigma_B$$

$$C_h = \frac{1}{2\pi} \oiint_{\Sigma_B} N_h(\bar{P}) \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_B$$

$$D_h = -\frac{1}{2\pi} \oiint_{\Sigma_B} N_h(\bar{P}) \frac{1}{R} \frac{\partial\hat{\Theta}}{\partial N} d\Sigma_B$$

$$F_n = \frac{1}{2\pi} \iint_{\Sigma_W} L_n(\bar{P}) \frac{\partial}{\partial N_u}\left(\frac{1}{R}\right) d\Sigma_W$$

$$G_n = -\frac{1}{2\pi} \iint_{\Sigma_W} L_n(\bar{P}) \frac{1}{R} \frac{\partial\hat{\Theta}}{\partial N_u} d\Sigma_W \tag{3-7}$$

and according to Equations (3-4) and (3-5)

$$\Delta\Phi_n(T - \Theta_n) = \sum_h S_{nh} \Phi_h (T - \Theta_n - \Pi_n) \tag{3-8}$$

Next consider, in particular, that $\bar{P}_*$ coincides with the node j on Σ_B ($\bar{P}_* = \bar{P}_j$). In this case $E = 1/2$ and, using Equation (3-8), Equation (3-6) reduces to

$$\begin{aligned}\Phi_j(T) &= \sum_h B_{jh} \Psi_h(T - \Theta_{jh}) + \sum_h C_{jh} \Phi_h(T - \Theta_{jh}) \\ &+ \sum_h D_{jh} \dot{\Phi}_h(T - \Theta_{jh}) + \sum_n \sum_h F_{jn} S_{nh} \Phi_h(T - \Theta_{jn} - \Pi_n) \\ &+ \sum_n \sum_h G_{jn} S_{nh} \dot{\Phi}_h(T - \Theta_{jn} - \Pi_n)\end{aligned} \tag{3-9}$$

where

$$\left(B_{jh}, C_{jh}, D_{jh}, F_{jn}, G_{jn}, \Theta_{jh}\right) \equiv \left(B_h, C_h, D_h, F_n, G_n, \Theta_h\right)\Big|_{\bar{P}_* = \bar{P}_j} \tag{3-10}$$

3.1.2 LAPLACE-DOMAIN ANALYSIS

Equation (3-9) indicates the nature of the aerodynamic operator relating potential and normal-wash as obtained by using finite-element representation to discretize the spatial problem. The operator is a linear differential-delay operator to which the methods of operational calculus can be applied. In this section $\tilde{\Phi}$ and $\tilde{\Psi}$ will indicate the Laplace transform of the unsteady part of the potential and normal wash, respectively, with initial conditions obtained from Equation (2-45).* However, before considering the Laplace transform of Equation (3-9), it is convenient to make some remarks about the contribution of the wake. It may be noted that, according to Equation (2-45), Φ_U is identically equal to zero for $T < 0$; therefore, according to Equation (3-4),

$$(\Delta \Phi_U)_n \equiv 0 \qquad (T < \Pi_n) \tag{3-11}$$

Hence, if the analysis is limited to $T < T_{max}$, only the contribution of elements with $\Pi_n < T_{max}$ need be considered, because $\Pi_n = T_{max}$ represents the full extent of the physical wake. The elements with $\Pi_n > T_{max}$ would contribute to the transfer function and thus to the transform of Φ but not to the final solution in the time domain for $T < T_{max}$. The advantage of deleting these latter elements is not only that less computational time is used (since fewer elements are employed) but also that the issue of convergence connected with the infinite wake (factors $e^{-p\Pi_n}$ with Real (p) < 0 and $\Pi_n \to \infty$) need not be addressed.**

---

*See Appendix B.4.
**This problem is analyzed further in Appendix B.2.

Next, taking the Laplace transform of Equation (3-9) yields*

$$\left[\tilde{Y}_{jh}\right]\left\{\tilde{\Phi}_h\right\} = \left[\tilde{Z}_{jh}\right]\left\{\tilde{\Psi}_h\right\} \tag{3-12}$$

where $\tilde{\Phi}_h$ and $\tilde{\Psi}_h$ are the Laplace transforms of the unsteady parts of Φ_h and Ψ_h whereas

$$\tilde{Y}_{jh} = \delta_{jh} - (C_{jh} + p\, D_{jh})\; e^{-p\,\Theta_{jh}}$$

$$- \sum_n (F_{jn} + p\, G_{jn})\; e^{-p\,(\Theta_{jn} + \Pi_n)}\; S_{nh} \tag{3-13}$$

and

$$\tilde{Z}_{jh} = B_{jh}\; e^{-p\,\Theta_{jh}} \tag{3-14}$$

In Equations (3-13) and (3-14), p is the complex reduced frequency (non-dimensional Laplace parameter), given by $p = (\gamma + i)k$, where $k = \omega\ell/U_\infty$ is the reduced frequency and $\gamma = \text{Real}\,(p) / \text{Imag}\,(p)$. Note that for simple harmonic motion, $p = ik$ (see also Appendix B). In particular, steady-state is obtained with $p = 0$.

Equation (3-12) is the desired numerical approximation of the subsonic integral equation relating (in the Laplace domain) the transformed vector of the potential $\left\{\tilde{\Phi}_h\right\}$ to the transformed vector of the normal wash $\left\{\tilde{\Psi}_h\right\}$. It is the form that is actually solved in the SOUSSA-P 1.1 program.

3.2 SUPERSONIC FORMULATION

Using Equation (3-1) and following the same procedure used for the subsonic case, one obtains the supersonic time-domain aerodynamic operator

*The same results are obtained for oscillatory flow (see Appendix B.3). For a discussion on the initial conditions see Appendix B.4.

$$2\,E(\bar{P}_*)\,\Phi(\bar{P}_*,\,T) = \sum_h B_h' \left[\Psi_h'(T-\Theta_h^+) + \Psi_h'(T-\Theta_h^-)\right]$$

$$+ \sum_h C_h' \left[\Phi_h(T-\Theta_h^+) + \Phi_h(T-\Theta_h^-)\right]$$

$$+ \sum_h \left[D_h^+ \dot{\Phi}_h(T-\Theta_h^+) + D_h^- \dot{\Phi}_h(T-\Theta_h^-)\right]$$

$$+ \sum_n F_n' \left[\Delta\Phi_n(T-\Theta_n^+) + \Delta\Phi_n(T-\Theta_n^-)\right]$$

$$+ \sum_n \left[G_n^+ \Delta\dot{\Phi}_n(T-\Theta_n^+) + G_n^- \Delta\dot{\Phi}_n(T-\Theta_n^-)\right] \qquad (3\text{-}15)$$

where

$$B_h' = -\frac{1}{2\pi} \oiint_{\Sigma_B} N_h(\bar{P}) \frac{H}{R'}\, d\Sigma_B$$

$$C_h' = \frac{1}{2\pi} \oiint_{\Sigma_B} N_h(\bar{P}) \frac{\partial}{\partial N^c}\left(\frac{H}{R'}\right) d\Sigma_B$$

$$D_h^{\pm} = -\frac{1}{2\pi} \oiint_{\Sigma_B} N_h(\bar{P}) \frac{H}{R'} \frac{\partial \hat{\dot{\Theta}}^{\pm}}{\partial N^c}\, d\Sigma_B$$

$$F_n' = \frac{1}{2\pi} \iint_{\Sigma_W} L_n(\bar{P}) \frac{\partial}{\partial N_u^c}\left(\frac{H}{R'}\right) d\Sigma_W$$

$$G_n^{\pm} = -\frac{1}{2\pi} \iint_{\Sigma_W} L_n(\bar{P}) \frac{H}{R'} \frac{\partial \hat{\Theta}^{\pm}}{\partial N_u^c}\, d\Sigma_W \qquad (3\text{-}16)$$

with

$$R' = \left[(X - X_*)^2 - (Y - Y_*)^2 - (Z - Z_*)^2\right]^{1/2} \qquad (3\text{-}17)$$

In particular, if $\bar{P}_*$ coincides with the node j, using Equation (3-8), Equation (3-15) reduces to

$$\begin{aligned}
\Phi_j(T) &= \sum_h B'_{jh} \left[\Psi'_h (T - \Theta^+_{jh}) + \Psi'_h (T - \Theta^-_{jh})\right] \\
&+ \sum_h C'_{jh} \left[\Phi_h (T - \Theta^+_{jh}) + \Phi_h (T - \Theta^-_{jh})\right] \\
&+ \sum_h \left[D^+_{jh} \dot{\Phi}_h (T - \Theta^+_{jh}) + D^-_{jh} \dot{\Phi}_h (T - \Theta^-_{jh})\right] \\
&+ \sum_h \sum_n F'_{jn} S_{nh} \left[\Phi_h (T - \Theta^+_{jn} - \Pi_n) + \Phi_h (T - \Theta^-_{jn} - \Pi_n)\right] \\
&+ \sum_h \sum_n \left[G^+_{jn} S_{nh} \dot{\Phi}_h (T - \Theta^+_{jn} - \Pi_n) + G^-_{jn} S_{nh} \dot{\Phi}_h (T - \Theta^-_{jn} - \Pi_n)\right]
\end{aligned} \qquad (3\text{-}18)$$

where

$$\left(B'_{jh}, C'_{jh}, D^{\pm}_{jh}, F'_{jn}, G^{\pm}_{jn}, \Theta^{\pm}_{jh}\right) = \left(B'_h, C'_h, D^{\pm}_h, F'_n, G^{\pm}_n, \Theta^{\pm}_h\right)\Bigg|_{\bar{P}_* = \bar{P}_j} \qquad (3\text{-}19)$$

Finally, taking the Laplace transform of Equation (3-18) one obtains*

$$\left[\tilde{Y}_{jh}\right]\left\{\tilde{\Phi}_h\right\} = \left[\tilde{Z}_{jh}\right]\left\{\tilde{\Psi}'_h\right\} \qquad (3\text{-}20)$$

*The same results are obtained for oscillatory flows (see Appendix B.3). For a discussion on the initial conditions see Appendix B.4.

where

$$
\begin{aligned}
\tilde{Y}_{jh} &= \delta_{jh} - (C'_{jh} + p D^{+}_{jh})\, e^{-p\Theta^{+}_{jh}} - (C'_{jh} + p D^{-}_{jh})\, e^{-p\Theta^{-}_{jh}} \\
&\quad - \sum_n (F'_{jn} + p G^{+}_{jn})\, e^{-p(\Theta^{+}_{jn} + \Pi_n)}\, S_{nh} \\
&\quad - \sum_n (F'_{jn} + p G^{-}_{jn})\, e^{-p(\Theta^{-}_{jn} + \Pi_n)}\, S_{nh} \\
\tilde{Z}_{jh} &= B'_{jh}\left(e^{-p\Theta^{+}_{jh}} + e^{-p\Theta^{-}_{jh}}\right)
\end{aligned}
\tag{3-21}
$$

Equation (3-20) is the desired numerical approximation of the supersonic integral equation relating (in the Laplace domain) the transformed vector of the potential $\left\{\tilde{\Phi}_h\right\}$ to the transformed vector of the conormal wash $\left\{\tilde{\Psi}'_h\right\}$. Note that, for supersonic trailing edges, $F_{jn} \equiv G^{+}_{jn} = G^{-}_{jn} \equiv 0$, i.e., the contribution of the wake is identically equal to zero. Note also that even if the trailing-edges are not completely supersonic (i.e., if a portion of the aircraft is contained in the Mach aftcone of any trailing-edge point), the wake can be truncated at finite distance from the aircraft, without any effect on the solution.

3.3 FORMULATION FOR SOUSSA-P 1.1

This section deals with the subsonic formulation used in SOUSSA-P 1.1. This formulation is a particular case of the general formulation presented above, and is briefly illustrated here. Divide the surface of the aircraft Σ_B into small elements, Σ_h. Consider the shape-function, N_h, equal to one inside Σ_h and equal to zero outside Σ_h, i.e.,

$$
\begin{aligned}
N_h(\bar{P}) &= 1 \qquad \text{if} \qquad \bar{P} \in \Sigma_h \\
&= 0 \qquad \text{otherwise}
\end{aligned}
\tag{3-22}
$$

A point located on the element Σ_h and identified as the center of the element will be designated as the point at which Φ and Ψ are evaluated. Equation (3-1) may thus be interpreted as saying that, within the element Σ_h, the normal wash and the potential are approximated with the values Ψ_h and Φ_h at the center, $\bar{P}_h$, of the element, Σ_h. (Note that the shape functions given by Equation (3-22) may be called zeroth-order shape functions. Therefore, the formulation presently used in SOUSSA-P 1.1 may be called zeroth-order finite-element formulation.)

Next, note that using Equation (3-22), Equation (3-7) yields for instance

$$B_h = -\frac{1}{2\pi} \iint_{\Sigma_h} \frac{1}{R} \, d\Sigma_h \tag{3-23}$$

$$C_h = -\frac{1}{2\pi} \iint_{\Sigma_h} \frac{1}{R^2} \frac{\partial R}{\partial N} \, d\Sigma_h \tag{3-24}$$

$$D_h = -\frac{1}{2\pi} \iint_{\Sigma_h} \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N} \, d\Sigma_h \tag{3-25}$$

If Σ_h is a quadrilateral element, then Σ_h may be approximated with a hyperboloidal element (see Ref. 7)

$$\bar{P} = \bar{P}_{00} + \bar{P}_{10}\,\xi + \bar{P}_{01}\,\eta + \bar{P}_{11}\,\xi\eta$$

$$= \sum_{n=0}^{1} \sum_{m=0}^{1} \bar{P}_{mn}\,\xi^m \eta^n \quad (-1 \le \xi \le 1;\ -1 \le \eta \le 1) \tag{3-26}$$

where ξ and η are local coordinates with origin at the element center, $\bar{P}_{00}$, and $\bar{P}_{mn}$ are obtained in terms of the locations of the four corner points as

$$\begin{Bmatrix} \bar{P}_{00} \\ \bar{P}_{10} \\ \bar{P}_{01} \\ \bar{P}_{11} \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_{++} \\ \bar{P}_{+-} \\ \bar{P}_{-+} \\ \bar{P}_{--} \end{Bmatrix} \qquad (3\text{-}27)$$

Note that quadrilateral hyperboloidal elements can be combined to yield a closed surface. Using these elements, the coefficients C_h and B_h can be evaluated analytically as (see Ref. 17)

$$C_h = I_D(1,1) - I_D(1,-1) - I_D(-1,1) + I_D(-1,-1) \qquad (3\text{-}28)$$

$$B_h = I_S(1,1) - I_S(1,-1) - I_S(-1,1) + I_S(-1,-1) \qquad (3\text{-}29)$$

with [using $-\pi/2 \leq \tan_p^{-1}(\;) \leq \pi/2$]

$$I_D(\xi,\eta) = (1/2\pi)\tan_p^{-1}(\bar{R} \times \bar{A}_1 \cdot \bar{R} \times \bar{A}_2 / |\bar{R}|\bar{R} \cdot \bar{A}_1 \times \bar{A}_2) \qquad (3\text{-}30)$$

and

$$I_S(\xi,\eta) = -\frac{1}{2\pi}\left\{-\bar{R}\times\bar{A}_1\cdot\bar{N}_c\,\frac{1}{|\bar{A}_1|}\sinh^{-1}\left(\frac{\bar{R}\cdot\bar{A}_1}{|\bar{R}\times\bar{A}_1|}\right)\right.$$

$$+\ \bar{R}\times\bar{A}_2\cdot\bar{N}_c\,\frac{1}{|\bar{A}_2|}\sinh^{-1}\left(\frac{\bar{R}\cdot\bar{A}_2}{|\bar{R}\times\bar{A}_2|}\right)$$

$$\left.+\ (\bar{R}\cdot\bar{N})_c\,\tan_p^{-1}\left(\frac{\bar{R}\times\bar{A}_1\cdot\bar{R}\times\bar{A}_2}{|\bar{R}|\bar{R}\cdot\bar{A}_1\times\bar{A}_2}\right)\right\} \qquad (3\text{-}31)$$

where

$$\bar{R}(\xi,\eta) = \bar{P}-\bar{P}_* = \bar{P}_{00}+\bar{P}_{10}\xi+\bar{P}_{01}\eta+\bar{P}_{11}\xi\eta-\bar{P}_* \qquad (3\text{-}32)$$

$$\bar{A}_1(\xi,\eta) = \partial\bar{R}/\partial\xi = \bar{P}_{10}+\eta\,\bar{P}_{11} \qquad (3\text{-}33)$$

$$\bar{A}_2(\xi,\eta) = \partial\bar{R}/\partial\eta = \bar{P}_{01}+\xi\,\bar{P}_{11} \qquad (3\text{-}34)$$

$$\bar{N}(\xi,\eta) = \bar{A}_1\times\bar{A}_2/|\bar{A}_1\times\bar{A}_2| \qquad (3\text{-}35)$$

and the subscript c indicates evaluation at the element center ($\xi = \eta = 0$).

If the values of $\bar{R}\cdot\bar{N}$ at the four corner points of the element do not have the same sign, then the use of the principal value of $\tan^{-1}$ is incorrect (Ref. 18). In this case the $\tan^{-1}$ term is evaluated by replacing the hyperboloidal element with two triangular elements: this is legitimate in view of the fact that the integrand in Equation (3-24) is the solid angle (see Ref. 3) which depends only upon the perimeter of the element, not the actual surface shape. A more complete analysis of the problem is given in Ref. 18.

Next consider the coefficient D_h which is evaluated as

$$D_h = -\frac{1}{2\pi} \iint_{\Sigma_h} \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N} d\Sigma_h = -\frac{M}{2\pi\beta} \iint_{\Sigma_h} \frac{1}{R} \left(\frac{\partial R}{\partial N} - MN_X \right) d\Sigma_h$$

$$\approx -\frac{M}{2\pi\beta} R_h \iint_{\Sigma_h} \frac{1}{R^2} \frac{\partial R}{\partial N} d\Sigma_h + \frac{1}{2\pi\beta} M^2 N_{X_h} \iint_{\Sigma_h} \frac{1}{R} d\Sigma_h$$

$$= (R_h C_h - MN_{X_h} B_h) M / \beta \tag{3-36}$$

where R_h and N_{X_h} are the values of R and $N_X = \bar{N} \cdot \bar{i}$ at the center, $\bar{P}_h$, of the element Σ_h.

Similarly the wake is divided into elements, Σ_n', and the shape-functions, $L_n(\bar{P})$, are defined as

$$\begin{aligned} L_n(\bar{P}) &= 1 \qquad \text{if} \qquad \bar{P} \in \Sigma_n' \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{3-37}$$

Hence the coefficients F_n and G_n are defined as

$$\begin{aligned} F_n &= -\frac{1}{2\pi} \iint_{\Sigma_n'} \frac{1}{R^2} \frac{\partial R}{\partial N_u} d\Sigma_n' \\ G_n &= -\frac{1}{2\pi} \iint_{\Sigma_n'} \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N_u} d\Sigma_n' \end{aligned} \tag{3-38}$$

The definition of F_n is identical with the one of C_h (with Σ_h replaced by Σ'_n). Therefore, the evaluation of F_n is identical to the one of C_h. For G_n, assuming that the wake has small inclination with respect to the X-axis (i.e., $N_X << 1$), the expression is

$$G_n = R_n F_n M / \beta \tag{3-39}$$

where R_n is the value of R at the center of element Σ'_n.

