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NONLINEAR AEROELASTIC EQUATIONS OF MOTION OF TWISTED, NONUNIFORM, FLEXIBLE HORIZONTAL-AXIS WIND TURBINE BLADES

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SUMMARY

The second-degree nonlinear equations of motion for a flexible, twisted, nonuniform, horizontal-axis wind turbine blade are developed using Hamilton's principle. The derivation of the equations has its basis in the geometric nonlinear theory of elasticity, and the final equations are consistent with the small deformation approximation in which the elongations and shears are negligible compared to unity and the square of the derivative of the extensional deformation of the elastic axis is negligible compared to the squares of the bending slopes. A mathematical ordering scheme which is consistent with the assumption of a slender beam is used to discard some higher-order elastic and inertial terms in the second-degree nonlinear equations. The blade aerodynamic loading which is employed accounts for both wind shear and tower shadow and is obtained from strip theory based on a quasi-steady approximation of two-dimensional, incompressible, unsteady, airfoil theory. The resulting equations have periodic coefficients and are suitable for determining the aeroelastic stability and response of large horizontal-axis wind turbine blades.

INTRODUCTION

The recently renewed efforts in wind power are due to its prospective uses as an alternative energy source. As a result of these efforts, several wind turbine projects have been initiated by NASA Lewis Research Center as a part of the Department of Energy's (DOE) overall wind energy program. To make wind energy cost effective, wind turbines substantially larger than the existing 100kW Mod-0 which has a rotor diameter of 38 meters are being studied. However, as the rotor diameter increases, blade flexibility and hence susceptibility to aeroelastic instability also increase. Furthermore, efficient construction and operation of wind turbines require that the vibratory loads and stresses on the rotor as well as on the combined rotor-tower system be reduced to the lowest possible levels. Thus, aeroelastic and structural dynamic considerations have a direct bearing on the manufacture, life, and operation of these large wind turbine systems. Although the structural dynamic and aeroelastic technology used to develop helicopter rotors appear to be adequate for the development of wind turbine machines, this technology has to be transformed from helicopter rotor applications to wind power applications, and additional studies have to be conducted to determine the effects of the parameters peculiar to wind power machines on the aeroelastic and structural dynamic behavior.

Several aeroelastic considerations are common to both the wind turbine and helicopter blades. These include flap-lag-torsion, flap-torsion and flap-lag instabilities, stall flutter, and torsional divergence. The wind velocity gradient due to the Earth's boundary layer and gravity loads in the case of a wind turbine rotor, and forward velocity in the case of a helicopter rotor lead to timewise periodic coefficients in the equations of motion. Several previous studies have considered the helicopter blade and developed the nonlinear aeroelastic equations of motion. Recently, reference 1, though primarily a study of the flap-lag dynamics of rigid articulated helicopter rotor blades, contains a cursory examination of the elastic blade. References 2 and 3 more completely examined the basis of the nonlinear aeroelastic equations. In particular, attention was directed at establishing the expressions for the nonlinear curvatures and the nonlinear transformation matrix between the undeformed and deformed blade coordinates. The resulting equations were compared with some of those in the literature. These comparisons indicated several descrepancies with the results of reference 3, particularly in the nonlinear terms. The reasons for these descrepancies were explained in reference 3.

For wind turbine blades, reference 4 presented a set of nonlinear equations of motion. An examination of these equations reveals that reference 4 fails to obtain several nonlinear elastic and aerodynamic terms which are of the same order as those retained. It appears that these terms were not obtained in reference 4 for two reasons: (1) an incorrect torsional curvature expression and (2) linearizing the resultant transformation matrix between the undeformed and deformed blade coordinates while developing nonlinear equations of motion.

Based on the considerations and discussions in references 1 and 2, reference 5 apparently sought to redevelop the nonlinear equations of equilibrium for rotor blades. However, the resulting equations are missing several elastic terms which were well established in the literature and some nonlinear terms as well. It is interesting that several third-degree nonlinear terms were retained. These general equilibrium equations were used to study aeroelastic stability of a single wind turbine blade in reference 6 and a coupled rotor/support system in reference 7. Nonlinear aeroelastic equations of motion of a single wind turbine blade were also developed in reference 8 using the Newtonian method. Several nonlinear elastic and aerodynamic terms are missing in this reference because of the use of an incorrect expression for the torsional curvature and a partial linearization of the resultant rotational transformation between the coordinates of the undeformed and deformed blade. In view of the differences in the equations existing in the literature, it is felt that a comprehensive development of the nonlinear aeroelastic equations of motion of an elastic horizontal-axis wind turbine blade is required. The basic ingredients of such a development were presented in reference 9. This report documents the details of the development.

The derivation of the nonlinear equations of motion herein follows essentially along the lines of reference 3. In this reference, the pretwist together with control inputs of the blades were combined with the elastic twist for simplicity, following a common practice in the helicopter blade literature. Physically, the pretwist is present in the blade even before the deformations. Thus, this report will include the rotation due to pretwist with the control inputs and impose this rotation first while imposing the rotational transformations.

SYMBOLS

a	airfoil lift-curve-slope
A	cross-sectional area
A_u, A_v, A_w	generalized aerodynamic forces per unit length in X, Y, Z directions, respectively
A_ϕ	generalized aerodynamic moment per unit length about elastic axis
b	number of blades; also blockage factor to account for tower shadow
$B_V, B_T, B_g, B_{\delta W_D}, B_A$	boundary terms arising from strain energy, kinetic energy, gravity, material damping, and aerodynamic forces, respectively
B_1, B_2	section constants
c	blade chord
c_{d_o}	airfoil profile drag coefficient
$C(k)$	Theodorsen's circulation function
C_1, C_2	aerodynamic constants
$d_i (i=1,2,\ldots 5)$	notation used in writing the virtual work associated with material damping in concise form
D	airfoil profile drag per unit length
e	chordwise offset of mass centroid from elastic axis (positive when in front of elastic axis)
e_A	chordwise distance of area centroid of cross section from elastic axis (positive when in front of elastic axis)
E	Young's modulus
E^*	coefficient of internal friction in tension

$\bar{e}_{x_3}, \bar{e}_{y_3}, \bar{e}_{z_3}$	unit vectors along x_3, y_3, z_3 axes, respectively
$\bar{e}_X, \bar{e}_Y, \bar{e}_Z$	unit vectors along X, Y, Z axes, respectively
$\bar{e}_{X_I}, \bar{e}_{Y_I}, \bar{e}_{Z_I}$	unit vectors along X_1, Y_1, Z_1 axes, respectively
$\bar{e}_x, \bar{e}_\eta, \bar{e}_\zeta$	unit vectors along x, η, ζ axes, respectively
$F_{x_3}, F_{y_3}, F_{z_3}$	components of aerodynamic forces per unit length in x_3, y_3, z_3 directions, respectively
F_x, F_η, F_ζ	components of aerodynamic forces per unit length in x, η, ζ directions, respectively
F_{gu}, F_{gv}, F_{gw}	components of gravitational forces per unit length in X, Y, Z directions, respectively
$F_{g\phi}$	gravitational moment per unit length about elastic axis
g	gravitational constant
$g_1, g_2, \ldots g_6$	notation used in writing the gravitational forces
G	shear modulus
G*	coefficient of internal friction in shear
$\dot{h}$	vertical velocity of two-dimensional section normal to free-stream
H	height of rotor hub center above ground
I_u, I_v, I_w	generalized inertia forces per unit length in X, Y, Z directions, respectively
I_ϕ	generalized inertia moment per unit length about elastic axis
$I_{\eta\eta}, I_{\zeta\zeta}$	area moments of inertia about η and ζ axes, respectively
J	torsional section constant
k	reduced frequency

k_A	polar radius of gyration of cross-sectional area about elastic axis
$k_i (i=1,2,...6)$	notation used in writing the variation of kinetic energy
k_m	polar radius of gyration of cross-sectional mass about elastic axis ($k_m^2 = k_{m_1}^2 + k_{m_2}^2$)
l_i, m_i, n_i	direction cosines (i = 1, 2, 3)
L	aerodynamic lift per unit length
M, M_ϕ, M_{x_3}	aerodynamic pitching moment per unit length about deformed elastic axis
m	mass of blade per unit length
$Q_{D_u}, Q_{D_v}, Q_{D_w}, Q_{D_\phi}$	generalized damping forces
R	length of the blade
$\bar{R}_1$	position vector of a point on the elastic axis after deformation
$\bar{R}_0$	position vector of a point on the elastic axis before deformation
$\bar{r}_1$	position vector of a point after deformation
$\bar{r}_0$	position vector of a point before deformation
s	coordinate along the undeformed elastic axis
s_1	coordinate along the deformed elastic axis
$s_i (i=1,2,...10)$	notation used in writing the variation of the strain energy
T	kinetic energy
T_A	blade tension due to aerodynamic forces
T_C	blade tension due to centrifugal forces
T_g	blade tension due to gravity