Finally

$$\Theta_h = \left[M(X_h - X_*) + |\bar{P}_h - \bar{P}_*| \right] M / \beta \tag{3-40}$$

is the propagation time from the center $\bar{P}_h$ of the element Σ_h to the control point $\bar{P}_*$. Similarly

$$\Theta_n = \left[M(X_n^{(W)} - X_*) + |\bar{P}_n^{(W)} - \bar{P}_*| \right] M/\beta \tag{3-41}$$

and

$$\Pi_n = \beta \left[X_n^{(W)} - X_{h(n)} \right] \tag{3-42}$$

where h(n) indicates the center of the trailing-edge element from which the wake strip containing the point $\bar{P}_n^{(W)}$ emanates. In the User's Manual (Reference 15), $\check{\Theta}_n$ indicates

$$\check{\Theta}_n = \Theta_n + \Pi_n \tag{3-43}$$

SECTION 4

NORMAL WASH

In this section the relationship between the normal wash Ψ (or the conormal wash Ψ') and the generalized coordinates, q_n, describing the deformation of the aircraft, is presented. It should be noted that in this report the motion of the aircraft is assumed to consist of very small (infinitesimal) perturbations around a steady-state configuration which itself generates a small perturbation from free stream. In particular, in defining the integral equation, the surface, Σ_B, of the body was assumed to be time independent (see Equation (1-10) with $\epsilon = 0$). However, as mentioned in Subsection 2.1, the unsteady contribution is retained in the boundary conditions which are considered in this Section. This point is analyzed in detail in Appendix C where it is shown that in order to take into account the motion of the surface, Ψ in Equation (2-34) is given by Equation (C.18), i.e.,

$$\Psi = \frac{1}{U_\infty} \psi_M \frac{|\bar{a}_1^{(t)} \times \bar{a}_2^{(t)}|}{|\bar{a}_1 \times \bar{a}_2|} \tag{4-1}$$

where the superscript (t) indicates evaluation with time-dependent surface ($\epsilon \neq 0$ in Equation (1-10)). The vectors $\bar{a}_\alpha$ are evaluated from the unperturbed surface ($\epsilon = 0$ in Equation (1-10)). The subscript M on ψ indicates that the modified expression for ψ (including the motion of the surface) is used here (see Appendix C).

4.1 PHYSICAL BOUNDARY CONDITION

The boundary condition on the body is obtained by imposing the condition that a fluid particle which is on the point $\bar{p}$ of the surface of the body at time t will remain on the surface, i.e., will be on the point $\bar{p} + \Delta\bar{p}$ (also on the surface of the body) at time $t + \Delta t$. If the surface of the body is described by

$$S(\bar{p}, t) = 0 \tag{4-2}$$

then it is also true that

$$S(\bar{p} + \Delta\bar{p},\ t + \Delta t) = 0 \tag{4-3}$$

with $\Delta\bar{p} = \bar{V}_F \Delta t$, where $\bar{V}_F$ is the velocity of the fluid, $\bar{V}_F = \nabla\phi$. Taking the Taylor series of Equation (4-3), and using Equation (4-2), one obtains

$$\left(\frac{\partial S}{\partial t} + \nabla S \cdot \bar{V}_F\right) \Delta t + O(\Delta t^2) = 0 \tag{4-4}$$

or, taking the limit as Δt goes to zero,

$$\frac{DS}{Dt} = 0 \tag{4-5}$$

where, using Equations (1-1) and (1-3)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{V}_F \cdot \nabla = \frac{\partial}{\partial t} + \nabla\phi \cdot \nabla$$

$$= \frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x} + \nabla\varphi \cdot \nabla = \frac{d}{dt} + \nabla\varphi \cdot \nabla \tag{4-6}$$

Note that ∇S can be assumed to be directed as the outwardly directed normal $\bar{n}$ (eventually Equation (4-2) must be multiplied by -1) so that

$$\bar{n}^{(t)} = \frac{\nabla S}{|\nabla S|} \tag{4-7}$$

The superscript (t) emphasizes that the normal is evaluated at t since $\bar{n}$ indicates the normal in the steady-state configuration.

Thus, the boundary condition, Equation (4-5), may be written as

$$\frac{1}{|\nabla S|} \frac{dS}{dt} + \nabla\varphi \cdot \frac{\nabla S}{|\nabla S|} = 0 \tag{4-8}$$

or

$$\Psi_M = -\frac{dS}{dt} \Big/ |\nabla S| = -\frac{1}{|\nabla S|}\left(\frac{\partial S}{\partial t} + U_\infty \frac{\partial S}{\partial x}\right) \tag{4-9}$$

4.2 RELATIONSHIP BETWEEN NORMAL WASH AND DISPLACEMENT

Next note that while Equation (4-9) is the classical form for the boundary conditions, here it is convenient to express the boundary condition in terms of the displacement, $\bar{u}$. Therefore, the surface of the body will be assumed to be given not by Equation (4-2) but rather as

$$\bar{x} = \bar{x}(t) = \bar{p}(\xi^\alpha) + \bar{u}(\xi^\alpha, t) \tag{4-10}$$

where $\bar{p}$ gives the reference configuration rigidly connected to the $\bar{i}$, $\bar{j}$, $\bar{k}$-space while $\bar{u}$ is a small steady or unsteady displacement relative to this configuration. In Equation (4-10), ξ^α ($\alpha = 1,2$) are convected curvilinear coordinates, moving with the surface of the body. The normal $\bar{n}^{(t)}$ is then given by

$$\bar{n}^{(t)} = \frac{\bar{a}_1^{(t)} \times \bar{a}_2^{(t)}}{|\bar{a}_1^{(t)} \times \bar{a}_2^{(t)}|} \tag{4-11}$$

with

$$\bar{a}_\alpha^{(t)} = \frac{\partial \bar{x}}{\partial \xi^\alpha} = \frac{\partial \bar{p}}{\partial \xi^\alpha} + \frac{\partial \bar{u}}{\partial \xi^\alpha} = \bar{a}_\alpha + \bar{u}_\alpha \tag{4-12}$$

where $\bar{u}_\alpha = \partial \bar{u}/\partial \xi^\alpha$.

In order to rewrite Equation (4-9) in a form compatible with Equation (4-10), introduce the velocity of the surface of the body

$$\bar{v}_B = \frac{\partial \bar{x}}{\partial t} = \frac{\partial \bar{u}}{\partial t} \qquad (4\text{-}13)$$

Then the component of $\bar{v}_B$ along the normal $\bar{n}^{(t)}$ is given by*

$$\bar{v}_B \cdot \bar{n}^{(t)} = -\frac{\partial S}{\partial t} \Big/ |\nabla S| \qquad (4\text{-}14)$$

Note also that the x-component of the normal $\bar{n}^{(t)}$ is given by

$$n_x^{(t)} = \frac{\partial S}{\partial x} \Big/ |\nabla S| \qquad (4\text{-}15)$$

Therefore Equation (4-9) may be rewritten as

$$\Psi_M = \left(-U_\infty \bar{i} + \bar{v}_B\right) \cdot \bar{n}^{(t)} \qquad (4\text{-}16)$$

Finally combining Equation (4-1), (4-11) and (4-16) one obtains

$$\Psi = \frac{1}{U_\infty} \left(-U_\infty \bar{i} + \bar{v}_B\right) \cdot \frac{\bar{a}_1^{(t)} \times \bar{a}_2^{(t)}}{|\bar{a}_1 \times \bar{a}_2|} \qquad (4\text{-}17)$$

Note that using Equation (4-12) one obtains

$$\frac{\bar{a}_1^{(t)} \times \bar{a}_2^{(t)}}{|\bar{a}_1 \times \bar{a}_2|} = \bar{n} + \Delta \bar{n} \qquad (4\text{-}18)$$

where

$$\Delta \bar{n} = \left(\bar{u}_1 \times \bar{a}_2 + \bar{a}_1 \times \bar{u}_2 + \bar{u}_1 \times \bar{u}_2\right) / |\bar{a}_1 \times \bar{a}_2| \qquad (4\text{-}19)$$

---

* Equation (4-14) may be obtained from Equation (4-5) written for a point moving with the surface, i.e., having velocity $\bar{v}_B$. This yields $\frac{\partial S_B}{\partial t} \Big/ |\nabla S_B| + \bar{v}_B \cdot \bar{n} = 0$ in agreement with Equation (4-14). Note that Equation (4-13) is valid only for a wind-axis formulation. The extension to body-axis formulation is given in Appendix D.

This yields

$$\Psi = \left(-\bar{i} + \frac{1}{U_\infty}\ \dot{\bar{u}}\right) \cdot (\bar{n} + \Delta\bar{n}) = \Psi_S + \Psi_U \tag{4-20}$$

where the steady part of the normal wash is given by

$$\Psi_S = -\bar{i} \cdot \bar{n} \tag{4-21}$$

and the unsteady part is given by

$$\Psi_U = \left[\frac{1}{U_\infty}\ \dot{\bar{u}} \cdot (\bar{n} + \Delta\bar{n}) - \bar{i} \cdot \Delta\bar{n}\right] \tag{4-22}$$

4.3 LINEAR RELATIONSHIP BETWEEN NORMAL WASH AND GENERALIZED COORDINATES

Let the displacement $\bar{u}$ be expressed as

$$\bar{u}(\xi^\alpha, t) = \ell \sum_{m=1}^{M} q_m(t)\ \bar{M}_m(\xi^\alpha) \tag{4-23}$$

where q_m are the nondimensional generalized (Lagrangian) coordinates and $\bar{M}_m(\xi^\alpha)$ are prescribed nondimensional modes (ordinarily the natural undamped modes of vibration of the aircraft). In SOUSSA-P 1.1 the motion is assumed to consist of small perturbations around a steady-state motion. Therefore q_m are assumed to be small. Then, to first order one obtains

$$\Delta\bar{n} = \sum_{m=1}^{M} q_m\ \Delta\bar{n}_m \tag{4-24}$$

$$\Delta\bar{n}_m = \ell\left(\frac{\partial \bar{M}_m}{\partial \xi^1} \times \bar{a}_2 + \bar{a}_1 \times \frac{\partial \bar{M}_m}{\partial \xi^2}\right) / |\bar{a}_1 \times \bar{a}_2| \tag{4-25}$$

and, again neglecting higher order terms,

$$\Psi_U = \frac{\ell}{U_\infty} \sum_{m=1}^{M} \bar{M}_m \cdot \bar{n}\, \dot{q}_m - \sum_{m=1}^{M} \bar{i} \cdot \Delta\bar{n}_m q_m \tag{4-26}$$

Next, taking the Laplace transform of Equation (4-26), one obtains*

$$\tilde{\Psi}_h = \sum_{m=1}^{M} \left[\frac{\ell}{U_\infty} s\bar{M}_m \cdot \bar{n} - \bar{i} \cdot \Delta\bar{n}_m \right]_{\bar{p}=\bar{p}_h} \tilde{q}_m \tag{4-27}$$

(where s is the Laplace parameter) or

$$\left\{ \tilde{\Psi}_h \right\} = \left[\tilde{E}^{(1)}_{hm} \right] \left\{ \tilde{q}_m \right\} \tag{4-28}$$

with

$$\left[\tilde{E}^{(1)}_{hm} \right] = \left[p\bar{M}_m \cdot \bar{n} - \bar{i} \cdot \Delta\bar{n}_m \right]_{\bar{p}=\bar{p}_h} \tag{4-29}$$

where

$$p = s\ell/U_\infty \tag{4-30}$$

is the nondimensional Laplace parameter (complex reduced frequency).

Equation (4-29) is used in SOUSSA-P 1.1 for the evaluation of the normal-wash matrix $\left[\tilde{E}^{(1)}_{hm}\right]$.

*It is assumed $q_n(0) = 0$ (see Appendix B.4).

SECTION 5

PRESSURE COEFFICIENTS AND GENERALIZED AERODYNAMIC FORCES

In this section the relationship between the pressure coefficient and the potential is considered first. For the sake of completeness the formulation for arbitrary finite-element representation is included here. The modifications necessary for the zeroth order formulation used in SOUSSA-P 1.1 are then outlined. These include a special definition of the shape functions (making use of the averaging scheme first introduced in Ref. 9) and the special evaluation of the pressure coefficient for trailing-edge elements (for both subsonic and supersonic flow). Finally the relationship between generalized aerodynamic forces and pressure coefficient is presented for an arbitrary as well as zeroth-order finite-element representation.

5.1 PRESSURE COEFFICIENT

The pressure coefficient is evaluated from the linearized Bernoulli theorem as

$$C_p = -\frac{2}{U_\infty^2}\frac{d\varphi}{dt} = -2\frac{\ell}{U_\infty}\left(\frac{\partial\Phi}{\partial t} + U_\infty\frac{\partial\Phi}{\partial x}\right) \qquad (5\text{-}1)$$

In order to evaluate $\frac{\partial\Phi}{\partial x}$, it is convenient to use the following procedure. Note that

$$\frac{\partial\Phi}{\partial x} \equiv \nabla\Phi\cdot\bar{i} \qquad (5\text{-}2)$$

and that $\nabla\Phi$ may be evaluated as

$$\nabla\Phi = \frac{\partial\Phi}{\partial\xi^1}\,\bar{a}^1 + \frac{\partial\Phi}{\partial\xi^2}\,\bar{a}^2 + \frac{\partial\Phi}{\partial n}\,\bar{n} \qquad (5\text{-}3)$$

where

$$\bar{a}^{\alpha} = a^{\alpha 1} \bar{a}_1 + a^{\alpha 2} \bar{a}_2 \tag{5-4}$$

with

$$\begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \Bigg/ \left(a_{11}\, a_{22} - a_{12}\, a_{21}\right) \tag{5-5}$$

The covariant components $a_{\alpha\beta}$ of the metric tensor are given by

$$a_{\alpha\beta} = \bar{a}_{\alpha} \cdot \bar{a}_{\beta} \tag{5-6}$$

where the base vectors $\bar{a}_{\alpha}$ are given by

$$\bar{a}_{\alpha} = \frac{\partial \bar{p}}{\partial \xi^{\alpha}} \tag{5-7}$$

Therefore,

$$C_p = -2\,\ell \left(\frac{1}{U_{\infty}} \frac{\partial \Phi}{\partial t} + \bar{i} \cdot \bar{a}^1 \frac{\partial \Phi}{\partial \xi^1} + \bar{i} \cdot \bar{a}^2 \frac{\partial \Phi}{\partial \xi^2} + \bar{i} \cdot \bar{n} \frac{\partial \Phi}{\partial n} \right) \tag{5-8}$$

Next note that, as with Equation (3-1)

$$\Phi\,(\xi^{\alpha}, T) = \sum_{h=1}^{H} \Phi_h(T)\; \hat{N}_h\,(\xi^{\alpha}) \tag{5-9}$$

(where $\hat{N}_h\,(\xi^{\alpha})$ does not necessarily coincide with N_h). Hence, the tangential derivatives are evaluated as

$$\frac{\partial \Phi}{\partial \xi^{\alpha}} = \sum_{h=1}^{H} \Phi_h \frac{\partial \hat{N}_h}{\partial \xi^{\alpha}} \tag{5-10}$$

Thus neglecting the term $n_x \frac{\partial \Phi}{\partial n}$ and combining Equations (5-8) and (5-10) one obtains

$$C_p(\bar{p}_k, t) = -2 \frac{\ell}{U_\infty} \frac{\partial \Phi_k}{\partial t} - 2 \sum_h \bar{i} \cdot \left[\bar{a}^1 \frac{\partial \hat{N}_h}{\partial \xi^1} + \bar{a}^2 \frac{\partial \hat{N}_h}{\partial \xi^2} \right]_{\bar{p}=\bar{p}_k} \Phi_h \tag{5-11}$$

or in the frequency-domain

$$\tilde{C}_p(\bar{p}_k) = -2p\tilde{\Phi}_k - 2 \sum_h \bar{i} \cdot \left[\bar{a}^1 \frac{\partial \hat{N}_h}{\partial \xi^1} + \bar{a}^2 \frac{\partial \hat{N}_h}{\partial \xi^2} \right]_{\bar{p}=\bar{p}_k} \tilde{\Phi}_h \tag{5-12}$$

i.e.

$$\left\{ \tilde{C}_{p,k} \right\} = \left[\hat{E}_{kh}^{(3)} \right] \left\{ \tilde{\Phi}_h \right\} \tag{5-13}$$

with

$$\hat{E}_{kh}^{(3)} = -2p\,\delta_{kh} - 2\,\bar{i} \cdot \left[\bar{a}^1 \frac{\partial \hat{N}_h}{\partial \xi^1} + \bar{a}^2 \frac{\partial \hat{N}_h}{\partial \xi^2} \right]_{\bar{p}=\bar{p}_k} \tag{5-14}$$

5.2 FORMULATION FOR SOUSSA-P 1.1

Next it should be noted that the formulation for the evaluation of the pressure as described in Section 5.1 needs modification in the case of zeroth-order formulation, for Equation (5-10) cannot be used in connection with Equation (3-22) unless it is interpreted in terms of the theory of distribution (or generalized functions). In order to avoid this problem, the following procedure is used in SOUSSA-P 1.1: from the values, Φ_h, of the potential at the centers $\bar{P}_h$ of the elements, Σ_h, the values, Φ_k', of the potential at the corners, $\bar{P}_k'$, of the elements, are evaluated by averaging. In other words

$$\{\Phi_k'\} = [\hat{A}_{kh}]\ \{\Phi_h\} \tag{5-15}$$

where $[\hat{A}_{kh}]$ is a weighted averaging matrix defined as

$$\hat{A}_{kh} = W_h \quad \text{if} \quad \bar{P}_k' \in \Sigma_h \tag{5-16}$$

(i.e., if the $\bar{P}_k'$ is one of the corner points of the element Σ_h) and

$$\hat{A}_{kh} = 0 \qquad \text{otherwise.} \tag{5-17}$$

In Equation (5-16), the weights W_h are proportional to a typical length of the element (which is assumed to be equal to the square root of the area of the element Σ_h). Having evaluated the values of Φ at the corner points, the potential is expressed as

$$\Phi = \sum_k \Phi_k' \hat{N}_k' (\xi^1, \xi^2) \tag{5-18}$$

where $\hat{N}_k'$ are first-order global shape functions obtained by assembling local shape functions of the type

$$\hat{N}_k^{(E)} = \frac{1}{4\xi_k \eta_k} (\xi + \xi_k)(\eta + \eta_k) \tag{5-19}$$

where $\xi_k = \pm 1$ and $\eta_k = \pm 1$ are the locations of the corners $\bar{P}_k'$ of the element Σ_h (ξ and η are the coordinates over the element; see Equation (3-26)).

Thus, Equation (5-10) is replaced by

$$\frac{\partial \Phi}{\partial \xi^\alpha} = \sum_k \Phi_k' \frac{\partial \hat{N}_k'}{\partial \xi^\alpha} = \sum_h \left(\sum_k \frac{\partial \hat{N}_k'}{\partial \xi^\alpha} \right) \hat{A}_{kh} \Phi_h \tag{5-20}$$

Equation (5-20) is formally equal to Equation (5-10) with

$$\frac{\partial \hat{N}_h}{\partial \xi^\alpha} = \sum_k \frac{\partial \hat{N}'_k}{\partial \xi^\alpha} \hat{A}_{kn} \tag{5-21}$$

However, note that the averaging scheme cannot be used to evaluate the value of the potential difference between upper and lower sides at the trailing-edge nodes. Therefore, a different method has to be used to evaluate the pressure at the centers of the elements adjacent to the trailing-edge.