U_R, U_T, U_P	radial, tangential, and perpendicular components of velocity for blade airfoil section
U	resultant of U_T and U_P
U_F	radial foreshortening of elastic axis
u, v, w	deformation of elastic axis in X, Y, and Z directions, respectively
u_g, v_g, w_g	components of gust velocity in X_I, Y_I, Z_I directions respectively
V	strain energy
V_H	mean wind velocity at height H
$V_m(x_I)$	mean wind velocity at x_I
$\overline{V}_{X_I Y_I Z_I}$	aerodynamic velocity vector expressed in $X_I Y_I Z_I$ coordinate axis system
$\overline{V}_{x_3 y_3 z_3}$	relative velocity of point on elastic axis expressed in $x_3 y_3 z_3$ coordinate system
$\overline{V}_{x\eta\zeta}$	relative velocity of point on elastic axis expressed in $x\eta\zeta$ coordinate system
v_i	induced velocity
W	work done by aerodynamic, structural damping, and gravitational forces
W_A	work done by aerodynamic loading
W_D	work done by structural damping
W_g	work done by gravitational forces
XYZ	coordinate system with the origin at the hub center which rotates with blade such that X-axis lies along initial undeformed position of the blade elastic axis

$X_I Y_I Z_I$	inertial axis system with the origin at hub centerline and Z_I-axis normal to the hub plane
$X_\Omega Y_\Omega Z_\Omega$	hub-fixed axis system rotating about Z_I-axis with angular velocity Ω
$x\eta\zeta$	principal axis system obtained by rotating about x_0-axis with an angle θ
$x_0 y_0 z_0$	blade-fixed axis system at arbitrary point on elastic axis before deformation
x_1, y_1, z_1	coordinates of a point (which was at $x\eta\zeta$ in the undeformed blade) in the deformed blade
$x_3 y_3 z_3$	blade-fixed orthogonal axis system in deformed configuration obtained by rotating $x\eta\zeta$; x_3-axis is tangent to the deformed elastic axis
$[T]$	transformation matrix between $x\eta\zeta$ and $x_3 y_3 z_3$ coordinate systems
$[T_R]$	transformation matrix between $x\eta\zeta$ and $X_I Y_I Z_I$ coordinate systems
$[\varepsilon_{ij}]$	Green's strain tensor
α	airfoil section angle of attack, $\alpha = \tan^{-1} U_P/U_T$
$\alpha_x, \alpha_y, \alpha_z$	notation used in writing the derivative of the displacement of a point
β_{pc}	angle of built-in coning (precone angle)
$\beta_1, \zeta_1, \theta_1$	Eulerian-type rotation angles between $x\eta\zeta$ and $x_3 y_3 z_3$
$\gamma_{xx}, \gamma_{x\eta}, \gamma_{x\zeta}$	engineering strain components
$\delta(\ )$	variation of ( )

ε	small parameter of the order of the bending slopes; the airfoil section pitch angle with respect to free-stream velocity; extensional component of Green's strain tensor on the elastic axis
$\varepsilon_{xx}, \varepsilon_{x\eta}, \varepsilon_{x\zeta}$	tensor strain components
η	sectional coordinate along major principal axis; also constant for mean velocity calculations
ζ	sectional coordinate normal to η-axis
χ	nondimensional coordinate along blade axis, $\chi = x/R$
θ	total geometric pitch angle, $\theta = \theta_{pt} + \theta_c$
θ_c	collective pitch
θ_{pt}	built-in twist (pretwist), positive when leading edge is upward
λ_i	induced flow ratio, $\lambda_i \Omega R = v_i$
λ_m	inflow ratio, $\lambda_m \Omega R = V_m(x_I)$
λ_H	inflow ratio, $\lambda_H \Omega R = V_H$
$\lambda_{ug}, \lambda_{vg}, \lambda_{wg}$	inflow ratios, $\lambda_{ug}\Omega R = u_g \sin \omega_g t$; $\lambda_{vg}\Omega R = v_g \sin \omega_g t$; $\lambda_{wg}\ \Omega R = w_g \sin \omega_g t$
ρ	mass density of the blade; also mass density of air
$\sigma_{xx}, \sigma_{x\eta}, \sigma_{x\zeta}$	engineering stresses
ϕ	angle of twisting deformation about elastic axis, positive when leading edge is upward
ψ	blade azimuth angle measured from downward position in the direction of rotation
ψ_0	angle between mean wind velocity V_m and rotor axis Z_I
$\omega_{x_3}, \omega_{y_3}, \omega_{z_3}$	torsional curvature (total rotation rate about x_3-axis) and bending curvatures, respectively

$\bar{\omega}_{x\eta\zeta}$	curvature vector before deformation
$\bar{\omega}_{x_3y_3z_3}$	curvature vector after deformation
$\bar{\omega}$	angular velocity of $x\eta\zeta$ coordinate system
Ω	rotational speed of the wind turbine rotor
$(\ )_C$	circulatory aerodynamic term
$(\ )_{NC}$	noncirculatory aerodynamic term
$[\ ]^T$	denotes transpose of matrix
$(\dot{\ })$	time derivative, $\frac{\partial}{\partial t}(\ )$
$(\ )'$	space derivative, $\frac{\partial}{\partial s}(\ ) = \frac{\partial}{\partial x}(\ )$

MATHEMATICAL MODEL AND COORDINATE SYSTEMS

The mathematical model chosen to represent the wind turbine blade consists of a straight, slender, variably twisted, nonuniform elastic blade. The elastic axis, the mass axis, and the tension axis are taken to be noncoincident; the elastic axis and the feathering axis are assumed coincident with the quarter chord of the blade. The effect of warping is assumed to be small and is neglected. The generalized aerodynamic forces account for tower shadow, wind shear, and gusts, and are calculated from strip theory based on a quasi-steady approximation of two-dimensional, incompressible, unsteady, airfoil theory.

Several orthogonal coordinate systems will be employed in the derivation of the equations of motion; those which are common to both the dynamic and aerodynamic aspects of the derivations are shown in figures 1 to 3. The axis system $X_I Y_I Z_I$ shown in figure 1(a), is fixed in an inertial frame with the origin at the center of the hub. The axis system XYZ, shown in figure 1(b), is obtained by rotating $X_I Y_I Z_I$ system about Z_I-axis by an angle $\psi = \Omega t$ and then $X_\Omega Y_\Omega Z_\Omega$ system about the negative Y_Ω-axis by an angle β_{pc}, the angle of built-in coning angle. The point on the cross section through which elastic axis passes is given by the intersection of the Y and Z axes. Let $x_o y_o z_o$ be axes fixed to the blade at an arbitrary point on the elastic axis of the blade so that before deformation $x_o y_o z_o$ are parallel to XYZ respectively. The η and ζ axes (fig. 2) with the origin at the elastic axis of the cross section are principal axes and are inclined to the Y and Z axes by an amount θ. The geometric pitch angle is given by $\theta = \theta_{pt} + \theta_c$ where θ_{pt} is the built-in twist angle (pretwist) and θ_c is the collective pitch angle.

The variables defining the configuration of the deformed blade are shown in figure 3. When the blade deforms, the elastic center of an arbitrary section deforms an amount u in the X direction, v in the Y direction, and w in the Z direction. The section rotates about the principal axes due to bending in addition to twisting an amount ϕ about the elastic axis. The final position of the triad $x\eta\zeta$ after deformation is denoted by $x_3y_3z_3$.

HAMILTON'S PRINCIPLE

The equations of motion will be derived using the extended Hamilton's principle as in reference 3 in the form

$$\int_{t_0}^{t_1} (\delta T - \delta V + \delta W)\, dt = 0 \tag{1}$$

where

$$\delta W = \delta W_D + \delta W_g + \delta W_A \tag{2}$$

In equation (1), T is the kinetic energy, V is the strain energy, and δW is the virtual work done by the damping, gravitational, and aerodynamic forces. In the following sections explicit expressions for δT, δV, and δW in terms of the variables u, v, w, and ϕ and the blade sectional properties will be developed.

STRAIN ENERGY

The expression for the strain energy of the blade in terms of stresses and engineering strains is

$$V = \frac{1}{2}\int_0^R \int_A (\sigma_{xx}\gamma_{xx} + \sigma_{x\eta}\gamma_{x\eta} + \sigma_{x\zeta}\gamma_{x\zeta})\, d\eta\, d\zeta\, dx \qquad (3)^*$$

where, using Hooke's law,

---

*The coordinates s and x are used interchangeably.

$$\sigma_{xx} = E\gamma_{xx}$$
$$\sigma_{x\eta} = G\gamma_{x\eta} \qquad (4)$$
$$\sigma_{x\zeta} = G\gamma_{x\zeta}$$

Assuming small strains, the engineering strains are related to the components of the strain tensor according to

$$\gamma_{xx} = \varepsilon_{xx}$$
$$\gamma_{x\eta} = 2\varepsilon_{x\eta} \qquad (5)$$
$$\gamma_{x\zeta} = 2\varepsilon_{x\zeta}$$

To develop the explicit expressions for strains, the expressions for the curvatures ω_{x_3}, ω_{y_3}, ω_{z_3} and the transformation matrix [T] between the coordinate systems $x\eta\zeta$ and $x_3y_3z_3$ are required in terms of the variables u, v, w, ϕ and the blade sectional properties. These expressions are developed in Appendix A following the procedure described in reference 2 and are given by

$$\omega_{x_3} = \phi' + \theta'_{pt}\left(1 - \frac{v'^2}{2} - \frac{w'^2}{2}\right) - (v' \cos\theta + w' \sin\theta) \cdot (-v' \sin\theta + w' \cos\theta)'$$

$$\omega_{y_3} = -w''(\cos\theta - \phi \sin\theta) + v''(\sin\theta + \phi \cos\theta) \qquad (6)$$

$$\omega_{z_3} = v''(\cos\theta - \phi \sin\theta) + w''(\sin\theta + \phi \cos\theta)$$

$$
[T] = \begin{bmatrix}
1 - \frac{v'^2}{2} - \frac{w'^2}{2} & v'\cos\theta + w'\sin\theta & -v'\sin\theta + w'\cos\theta \\
-v'(\cos\theta - \phi\sin\theta) - w'(\sin\theta + \phi\cos\theta) & 1 - \frac{(v'\cos\theta + w'\sin\theta)^2}{2} - \frac{\phi^2}{2} & \phi - (v'\cos\theta + w'\sin\theta)\cdot(-v'\sin\theta + w'\cos\theta) \\
v'(\sin\theta + \phi\cos\theta) - w'(\cos\theta - \phi\sin\theta) & -\phi & 1 - \frac{(-v'\sin\theta + w'\cos\theta)^2}{2} - \frac{\phi^2}{2}
\end{bmatrix} \quad (7)
$$

Using equations (6) and (7), the second-degree nonlinear expressions for the strains are developed in Appendix B, and are

$$
\gamma_{xx} = \varepsilon_{xx} = u' + (\eta^2 + \zeta^2)\left(\frac{\phi^2}{2} + \theta'_{pt}\,\phi'\right) - (v'' + w''\phi)(\eta\cos\theta - \zeta\sin\theta)
$$

$$
- (w'' - v''\phi)(\eta\sin\theta + \zeta\cos\theta) \quad (8a)
$$

$$
\gamma_{x\eta} = 2\varepsilon_{x\eta} = -\zeta\ [\phi' - (v'\cos\theta + w'\sin\theta)\ (-v'\sin\theta + w'\cos\theta)'
$$

$$
-\theta'_{pt}\ \left(\frac{v'^2}{2} + \frac{w'^2}{2}\right)] \quad (8b)
$$

$$
\gamma_{x\zeta} = 2\varepsilon_{x\zeta} = \eta\ [\phi' - (v'\cos\theta + w'\sin\theta)\ (-v'\sin\theta + w'\cos\theta)'
$$

$$
-\theta'_{pt}\ \left(\frac{v'^2}{2} + \frac{w'^2}{2}\right)] \quad (8c)
$$

In the above expressions, several higher-order terms have been discarded based either on considerations related to the small deformations Level I approximation, as discussed in reference 2, or on considerations related to the approximations which can be made because of the assumed slenderness of the blade, as discussed in Appendix C. Formal retention of higher-order terms in the expression for strain components is not a problem. However, these higher-order terms will lead to higher-order elastic terms in the final equations of motion. Thus, discarding these higher-order terms in the expressions using the considerations of Appendix C simplifies the subsequent algebraic manipulations.

Taking the first variation of V as given in equation (3) and using equation (4), yields

$$\delta V = \int_0^R E \iint_A \gamma_{xx}\delta\gamma_{xx} d\eta\, d\zeta\, dx + \int_0^R G \iint_A (\gamma_{x\eta}\delta\gamma_{x\eta} + \gamma_{x\zeta}\delta\gamma_{x\zeta})\, d\eta\, d\zeta\, dx \tag{9}$$

Substituting equation (8) into equation (9), taking the variations, and integrating over the cross section leads to

$$\delta V = \int_0^R (s_1\delta u' + s_2\delta\phi + s_3\delta\phi' + s_4\delta v'' + s_5\delta w'' + s_6\delta\phi' + s_7\delta v' + s_8\delta v'' + s_9\delta w' + s_{10}\delta w'')\, dx \tag{10}$$

where

$$s_1 = EA\,[u' + k_A^2\,(\phi'\theta'_{pt} + \frac{1}{2}\phi'^2) - e_A\,(v'' + \phi w'')\cos\theta + e_A\,(\phi v'' - w'')\sin\theta]$$

$$s_2 = EAe_A u'(v''\sin\theta - w''\cos\theta) + EB_2(v''\theta'_{pt}\phi'\sin\theta - w''\theta'_{pt}\phi'\cos\theta) + v''w''(EI_{\zeta\zeta} - EI_{\eta\eta})\cos 2\theta + w''^2(EI_{\zeta\zeta} - EI_{\eta\eta})\sin\theta\cos\theta + v''^2(EI_{\eta\eta} - EI_{\zeta\zeta})\sin\theta\cos\theta$$

$$s_3 = EAk_A^2 u'(\theta'_{pt} + \phi') + EB_1 [\theta'_{pt} \phi'^2 + \theta'_{pt} (\phi'\theta'_{pt} + \frac{1}{2} \phi'^2)]$$