Consider the subsonic case first. In this case the trailing edge evaluation is based upon the Kutta condition, i.e., that the pressure discontinuity goes to zero like the square root of the distance from the trailing edge. Therefore, near the trailing edge,

$$\tilde{C}_{p_u} \simeq k_A + k_D \sqrt{x_{TE} - x}$$

$$\tilde{C}_{p_\ell} \simeq k_A - k_D \sqrt{x_{TE} - x} \tag{5-22}$$

Noe that the average value of the potential $(\varphi_u + \varphi_\ell)/2$, at the trailing edge can be evaluated correctly using the averaging scheme. Hence, indicating with $\tilde{C}'_p$ the values obtained with the averaging scheme, k_A may be obtained from the values of $\tilde{C}'_p$ at $\bar{p}_{i_1}$ and $\bar{p}_{i_2}$ (assuming that $x_{i_1} \simeq x_{i_2}$, see Figure 5-1), as

$$\tilde{C}_{p_u} + \tilde{C}_{p_\ell} = \text{constant} = 2k_A \simeq \tilde{C}'_{p_{i_1}} + \tilde{C}'_{p_{i_2}} \tag{5-23}$$

Note that, according to Equation (5-22), k_D may be evaluated from the values of $\tilde{C}'_p$ at $\bar{p}_{i_3}$ and $\bar{p}_{i_4}$ (assuming that $x_{i_3} \simeq x_{i_4}$, see Figure 5-1), as

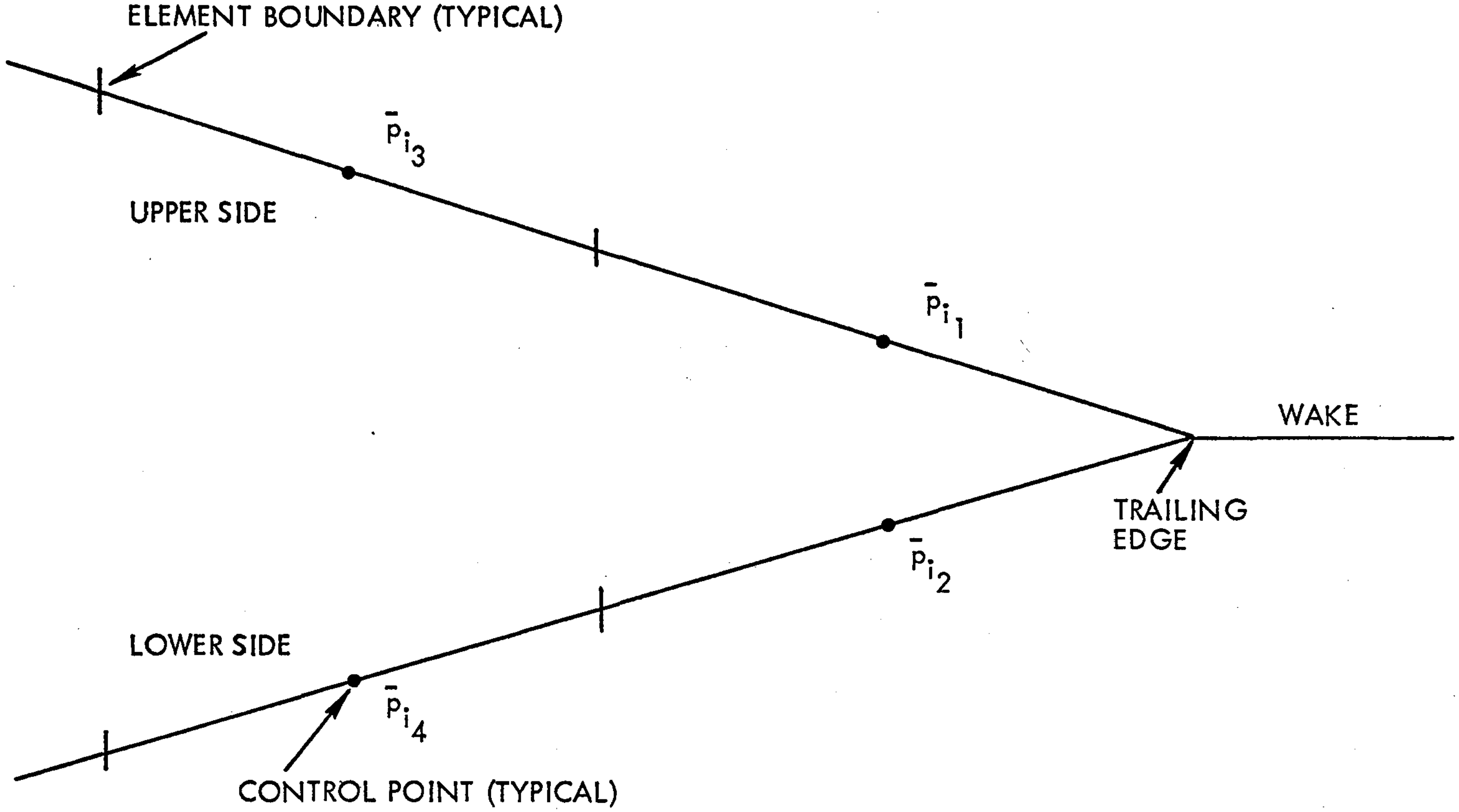

Figure 5-1. Geometry for Trailing Edge Evaluation

$$(\tilde{C}_{p_u} - \tilde{C}_{p_\ell})(x_{TE} - x)^{-1/2} = \text{constant} = 2k_D$$

$$\approx (\tilde{C}'_{p_{i_3}} - \tilde{C}'_{p_{i_4}}) \left(x_{TE} - \frac{x_{i_3} + x_{i_4}}{2} \right)^{-1/2} \tag{5-24}$$

Equations (5-23) and (5-24) yield

$$k_D = \frac{1}{2} \left(\tilde{C}'_{p_{i_3}} - \tilde{C}'_{p_{i_4}} \right) \left(x_{TE} - \frac{x_{i_3} + x_{i_4}}{2} \right)^{-1/2}$$

$$k_A = \frac{1}{2} \left(\tilde{C}'_{p_{i_1}} + \tilde{C}'_{p_{i_2}} \right) \tag{5-25}$$

Equations (5-24) and (5-25) may be used to evaluate $\tilde{C}_{p_{i_1}}$ and $\tilde{C}_{p_{i_2}}$ (assuming that $x_{i_1} \simeq x_{i_2}$) as

$$\tilde{C}_{p_{i_1}} = \frac{1}{2} (\tilde{C}'_{p_{i_1}} + \tilde{C}'_{p_{i_2}} + \hat{\rho}\, \tilde{C}'_{p_{i_3}} - \hat{\rho}\, \tilde{C}'_{p_{i_4}}) \tag{5-26}$$

$$\tilde{C}_{p_{i_2}} = \frac{1}{2} (\tilde{C}'_{p_{i_1}} + \tilde{C}'_{p_{i_2}} - \hat{\rho}\, \tilde{C}'_{p_{i_3}} + \hat{\rho}\, \tilde{C}'_{p_{i_4}}) \tag{5-27}$$

where

$$\hat{\rho} = \left(\frac{2x_{TE} - x_{i_1} - x_{i_2}}{2x_{TE} - x_{i_3} - x_{i_4}} \right)^{1/2} \tag{5-28}$$

Note that Equation (5-26) together with

$$\tilde{C}_{p_i} = \tilde{C}'_{p_i} \quad (\Sigma_i \text{ not on trailing edge}) \tag{5-29}$$

may be rewritten as

$$\left\{\tilde{C}_{p_i}\right\} = \left[E_{ij}^{(TE)}\right]\left\{\tilde{C}'_{p_i}\right\} \tag{5-30}$$

where the matrix $E_{ij}^{(TE)}$ is obtained from the unit matrix (see Equation 5-29) by replacing the row corresponding to the upper-side elements Σ_{i_1} which are in contact with the trailing edge (see Equation (5-26)), i.e., for $i = i_1$, with

$$\begin{aligned} E_{ij}^{(TE)} &= 1/2 && \text{for} \quad j = i_1 \\ &= 1/2 && \text{for} \quad j = i_2 \\ &= \hat{p}/2 && \text{for} \quad j = i_3 \\ &= -\hat{p}/2 && \text{for} \quad j = i_4 \\ &= 0 && \text{otherwise} \end{aligned} \tag{5-31}$$

Similarly for the lower-side elements Σ_{i_2}, i.e., for $i = i_2$,

$$\begin{aligned} E_{ij}^{(TE)} &= 1/2 && \text{for} \quad j = i_1 \\ &= 1/2 && \text{for} \quad j = i_2 \\ &= -\hat{p}/2 && \text{for} \quad j = i_3 \\ &= \hat{p}/2 && \text{for} \quad j = i_4 \\ &= 0 && \text{otherwise} \end{aligned} \tag{5-32}$$

This is easily accomplished in the program by introducing an index function

$$i_h = i_{Kutta}(h, m) \tag{5-33}$$

(where m ranges over all the trailing edge segments and h = 1, 2, 3, 4) which gives the element numbers i_1, i_2, i_3 and i_4 relative to the m^{th} trailing edge segment.

The above results may be combined by noting that

$$\left[\tilde{E}_{kh}^{(3)}\right] = \left[E_{ki}^{(TE)}\right]\left[\hat{E}_{ih}^{(3)}\right] \tag{5-34}$$

gives the proper values for $\tilde{C}_p$ at all the element centers as

$$\left\{\tilde{C}_{p,k}\right\} = \left[\tilde{E}_{kh}^{(3)}\right]\left\{\tilde{\Phi}_h\right\} \tag{5-35}$$

5.3 GENERALIZED AERODYNAMIC FORCES

Next consider the generalized aerodynamic forces e_n given by Equation (1-13). In order to obtain the relationship between e_n and $C_{p,h}$ set, in analogy to Equation (3-1)

$$C_p = \sum_{h=0}^{H} C_{p,h} N_h(\xi^\alpha) \tag{5-36}$$

Combining Equations (1-13) and (5-36) yields

$$e_n = \frac{-1}{\ell^2} \oiint_\sigma \left(\sum_{h=0}^{H} C_{p,h} N_h \right) \bar{n} \cdot \bar{M}_n \, d\sigma \tag{5-37}$$

Equation (5-37) may be rewritten as

$$\{e_n\} = [E^{(4)}_{nh}] \ \{C_{p,h}\} \tag{5-38}$$

where

$$E^{(4)}_{nh} = \frac{-1}{\ell^2} \oiint_{\sigma} N_h \, \bar{n} \cdot \bar{M}_n \, d\sigma \tag{5-39}$$

In SOUSSA-P 1.1, in approximating Equation (1-13) and hence Equation (5-39), the pressure is assumed to be constant within each element, i.e., the pressure mode shapes are given in Equation (3-22). (This is consistent with the numerical approximation of the integral equation.) Combining Equation (3-22) and (5-37) one obtains

$$\begin{aligned} E^{(4)}_{nh} &= \frac{-1}{\ell^2} \iint_{\sigma_h} \bar{n} \cdot \bar{M}_n \, d\sigma \\ &= \frac{-1}{\ell^2} \int_{-1}^{1}\int_{-1}^{1} \bar{a}_1 \times \bar{a}_2 \cdot \bar{M}_n \, d\xi^1 \, d\xi^2 \end{aligned} \tag{5-40}$$

or, approximating the integrand with its value at the center,

$$E^{(4)}_{nh} = \frac{-4}{\ell^2} \left(\bar{a}_1 \times \bar{a}_2 \cdot \bar{M}_n\right)\Big|_{\bar{P} = \bar{P}_h} \tag{5-41}$$

Note that $\bar{a}_\alpha$ are the base vectors in the physical space, not in the Prandtl-Glauert space. It should also be noted that the formulation presented in this subsection is valid for both subsonic and supersonic flows.

SECTION 6

APPLICATIONS

In this Section the formulation developed in the preceding Sections is applied to two specific problems: aerodynamic analysis (with emphasis on flutter analysis) and aerodynamic design.

6.1 AERODYNAMIC ANALYSIS

For the sake of clarity matrix notations will be used throughout this Section. For instance, Equation (4-28) may be written as

$$\underline{\tilde{\Psi}} = \underline{\tilde{E}}_1 \ \underline{\tilde{q}} \tag{6-1}$$

where

$$\underline{\tilde{\Psi}} = \left\{ \tilde{\Psi}_h \right\} \tag{6-2}$$

$$\underline{\tilde{q}} = \left\{ \tilde{q}_m \right\} \tag{6-3}$$

and

$$\underline{\tilde{E}}_1 = \left[\tilde{E}^{(1)}_{hm} \right] \tag{6-4}$$

Similarly Equations (3-12), (3-20), (5-35) and (5-38) may be written as

$$\underline{\tilde{Y}} \ \underline{\tilde{\Phi}} = \underline{\tilde{Z}} \ \underline{\tilde{\Psi}} \tag{6-5}$$

$$\underline{\tilde{C}}_p = \underline{\tilde{E}}_3 \ \underline{\tilde{\Phi}} \tag{6-6}$$

$$\underline{\tilde{e}} = \underline{E}_4 \ \underline{\tilde{C}}_p \tag{6-7}$$

where $\tilde{\underline{\Psi}}$ is given by Equation (6-2), $\tilde{\underline{\Phi}} = \{\tilde{\Phi}_n\}$, $\tilde{\underline{C}}_p = \{\tilde{C}_{p,k}\}$, and $\tilde{\underline{e}} = \{\tilde{e}_n\}$, whereas

$$\tilde{\underline{Y}} = \left[\tilde{Y}_{jh}\right]; \qquad \tilde{\underline{Z}} = \left[\tilde{Z}_{jh}\right] \tag{6-8}$$

$$\tilde{\underline{E}}_3 = \left[\tilde{E}^{(3)}_{kh}\right] \tag{6-9}$$

$$\underline{E}_4 = \left[E^{(4)}_{nh}\right] \tag{6-10}$$

Combining Equations (6-1), (6-5), (6-6) and (6-7) one obtains

$$\tilde{\underline{\Phi}} = \tilde{\underline{E}}_{21} \tilde{\underline{q}} \tag{6-11}$$

$$\tilde{\underline{C}}_p = \tilde{\underline{E}}_{321} \tilde{\underline{q}} \tag{6-12}$$

$$\tilde{\underline{e}} = \tilde{\underline{E}} \tilde{\underline{q}} \tag{6-13}$$

with

$$\tilde{\underline{E}}_{21} = \tilde{\underline{E}}_2 \tilde{\underline{E}}_1 \tag{6-14}$$

$$\tilde{\underline{E}}_{321} = \tilde{\underline{E}}_3 \tilde{\underline{E}}_2 \tilde{\underline{E}}_1 \tag{6-15}$$

$$\tilde{\underline{E}} = \underline{E}_4 \tilde{\underline{E}}_3 \tilde{\underline{E}}_2 \tilde{\underline{E}}_1 \tag{6-16}$$

where

$$\tilde{\underline{E}}_2 = \tilde{\underline{Y}}^{-1} \tilde{\underline{Z}} \tag{6-17}$$

6.1.1. PHYSICAL MEANING OF $\tilde{\underline{E}}_1$, $\tilde{\underline{E}}_{21}$, $\tilde{\underline{E}}_{321}$ AND $\tilde{\underline{E}}$

Note that the nth columns of the matrices $\tilde{\underline{E}}_{21}$, $\tilde{\underline{E}}_{321}$ and $\tilde{\underline{E}}$ give the vectors of the values of the potential, pressure coefficients, and generalized forces, respectively, due to the nth boundary-condition mode (nth column of $\tilde{\underline{E}}_1$). This is apparent from Equations (6-11), (6-12) and (6-13) by setting

$$\tilde{q}_m = 1 \qquad m = n$$

$$= 0 \qquad m \neq n \tag{6-18}$$

Finally, note that the matrix $\tilde{\underline{E}}$ given by Equation (6-12) is the matrix of the generalized aerodynamic forces used, for instance, in flutter analysis. (This is apparent from Equation (6-13).) Other applications for the matrix $\tilde{\underline{E}}$ include evaluation of aerodynamic coefficients, stability derivatives and forces due to turbulence.

6.2 AERODYNAMIC DESIGN*

Consider the aerodynamic-design problem (also known as the inverse problem) of determining the aircraft surface shape which will generate a desired (prescribed) steady-state ($p = 0$) pressure distribution.

The problem may be solved using the procedure presented here. Let $\bar{p}^{(0)}$ and $\bar{p}^{(1)}$ indicate an initial-estimate shape and the revised shape, respectively. Let the displacement $\bar{u} = \bar{p}^{(1)} - \bar{p}^{(0)}$ be expressed as

$$\bar{u} = \sum_{1}^{N} q_n \bar{M}_n(\xi^{\alpha}) \tag{6-19}$$

The difference in pressure between initial and new shape is given by

$$\underline{\delta} = \underline{C}_p^{(1)} - \underline{C}_p^{(0)} = \hat{\underline{E}}\, \underline{q} \tag{6-20}$$

where ($p = 0$, for steady state)

$$\hat{\underline{E}} = \underline{E}_{321}\Big|_{p=0} \tag{6-21}$$

If the pressure is prescribed at N points, solving Equation (6-20) one obtains the values q_n and hence the displacement $\bar{u}$ and from this the new shape. If necessary the procedure might be repeated until the iteration converges. (At any iteration a

*This option is not available in the version 1.1 of the program SOUSSA-P.

new geometry is considered: this implies re-evaluating the matrices $\tilde{\underline{E}}_1$, $\tilde{\underline{E}}_2$ and $\tilde{\underline{E}}_3$, which depend upon the geometry.)

Note, however, that the above procedure is likely to yield wiggly shapes. Hence it is convenient to use few mode shapes and satisfy Equation (6-20) in a least-square sense, i.e.*

$$\left(\underline{\delta}^T - \underline{q}^T \hat{\underline{E}}^T\right)\left(\hat{\underline{E}}\,\underline{q} - \underline{\delta}\right) = \underset{\underline{q}}{\text{minimum}} \tag{6-22}$$

i.e.

$$\hat{\underline{E}}^T\hat{\underline{E}}\,\underline{q} = \hat{\underline{E}}^T\underline{\delta} \tag{6-23}$$

which yields

$$\underline{q} = \left(\hat{\underline{E}}^T\hat{\underline{E}}\right)^{-1} \hat{\underline{E}}^T\underline{\delta} \tag{6-24}$$

Equation (6-24) may be used to obtain the values of $\underline{q}$ from the desired (prescribed) change $\underline{\delta}$, in pressure distribution.

*Smoothing between iterations is an alternative procedure.

SECTION 7

CONCLUDING REMARKS

A general formulation for subsonic and supersonic, steady, oscillatory and unsteady aerodynamics for complex aircraft configuration has been presented. The formulation is used in the computer program SOUSSA-P 1.1. A few comments about the present formulation are presented in this Section.

7.1 ADVANTAGES OF FORMULATION

The formulation presented here offers a unified approach to solve steady, oscillatory and unsteady problems for both subsonic and supersonic flow. The formulations for subsonic and supersonic are very similar and highly "compatible" from a programming point of view. The formulation is such that very little human intervention is necessary; each surface element is treated the same way.

The formulation is very modular. The matrices $\tilde{\underset{\sim}{E}}_1$, $\tilde{\underset{\sim}{E}}_2$, $\tilde{\underset{\sim}{E}}_3$ and $\underset{\sim}{E}_4$ are obtained in a totally independent way. Note that the formulation is particularly advantageous for multiple frequency evaluation. For, note that, comparing Equations (4-29) and (4-30), it is possible to write

$$\tilde{\underset{\sim}{E}}_1 = \tilde{\underset{\sim}{E}}_1^{(0)} + p\,\tilde{\underset{\sim}{E}}_1^{(1)} \tag{7-1}$$

where $\tilde{\underset{\sim}{E}}_1^{(0)}$ and $\tilde{\underset{\sim}{E}}_1^{(1)}$ are independent of p. Similarly (see Equations (5-14) and (5-34))

$$\tilde{\underset{\sim}{E}}_3 = \tilde{\underset{\sim}{E}}_3^{(0)} + p\,\tilde{\underset{\sim}{E}}_3^{(1)} \tag{7-2}$$

where $\tilde{\underset{\sim}{E}}_3^{(0)}$ and $\tilde{\underset{\sim}{E}}_3^{(1)}$ are independent of p. Also (see Equation (5-39)), $\underset{\sim}{E}_4$ is independent of p. Finally,

$$\tilde{\underline{E}}_2 = \tilde{\underline{Y}}^{-1} \tilde{\underline{Z}} \tag{7-3}$$

where for subsonic flow $\tilde{Y}_{jh}$ and $\tilde{Z}_{jh}$ are given by Equation (3-13) where δ_{jh}, C_{jh}, D_{jh}, Θ_{jh}, F_{jn}, G_{jn}, Θ_{jn}, Π_n and S_{nh} are independent of p.* Therefore, once the above p-independent arrays are evaluated, the evaluation for multiple frequency requires only the assembly of the matrices $\tilde{\underline{E}}_1$, $\tilde{\underline{Y}}$, $\tilde{\underline{Z}}$ and $\tilde{\underline{E}}_3$ and the evaluation of

$$\tilde{\underline{E}} = \underline{E}_4 \tilde{\underline{E}}_3 \tilde{\underline{E}}_{21} \tag{7-4}$$

where $\tilde{\underline{E}}_{21}$ is obtained by solving the system of algebraic equations

$$\tilde{\underline{Y}} \tilde{\underline{E}}_{21} = \tilde{\underline{Z}} \tilde{\underline{E}}_1 \tag{7-5}$$

In addition, calculations for updated sets of modes (as necessary in optimal design where a different set of modes is used at each iteration, but geometry and Mach number are not changed) are greatly facilitated by the modularity of the formulation. For, only the matrices $\tilde{\underline{E}}_1$ and $\underline{E}_4$ are mode-dependent, therefore, the matrices $\tilde{\underline{Y}}$, $\tilde{\underline{Z}}$ and $\tilde{\underline{E}}_3$ need not be reevaluated at each iteration.

The above features make the program SOUSSA-P 1.1 not only general and flexible, but also modular, simple to use and efficient, especially for calculation involving multiple frequencies or mode changes.

7.2 WORK IN PROGRESS

While the program SOUSSA-P is the most advanced program for unsteady aerodynamics for complex configurations, the Green's function method covers cases which are not included in the present version 1.1 of the program. In particular, the method can be used to solve nonlinear subsonic unsteady flow in the time domain.

*Similarly, in supersonic flow, $\tilde{Y}_{jh}$ and $\tilde{Z}_{jh}$ are given by Equation (3-21) where all the coefficients are frequency independent.

This work is now being completed. Preliminary results are presented in Reference 22. The formulation is now being extended to unsteady transonic flow (Reference 23). (The Green's function method is applicable to fully nonlinear unsteady transonic flow with moving shock waves (see Reference 3).)

Additional work now underway includes a higher-order finite-element formulation, special-purpose elements (such as hinge-line elements), wake roll-up and jet engine flow modeling. Finally, preliminary work for the inclusion of the rotational part of the velocity flow field (for use in boundary-layer analysis) is also underway.

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APPENDIX A

THE VALUE OF THE FUNCTION E ON THE SURFACE

Consider Equations (2-52) and (2-63) which give (for subsonic and supersonic flow, respectively) a representation of the potential Φ, anywhere in the fluid volume, in terms of the values of Φ and $\frac{\partial \Phi}{\partial N}$ on the surface of the body and $\Delta\Phi$ on the surface of the wake. The values of $\partial\Phi/\partial N$ on the surface of the body are given by the boundary conditions but the values of Φ are not known. In order to solve the problem, it is thus necessary to obtain first the values of Φ on the surface. This can be accomplished by letting the point $\bar{P}_*$ of the fluid volume V approach a point $\bar{P}_o$ of the surface. Then Equation (2-52) yields

$$4\pi E_* \Phi(\bar{P}_o, T) = \lim_{\bar{P}_* \to \bar{P}_o} \left\{ - \oiint_{\Sigma_B} [\Psi]^{\Theta} \frac{1}{R} \, d\Sigma_B + \oiint_{\Sigma_B} [\Phi]^{\Theta} \frac{\partial}{\partial N} \left(\frac{1}{R}\right) d\Sigma_B - \oiint_{\Sigma_B} \left[\frac{\partial \Phi}{\partial T}\right]^{\Theta} \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N} \, d\Sigma_B + I_W \right\} \tag{A-1}$$

where

$$I_W = \iint_{\Sigma_W} \left\{ [\Delta\Phi]^{\Theta} \frac{\partial}{\partial N_u} \left(\frac{1}{R}\right) - \left[\frac{\partial \Delta\Phi}{\partial T}\right]^{\Theta} \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N_u} \right\} d\Sigma_W \tag{A-2}$$

is the contribution of the wake and

$$E_* = 1 \qquad \bar{P}_* \text{ outside } \Sigma_B$$

$$E_* = 0 \qquad \bar{P}_* \text{ inside } \Sigma_B \tag{A-3}$$

A similar expression is obtained from Equation (2-63) for supersonic flow.

In this Appendix, it is shown that, in the limit, Equation (2-52) is still valid if the definition of the function E is generalized as follows:

$$E_* = 1 \qquad \bar{P}_* \text{ outside } \Sigma_B$$

$$E_* = 1/2 \qquad \bar{P}_* \text{ on } \Sigma_B \text{ (regular point)}$$

$$E_* = 0 \qquad \bar{P}_* \text{ inside } \Sigma_B \tag{A-4}$$

In order to simplify the discussion of the limit as $\bar{P}_* \rightarrow \bar{P}_o$, the steady subsonic case is considered first. The results are then extended to the unsteady subsonic case. The supersonic case is considered last.

A.1 STEADY SUBSONIC FLOW

Consider the steady incompressible flow first. By letting $\bar{P}_* \rightarrow \bar{P}_o$, the integrands become singular in the neighborhood of $\bar{P}_o$. Thus, it is convenient to separate the contribution of a small neighborhood of $\bar{P}_o$, which will be indicated as Σ_ϵ. The neighborhood Σ_ϵ is a small circular element of the surface Σ_B with center $\bar{P}_o$ and radius ϵ. Thus, Equation (2-52) may be rewritten as:

$$4\pi E_* \Phi_* = - \iint_{\Sigma_B - \Sigma_\epsilon} \Psi \frac{1}{R} \, d\Sigma_B + \iint_{\Sigma_B - \Sigma_\epsilon} \Phi \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_B + I_\epsilon + I_W \tag{A-5}$$

where I_ϵ is the contribution of the neighborhood of $\bar{P}_o$, given by

$$I_\epsilon = - \iint_{\Sigma_\epsilon} \Psi \frac{1}{R} \, d\Sigma_B + \iint_{\Sigma_\epsilon} \Phi \frac{\partial}{\partial N} \left(\frac{1}{R}\right) d\Sigma_B \qquad \text{(A-6)}$$

A.1.1 REGULAR POINT

Consider first the case in which $\bar{P}_o$ is a regular point of the surface Σ_B, i.e., a point in which there exists a unique tangent plane to the surface Σ_B. For notational simplicity, the analysis is performed by assuming that the origin is located at $\bar{P}_o$ and that the Z-axis is normal to the tangent plane and directed from E = 0 to E = 1, i.e., parallel to the normal $\bar{N}$. The point $\bar{P}_*$ is on the Z-axis and therefore $\bar{P}_* \equiv (0, 0, Z_*)$. (The extension to an arbitrary element is obtained by replacing the coordinates X, Y, and Z with a local coordinate system $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ with $\bar{P}_o$ located at $\hat{X} = \hat{Y} = \hat{Z} = 0$ and with $\hat{Z}$ normal to the tangent plane and directed from $E_* = 0$ to $E_* = 1$.) Separating terms of order ϵ and noting that $X_* = Y_* = 0$, $Z = 0$ and $\left.\frac{\partial R}{\partial Z}\right|_{Z=0} = - \left.\frac{\partial R}{\partial Z_*}\right|_{Z=0}$, Equation (A-6) reduces to

$$I_\epsilon = - \Psi_o \iint_{X^2+Y^2<\epsilon^2} \frac{1}{\sqrt{X^2+Y^2+Z_*^2}} \, dX dY$$

$$- \Phi_o \iint_{X^2+Y^2<\epsilon^2} \frac{\partial}{\partial Z_*} \frac{1}{\sqrt{X^2+Y^2+Z_*^2}} \, dX dY + O(\epsilon) \qquad \text{(A-7)}$$

where the subscript o indicates evaluation at $\bar{P}_o$. By using polar coordinates

$$\hat{R} = \sqrt{X^2 + Y^2}$$

$$\theta = \tan^{-1} \frac{Y}{X} \qquad \text{(A-8)}$$

one obtains

$$I_\epsilon = -2\pi \Psi_o \int_o^\epsilon \frac{1}{\sqrt{\hat{R}^2 + Z_*^2}} \hat{R} d\hat{R}$$

$$- 2\pi \Phi_o \int_o^\epsilon \frac{\partial}{\partial Z_*} \frac{1}{\sqrt{\hat{R}^2 + Z_*^2}} \hat{R} d\hat{R} + O(\epsilon) \qquad \text{(A-9)}$$

Noting that

$$\frac{1}{Z_*} \frac{\partial}{\partial Z_*} \left(\frac{1}{\sqrt{\hat{R}^2 + Z_*^2}} \right) = - \left(\hat{R}^2 + Z_*^2 \right)^{-\frac{3}{2}} = \frac{1}{\hat{R}} \frac{\partial}{\partial \hat{R}} \left(\frac{1}{\sqrt{\hat{R}^2 + Z_*^2}} \right) \qquad \text{(A-10)}$$

Equation (A-9) becomes

$$I_\epsilon = -2\pi \Psi_o \left[\sqrt{\hat{R}^2 + Z_*^2} \right]_o^\epsilon$$

$$- 2\pi \Phi_o Z_* \left[\frac{1}{\sqrt{\hat{R}^2 + Z_*^2}} \right]_o^\epsilon + O(\epsilon) \qquad \text{(A-11)}$$

Finally, by letting $\vec{P}_*$ go to $\vec{P}_o$, (that is, $Z_* \to 0$), one obtains

$$\lim_{P_* \to P_o} I_\epsilon = \lim_{Z_* \to 0} \left[-2\pi \Psi_o \left(\sqrt{\epsilon^2 + Z_*^2} - |Z_*| \right) \right.$$

$$\left. - 2\pi \Phi_o Z_* \left(\frac{1}{\sqrt{\epsilon^2 + Z_*^2}} - \frac{1}{|Z_*|} \right) \right] + O(\epsilon)$$

$$= \left[-2\pi \Psi_o \epsilon + 2\pi \Phi_o \operatorname{sgn}(Z_*) \right] + O(\epsilon)$$

$$= \pm 2\pi \Phi_o + O(\epsilon) \qquad \text{(A-12)}$$

where the upper (lower) sign holds for $Z_* > 0$ $(Z_* < 0)$, that is, when $\bar{P}_*$ is located outside (inside) the surface Σ_B ; correspondingly, the function E assumes the values $E_* = 1$ $(E_* = 0)$.

Finally, using this result in Equation (A-1) one obtains, for $\bar{P}_*$ on Σ_B (i.e., $\Phi_o = \Phi_*$),

$$4\pi \left(E_* \mp \frac{1}{2}\right) \Phi_o = - \iint_{\Sigma_B - \Sigma_\epsilon} \Psi \frac{1}{R} \, d\Sigma$$

$$+ \iint_{\Sigma_B - \Sigma_\epsilon} \Phi \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma + I_W + O(\epsilon) \qquad \text{(A-13)}$$

Note that, in both cases ($\bar{P}_*$ inside or outside Σ_B),

$$E_o = E_* \mp \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2} \quad \bar{P}_* \text{ outside } \Sigma_B$$

$$= 0 + \frac{1}{2} = \frac{1}{2} \quad \bar{P}_* \text{ inside } \Sigma_B \qquad \text{(A-14)}$$

Furthermore, R is the distance between the dummy point, $\bar{P}$, and the control point (on the surface Σ_B), $\bar{P}_o$. Hence, by letting ϵ go to zero, Equation (A-13) yields

$$4\pi E_o \Phi_o = - \oiint_{\Sigma_B} \Psi \frac{1}{R} \, d\Sigma_B + \oiint_{\Sigma_B} \Phi \frac{\partial}{\partial N} \left(\frac{1}{R}\right) d\Sigma_B + I_W \qquad \text{(A-15)}$$

Note that for steady subsonic flow, Equation (2-25) yields

$$4\pi E_* \Phi_* = - \oiint_{\Sigma_B} \Psi \frac{1}{R} \, d\Sigma_B + \oiint_{\Sigma_B} \Phi \frac{\partial}{\partial N} \left(\frac{1}{R}\right) d\Sigma_B + I_W \qquad \text{(A-16)}$$

Note that Equation (A-16) must be used if $\bar{P}_*$ is outside or inside the surface, whereas Equation (A-15) must be used if $\bar{P}_o$ is on the surface. However, by comparing Equations (A-15) and (A-16), it is easily seen that Equation (A-16) is valid everywhere (outside, inside and on the surface Σ_B), if the convention is made that E is given by Equation (A-4).

A.1.2 COMMENTS

It should be emphasized that in obtaining Equation (A-15) from Equation (A-13) the limit $\epsilon \to 0$ is performed with $\bar{P}_*$ on the surface Σ_B. This implies that the contribution of Σ_ϵ is now of order ϵ. In order to clarify this point, consider the quantity

$$I_{\epsilon_o}^{\epsilon} = \int_{\epsilon_o}^{\epsilon} \frac{\partial}{\partial Z_*} \left(\frac{1}{\sqrt{\hat{R}^2 + Z_*^2}} \right) \hat{R}\, d\hat{R}$$

$$= \left[\frac{Z_*}{\sqrt{\hat{R}^2 + Z_*^2}} \right]_{\epsilon_o}^{\epsilon} = \frac{Z_*}{\sqrt{\epsilon^2 + Z_*^2}} - \frac{Z_*}{\sqrt{\epsilon_o^2 + Z_*^2}} \tag{A-17}$$

and note that

$$\lim_{Z_* \to 0} \left\{ \lim_{\epsilon_o \to 0} I_{\epsilon_o}^{\epsilon} \right\} = \lim_{Z_* \to 0} \left\{ \frac{Z_*}{\sqrt{\epsilon^2 + Z_*^2}} - \frac{Z_*}{|Z_*|} \right\} = -\mathrm{sgn}(Z_*) \tag{A-18}$$

whereas

$$\lim_{\epsilon_o \to 0} \left\{ \lim_{Z_* \to 0} I_{\epsilon_o}^{\epsilon} \right\} = \lim_{\epsilon_o \to 0} 0 = 0 \tag{A-19}$$

The difference between these two limits is due to the fact that, in the limit (as $Z_* \rightarrow 0$), the integrand of $I^{\epsilon}_{\epsilon_o}$ behaves like a Dirac delta function and hence, its contribution for a domain which excludes the singular point is zero.

It is apparent that the sequence of limits as indicated in Equation (A-18) is performed for Equation (A-12), whereas the one indicated in Equation (A-19) is used for Equation (A-15). The above results may be restated by saying that the limit as $\bar{P}_* \rightarrow \bar{P}_o$ of a doublet is a generalized function, i.e.

$$\lim_{\bar{P}_* \rightarrow \bar{P}_o} \frac{\partial}{\partial N}\left(\frac{-1}{4\pi R}\right) = \frac{\partial}{\partial N}\left(\frac{-1}{4\pi R}\right)\bigg|_{\bar{P}_* = \bar{P}_o} \mp \frac{1}{2}\,\delta(\bar{P} - \bar{P}_o) \qquad \text{(A-20)}$$

where the upper (lower) sign holds for $\bar{P}_o$ outside (inside) Σ_B.

A.1.3 NON-REGULAR POINT

The above results may be generalized to non-regular points (i.e., points of Σ_B in which there is not a unique tangent plane to Σ_B) if they are expressed in terms of the classical concept of solid angle. Note that by definition, the solid angle, $d\Omega$, is

$$d\Omega = \frac{d\Sigma_P}{R^2} = \frac{d\Sigma_B}{R^2}\cos\alpha_{RN} = \frac{\bar{R}\cdot\bar{N}}{R^3}\,d\Sigma_B$$

$$= -\frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_B \qquad \text{(A-21)}$$

where $d\Sigma_P$ is the projection of $d\Sigma_B$ into a plane normal to $\bar{R}$ (see Figure A-1).

Using Equation (A-21) one obtains

$$\oiint_{\Sigma_B} \frac{\partial}{\partial N}\left(\frac{-1}{4\pi R}\right) d\Sigma_B = \frac{1}{4\pi}\oiint_{\Sigma_B} d\Omega = \Omega_*/4\pi \qquad \text{(A-22)}$$

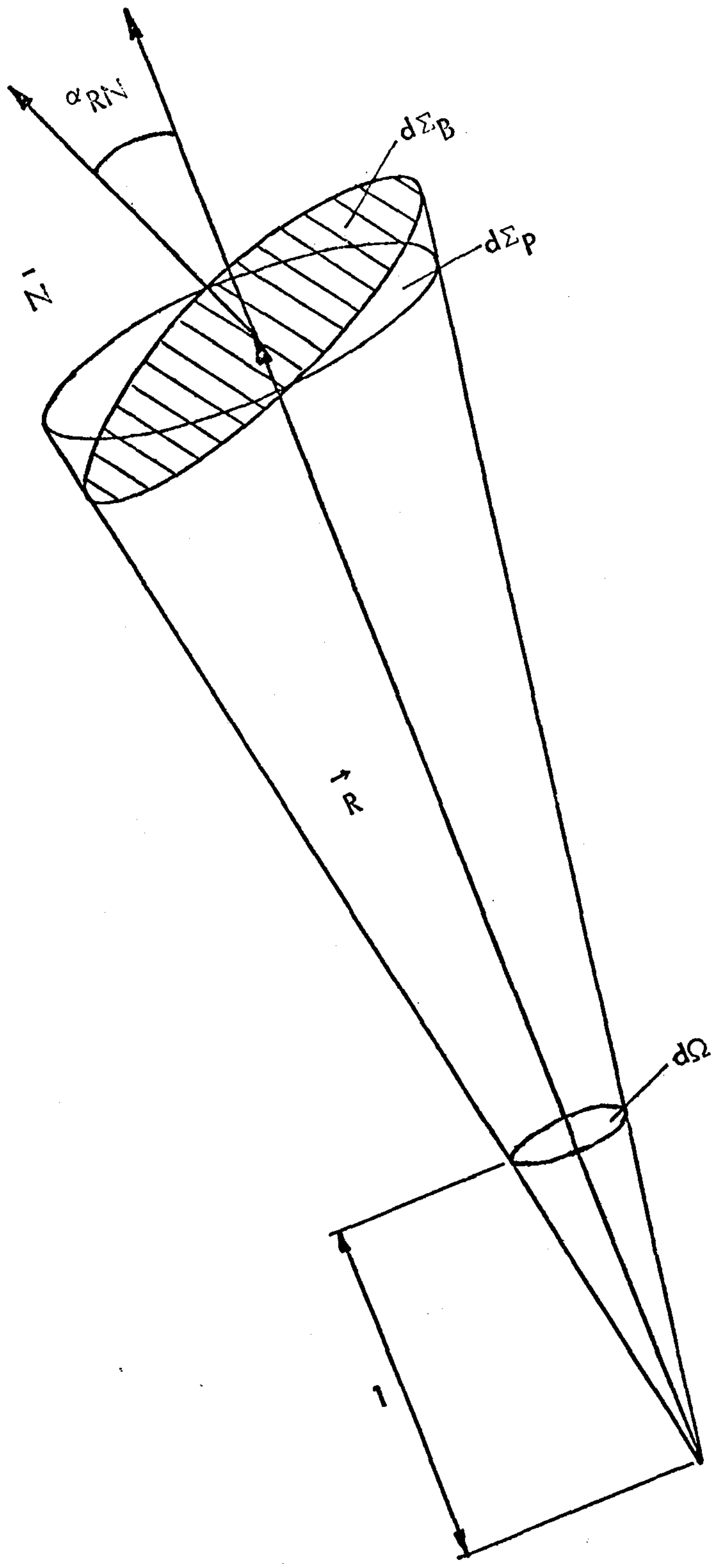


Figure A-1. Solid Angle.

where

$$\Omega_* = 0 \qquad \bar{P}_* \text{ inside } \Sigma_B$$

$$= 2\pi \qquad \bar{P}_* \text{ on } \Sigma_B \text{ (regular point)}$$

$$= 4\pi \qquad \bar{P}_* \text{ outside } \Sigma_B \tag{A-23}$$

Next, note that using Equation (A-22), Equation (A-16) may be written as

$$4\pi\Phi_* = -\oint\!\!\oint_{\Sigma_B} \Psi \frac{1}{R} \, d\Sigma_B$$

$$+ \oint\!\!\oint_{\Sigma_B} (\Phi - \Phi_*) \frac{\partial}{\partial N} \left(\frac{1}{R}\right) d\Sigma_B + I_W \tag{A-24}$$

since, according to Equations (A-22) and (A-23)

$$4\pi E_* - \oint\!\!\oint_{\Sigma_B} \frac{\partial}{\partial N} \left(\frac{1}{R}\right) d\Sigma_B = 4\pi E_* + \Omega_* = 4\pi \tag{A-25}$$

for $\bar{P}_*$ outside and inside Σ_B. Note that the right-hand side of Equation (A-24) is continuous because the Dirac delta function in Equation (A-20) is now multiplied by

$$(\Phi - \Phi_*) \big|_{\bar{P} = \bar{P}_*} = 0 \tag{A-26}$$

Therefore, Equation (A-24) is valid for all cases, i.e., for $\bar{P}_*$ outside Σ_B, inside Σ_B as well as on Σ_B (for both regular and non-regular points). Finally, Equation (A-24) may be rewritten as Equation (A-16), if E_* is defined as

$$E_* = 1 + \oint\!\!\oint_{\Sigma_B} \frac{\partial}{\partial N} \left(\frac{1}{R}\right) d\Sigma_B = 1 - \frac{\Omega_*}{4\pi} \tag{A-27}$$

Note that the above Equation is a generalization of Equation (A-4) in that it is valid for non-regular points as well.