$$+ EB_2[\theta'_{pt} (\phi v'' - w'')\sin\theta - \phi' v'' \cos\theta - \phi' w'' \sin\theta$$

$$-\theta'_{pt} (v'' + \phi w'') \cos\theta]$$

$$s_4 = EAe_A u'(\phi \sin\theta - \cos\theta) - EB_2 \phi'\theta'_{pt} \cos\theta$$

$$+ v''[EI_{\eta\eta}(\sin^2\theta + \phi \sin 2\theta) + EI_{\zeta\zeta} (\cos^2\theta - \phi \sin 2\theta)]$$

$$+ w''[(EI_{\zeta\zeta} - EI_{\eta\eta})(\sin\theta \cos\theta + \phi \cos 2\theta)]$$

$$s_5 = -EAe_A u'(\phi \cos\theta + \sin\theta) - EB_2 \phi'\theta'_{pt} \sin\theta$$

$$+ v''[(EI_{\zeta\zeta} - EI_{\eta\eta})\sin\theta \cos\theta + \phi(EI_{\zeta\zeta} - EI_{\eta\eta})\cos 2\theta]$$

$$+ w''[EI_{\eta\eta} \cos^2\theta + EI_{\zeta\zeta} \sin^2\theta + \phi(EI_{\zeta\zeta} - EI_{\eta\eta})\sin 2\theta]$$

$$s_6 = GJ[\phi' + (v'v'' - w'w'') \cos\theta \sin\theta - v'w'' \cos^2\theta$$

$$+ w'v'' \sin^2\theta + \theta'_{pt} (\frac{v'^2}{2} \cos 2\theta - \frac{w'^2}{2} \cos 2\theta + v'w' \sin 2\theta)]$$

$$s_7 = GJ\phi' (v'' \cos\theta \sin\theta - w'' \cos^2\theta + v'\theta'_{pt} \cos 2\theta + w'\theta'_{pt} \sin 2\theta)$$

$$s_8 = GJ\phi'(v' \cos\theta \sin\theta + w' \sin^2\theta)$$

$$s_9 = GJ\phi'(-w'' \sin\theta \cos\theta + v'' \sin^2\theta + v'\theta'_{pt} \sin 2\theta - \theta'_{pt} w' \cos 2\theta)$$

$$s_{10} = -GJ\phi'(w' \cos\theta \sin\theta + v' \cos^2\theta)$$

The sectional properties appearing in equation (11) are defined as follows:

$$A = \iint d\eta\, d\zeta \qquad Ae_A = \iint \eta\, d\eta\, d\zeta$$

$$I_{\eta\eta} = \iint \zeta^2\, d\eta\, d\zeta \qquad I_{\zeta\zeta} = \iint \eta^2\, d\eta\, d\zeta$$

$$Ak_A^2 = \iint (\eta^2 + \zeta^2)\, d\eta\, d\zeta \qquad J = \iint (\eta^2 + \zeta^2)\, d\eta\, d\zeta \qquad (12)$$

$$B_1 = \iint (\eta^2 + \zeta^2)^2\, d\eta\, d\zeta \qquad B_2 = \iint \eta(\eta^2 + \zeta^2)\, d\eta\, d\zeta$$

Since the cross section is assumed symmetrical about the η axis, the following integrals are zero:

$$\iint \zeta \, d\eta \, d\zeta = 0 \qquad \iint \eta\zeta \, d\eta \, d\zeta = 0$$

$$\iint \zeta \, (\eta^2 + \zeta^2) \, d\eta \, d\zeta = 0 \tag{13}$$

Integrating equation (10) by parts, the resulting expressions can be put in the form

$$\delta V = \int_0^R (S_u \, \delta u + S_v \, \delta v + S_w \, \delta w + S_\phi \, \delta\phi) \, dx + B_V \tag{14}$$

where the generalized elastic forces S_u, S_v, S_w, and S_ϕ, to second-degree, are given by

$$\begin{aligned} S_u &= -s_1' \\ S_v &= s_4'' - s_7' + s_8'' \\ S_w &= s_5'' - s_9' + s_{10}'' \\ S_\phi &= s_2 - s_3' - s_6' \end{aligned} \tag{15}$$

and the boundary term B_V is given by

$$\begin{aligned} B_V = [&s_1 \, \delta u + (s_4 + s_8)\delta v' + (-s_4' + s_7 - s_8')\delta v \\ &+ (s_5 + s_{10})\delta w' + (-s_5' + s_9 - s_{10}')\delta w + (s_3 + s_6)\delta\phi]\Big|_0^R \end{aligned} \tag{16}$$

KINETIC ENERGY

The expression for kinetic energy T is given by

$$T = \frac{1}{2} \int_0^R \iint_A \rho \, \frac{d\bar{r}_1}{dt} \cdot \frac{d\bar{r}_1}{dt} \, d\eta \, d\zeta \, dx \tag{17}$$

and its variation, integrated between t_0 and t_1, is given by

$$\int_{t_0}^{t_1} \delta T = \int_{t_0}^{t_1} \int_0^R \iint_A \rho \frac{d\bar{r}_1}{dt} \cdot \delta \frac{d\bar{r}_1}{dt} \, d\eta \, d\zeta \, dx \, dt \tag{18}$$

In the above equation, the absolute velocity of the mass point is $d\bar{r}_1/dt$ and is defined by

$$\frac{d\bar{r}_1}{dt} = \dot{\bar{r}}_1 + \bar{\omega} \times \bar{r}_1 \tag{19}$$

where $\bar{\omega}$ is the angular velocity of the $x\eta\zeta$ coordinate system and $\bar{r}_1$ is the position vector of the mass point expressed in terms of the unit vectors of the $x\eta\zeta$ system. The angular velocity $\bar{\omega}$ is obtained by projecting Ω along the x, η, and ζ directions and is given by

$$\bar{\omega} = \Omega\beta_{pc}\bar{e}_x + \Omega \sin\theta \, \bar{e}_\eta + \Omega \cos\theta \, \bar{e}_\zeta \tag{20}$$

In the above equation, the precone angle β_{pc} is assumed to be small. The position vector $\bar{r}_1$ is given by

$$\bar{r}_1 = x\bar{e}_x + (u - U_F)\bar{e}_x + (v \cos\theta + w \sin\theta)\bar{e}_\eta + (-v \sin\theta + w \cos\theta)\bar{e}_\zeta + [T]^T \begin{Bmatrix} 0 \\ \eta \\ \zeta \end{Bmatrix} \tag{21}$$

Substituting equation (A38) into equation (21), the components of the position vector to second-degree are given by

$$x_1 = x + u - U_F - (v' + w'\phi)(\eta \cos\theta - \zeta \sin\theta) - (w' - v'\phi)(\eta \sin\theta + \zeta \cos\theta)$$

$$y_1 = v \cos\theta + w \sin\theta + \left[1 - \frac{(v' \cos\theta + w' \sin\theta)^2}{2} - \frac{\phi^2}{2}\right] \eta - \zeta\phi$$

$$z_1 = -v \sin\theta + w \cos\theta \tag{22}$$

$$+ [\phi - (v' \cos\theta + w' \sin\theta)(-v' \sin\theta + w' \cos\theta)]\,\eta$$

$$+ [1 - \frac{(-v' \sin\theta + w' \cos\theta)^2}{2} - \frac{\phi^2}{2}]\,\zeta$$

Differentiating $\bar{r}_1$ with respect to time according to equation (19), the absolute velocity of the mass point is

$$\frac{d\bar{r}_1}{dt} = (\dot{x}_1 + z_1\Omega \sin\theta - y_1 \cos\theta\ \Omega)\bar{e}_x$$

$$+ (\dot{y}_1 - z_1\Omega\beta_{pc} + x_1 \cos\theta\ \Omega)\bar{e}_\eta + (\dot{z}_1 + y_1\Omega\beta_{pc} - x_1\Omega \sin\theta)\bar{e}_\zeta \tag{23}$$