In SOUSSA-P the control point is always located at a regular point (center of the hyperboloidal element described in Subsection 3.3). Therefore, in the rest of this report it is always assumed $E = 1/2$ on Σ_B.

A.2 UNSTEADY SUBSONIC FLOW

The results of the Subsection A.1.3 are immediately extended to unsteady subsonic case by noting that, using Equation (A-25), Equation (2-52) may be rewritten as

$$
\begin{aligned}
4\pi E_* \Phi_* = & - \oiint_{\Sigma_B} [\Psi]^{\Theta} \frac{1}{R} \, d\Sigma_B \\
& + \oiint_{\Sigma_B} \left([\Phi]^{\Theta} - \Phi_* \right) \frac{\partial}{\partial N} \left(\frac{1}{R} \right) d\Sigma_B \\
& + \oiint_{\Sigma_B} \left[\frac{\partial \Phi}{\partial T} \right]^{\Theta} \frac{1}{R} \frac{\partial \hat{\Theta}}{\partial N} \, d\Sigma_B + I_W
\end{aligned}
\tag{A-28}
$$

with I_W given by Equation (A-2). Equation (A-28) is valid for $\bar{P}_*$ outside and inside Σ_B. In addition, there are no discontinuities as $\bar{P}_*$ approaches Σ_B since the Dirac delta function due to the doublet distribution is multipled by zero. Therefore, the equation is still valid in the limit as $\bar{P}_*$ approaches the surface Σ_B. This implies that Equation (2-52) is valid on the surface Σ_B as well if E_* is defined by Equation (A-27), or in particular, by Equation (A-3).

A.3 SUPERSONIC FLOW

Consider the steady supersonic case. For steady state Equation (2-63) reduces to *

$$4\pi E_* \Phi_* = - \oiint_{\Sigma_B} \Psi' \frac{2H}{R'} d\Sigma_B + \oiint_{\Sigma_B} \Phi \frac{\partial}{\partial N^c} \left(\frac{2H}{R'}\right) d\Sigma_B + I_W \qquad \text{(A-29)}$$

where I_W is given by

$$I_W = \iint_{\Sigma_W} \Delta\Phi \frac{\partial}{\partial N_1^c} \left(\frac{2H}{R'}\right) d\Sigma_W \qquad \text{(A-30)}$$

A.3.1 SUPERSONIC SOLID ANGLE

Note that (see Equations (2-64), (2-66) and (A-21))

$$- \frac{\partial}{\partial N^c} \left(\frac{2H}{R'}\right) d\Sigma_B$$

$$= \left[N_X \frac{\partial}{\partial X} \left(\frac{1}{R'}\right) - N_Y \frac{\partial}{\partial Y} \left(\frac{1}{R'}\right) - N_Z \frac{\partial}{\partial Z} \left(\frac{1}{R'}\right)\right] 2H \, d\Sigma_B$$

$$= \frac{\bar{R} \cdot \bar{N}}{R'^3} 2H \, d\Sigma_B = \left(\frac{R}{R'}\right)^3 2H \, d\Omega \qquad \text{(A-31)}$$

---

*All the integrals in this subsection are defined within the theory of distributions or generalized functions. Therefore, the concept of the finite part of the integral is considered whenever applicable: with this convention the derivative $\partial H/\partial N^c$ is identically equal to zero.

Therefore, the analysis for the value of E_* on the surface is facilitated by introducing the concept of supersonic solid angle defined by

$$d\Omega' = \left(\frac{R}{R'}\right)^3 2H\, d\Omega = -\frac{\partial}{\partial N^c}\left(\frac{2H}{R'}\right) d\Sigma_B \tag{A-32}$$

Note that

$$\left(\frac{R'}{R}\right)^2 = \left[\frac{(X-X_*)^2}{(Y-Y_*)^2 + (Z-Z_*)^2} - 1\right]\left[\frac{(X-X_*)^2}{(Y-Y_*)^2 + (Z-Z_*)^2} + 1\right]^{-1} \tag{A-33}$$

Therefore, $(R/R')^3 H$ depends only upon $(X - X_*)^2/[(Y - Y_*)^2 + (Z - Z_*)^2]$. Hence, $d\Omega'$ is independent of the distance. This implies that the integral

$$\begin{aligned} \Omega'_* &= \oint\!\!\oint_{\Sigma_B} d\Omega' = \oint\!\!\oint_{\Sigma_B} \left(\frac{R}{R'}\right)^3 2H\, d\Omega \\ &= -\oint\!\!\oint_{\Sigma_B} \frac{\partial}{\partial N^c}\left(\frac{2H}{R'}\right) d\Sigma_B \end{aligned} \tag{A-34}$$

is independent of the actual surface Σ_B but depends only upon the topological relationship between P_* and Σ_B. In particular, it is apparent that

$$\Omega'_* = 0 \qquad P_* \text{ outside } \Sigma_B \tag{A-35}$$

If P_* is a regular point (as defined in Subsection A.1.1) of the surface Σ_B, the value for Ω'_* may be obtained by replacing the surface Σ_B with the surface Σ'_B, as indicated in Figure A-2. For notational simplicity the point $\bar{P}_*$ is assumed to coincide with the origin and the normal $\bar{N}$ to Σ_B is assumed to be normal to the

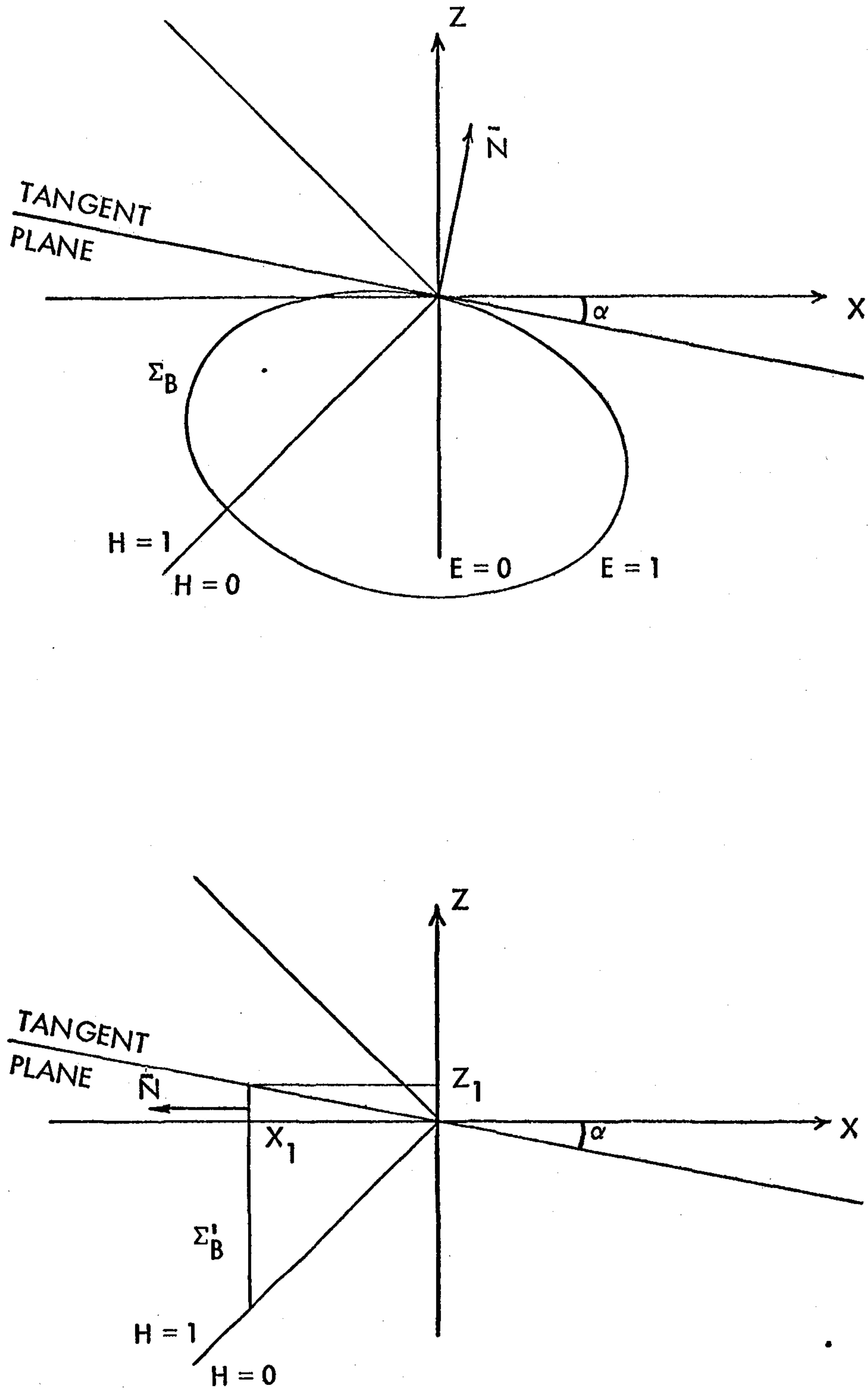


Figure A-2. Supersonic Solid Angle for $\bar{P}_*$ on Σ_B.

Z-axis. (If this is not the case, the analysis can be repeated with a local coordinate system satisfying the above requirements.) With these assumptions one obtains (see Figure A-3)

$$\Omega'_* = - \oiint_{\Sigma_B} \frac{\partial}{\partial N^c} \left(\frac{2H}{R'} \right) d\Sigma_B$$

$$= -2 \int_{-|X_1|}^{Z_1} dZ \int_{-\sqrt{X_1^2 - Z^2}}^{\sqrt{X_1^2 - Z^2}} \frac{\partial}{\partial X_1} \left(\frac{1}{R'_1} \right) dY \qquad \text{(A-36)}$$

where $R'_1 = (X_1^2 - Y^2 - Z^2)^{1/2}$.

Consider the rule for the integral of the distribution $u^{3/2}$

$$\int_0^a \frac{f}{u^{3/2}} du = \lim_{\epsilon_o \to 0} \left[\int_{\epsilon_o}^a \frac{f du}{u^{3/2}} - \frac{2 f(o)}{\epsilon_o^{1/2}} \right] \qquad \text{(A-37)}$$

This yields, for $|g'(0)| < \infty$,

$$\int_0^a \frac{\partial}{\partial u} \left(\frac{g}{u^{1/2}} \right) du = \int_0^a \left(u g' - \frac{1}{2} g \right) u^{-3/2} du$$

$$= \lim_{\epsilon_o \to 0} \left\{ g u^{-1/2} \Big|_{\epsilon_o}^a - (2ug' - g) \Big|_{u=0} \epsilon_o^{-1/2} \right\} = g(a)/\sqrt{a} \qquad \text{(A-38)}$$

It is convenient to rewrite Equation (A-36) as

$$\Omega'_* = 4 \lim_{\epsilon \to 0} I'_\epsilon \qquad \text{(A-39)}$$

where

$$I'_\epsilon = \int_{-\sqrt{X_1^2 - \epsilon^2}}^{Z_1} dZ \int_{\epsilon}^{\sqrt{X_1^2 - Z^2}} \frac{X_1}{R_1'^3} dY \qquad \text{(A-40)}$$

or, using Equation (A-38) to perform the integration with respect to Y,

$$I'_\epsilon = - \int_{-\sqrt{X_1^2 - \epsilon^2}}^{Z_1} \left[\frac{X_1}{X_1^2 - Z^2} \quad \frac{Y}{R_1'} \right]_{Y = \epsilon} dZ$$

$$= - \left[\tan^{-1} \left(\frac{\epsilon Z}{X_1 \sqrt{X_1^2 - \epsilon^2 - Z^2}} \right) \right]_{Z = -\sqrt{X_1^2 - \epsilon^2}}^{Z = Z_1}$$

$$= \frac{\pi}{2} - \tan^{-1} \left(\frac{\epsilon Z_1}{X_1 \sqrt{X_1^2 - \epsilon^2 - Z_1^2}} \right) \tag{A-41}$$

Combining Equations (A-39) and (A-41) one obtains

$$\Omega'_* = 2\pi \qquad \bar{P}_* \text{ on } \Sigma_B \text{ (regular point)} \tag{A-42}$$

(It may be worth noting that if ϵ is replaced with zero, the integrand is equal to zero except at $Z = |X_1|$. This indicates that as ϵ goes to zero the integrand in Equation (A-41) tends to $\pi/2$ times a Dirac delta function located at $Z = - |X_1|$.)

It is important to note that the value of Ω'_*, besides being independent of X_1 (which is to be expected), is also independent of Z_1, i.e., the "angle of attack", $\alpha = \tan^{-1} (Z_1/X_1)$, of the tangent plane, as long as $Z_1 < |X_1|$.

The case $Z_1 \geq |X_1|$ corresponds to the case of $\bar{P}_*$ inside Σ_B, for in this case the domain of integration in Equation (A-36) is given by the whole area inside the circle in Figure A-3. In other words, if $\bar{P}_*$ is inside Σ_B, Ω'_* is given by Equation (A-36) with Z_1 replaced by $|X_1|$. Accordingly I'_ϵ is given by Equation (A-40) with Z_1 replaced by $\sqrt{X_1^2 - \epsilon^2}$. This yields $I'_\epsilon = \pi$ and

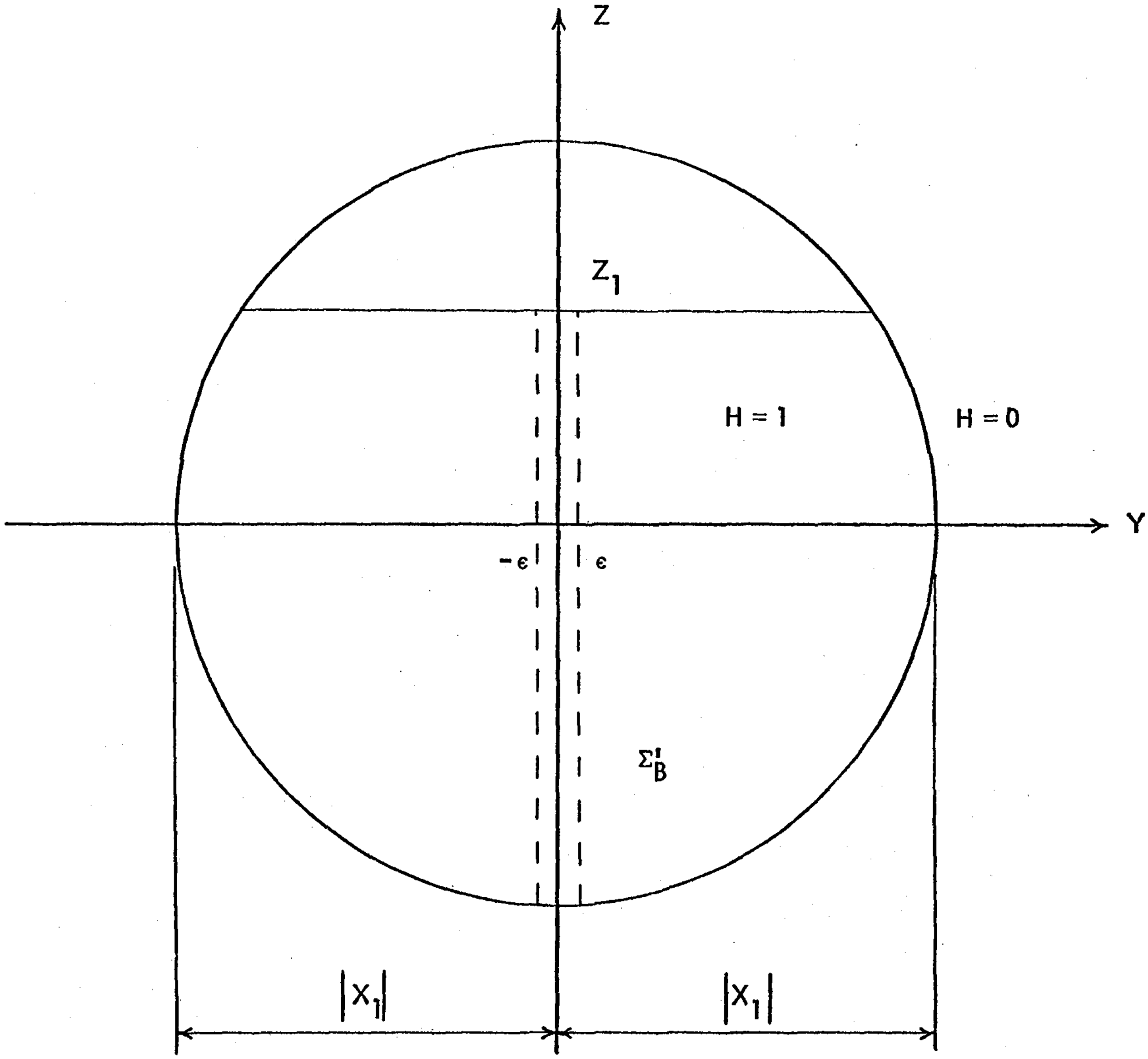


Figure A-3. Domain of Integration for Supersonic Solid Angle for $\bar{P}_*$ on Σ_B.

$$\Omega'_* = 4\pi \qquad \bar{P}_* \text{ inside } \Sigma_B \tag{A-43}$$

(Note the presence of a second Dirac delta function at $Z = |X_1|$.)

In summary

$$\Omega'_* = - \oiint_{\Sigma_B} \frac{\partial}{\partial N_c} \left(\frac{2H}{R'} \right) d\Sigma_B$$

$$\begin{aligned} &= 0 && \bar{P}_* \text{ outside } \Sigma_B \\ &= 2\pi && \bar{P}_* \text{ on } \Sigma_B \text{ (regular point)} \\ &= 4\pi && \bar{P}_* \text{ inside } \Sigma_B \end{aligned} \tag{A-44}$$

A.3.2 VALUE OF E_* FOR SUPERSONIC

Note that

$$4\pi E_* + \Omega'_* = 4\pi \tag{A-45}$$

for $\bar{P}_*$ inside and outside Σ_B; therefore, Equation (A-29) may be rewritten as

$$4\pi \Phi_* = - \oiint_{\Sigma_B} \Psi' \frac{2H}{R'} d\Sigma_B + \oiint_{\Sigma_B} (\Phi - \Phi_*) \frac{\partial}{\partial N^c} \left(\frac{2H}{R'} \right) d\Sigma_B + I_W \tag{A-46}$$

Repeating the considerations presented in Subsection A.1.3 one obtains that Equation (A-29) is valid for $\bar{P}_*$ outside, inside and on Σ_B (for both regular and non-regular points) if E_* is defined as

$$E_* = 1 - \Omega_*'/4\pi \tag{A-47}$$

In particular,

$$E_* = \frac{1}{2} \qquad \bar{P}_* \text{ on } \Sigma_B \text{ (regular point)} \tag{A-48}$$

is the value used in SOUSSA-P.

The extension to unsteady flow is easily obtained using the same procedure used in Subsection A.2 for subsonic flow.

APPENDIX B

LAPLACE TRANSFORM IN UNSTEADY AERODYNAMICS

In this Appendix, issues related to the use of Laplace transform are examined. In particular, the question of the truncation of the wake (its rationale and its advantages) are examined in detail. Finally, the relationship between the Laplace-transform analysis for unsteady flow and the complex exponential-analysis for oscillatory flow is examined; it is shown that Laplace transform results for Real $(p) > 0$ may be used to study divergent oscillatory flow, whereas convergent oscillatory flows (of the type e^{pt} with $-\infty \leq t \leq \infty$) are "physically impossible" (the Laplace transform for Real $(p) < 0$ exists only for truncated-wake analysis).

B.1 LAPLACE TRANSFORM

Consider a function $f(t)$ such that

$$f(t) = 0 \qquad (t < 0) \tag{B-1}$$

and otherwise arbitrary. The Laplace transform of $f(t)$ is defined by

$$\tilde{f}(s) = L(f) = \int_0^\infty f(t)\, e^{-st}\, dt \qquad (s \in D_E) \tag{B-2}$$

where s is the Laplace parameter (complex frequency) and D_E is the domain of convergence of the integral.

Note that $\tilde{f}(s)$ is a function of a complex variable: the definition of this function outside the domain D_E is obtained by analytic continuation. However, the Laplace transform of $f(t)$ is only defined in the domain D_E.

The inverse Laplace transform yields

$$f(t) = L^{-1}(\tilde{f}) = \frac{1}{2\pi i} \int_{\alpha_1 - i\infty}^{\alpha_1 + i\infty} \tilde{f}(s)\, e^{st}\, ds \tag{B-3}$$

where the path of integration is in the domain D_E. In order to evaluate this integral, it is convenient to use Jordan's lemma (Reference 24, p. 81): given a single valued function such that $|g(Re^{i\theta})| \to 0$, as $R \to \infty$, then

$$\lim_{R \to \infty} \int_{\Gamma} g(s)\, e^{st}\, ds = 0 \tag{B-4}$$

where t is a real constant and Γ is a semi-circular path with $s = R\, e^{i\theta}$, where $-\pi/2 \leq \theta \leq \pi/2$ for $t < 0$ and $\pi/2 \leq \theta \leq 3\pi/2$ for $t > 0$. Thus, the path of integration may be closed on the left-hand side of the plane for $t > 0$ (right-hand side for $t < 0$). It may be worth noting that if $\tilde{f}(s)$ does not have singularities for $s_R > \alpha$, then Equations (B-3) and (B-4) yield immediately $f(t) = 0$ for $t < 0$ in agreement with Equation (B-1).

For nondimensional time $T = U_\infty t/\ell$, one obtains

$$\tilde{F}(p) = \int_0^\infty F(T)\, e^{-pT}\, dT \tag{B-5}$$

where

$$p = s\ell/U_\infty \tag{B-6}$$

is the complex reduced frequency (nondimensional Laplace parameter).

B.2 TRUNCATION OF WAKE

As mentioned in Section 3, the wake is truncated at a finite distance from the body. This point is examined in detail in this Subsection.

B.2.1 LAPLACE TRANSFORM OF TRUNCATED FUNCTIONS

Consider the "truncated functions" (see Figure B-1)

$$
\begin{aligned}
f_1(t) &= f(t) && t < t_o \\
f_1(t) &= 0 && t > t_o
\end{aligned}
\tag{B-7}
$$

and

$$
\begin{aligned}
f_2(t) &= 0 && t < t_o \\
f_2(t) &= f(t) && t > t_o
\end{aligned}
\tag{B-8}
$$

where $t_o > 0$. Note that $f_1(t) = f_2(t) = 0$ for $t < 0$. The transforms of f_1 and f_2 are given by

$$
\begin{aligned}
\tilde{f}_1(s) &= \int_0^{t_o} f(t)\, e^{-st}\, dt \\
\tilde{f}_2(s) &= \int_{t_o}^{\infty} f(t)\, e^{-st}\, dt
\end{aligned}
\tag{B-9}
$$

Note that

$$\tilde{f}(s) = \tilde{f}_1(s) + \tilde{f}_2(s) \tag{B-10}$$

However, if one is interested only in the interval from 0 to t_o then replacing $\tilde{f}$ with $\tilde{f}_1$ yields no difference in the response (since $\tilde{f}_2$ contributes only for $t > t_o$).

In order to clarify this point further, consider, as an illustration, the function

$$
\begin{aligned}
f(t) &= e^{-\alpha t} && t > 0 \\
&= 0 && t < 0
\end{aligned}
\tag{B-11}
$$

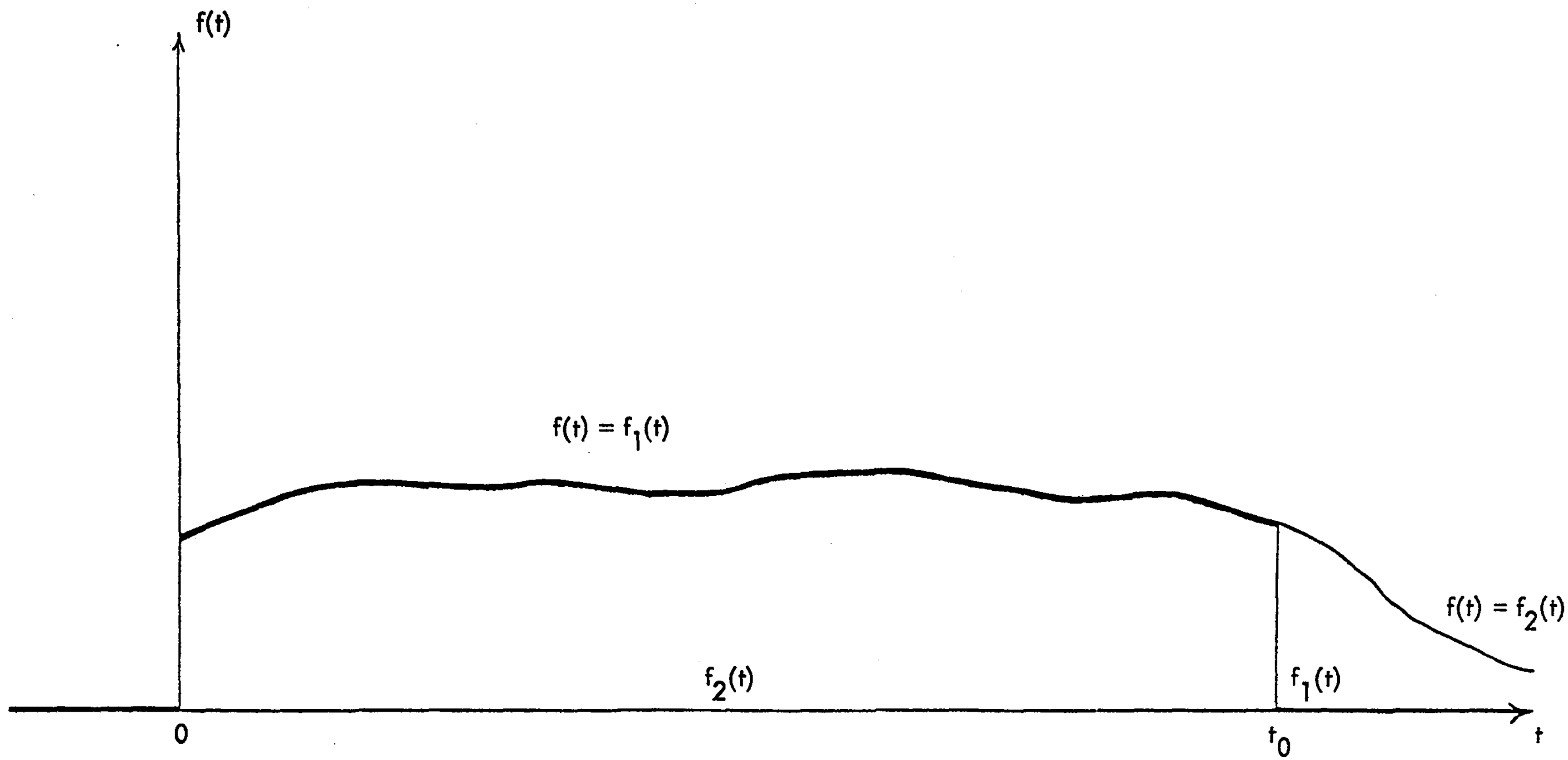


Figure B-1. Truncated Functions.

The Laplace transform is

$$\tilde{f} = \int_0^{\infty} e^{-\alpha t} e^{-st}\, dt = \frac{1}{s+\alpha} \qquad [\,\text{Real } (s) > \alpha\,] \tag{B-12}$$

Next consider

$$\begin{aligned} f_1(t) &= f(t) & t &< t_o \\ &= 0 & t &> t_o \end{aligned} \tag{B-13}$$

The Laplace transform of f_1 is*

$$\begin{aligned} \tilde{f}_1 &= \int_0^{t_o} e^{-\alpha t} e^{-st}\, dt = \frac{1}{s+\alpha}\left(1 - e^{-(s+\alpha)t_o}\right) \\ &= \tilde{f} - \tilde{f}_2 \end{aligned} \tag{B-14}$$

where

$$\begin{aligned} \tilde{f} &= \frac{1}{s+\alpha} & &[\,\text{Real } (s) > \alpha\,] \\ \tilde{f}_2 &= \frac{-1}{s+\alpha}\, e^{-(s+\alpha)t_o} & &[\,\text{Real } (s) > \alpha\,] \end{aligned} \tag{B-15}$$

Using Equations (B-3) and (B-4) and the residue theorem yields

$$\begin{aligned} L^{-1}(\tilde{f}) = \frac{1}{2\pi i} \int_{\alpha_1 - i\infty}^{\alpha_1 + i\infty} \frac{1}{s+\alpha}\, e^{st}\, ds &= 0 & t &< 0 \\ &= e^{-\alpha t} & t &> 0 \end{aligned} \tag{B-16}$$

and, using $t - t_o$ instead of t in Equation (B-4),

$$\begin{aligned} L^{-1}(\tilde{f}_2) = \frac{1}{2\pi i} \int_{\alpha_1 - i\infty}^{\alpha_1 + i\infty} \frac{e^{-(s+\alpha)t_o}}{s+\alpha}\, e^{st}\, ds &= 0 & t &< t_o \\ &= e^{-\alpha t} & t &> t_o \end{aligned} \tag{B-17}$$

---

*Note that the integral in Equation (B-14) exists for all finite values of s. Hence, D_E coincides with the finite s-plane ($\tilde{f}_1$ is not singular at $s = -\alpha$).

i.e.,

$$
\begin{aligned}
L^{-1}(\tilde{f}_1) &= 0 & -\infty < t < 0 \\
&= e^{\alpha t} & 0 < t < t_o \\
&= 0 & t_o < t < \infty
\end{aligned}
\tag{B-18}
$$

It is apparent, comparing Equations (B-16) and (B-18), that the use of $\tilde{f}_1$ instead of $\tilde{f}$ yields wrong results only for $t > t_o$. Therefore, if $\tilde{f}_1$ is simpler to use than $\tilde{f}$, it is legitimate (and convenient) to use $\tilde{f}_1$ instead of $\tilde{f}$ in order to obtain the response for $t < t_o$.

B.2.2 RATIONALE FOR WAKE TRUNCATION

In order to understand better the rationale for the truncation of the wake it is convenient to consider the Laplace transform of the subsonic integral equation (as mentioned in Subsection 3.2, the problem of the truncation of the wake is limited to the subsonic flow). For the sake of clarity a simple case is considered here. Consider a wing (with straight trailing edge located on the Y-axis) in incompressible flow. Then Equation (2-32) for $M = 0$, yields $\Theta = \hat{\Theta} = 0$ and Equation (2-52) reduces to

$$
4\pi E(\bar{P}_*)\,\Phi(\bar{P}_*, T) = -\oint\!\!\oint_{\Sigma_B} \left[\Psi \frac{1}{R} - \Phi \frac{\partial}{\partial N}\left(\frac{1}{R}\right)\right] d\Sigma_B + \int_{-b/2}^{b/2} J_W \, dY \tag{B-19}
$$

where

$$
J_W = \int_o^\infty \Delta\Phi \frac{\partial}{\partial N_u}\left(\frac{1}{R_o}\right) dX \tag{B-20}
$$

with

$$R_o = R\big|_{Z=0} \tag{B-21}$$

Using Equations (2-53) and (2-54) with $\beta = 1$, i.e., $\Pi = X$. Equation (B-20) may be rewritten as

$$J_W(Y,T) = \int_0^T \Delta\Phi(Y,T)\, \frac{\partial}{\partial N_u}\left(\frac{1}{R_o}\right) dX \tag{B-22}$$

The upper limit on the integral is T instead of ∞ because, according to Equation (2-45)

$$\Delta\Phi(Y,T) = \Delta\Phi_{TE}(Y, T-X) \tag{B-23}$$

is equal to zero for $X > T$. The Laplace transform of Equation (B-19) is

$$4\pi E_* \tilde{\Phi}_*(\bar{P}) = - \oint\!\!\oint_{\Sigma_B} \left[\tilde{\Psi}\, \frac{1}{R} - \tilde{\Phi}\, \frac{\partial}{\partial N}\left(\frac{1}{R}\right) \right] d\Sigma_B$$

$$+ \int_{-b/2}^{b/2} \tilde{J}_W\, dY \tag{B-24}$$

where, using the convolution-integral formula

$$L\left(\int_0^T F(T-X)\, G(X)\, dX\right) = L(F)\, L(G) \tag{B-25}$$

one obtains, from Equations (B-22) and (B-23)

$$\tilde{J}_W(Y) = \Delta\tilde{\Phi}_{TE} \int_0^\infty \frac{\partial}{\partial N_u}\left(\frac{1}{R_o}\right) e^{-pX}\, dX \tag{B-26}$$

Note that if the wake is truncated at $X = T_o$, Equation (B-20) yields

$$J_W^{(o)} = \int_0^{T_o} \Delta\Phi \frac{\partial}{\partial N_u}\left(\frac{1}{R_o}\right) dX \tag{B-27}$$

which yields, using Equation (B-23)

$$\tilde{J}_W^{(o)} = \int_0^{T_o} \Delta\tilde{\Phi}_{TE}\, e^{-pX} \frac{\partial}{\partial N_u}\left(\frac{1}{R_o}\right) dX \tag{B-28}$$

It is apparent that if $\Delta\Phi_{TE}(Y, T-X) = 0$ for $X > T$, $J_W^{(o)}$ is equal to J_W for $T < T_o$. Therefore, according to the analysis presented in Subsection B.2.1, the truncated wake integral equation may be used for $T < T_o$ without any effect on the results. This is the rationale for introducing the truncation of the wake; the Laplace transform of the solution will depend upon the value of T_o, but the solution in the time domain is identical to the infinite-wake solution for $T < T_o$.

It may be worth noting that $\tilde{J}_W^{(o)} \rightarrow \tilde{J}_W$ as $T_o \rightarrow \infty$ indicating that the infinite-wake analysis may be obtained, in the limit, from the truncated-wake analysis. In this case however, the wake integral is expressed in terms of a multivalued function (see next subsection).

B.2.3 ADVANTAGES OF WAKE TRUNCATION

The integral over the wake is evaluated in Reference 3 for the case $p = ik$. The procedure is briefly outlined here for the more general case p_R = Real (p) $\neq 0$, in order to show that the integral exists only for $p_R > 0$ and also that the function obtained from the analytic continuation of the Laplace transform is a multivalued function which requires the use of branch cuts for the evaluation of the inverse Laplace transform. It is also indicated that these problems are eliminated by the use of the truncated wake which therefore has the following advantages:

first, the Laplace transform of the integral equation exists in the whole plane (therefore, analytic continuation is not required); and second, it is a single valued function

Consider the wake integral in Equation (2-52)

$$I_W = \iint_{\Sigma_W} \left\{ [\Delta\Phi]^{\Theta} \frac{\partial}{\partial N_u}\left(\frac{1}{R}\right) - [\Delta\dot{\Phi}]^{\Theta} \frac{\partial\hat{\Theta}}{\partial N_u} \frac{1}{R} \right\} d\Sigma_W \tag{B-29}$$

For simplicity, assume, in line with the small perturbation assumption, that the trajectory of the points of the wake are composed of straight lines parallel to the X axis. Then, using Equation (2-53), Equation (B-29) yields

$$I_W = \int_{TE} dY \int_{X_{TE}}^{\infty} \left[\Delta\Phi_{TE}(Y, T - \Theta - \Pi) \frac{\partial}{\partial N_u}\left(\frac{1}{R}\right) \right.$$

$$\left. - \Delta\dot{\Phi}_{TE}(Y, T - \Theta - \Pi) \frac{\partial\hat{\Theta}}{\partial N_u} \frac{1}{R} \right] dX \tag{B-30}$$

The Laplace transform of Equation (B-30) is

$$\tilde{I}_W = \int_{TE} \Delta\tilde{\Phi}_{TE}\, \tilde{I}'_W \, dY \tag{B-31}$$

where

$$\tilde{I}'_W = \int_{X_{TE}}^{\infty} e^{-p(\Theta+\Pi)} \left[\frac{\partial}{\partial N_u}\left(\frac{1}{R}\right) - p\, \frac{\partial\hat{\Theta}}{\partial N_u} \frac{1}{R} \right] dX \tag{B-32}$$

or using for $\bar{N}_u$ the unit vector in the direction of the positive Z axis and Equation (2-32)

$$\tilde{I}'_W = \int_{X_{TE}}^{\infty} e^{-p(\Theta+\Pi)} \left[\frac{\partial}{\partial Z}\left(\frac{1}{R}\right) - p\, \frac{M}{\beta} \frac{\partial R}{\partial Z} \frac{1}{R} \right] dX \tag{B-33}$$

Note that according to Equations (2-31) and (2-54)

$$\Theta + \Pi = \frac{1}{\beta}\left[M^2(X - X_*) + MR + \beta^2(X - X_{TE})\right]$$

$$= \frac{1}{\beta}(X - X_* + MR) + \beta(X_* - X_{TE}) \qquad \text{(B-34)}$$

and that

$$\frac{\partial}{\partial Z} e^{-p(\Theta+\Pi)} = - e^{-p(\Theta+\Pi)} \, p \, \frac{M}{\beta} \frac{\partial R}{\partial Z} \qquad \text{(B-35)}$$

Hence, Equation (B-33) may be rewritten as

$$\tilde{I}'_W = e^{-p\beta(X_* - X_{TE})} \frac{\partial}{\partial Z} \int_{X_{TE}}^{\infty} e^{-\frac{p}{\beta}(X - X_* + MR)} \frac{1}{R} \, dX \qquad \text{(B-36)}$$

This integral is of the same type as the one given in Equation (D-11) of Reference 3.