Substituting equations (22) into (23) and the result into equation (18), integrating by parts over time where necessary, and then integrating over the cross-section, the variation of T can be put in the form

$$\delta T = \int_0^R (k_1\delta u - k_1\delta U_F$$

$$+ k_2\delta v' + k_3\delta v - k_4\delta w' - k_5\delta w + k_6\delta\phi)\,dx \tag{24}$$

where, consistent with the ordering scheme given in Appendix C, are

$$k_1 = - m\ddot{u} + 2m\Omega(\dot{v} - e\dot{\Phi} \sin\theta)$$

$$+ m\Omega^2 (x + u - U_F - ev' \cos\theta - ew' \sin\theta)$$

$$- m\Omega^2\beta_{pc} (w + e \sin\theta + e\phi \cos\theta) + me (\ddot{v}' \cos\theta + \ddot{w}' \sin\theta)$$

$$k_2 = m\Omega^2 e\phi x \sin\theta - 2me\Omega\dot{v} \cos\theta - me\Omega^2 x \cos\theta$$

$$k_3 = m\Omega^2(v + e \cos\theta - e\phi \sin\theta) - m\ddot{v} + me\ \ddot{\phi} \sin\theta + 2m\Omega\beta_{pc}\dot{w}$$

$$- 2m\Omega(\dot{u} - \dot{U}_F - e\dot{v}' \cos\theta - e\dot{w}' \sin\theta)$$

$$k_4 = m\Omega^2 e\phi x \cos\theta + 2m\Omega e\dot{v} \sin\theta + m\Omega^2 ex \sin\theta$$

$$k_5 = m\ddot{w} + me\,\ddot{\phi}\cos\theta + 2m\Omega\beta_{pc}\,\dot{v} + m\Omega^2\beta_{pc}x$$

$$\begin{aligned}
k_6 = & -w'\,(2m\Omega e\dot{v}\cos\theta + m\Omega^2\, ex\cos\theta) - m\Omega^2 e\phi v\cos\theta \\
& - m\Omega^2\phi(k_{m_2}^2 - k_{m_1}^2)\cos 2\theta - m\Omega^2(k_{m_2}^2 - k_{m_1}^2)\sin\theta\cos\theta \\
& + me\ddot{\phi}v\cos\theta + 2m\Omega ev'\dot{v}\sin\theta + m\Omega^2 exv'\sin\theta \\
& - m\Omega^2 ev\sin\theta + me\phi\ddot{w}\sin\theta + m\Omega^2\beta_{pc}e\phi x\sin\theta \\
& + 2m\Omega\,[e\sin\theta\,\dot{u} - \dot{U}_F) - (k_{m_2}^2 - k_{m_1}^2)\dot{v}'\sin\theta\cos\theta \\
& - \dot{w}'(k_{m_2}^2\sin^2\theta + k_{m_1}^2\cos^2\theta)] + me\ddot{v}\sin\theta \\
& - 2m\Omega\beta_{pc}e\dot{w}\sin\theta - mk_m^2\,\ddot{\phi} - m\Omega^2\beta_{pc}ex\cos\theta \\
& - me\ddot{w}\cos\theta - 2m\Omega\beta_{pc}e\dot{v}\cos\theta
\end{aligned}$$

The sectional properties appearing in equation (25) are defined as follows:

$$m = \iint_A \rho\, d\eta\, d\zeta \qquad me = \iint_A \rho\eta\, d\eta\, d\zeta$$

$$mk_{m_1}^2 = \iint_A \rho\zeta^2\, d\eta\, d\zeta \qquad mk_{m_2}^2 = \iint_A \rho\eta^2\, d\eta\, d\zeta \tag{26}$$

$$k_m^2 = k_{m_1}^2 + k_{m_2}^2$$

From symmetry of the cross-section about the η axis, the following integrals have been set to zero:

$$\iint \rho\zeta\, d\eta\, d\zeta = 0 \qquad \iint \rho\eta\zeta\, d\eta\, d\zeta = 0 \tag{27}$$

Since U_F is a function of v' and w', the term involving δU_F in equation (24) requires separate treatment. Using equation (A16), the second term in equation (24) can be written in the expanded form

$$\int_0^R k_1 \delta U_F \, dx = \int_0^R k_1 \left[\int_0^x (w'\delta w' + v'\delta v') \, dx\right] dx \tag{28}$$

which can be further rewritten as

$$\int_0^R k_1 \delta U_F \, dx = \int_0^R \left(\int_x^R k_1 \, dx\right) (w'\delta w' + v'\delta v') \, dx \tag{29}$$

Defining the tension T_C as

$$T_C = \int_x^R k_1 \, dx \tag{30}$$

equation (29) can be written as

$$\int_0^x k_1 \delta U_F \, dx = \int_0^R T_C (w'\delta w' + v'\delta v') \, dx \tag{31}$$

Integrating equation (24) by parts, the resulting expression can be put in the form

$$\delta T = \int_0^R (I_u \delta u + I_v \delta v + I_w \delta w + I_\phi \delta\phi) \, dx + B_T \tag{32}$$

where the generalized inertia forces I_u, I_v, I_w, and I_ϕ are given by

$$
\begin{aligned}
I_u &= k_1 = -T_C' \\
I_v &= -k_2' + k_3 + (T_C v')' \\
I_w &= k_4' - k_5 + (T_C w')' \\
I\phi &= k_6
\end{aligned}
\tag{33}
$$

and the boundary term B_T by

$$B_T = (k_2 - T_C v')\delta v \Big|_0^R - (k_4 + T_C w')\delta w \Big|_0^R \tag{34}$$

VIRTUAL WORK DUE TO MATERIAL DAMPING

The virtual work due to the dissipative forces associated with structural (material) damping can be expressed in the form

$$\delta W_D = \sum_{k=1}^{4} Q_{D_k} \delta q_k \tag{35}$$

where Q_{D_k} is the generalized damping force associated with the k^{th} dependent variable and δq_k is the variation of the k^{th} dependent variable. In the present development the generalized damping forces accounting for the dissipation of energy due to material damping will be taken to be those consistent with the assumption of a material which exhibits a linear viscoelastic behavior. This theory assumes that the stresses are linear functions of the strains and strain rates. Such a behavior is analogous to a spring and a dashpot in parallel, and a model which exhibits such a behavior is often termed a Kelvin-Voigt solid in the literature. This model was used in reference 3. For the stresses and strains of interest herein, these constitutive relations have the form

$$\begin{aligned} \tau_{xx} &= E\gamma_{xx} + E^*\dot{\gamma}_{xx} \\ \tau_{x\eta} &= G\gamma_{x\eta} + G^*\dot{\gamma}_{x\eta} \\ \tau_{x\zeta} &= G\gamma_{x\zeta} + G^*\dot{\gamma}_{x\zeta} \end{aligned} \tag{36}$$

where E and G are Young's modulus and the shear modulus, respectively, and E^* and G^* are coefficients which take into account internal damping of the material in tension and shear, respectively. The first term on the right hand side of each of equations (36) contributes to the usual elastic strain energy and have already been treated in an earlier section. Considering only the dissipative terms in equations (36), the virtual work of the structural dissipative forces can be written as

$$\delta W_D = -\int_0^R E^* \iint_A \dot{\gamma}_{xx}\delta\gamma_{xx}\, d\eta\, d\zeta\, dx$$
$$- \int_0^R G^* \iint_A (\dot{\gamma}_{x\eta}\delta\gamma_{x\eta} + \dot{\gamma}_{x\zeta}\delta\gamma_{x\zeta})\, d\eta\, d\zeta\, dx \qquad (37)$$

The result given in equation (37) is general. However, because of the lack of knowledge as to the distribution of damping, only the direct damping terms are generally retained in practice. Thus, off-diagonal terms accounting for damping coupling between the dependent variables which arise from equation (37) are taken to be zero and only the direct damping terms associated with the dependent variables are retained. In addition to adopting this expedient in the present development, it will also be assumed that a first approximation to the direct damping terms can be obtained by retaining only the linear damping terms in the final equations of motion. Thus, it is sufficient to retain terms up to only the first-degree in the expressions for the strains. To first degree the resulting strain expressions are

$$\gamma_{xx} = u' + (\eta^2 + \zeta^2)\phi'\theta'_{pt} - v''(\eta \cos\theta - \zeta \sin\theta) - w''(\eta \sin\theta + \zeta \cos\theta)$$

$$\gamma_{x\eta} = \zeta\phi' \qquad (38)$$

$$\gamma_{x\zeta} = \eta\phi'$$

Substituting equations (38) into equation (37), integrating over the cross section, and retaining only the linear direct damping terms leads to

$$\delta W_D = -\int_0^R (d_1\delta u' + d_2\delta\phi' + d_3\delta w'' + d_5\delta\phi')dx \qquad (39)$$

where

$$d_1 = E^*A\dot{u}'$$
$$d_2 = E^*B_1\theta'^2_{pt}\,\dot{\phi}'$$
$$d_3 = E^*(I_{\zeta\zeta}\cos^2\theta + I_{\eta\eta}\sin^2\theta)\dot{v}'' \qquad (40)$$
$$d_4 = E^*(I_{\zeta\zeta}\sin^2\theta + I_{\eta\eta}\cos^2\theta)\dot{w}''$$
$$d_5 = G^*J\dot{\phi}'$$

Integrating equation (39) by parts, the generalized damping forces Q_{D_u}, Q_{D_v}, Q_{D_w}, and Q_{D_ϕ} become

$$Q_{D_u} = d_1' = (E^*A\dot{u}')'$$
$$Q_{D_v} = -d_3'' = -[E^*(I_{\zeta\zeta}\cos^2\theta + I_{\eta\eta}\sin^2\theta)\dot{v}'']''$$
$$Q_{D_w} = -d_4'' = -[E^*(I_{\zeta\zeta}\sin^2\theta + I_{\eta\eta}\cos^2\theta)\dot{w}'']'' \qquad (41)$$
$$Q_{D_\phi} = d_2' + d_5' = (E^*B_1\theta'^2_{pt}\dot{\phi}')' + (G^*J\dot{\phi}')'$$

and the boundary term $B_{\delta W_D}$ becomes

$$B_{\delta W_D} = -d_1\delta u\Big|_0^R + d_3'\delta v\Big|_0^R - d_3\delta v'\Big|_0^R + d_4'\delta w\Big|_0^R$$
$$- \; d_4\delta w'\Big|_0^R - (d_2 + d_5)\delta\phi\Big|_0^R \qquad (42)$$

VIRTUAL WORK DUE TO GRAVITY

The virtual work due to the gravitational forces of the blade can be expressed in the form

$$\delta W_g = \int^R \iint_A \rho\bar{g} \cdot \delta\bar{r}_1 \, d\eta \, d\zeta \, dx \qquad (43)$$

where $\bar{g}$ is the gravitational acceleration vector. It is obtained by projecting the gravity, $\bar{g}$, which acts along the X_I axis as shown in figure 1(b), along the x, η, and ζ, and is given by

$$\bar{g} = g\ [\cos\psi \bar{e}_x - (\cos\theta \sin\psi + \beta_{pc} \sin\theta \cos\psi)\bar{e}_\eta$$

$$+ (\sin\theta \sin\psi - \beta_{pc} \cos\theta \cos\psi)\bar{e}_\zeta] \qquad (44)$$

In the above equation the precone angle β_{pc} is assumed small and hence $\cos\beta_{pc}$ is replaced by one and $\sin\beta_{pc}$ by β_{pc}.

The position vector, $\bar{r}_1$ of a point on the blade after deformation is given by equation (21). Taking the virtual variation of equation (21), substituting the result into equation (43), and integrating the result by parts, one obtains

$$\delta W_g = \int_0^R (F_{gu}\delta u + F_{gv}\delta v + F_{gw}\delta w + F_{g\phi}\delta\phi)\ dx + B_g \qquad (45)$$

where

$$F_{gu} = g_1$$

$$F_{gv} = g_2 + (T_g v') - g_5'$$

$$F_{gw} = g_3 + (T_g w') - g_6'$$

$$F_{g\phi} = g_4$$

$$T_g = \int_x^R g_1\ dx$$

$$g_1 = mg \cos\psi$$

$$g_2 = -mg \sin\psi$$

$$g_3 = -mg\beta_{pc} \cos\psi$$

$$g_4 = mge\cos\psi(v'\sin\theta - w'\cos\theta) + mge\phi(\cos\theta\sin\psi + \beta_{pc}\sin\theta\cos\psi) + mge(\sin\theta\sin\psi - \beta_{pc}\cos\theta\cos\psi)$$

$$g_5 = mge\cos\psi(-\cos\theta + \phi\sin\theta) + mge(\cos\theta\sin\psi + \beta_{pc}\sin\theta\sin\psi)\cdot(v'\cos^2\theta + w'\sin\theta\cos\theta)$$

$$g_6 = mge\cos\psi(-\phi\cos\theta + \sin\theta) + mge(\cos\theta\sin\psi + \beta_{pc}\sin\theta\cos\psi)\cdot(w'\sin^2\theta + v'\sin\theta\cos\theta) \quad (46)$$

The boundary terms B_g are given by

$$B_g = (g_5 - T_g v')\delta v\Big|_0^R + (g_6 - T_g w')\delta w\Big|_0^R \quad (47)$$

It should be pointed out that all the terms of order $0(\varepsilon^3)$ and higher are neglected in the expressions for gravitational forces.

GENERALIZED AERODYNAMIC FORCES

The aerodynamic forces will be generated from two-dimensional, incompressible, quasi-steady, strip theory in which only the velocity components perpendicular to the spanwise axis of the deformed blade (the x_3-axis) are assumed to influence the aerodynamic loading. In calculating the velocity components, the effects of wind shear and tower shadow as well as gusts are included. Account is taken of the pulsating free-stream velocity V(t) associated with a rotating blade employing Greenberg's extension of Theodorsen's unsteady theory (ref. 10) for determining the aerodynamic lift and pitching moment acting on the blade. The resulting expressions are specialized to the case of quasi-steady flow by setting Theodorsen's circulation function to unity. The classical blade element momentum theory can be used to calculate the steady flow induced by the rotor.

In the present application of Greenberg's theory, the airfoil is taken to be pivoted in pitch about the aerodynamic center at the quarter chord and to be executing harmonic motions in pitch ($\varepsilon(t)$) and plunge ($\dot{h}(t)$) while immersed in a pulsating airstream $V(t)$, as shown in figure 4. The lift and moment acting on an elemental section of the blade may be expressed in terms of the circulatory and noncirculatory components as

$$L = L_C + L_{NC} \qquad (48)$$

$$M = M_C + M_{NC}$$

Assuming that the blade elastic axis is coincident with the aerodynamic center at the quarter chord, the individual components of equation (48) follow from reference 10 and can be written as

$$L_{NC} = \frac{1}{2} \rho a \frac{c^2}{4} (\ddot{h} + \dot{V}\varepsilon + V\varepsilon + \frac{c}{4} \ddot{\varepsilon}) \qquad (49a)$$

$$L_C = \frac{1}{2} \rho a c V(\dot{h} + V\varepsilon + \frac{c}{2} \dot{\varepsilon}) \qquad (49b)$$

$$M_{NC} = - \frac{1}{2} \rho a c \left(\frac{c}{4}\right)^2 (\dot{V}\varepsilon + \ddot{h} + \frac{3c}{8} \ddot{\varepsilon}) \qquad (49c)$$

$$M_C = - \frac{1}{2} \rho a c \left(\frac{c}{4}\right)^2 2V\dot{\varepsilon} \qquad (49d)$$

In the course of arriving at the circulatory terms in equation (49), the quasi-steady approximation has been introduced by setting the reduced frequency k to zero, in consequence of which Theodorsen's circulation function $C(k)$ assumes the value of unity. The noncirculatory lift and moment are associated with apparent mass forces and are oftentimes discarded in rotor blade applications. Note that Greenberg's modification (i.e., a pulsating stream in which $\dot{V} \neq 0$) appears only in the noncirculatory expressions for the lift moment. Hence, if one assumes, a priori, that apparent mass forces will be neglected, there is no Greenberg's modification.