Using the classical transformation

$$U = \frac{\Lambda + MR}{\beta R_o} \qquad \text{(B-37)}$$

where $\Lambda = X - X_*$ and

$$R_o = \left[(Y - Y_*)^2 + (Z - Z_*)^2\right]^{1/2} = \sqrt{R^2 - \Lambda^2} \qquad \text{(B-38)}$$

and noting that

$$\sqrt{1 + U^2} = \frac{1}{\beta R_o}\left[\beta^2 R_o^2 + \Lambda^2 + 2\Lambda MR + M^2(\Lambda^2 + R_o^2)\right]^{1/2} \qquad \text{(B-39)}$$

or

$$\sqrt{1+U^2} = \frac{1}{\beta R_o}\left[R_o^2 + \Lambda^2 + 2\Lambda M R + M^2\Lambda^2\right]^{1/2}$$

$$= \frac{1}{\beta R_o}\,(M\Lambda + R) \tag{B-40}$$

and

$$\frac{dU}{dX} = \frac{1}{\beta R_o}\left(1 + M\frac{\Lambda}{R}\right) = \frac{1}{R}\sqrt{1+U^2} \tag{B-41}$$

one obtains

$$\int_{X_{TE}}^{\infty} e^{-\frac{p}{\beta}(X - X_* + MR)}\,\frac{1}{R}\,dX = \int_{U_{TE}}^{\infty}\frac{e^{-pR_oU}}{\sqrt{1+U^2}}\,dU \tag{B-42}$$

where

$$U_{TE} = \left.\frac{X - X_* + MR}{\beta R_o}\right|_{X = X_{TE}} . \tag{B-43}$$

Next, consider the contour indicated in Figure B-2, and note that

$$I_C = \oint_C \frac{e^{-ikR_oU}}{\sqrt{1+U^2}}\,dU = 0 \tag{B-44}$$

since the integrand is analytic inside C. Note that using Jordan's lemma, Equation (B-4), if the radius of C_2 goes to infinity, the contribution of C_2 goes to zero. Also, if the radius of C_4 goes to zero, the contribution of C_4 goes to zero. Also, note that in the limit, the contribution of C_3 tends to (using $V = iU = \cosh\alpha$)

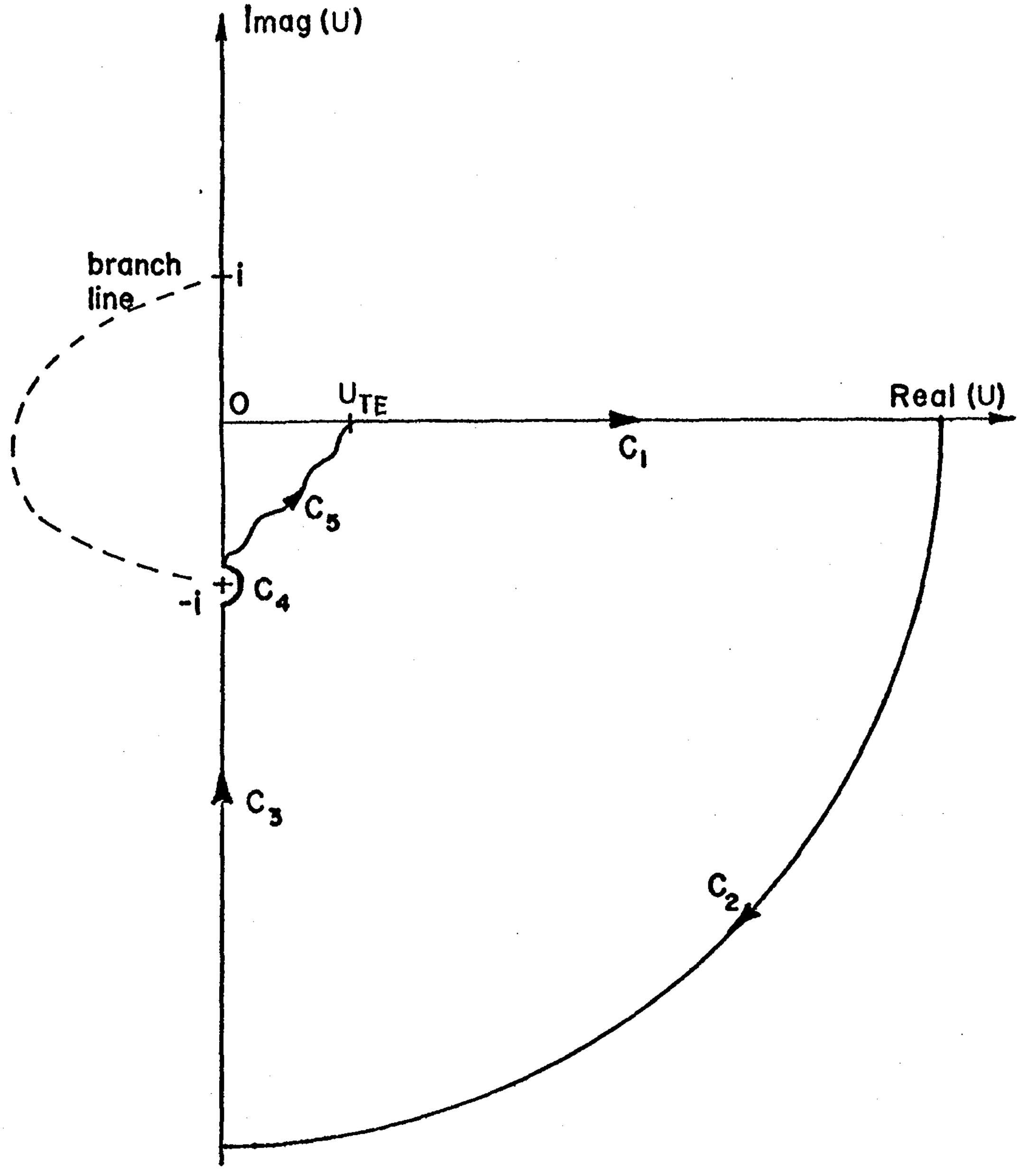


Figure B-2. The Contour of Integration.

$$\int_{-i\infty}^{-i} \frac{e^{-ikR_oU}}{\sqrt{1+U^2}}\, dU = -\int_1^{\infty} \frac{e^{-kR_oV}}{\sqrt{V^2-1}}\, dV$$

$$= -\int_0^{\infty} e^{-kR_o\cosh\alpha}\, d\alpha = -K_o(kR_o) \qquad \text{(B-45)}$$

where K_o is the modified Bessel function of second kind of order zero (see Reference 25). Therefore, Equation (B-44) may be rewritten as

$$\int_{U_{TE}}^{\infty} \frac{e^{-ikR_oU}}{\sqrt{1+U^2}}\, dU = \int_{U_{TE}}^{-i} \frac{e^{-ikR_oU}}{\sqrt{1+U^2}}\, dU + K_o(kR_o) \qquad \text{(B-46)}$$

Combining Equations (B-36), (B-42) and (B-46) and noting that (Reference 25)

$$\frac{dK_o(\alpha)}{d\alpha} = -K_1(\alpha) \qquad \text{(B-47)}$$

(where K_1 is the modified Bessel function of second kind of first order) one obtains, for $p = ik$,

$$\tilde{I}'_W = e^{-ik\beta(X_* - X_{TE})}\left[\frac{\partial}{\partial Z}\int_{U_{TE}}^{-i} \frac{e^{-ikR_oU}}{\sqrt{1+U^2}}\, dU - K_1(kR_o)\, k\, \frac{\partial R_o}{\partial Z}\right] \qquad \text{(B-48)}$$

The first term in the bracket can be easily evaluated numerically or analytically (an analytical expression is obtained in Reference 3). However, (Reference 25)

$$K_1(\alpha) = \left(\gamma + \ell n \frac{\alpha}{2}\right) I_1(\alpha) + \frac{1}{2}$$

$$- \frac{\alpha}{2} \sum_{m=0}^{\infty} \frac{1}{m!(m+1)!} \left[\left(1 + \frac{1}{2} + \ldots + \frac{1}{m+1}\right)\right.$$

$$\left. - \frac{1}{2} \frac{1}{m+1}\right] \left(\frac{\alpha}{2}\right)^{2m} \tag{B-49}$$

(where γ is the Euler constant) is a multivalued function.

Furthermore, using the same contour of integration, one obtains (using $V = i\,U$)

$$\int_{U_{TE}}^{\infty} \frac{e^{-pR_oU}}{\sqrt{1 + U^2}} dU = \int_{U_{TE}}^{-i} \frac{e^{-pR_oU}}{\sqrt{1 + U^2}} dU$$

$$+ \int_{1}^{\infty} \frac{e^{ipR_oV}}{\sqrt{V^2 - 1}} dV \tag{B-50}$$

only for

$$\text{Real } p > 0 \tag{B-51}$$

If Real $p < 0$, Jordan's lemma is not applicable. The contribution of C_2 goes to infinity (while the integrals over C_3, C_4 and C_5 remain finite) indicating that the integral over C_1 does not exist.

Finally, it is worth noting that if the wake is truncated, then I_W' is given by (see Equations (B-36) and (B-42))

$$\tilde{I}'_W = e^{-p\beta(X_* - X_{TE})} \frac{\partial}{\partial Z} \int_{U_{TE}}^{U_o} \frac{e^{-pR_o U}}{\sqrt{1+U^2}} \, dU$$

$$= e^{-p\beta(X_* - X_{TE})} \sum_{n=0}^{\infty} \frac{\partial}{\partial Z} \left(R_o^n \int_{U_{TE}}^{U_o} \frac{U^n}{\sqrt{1+U^2}} \, dU \right) \frac{(-p)^n}{n!} \qquad \text{(B-52)}$$

where the integrals may be easily evaluated by the procedure used in Appendix D of Reference 3. (The summation sign and the integral sign may be interchanged because the exponential series is uniformly convergent in the interval of integration; this would not be legitimate for the infinite wake.)

Equation (B-52) indicates that $\tilde{I}'_W$ exists for any value of p, since the integral of a uniformly convergent series is a uniformly convergent series.

In conclusion $\tilde{I}'_W$ is a single-valued analytic function of p in the whole p plane for the truncated wake, whereas it exists only for Real $p > 0$ for the infinite wake. In this last case the use of analytic continuation is necessary to obtain the function in the whole p-plane and this function is multi-valued.

B.3 OSCILLATORY AND STEADY FLOW

In this section is shown the relationship between Laplace-transform analysis and oscillatory-flow (in particular steady-state) analysis. Consider Equation (3-9) and assume Ψ_i and Φ_i to be of the type

$$\Psi_i(T) = \hat{\Psi}_i \, e^{ikT} \qquad \text{(B-53)}$$

and

$$\Phi_i(T) = \hat{\Phi}_i \, e^{ikT} \qquad \text{(B-54)}$$

where

$$k = \omega \ell / U_\infty \tag{B-55}$$

In particular if $k = 0$ one obtains the steady-state analysis.

Combining with Equation (3-9) one obtains an expression similar to Equation (3-12) with $\tilde{\Psi}_i$, $\tilde{\Phi}_i$ and p replaced by $\hat{\Psi}_i$, $\hat{\Phi}_i$ and ik. Therefore, the formulation presented here for transient analysis ($\Phi_i = 0$ for $T < 0$) in the Laplace domain is valid for oscillatory flow as well $(-\infty \leq T \leq \infty)$, with the amplitudes of the oscillatory solution given by

$$\hat{\Psi}_i = \tilde{\Psi}_i \Big|_{p = ik}$$

$$\hat{\Phi}_i = \tilde{\Phi}_i \Big|_{p = ik} \tag{B-56}$$

This is a well-known relationship between the two types of analysis (the same results hold for supersonic flow; see Equation (3-15)): as pointed out in References 26 and 27, any code developed for oscillatory flow (imaginary analysis) can be used for Laplace-domain analysis (with zero initial conditions) simply by replacing k with $-ip$.

However, in Equations (B-53) and (B-54)

$$-\infty < T < \infty \tag{B-57}$$

Therefore, the truncation of the wake (which is based upon Equation (3-11); see also Subsection B.2), is not applicable in this case. Hence, for oscillatory flows Equation (3-13) must contain the complete wake.

More subtle is the relationship for exponentially growing or exponentially damped oscillations; in this case, setting

$$\Psi_i(T) = \tilde{\Psi}_i \; e^{pT}$$

$$\Phi_i(T) = \tilde{\Phi}_i \; e^{pT} \tag{B-58}$$

and combining with Equation (3-9) yields Equation (3-12). This yields no problem for divergent oscillations, i.e.,

$$\text{Real } (p) > 0 \tag{B-59}$$

since the terms $e^{-p\Pi_n} \rightarrow 0$ as $\Pi_n = \beta\,(X_n - X_{TE}) \rightarrow \infty$ (note that Π_n increases with the distance of the wake element, Σ_n, from the trailing edge).

However, for convergent oscillations, i.e.,

$$\text{Real } (p) < 0 \tag{B-60}$$

one obtains

$$e^{-p\Pi_n} \rightarrow \infty \qquad \text{as } \Pi_n \rightarrow \infty \tag{B-61}$$

i.e., the solution does not necessarily exist. (The results of Subsection B.2.3 indicate that indeed the Laplace transform of the wake integral does not exist for Real $(p) < 0$.)

It should be noted that this phenomenon is not in contradiction with the physical model; for, if the amplitude of Ψ_i at $t = 0$ is finite, then

$$\Psi_i(-\infty) = \infty \tag{B-62}$$

If $\Psi_i(-\infty)$ has a "finite effect" at $t = 0$, then the solution is infinite.

B.4 INITIAL CONDITIONS

Note that for any arbitrary function $f(t)$

$$L(\dot{f}) = s\tilde{f} - f(0) \tag{B-63}$$

In relationship to Equation (B-63), the question of what restrictions have been used on the initial conditions is examined here.

The flow is assumed to be in steady state for time $t < 0$. This implies, (see Equation 2-45))

$$\Phi_U = 0 \qquad (t < 0) \tag{B-64}$$

and

$$\Psi_U = 0 \qquad (t < 0) \tag{B-65}$$

At time $t = 0$ it is assumed

$$q_n(0) = 0 \tag{B-66}$$

in order to obtain Equation (4-27) from Equation (4-26).

It is not necessary to assume however that $\dot{q}_m(0) = 0$. This implies, from Equation (4-26), that in general

$$\Psi_U(0) \neq 0 \tag{B-67}$$

Equations (B-65) and (B-67) imply, from Equation (2-48) that

$$\Phi_U(0) = 0 \tag{B-68}$$

if and only if $\Theta \neq 0$ (except at $P = P_*$), i.e., for $M \neq 0$ (see Equation (2-31)). However, in general

$$\Phi_U(0) \neq 0 \tag{B-69}$$

for $M = 0$.

Finally, consider the effect of the initial conditions in obtaining Equation (3-12) from Equation (3-9). If $M \neq 0$, Equation (B-68) applies and therefore Equation (3-12) is correct, since the last term in Equation (B-63) is equal to zero. On the other hand if $M = 0$ then $\hat{\Theta} \equiv 0$ (see Equation 2-32)) and therefore Equation (3-12) is still correct, since in this case

$$D_{jh} = G_{jn} = 0$$

(see Equations (3-7) and (3-10)).

It should be noted also that the case in which Equation (B-64) and (B-65) are not valid (i.e., when the flow is not in steady state for $t < 0$) is examined in Appendix F of Reference 3, where it is shown that the formulation is still valid if the term given by Equation (F-2) of Reference 3 is added to Equation (2-16).

APPENDIX C

INFLUENCE OF MOTION OF SURFACE

As mentioned above, in deriving the integral equation, the surface of the body has been assumed to be fixed ($\epsilon = 0$ in Equation (1.10)). The correct derivation of the integral equation for time dependent surface is given in Reference 3. Such derivation is not needed here with the only exception of a minor point, which is important in deriving the boundary conditions. For, consider the normal-wash integral in Equation (2-34)

$$I_\Psi = -\oiint [\Psi]^\Theta \frac{1}{R} \, d\Sigma$$

$$= -\oiint \left[\frac{\nabla_o \Phi \cdot \bar{A}_1 \times \bar{A}_2}{|\bar{A}_1 \times \bar{A}_2|} \right]^\Theta \frac{1}{R} \, |\bar{A}_1 \times \bar{A}_2| \; d\Xi^1 \, d\Xi^2 \qquad \text{(C-1)}$$

where Ξ^α are the convected curvilinear coordinates (moving with the body surface) and

$$\bar{A}_\alpha = \frac{\partial \bar{P}}{\partial \Xi^\alpha} \qquad \text{(C-2)}$$

are the surface base vectors.

Note that if the surface is time independent then

$$|\bar{A}_1 \times \bar{A}_2|^\Theta = |\bar{A}_1 \times \bar{A}_2| \qquad \text{(C-3)}$$

However, if the surface is time dependent the two factors are different. Since Equation (2-34) was obtained under the hypothesis of time-independent surface, it is impossible to determine what is the appropriate value of $|\bar{A}_1 \times \bar{A}_2|$ to be used. Therefore, in order to obtain the correct interpretation of Equation (C-1) it is necessary to make use of the results of Ref. 3, where it is shown that the correct expression is (Equation (6-40) of Ref. 3)

$$I_{\Psi} = -\oiint \left[\frac{\Psi_M}{R} \frac{|\nabla_o S|}{|\nabla_o S^{\Theta}|} \right]^{\Theta} d\Sigma^{\Theta} \tag{C-4}$$

where Ψ_M is the normal wash in the modified expression of I_{Ψ}, whereas Σ^{Θ} is the surface described by the equation

$$S^{\Theta} = S(P, T - \Theta) = 0 \tag{C-5}$$

or

$$\bar{P}^{\Theta}(\Xi^1, \Xi^2) = \bar{P}(\Xi^1, \Xi^2, T - \Theta) \tag{C-6}$$

with Θ given by Equation (2-31).

Note that the base vectors $\bar{A}_{\alpha}^{\Theta}$ of the surface Σ^{Θ} are given by

$$\bar{A}_{\alpha}^{\Theta} = \frac{\partial \bar{P}^{\Theta}}{\partial \Xi^{\alpha}} = \left[\bar{A}_{\alpha}^{(t)} - \dot{\bar{P}} \, \nabla_o \Theta \cdot \bar{A}_{\alpha}^{\Theta} \right]_{T-\Theta} \tag{C-7}$$

(where $\bar{A}_{\alpha}^{(t)} \equiv \dfrac{\partial \bar{P}}{\partial \Xi^{\alpha}}$, $\dot{\bar{P}} \equiv \partial \bar{P}/\partial T$ and $\nabla_o \Theta \cdot \bar{A}_{\alpha}^{\Theta} \equiv \dfrac{\partial \Theta}{\partial \Xi^{\alpha}}$) or, solving for $\bar{A}_{\alpha}^{\Theta}$,*

$$\bar{A}_{\alpha}^{\Theta} = \left[\bar{A}_{\alpha}^{(t)} - \dot{\bar{P}} \; \nabla_o \Theta \cdot \bar{A}_{\alpha}^{(t)} \left(1 + \dot{\bar{P}} \cdot \nabla_o \Theta\right)^{-1} \right]_{T-\Theta} \tag{C-8}$$

This yields (using $\dot{\bar{P}}$ as third base vector, $\bar{A}_3^{(t)}$)

$$\bar{A}_1^{\Theta} \times \bar{A}_2^{\Theta} = \left[\bar{A}_1^{(t)} \times \bar{A}_2^{(t)} - \left(1 + \dot{\bar{P}} \cdot \nabla_o \Theta\right)^{-1} \left(\bar{A}_1^{(t)} \times \dot{\bar{P}} \, \Theta_{/2} - \bar{A}_2^{(t)} \times \dot{\bar{P}} \, \Theta_{/1} \right) \right]_{T-\Theta}$$

$$= \left[\left(1 + \dot{\bar{P}} \cdot \nabla_o \Theta\right)^{-1} \left(\bar{A}_1^{(t)} \times \bar{A}_2^{(t)} \left(1 + \dot{\bar{P}} \cdot \nabla_o \Theta\right) - \bar{A}_1^{(t)} \times \dot{\bar{P}} \, \Theta_{/2} + \bar{A}_2^{(t)} \times \dot{\bar{P}} \, \Theta_{/1} \right) \right]_{T-\Theta}$$

*Note that, if $\bar{a} = \bar{b} + \bar{c}\,\bar{d} \cdot \bar{a}$, then $\bar{d} \cdot \bar{a} = \bar{d} \cdot \bar{b}/(1 - \bar{c} \cdot \bar{d})$ and $\bar{a} = \bar{b} + \bar{c}\,\bar{d} \cdot \bar{b}/(1 - \bar{c} \cdot \bar{d})$.

$$= \left[(1 + \dot{\bar{P}} \cdot \nabla_o \Theta)^{-1} \; (\bar{A}_1^{(t)} \times \bar{A}_2^{(t)} + \dot{\bar{P}} \cdot \bar{A}_1^{(t)} \times \bar{A}_2^{(t)} \nabla_o \Theta) \right]_{T-\Theta}$$

$$= \left[(1 + \dot{\bar{P}} \cdot \nabla_o \Theta)^{-1} \; |A_1^{(t)} \times A_2^{(t)}| \; (\bar{N}^{(t)} + \dot{\bar{P}} \cdot N^{(t)} \nabla_o \Theta) \right]_{T-\Theta} \qquad \text{(C-9)}$$

(where $\Theta_{/\alpha} = \nabla_o \Theta \cdot \bar{A}_\alpha^{(t)}$). Note that (see Equation (4-14))

$$\frac{|\nabla_o S^\Theta|}{|\nabla_o S|^\Theta} = \left| \frac{\nabla_o S}{|\nabla_o S|} - \frac{\partial S}{\partial T} \frac{1}{|\nabla_o S|} \nabla_o \Theta \right|_{T-\Theta}$$

$$= \left| \bar{N}^{(t)} + \dot{\bar{P}} \cdot \bar{N}^{(t)} \nabla_o \Theta \right|_{T-\Theta} \qquad \text{(C-10)}$$

Hence

$$\frac{|\nabla_o S|^\Theta}{|\nabla_o S^\Theta|} d\Sigma^\Theta = \left| \bar{N}^{(t)} + \dot{\bar{P}} \cdot \bar{N}^{(t)} \nabla_o \Theta \right|_{T-\Theta}^{-1} \left| \bar{A}_1^\Theta \times \bar{A}_2^\Theta \right| \, d\Xi^1 \, d\Xi^2$$

$$= \frac{|\bar{A}_1^{(t)} \times \bar{A}_2^{(t)}|^\Theta}{\left[1 + \dot{\bar{P}} \cdot \nabla_o \Theta \right]^\Theta} \, d\Xi^1 d\Xi^2 \qquad \text{(C-11)}$$

Note that in the common case of a zero-thickness wing, at zero angle of attack, with motion in the direction of the normal, $\nabla_o \Theta$ is in the plane of the wing, whereas $\dot{\bar{P}}$ is along the normal $\bar{N}$ and hence $\dot{\bar{P}} \cdot \nabla_o \Theta \equiv 0$. In general for small-amplitude oscillations at reduced frequencies of order one, one obtains

$$|\dot{\bar{P}} \cdot \nabla_o \Theta| << 1 \qquad \text{(C-12)}$$

This approximation is used in this report.