The lifts and moments given in equation (49) must now be expressed in terms of U_R, U_T, and U_P, the radial, tangential, and perpendicular velocity components relative to a point on the elastic axis of the airfoil (fig. 5). Now the expression in parentheses of equation (49a) for L_{NC} is the downward acceleration of the mid-chord point of the airfoil, and the expression in the parentheses of equation (49b) for L_C is the downward velocity of the three-quarter-chord point of the airfoil. Since U_P is the relative velocity component perpendicular to the quarter-chord, the sectional lift can also be written as

$$L_{NC} = \frac{1}{2} \rho a \frac{c^2}{4} \left(-\dot{U}_P + \frac{c}{4} \ddot{\varepsilon}\right) \qquad (50a)$$

$$L_C = \frac{1}{2} \rho a c U \left(-U_P + \frac{c}{2} \dot{\varepsilon}\right) \qquad (50b)$$

where V(t), appearing outside the parentheses of equation (49b), has been approximated by the resultant of only the tangential and perpendicular velocity components and is given by

$$V \simeq U \simeq U_T^2 + U_P^2 \qquad (51)$$

As indicated in figure 5, the noncirculatory lift acts normal to the section chordline* and the circulatory lift acts normal to the resultant velocity U. The profile drag force acts parallel to U and is given by

---

*A portion of L_{NC} acts at the 3/4-chord point and another at the 1/2-chord point. However, the resultant of these two components is shown along the z_3 axis in figure 5 only for pictorial convenience.

$$D = \frac{1}{2} \rho a c \frac{C_{d_o}}{a} U^2 \tag{52}$$

where C_{d_o} is the (constant) profile drag coefficient.

The components of the aerodynamic forces in the direction of the y_3 and z_3 axes are given by

$$F_{y_3} = -L_C \sin \alpha - D \cos \alpha \tag{53a}$$

$$F_{z_3} = L_C \cos \alpha + L_{NC} - D \sin \alpha \tag{53b}$$

where, from figure 5,

$$\sin \alpha = U_P/U \tag{54}$$

$$\cos \alpha = U_T/U$$

and U is given by equation (51). The aerodynamic force in the x_3 direction is given by F_{x_3} and is a profile drag force which is a function of the radial velocity component U_R. Following usual practice, this force component is taken to be zero.

Substituting equations (50), (52), and (54) into equation (53) leads to

$$F_{y_3} = \frac{1}{2} \rho a c \left(U_P^2 - \frac{c}{2} U_P \dot{\varepsilon} - \frac{C_{d_o}}{a} U_T U\right) \tag{55a}$$

$$F_{z_3} = \frac{1}{2} \rho a c \left[-U_P U_T + \frac{c}{2} U_T \dot{\varepsilon} - \frac{c}{4} U \dot{P} + \left(\frac{c}{4}\right)^2 \ddot{\varepsilon} - \frac{C_{d_o}}{a} U_P U\right] \tag{55b}$$

The noncirculatory and circulatory moments in equation (49c) and (49d) can be written in terms of U_T, U_P, U, and ε and assume the form

$$M_{NC} = -\frac{1}{2} \rho a c \left(\frac{c}{4}\right)^2 \left(-\dot{U}_P - U\dot{\varepsilon} + \frac{3c}{8} \ddot{\varepsilon}\right) \tag{56a}$$

$$M_C = -\frac{1}{2} \rho a c \left(\frac{c}{4}\right)^2 2U\dot{\varepsilon} \tag{56b}$$

from which the total pitching moment M_{x_3} is given by the sum of equations (56a) and (56b) as

$$M_{x_3} = -\frac{1}{2} \rho a c \left(\frac{c}{4}\right)^2 \left(U\dot{\varepsilon} - \dot{U}_P + \frac{3c}{8} \ddot{\varepsilon}\right) \quad (57)$$

The virtual work of the aerodynamic forces can be written as

$$\delta W_A = \int_0^R [(F_x \bar{e}_x + F_\eta \bar{e}_\eta + F_\zeta \bar{e}_\zeta) \cdot \delta \bar{R}_1$$

$$+ M_{x_3} [\delta\phi - (v' \cos\theta + w' \sin\theta)(-\delta v' \sin\theta + \delta w' \cos\theta)]]dx \quad (58)$$

where $\delta\bar{R}_1$ is the variation of the position vector of a point on the elastic axis and is given by

$$\delta\bar{R}_1 = (\delta u - \delta U_F)\bar{e}_x$$

$$+ (\delta v \cos\theta + \delta w \sin\theta)\bar{e}_\eta + (-\delta v \sin\theta + \delta w \cos\theta)\bar{e}_\zeta \quad (59)$$

The components of the aerodynamic force vector F_x, F_η, and F_ζ are given by

$$\begin{Bmatrix} F_x \\ F_\eta \\ F_\zeta \end{Bmatrix} = [T]^T \begin{Bmatrix} 0 \\ F_{y_3} \\ F_{z_3} \end{Bmatrix} \quad (60)$$

where [T] is the rotational transformation matrix which relates the coordinate axes of the deformed and undeformed blade and F_{x_3} has been set to zero. Substituting equation (59) into equation (58), and integrating by parts, one obtains

$$\delta W_A = \int_0^R (A_u \delta u + A_v \delta v + A_w \delta w + A_\phi \delta\phi)dx + B_A \quad (61)$$

where

$$A_u = F_x$$

$$A_v = F_\eta \cos\theta - F_\zeta \sin\theta + (T_A v')'$$

$$+ [-M_{x_3}(v' \cos\theta \sin\theta + w' \sin^2\theta)]'$$

$$A_w = F_\eta \sin\theta + F_\zeta \cos\theta + (T_A w')' \tag{62}$$

$$+ [M_{x3}(v' \cos^2\theta + w' \sin\theta \cos\theta)]'$$

$$A_\phi = M_{x3}$$

$$T_A = \int_x^R A_u \, dx$$

$$B_A = -T_A v' \delta v \Big|_0^R - T_A W' \delta w \Big|_0^R + M_{x3}(v' \cos\theta + w' \sin\theta)\sin\theta \delta u \Big|_0^R$$

$$-M_{x3}(v' \cos\theta + w' \sin\theta)\cos\theta \delta w \Big|_0^R$$

In order to obtain the explicit expressions for the generalized aerodynamic forces, the quantities F_{y3}, F_{z3}, and M_{x3} must be known in terms of the dependent variables u, v, w, and ϕ, and the geometric pitch angle θ. This requires that U_T, U_P, and $\dot{\varepsilon}$ first be obtained in terms of these variables. These expressions are developed in Appendix D, and are given by the equations (D16) and (D20). The expressions for U_T and U_P in equations (D16) include the induced velocity v_i which can be calculated using blade element and momentum theories. Equations (D16) and (D20) in combination with equations (51), (55), and (57) are sufficient to obtain the generalized aerodynamic forces from equations (60), (61), and (62).