Combining Equations (C-4), (C-11) and (C-12) and noting that, for small-amplitude motion, $R^\Theta \simeq R$, one obtains

$$I_\Psi = -\oint\!\!\oint \left[\Psi_M^{(t)} \, |\bar{A}_1^{(t)} \times \bar{A}_2^{(t)}| \right]^{\Theta} \frac{1}{R} \, d\Xi^1 \, d\Xi^2 \tag{C-13}$$

or

$$I_\Psi = -\oint\!\!\oint \left[\Psi_M^{(t)} \, |\bar{A}_1^{(t)} \times \bar{A}_2^{(t)}| \right]^{\Theta} \frac{1}{R} \, \frac{d\Sigma}{|\bar{A}_1 \times \bar{A}_2|} \tag{C-14}$$

where R, $|\bar{A}_1 \times \bar{A}_2|$ and $d\Sigma$ are evaluated at time T or, more conveniently in the undisturbed configuration.

The results of this subsection may be summarized by saying that Equation (C-1) is still valid if Ψ^{Θ} is defined as

$$\Psi^{\Theta} = \frac{|\bar{A}_1 \times \bar{A}_2|^{\Theta}}{|\bar{A}_1 \times \bar{A}_2|} \, \Psi_M^{\Theta} \tag{C-15}$$

where Ψ_M has been introduced in Equation (C-4). Note that Equation (2-44) applies to Ψ_M as well, and therefore

$$\Psi_M = \frac{1}{U_\infty} \, \psi \tag{C-16}$$

Note also that

$$\frac{|\bar{A}_1 \times \bar{A}_2|^{\Theta}}{|\bar{A}_1 \times \bar{A}_2|} \simeq \frac{|\bar{a}_1 \times \bar{a}_2|^{\Theta}}{|\bar{a}_1 \times \bar{a}_2|} \tag{C-17}$$

Therefore in order to take into account the motion of the surface it is convenient to retain Equation (C-1) with Ψ given by

$$\Psi = \frac{1}{U_\infty} \psi_M \frac{|\bar{a}_1^{(t)} \times \bar{a}_2^{(t)}|}{|\bar{a}_1 \times \bar{a}_2|} \qquad \text{(C-18)}$$

In Equation (C-14) the superscript (t) is used to indicate evaluation with time-dependent surface ($\epsilon \neq 0$) in Equation (1-10)). The base vectors $\bar{a}_\alpha$ are evaluated from the time-independent surface ($\epsilon = 0$ in Equation (1-10)).

APPENDIX D

BODY-AXIS BOUNDARY CONDITIONS

In order to extend the formulation of Section 4 to the body-axis formulation (which is used in flight dynamics, see for instance Reference 28), it is necessary to express the downwash in terms of the generalized coordinates and generalized velocities. This relationship can be obtained by using the boundary conditions as obtained by combining Equations (4-16) and (4-18) to obtain

$$\Psi = \left(-\bar{i} + \frac{\bar{v}_B}{U_\infty} \right) \cdot (\bar{n} + \Delta\bar{n}) \tag{D-1}$$

or

$$\Psi = \Psi_S + \Psi_U \tag{D-2}$$

where

$$\Psi_S = -\bar{i} \cdot \bar{n} \tag{D-3}$$

and, neglecting higher order terms

$$\Psi_U = \frac{\bar{\bar{v}}_B}{U_\infty} \cdot \bar{n} - \bar{i} \cdot \Delta\bar{n} \tag{D-4}$$

It should be noted that in the body-axis formulation $\bar{u}$ is the displacement (of a point $\bar{P}$ on the surface of the body) evaluated with respect to the frame of reference $\bar{i}_B$, $\bar{j}_B$, $\bar{k}_B$. Therefore, including the motion of the body axes to Equation (4-13) with $\bar{u}$ given by Equation (4-23) one obtains (see Reference 28)

$$\bar{v}_B = U_\infty \sum_{n=1}^{6} v_n \bar{M}_n^{(R)} (\xi^\alpha) + \ell \sum_{m=1}^{M} \dot{q}_m M_m (\xi^\alpha) \tag{D-5}$$

where typically $\bar{M}_m$ are the nondimensional natural modes of vibration and q_m are the corresponding nondimensional generalized coordinates, whereas v_n are the nondimensional components of the translational and angular velocities of the body-axis frame of reference whereas $\bar{M}_m^{(R)}$ are the nondimensional six rigid-body mode shapes. Note that the nondimensional translational velocities are obtained from the translational velocities by dividing by U_∞ (the corresponding modes are already nondimensional). On the other hand, the nondimensional angular velocities are obtained from the angular velocities by dividing by U_∞/ℓ (the corresponding modes are made nondimensional by dividing by ℓ).

Combining Equations (D-4), (D-5), and (4-24) yields

$$\Psi_U = \sum_{n=1}^{6} v_n \bar{M}_n^{(R)} \cdot \bar{n} + \sum_{m=1}^{M} \left[\frac{\ell}{U_\infty} \bar{M}_m \cdot \bar{n} \dot{q}_m - \Delta\bar{n}_m \cdot \bar{i} q_m \right] \tag{D-6}$$

The Laplace transform of Equation (D-6) is*

$$\tilde{\Psi} = \sum_{n=1}^{6} \bar{M}_n^{(R)} \cdot \bar{n} \tilde{v}_n + \sum_{n=1}^{M} [p \bar{M}_n \cdot \bar{n} - \Delta\bar{n}_m \cdot \bar{i}] \tilde{q}_m \tag{D-7}$$

Equation (D-7) may be rewritten as (the subscript h indicates the evaluation at the center $\bar{P}_h$ of the element Σ_h)

$$\{\tilde{\Psi}_h\} = \left[E_{hn}^{(1,V)}\right] \{\tilde{v}_n\} + \left[E_{hm}^{(1,Q)}\right] \{\tilde{q}_m\} \tag{D-8}$$

*It is assumed $q_m(0) = 0$; see Section B.4.

where

$$\tilde{E}^{(1,V)}_{hn} = \left[\bar{M}^{(R)}_{n} \cdot \bar{n}\right]_{\bar{P} = \bar{P}_h} \tag{D-9}$$

and

$$\tilde{E}^{(1,Q)}_{hm} = [p\,\bar{M}_m \cdot \bar{n} - \Delta\,\bar{n}_m \cdot \bar{i}\,]_{\bar{P} = \bar{P}_h} \tag{D-10}$$

APPENDIX E

CLOSED-WAKE PHENOMENON

Consider, for instance, the wake emanating from the perimeter of the base of a truncated cylinder. This wake separates the flow field into two parts: one inside the wake and one outside the wake. This type of wake will be referred to as a "closed wake". In this Appendix a few mathematical details relative to the analysis of a closed-wake configuration are discussed. It will be shown that in this case the integral equation (Equation (2-52) is singular (i.e., the homogeneous integral equation has a nontrivial solution, and hence the nonhomogeneous integral equation has a nonunique solution). For simplicity only the steady incompressible flow is considered here.

The integral representation of the potential for steady incompressible flow is given by (see Equation (2-52))

$$4\pi E_* \varphi_* = - \oint\!\!\oint_{\Sigma_B} \left[\frac{\partial \varphi}{\partial n} \left(\frac{1}{r} \right) - \varphi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) \right] d\Sigma$$

$$+ \iint_{\Sigma_W} \Delta\varphi \frac{\partial}{\partial n_u} \left(\frac{1}{r} \right) d\Sigma \qquad \text{(E-1)}$$

If $\bar{p}_*$ is on Σ_B, the function $E(\bar{p})$ assumes the value $E_* = 1/2$ and Equation (E-1) reduces to an integral equation for the potential on the surface of the body.

As previously mentioned, whenever the wake separates the flow field into two separate regions, one inside and one outside the wake, the integral equation is singular. This problem occurs in the formulation for bodies with a blunt back such as a projectile or a fuselage with truncated base. A similar problem occurs in the formulation for building aerodynamics and actuator-disk aerodynamics.

In order to show that the integral equation for closed-wake configurations is singular, it is convenient to obtain Equation (E-1) a different way. Consider Figure Figure E-1(c), and note that the value $E = 1$ outside the body may be obtained as the sum of the function E_i for inner flow (i.e., inside the wake) and the function E_o for the outer flow (i.e., outside the wake and the body) or

$$E(\bar{p}) = E_i(\bar{p}) + E_o(\bar{p}) \tag{E-2}$$

Applying Green's theorem to the regions shown in Figures E-1(a) and E-1(b) yields

$$4\pi E_i(\bar{p}_*)\,\varphi(\bar{p}_*) = -\oint\!\!\oint \left[\frac{\partial\varphi}{\partial n_i}\left(\frac{1}{r}\right) - \varphi\frac{\partial}{\partial n_i}\left(\frac{1}{r}\right)\right] d\Sigma_i \tag{E-3}$$

and

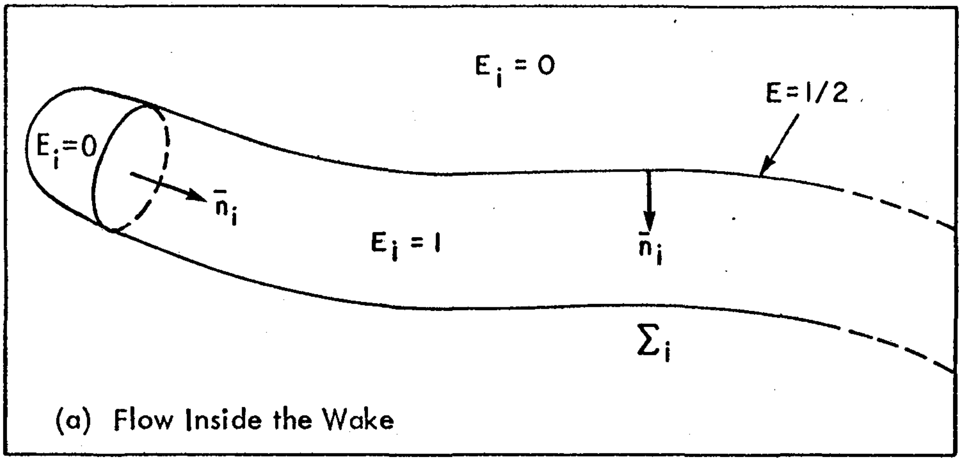
$$4\pi E_o(\bar{p}_*)\,\varphi(\bar{p}_*) = -\oint\!\!\oint \frac{\partial\varphi}{\partial n_o}\left(\frac{1}{r}\right) - \varphi\frac{\partial}{\partial n_o}\left(\frac{1}{r}\right) \quad d\Sigma_o \tag{E-4}$$

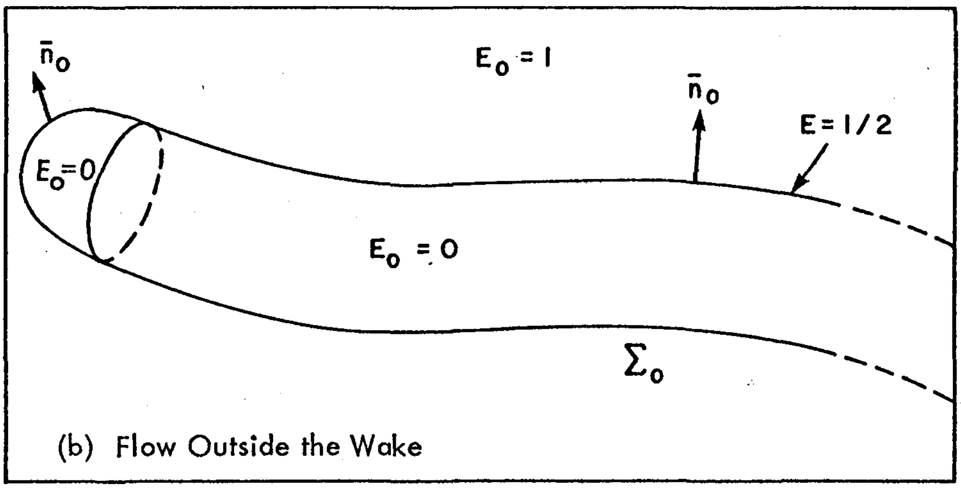
Note that

$$\begin{aligned} \bar{n}_i &= \bar{n};\ \bar{n}_o = \bar{n} && \text{on } \Sigma_B \\ \bar{n}_i &= -\bar{n}_u;\ \bar{n}_o = \bar{n}_u && \text{on } \Sigma_W \end{aligned} \tag{E-5}$$

Thus, adding Equations (E-3) and (E-4) and using Equations (E-2) and (E-5) yields

$$\begin{aligned} 4\pi E(\bar{p}_*)\varphi(\bar{p}_*) = &-\oint\!\!\oint_{\Sigma_B} \left[\frac{\partial\varphi}{\partial n}\left(\frac{1}{r}\right) - \varphi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] d\Sigma \\ &+ \iint_{\Sigma_W} (\varphi_o - \varphi_i)\,\frac{\partial}{\partial n_u}\left(\frac{1}{r}\right) d\Sigma \end{aligned} \tag{E-6}$$





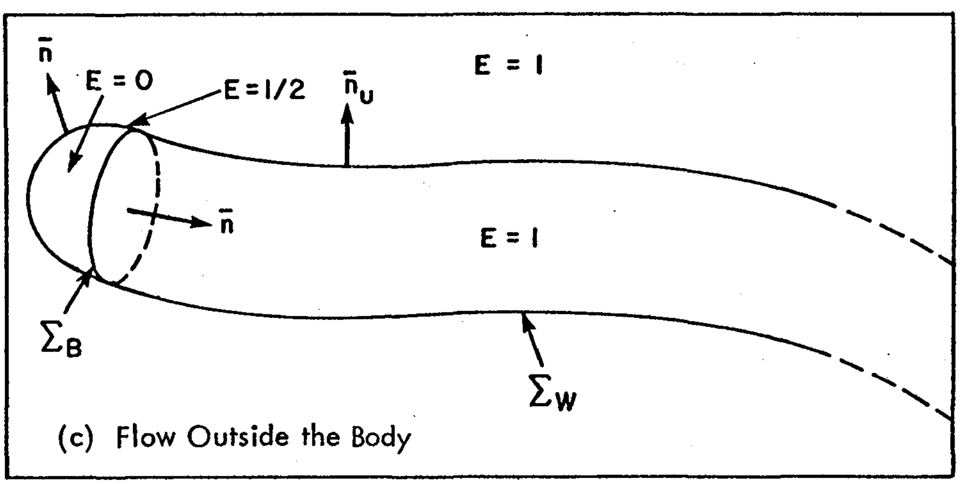


Figure E-1. Geometry for Green's Theorem for Closed-Wake Configuration.

Note that Equation (E-6) coincides with Equation (E-1) with $\Delta\varphi \equiv \varphi_o - \varphi_i$. In order to show that the operator in Equation (E-6) is singular, it is sufficient to show that the homogeneous equation (i.e., with $\partial\varphi/\partial n = 0$)

$$4\pi E(\bar{p}_*)\varphi(\bar{p}_*) = \oiint_{\Sigma_B} \varphi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\Sigma + \iint_{\Sigma_W} \Delta\varphi \frac{\partial}{\partial n_u}\left(\frac{1}{r}\right) d\Sigma \tag{E-7}$$

has a nontrivial solution. In order to show this, consider the function

$$\begin{aligned} \varphi_{NT}(\bar{p}) &= 0 \quad \bar{p} \text{ outside } \Sigma_i \\ &= 1 \quad \bar{p} \text{ inside } \Sigma_i \end{aligned} \tag{E-8}$$

If $\partial\varphi/\partial n = 0$ (homogeneous problem), this function is a trival solution (i.e., $0 = 0$) of Equation (E-4), whereas for Equation (E-3) one obtains (note that $E_i\varphi_{NT} = E_i$)

$$4\pi E_i = \oiint_{\Sigma_i} \frac{\partial}{\partial n_i}\left(\frac{1}{r}\right) d\Sigma_i \tag{E-9}$$

which is identically satisfied, since (indicating with Ω the solid angle)

$$\begin{aligned} \oiint_{\Sigma_i} \frac{\partial}{\partial n_i}\left(\frac{1}{r}\right) d\Sigma_i \equiv \Omega &= 0 \text{ outside } \Sigma_i \\ &= 2\pi \text{ on } \Sigma_i \\ &= 4\pi \text{ inside } \Sigma_i \end{aligned} \tag{E-10}$$

Since φ_{NT} is a nontrivial solution for both Equations (E-3) and (E-4) with $\partial\varphi/\partial n = 0$, it is a nontrivial solution for their sum, Equation (E-7). Hence, the operator in Equation (E-1) is singular; therefore, if $\hat{\varphi}$ satisfies Equation (E-1), then

$$\varphi = \hat{\varphi} + C\varphi_{NT} \tag{E-11}$$

(where C is an arbitrary constant) also satisfies Equation (E-1).

Finally note that in the numerical formulation, Equation (E-1) is replaced by Equation (3-12) with $p = 0$. However, the doublet integral represents solid angles and the solid angle is evaluated exactly with the use of the hyperboloidal elements. Thus the discrete form of Equation (E-10) is still valid exactly. Therefore, even the discrete system is singular, i.e., the determinant of the system given by Equation (3-12) with $p = 0$ is equal to zero. This implies that the vector

$$\left\{\varphi_i^{NT}\right\} = \left\{\varphi_{NT}(\bar{p}_i)\right\} \tag{E-12}$$

is a nontrivial solution for Equation (3-12) with right-hand-side equal to zero, or that if $\hat{\varphi}_i$ is a solution to Equation (3-12), then

$$\left\{\varphi_i\right\} = \left\{\hat{\varphi}_i\right\} + C\left\{\varphi_i^{NT}\right\} \tag{E-13}$$

(where C is an arbitrary constant) is also a solution to Equation (3-12) (i.e., if the solution exists, it is not unique).

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16. Abstract

A review and summary of recent developments of the Green's function method and the computer program SOUSSA (Steady, Oscillatory, and Unsteady Subsonic and Supersonic Aerodynamics) are presented. Applying the Green's function method to the fully unsteady (transient) potential equation yields an integro-differential-delay equation. With spatial discretization by the finite-element method, this equation is approximated by a set of differential-delay equations in time. Time solution by Laplace transform yields a matrix relating the velocity potential to the normal wash. Premultiplying and postmultiplying by the matrices relating generalized forces to the potential and the normal wash to the generalized coordinates, one obtains the matrix of the generalized aerodynamic forces. The frequency and mode-shape dependence of this matrix makes the program SOUSSA very useful for multiple frequency and repeated mode-shape evaluations. The program SOUSSA is general, flexible, easy to use, and accurate. Applications to aerodynamic design are also discussed. The user/programmer manual for SOUSSA-P 1.1 is presented in Volume 2 of this report (NASA CR-159131).

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