SUMMARY OF EQUATIONS

In the previous sections, expressions for δV, δT, and δW have been obtained. Substituting these expressions and their associated boundary terms into equation (1), there results the expression of the form

$$\int_{t_0}^{t_1} \{\int_0^R [(\ )\delta u + (\ )\delta v + (\ )\delta w + (\ )\delta\phi]\, dx + B\}\, dt = 0 \tag{63}$$

For arbitrary admissible variations δu, δv, δw, and $\delta\phi$, the four expressions in parentheses must vanish individually as must the assembly boundary terms denoted by B. The first condition will yield the four governing nonlinear partial differential equations for u, v, w, and ϕ and the second condition gives the associated boundary conditions at the ends of the beam. Since the control angle θ_C is assumed to be given, the equation associated with the control will not appear. The governing equations of motion and boundary conditions are summarized below.

Extension:

$$
\begin{aligned}
& m\ddot{u} - me(\ddot{v}' \cos\theta + \ddot{w}' \sin\theta) - 2m\Omega(\dot{v} - e\dot{\phi}\sin\theta) \\
& \quad - m\Omega^2 (x + u - U_F - ev' \cos\theta - ew' \sin\theta) + m\Omega^2\beta_{pc} (w + e\sin\theta \\
& \quad + e\phi\cos\theta) - \{EA[u' + k_A^2 \phi'\theta'_{pt} - e_A(v'' + \phi w'')\cos\theta \\
& \quad + e_A(\phi v'' - w'')\sin\theta] + E^*A\dot{u}'\}' = mg\cos\psi + A_u
\end{aligned} \tag{64a}
$$

Chordwise bending:

$$
\begin{aligned}
& m\ddot{v} - me\ddot{\phi}\sin\theta - 2m\Omega\beta_{pc}\dot{w} - m\Omega^2 (v + e\cos\theta - e\phi\sin\theta) \\
& \quad - \{me[\Omega^2 x(\cos\theta - \phi\sin\theta) + 2\Omega\dot{v}\cos\theta]\}' \\
& \quad + 2m\Omega(\dot{u} - \dot{U}_F - e\dot{v}'\cos\theta - e\dot{w}'\sin\theta) \\
& \quad - (T_C v')' - [GJ\phi(v''\cos\theta\sin\theta - w''\cos^2\theta \\
& \quad + v'\theta'_{pt}\cos 2\theta + w'\theta'_{pt}\sin 2\theta)]' + \{EAe_A u'(\phi\sin\theta - \cos\theta) \\
& \quad - EB_2\phi'\theta'_{pt}\cos\theta + v''[EI_{\eta\eta}(\sin^2\theta + \phi\sin 2\theta) + EI_{\zeta\zeta}(\cos^2\theta \\
& \quad - \phi\sin 2\theta)] + w''[(EI_{\zeta\zeta} - EI_{\eta\eta})(\sin\theta\cos\theta + \phi\cos 2\theta)] \\
& \quad + GJ\phi'(v'\cos\theta\sin\theta + w'\sin^2\theta) + E^*(I_{\zeta\zeta}\cos^2\theta + I_{\eta\eta}\sin^2\theta)\dot{v}''\}'' \\
& \quad - (T_g v')' + [mge\cos\psi(-\cos\theta + \phi\sin\theta) + mge(\cos\theta\sin\psi \\
& \quad + \beta_{pc}\sin\theta\sin\psi)(v'\cos^2\theta + w'\sin\theta\cos\theta)]' = -mg\sin\psi + A_v
\end{aligned} \tag{64b}
$$

Flapwise bending:

$$m\ddot{w} + me\ddot{\phi}\cos\theta + 2m\Omega\beta_{pc}\dot{v} - (T_C w')'$$

$$- \{me\,[\Omega^2 x(\sin\theta + \phi\cos\theta) + 2\Omega\dot{v}\sin\theta]\}'$$

$$- [GJ\phi'(-w''\sin\theta\cos\theta + v''\sin^2\theta + v'\theta'_{pt}\sin 2\theta$$

$$- \theta'_{pt} w'\cos 2\theta)]' + \{-EAe_A u'(\phi\cos\theta + \sin\theta)$$

$$- EB_2\phi'\theta'_{pt}\sin\theta + w''\,[EI_{\eta\eta}\cos^2\theta + EI_{\zeta\zeta}\sin^2\theta$$

$$+ \phi(EI_{\zeta\zeta} - EI_{\eta\eta})\sin 2\theta] + v''\,[(EI_{\zeta\zeta} - EI_{\eta\eta})\sin\theta\cos\theta$$

$$+ \phi(EI_{\zeta\zeta} - EI_{\eta\eta})\cos 2\theta] - GJ\phi'(w'\cos\theta\sin\theta + v'\cos^2\theta)\}''$$

$$+ [E^*(I_{\zeta\zeta}\sin^2\theta + I_{\eta\eta}\cos^2\theta)\dot{w}'']'' - (T_g w')' - (T_C w')'$$

$$+ [mge\cos\psi(-\phi\cos\theta + \sin\theta) + mge(\cos\theta\sin\psi$$

$$+ \beta_{pc}\sin\theta\cos\psi)(w'\sin^2\theta + v'\sin\theta\cos\theta)]' = -mg\beta_{pc}\cos\psi$$

$$- m\Omega^2\beta_{pc}x + A_w \qquad (64c)$$

Torsion:

$$mk_m^2\ddot{\phi} + m\Omega^2\phi(k_{m_2}^2 - k_{m_1}^2)\cos 2\theta + me\,[\Omega^2(w'\cos\theta - v'\sin\theta)$$

$$- (\ddot{v} - \Omega^2 v)\sin\theta + \ddot{w}\cos\theta] + m\Omega^2 e\phi\,(v\cos\theta - x\beta_{pc}\sin\theta)$$

$$- 2m\Omega\,[e\sin\theta(\dot{u} - \dot{U}_F) - (k_{m_2}^2 - k_{m_1}^2)\dot{v}'\sin\theta\cos\theta$$

$$- \dot{w}'(k_{m_2}^2\sin^2\theta + k_{m_1}^2\cos^2\theta) - e\beta_{pc}(\dot{v}\cos\theta + \dot{w}\sin\theta)]$$

$$+ 2m\Omega e\dot{v}(w'\cos\theta - v'\sin\theta) - me\phi(\ddot{v}\cos\theta + \ddot{w}\sin\theta)$$

$$- \{\underline{EAk_A^2\,u'(\theta'_{pt} + \phi')} + EB_1\,\theta'^2_{pt}\,\phi' + EB_2[\theta'_{pt}(\phi v'' - w'')\sin\theta$$

$$- \phi' v''\cos\theta - \phi' w''\sin\theta - \theta'_{pt}(v'' + \phi w'')\cos\theta] + GJ\,[\phi'$$

$$+ (v'v'' - w'w'')\cos\theta\sin\theta - v'w''\cos^2\theta + w'v''\sin^2\theta$$

$$+ \theta'_{pt}\left(\frac{v'^2}{2}\cos 2\theta - \frac{w'^2}{2}\cos 2\theta + v'w'\sin 2\theta)\right]\}'$$

$$+ EAe_A u'(v''\sin\theta - w''\cos\theta) + EB_2(v''\theta'_{pt}\phi'\sin\theta -$$

$$- w''\theta'_{pt}\phi'\cos\theta) + v''w''(EI_{\zeta\zeta} - EI_{\eta\eta})\cos 2\theta + w''^2(EI_{\zeta\zeta}$$

$$- EI_{\eta\eta})\sin\theta\cos\theta + v''^2(EI_{\eta\eta} - EI_{\zeta\zeta})\sin\theta\cos\theta - [G^*J\dot{\phi}'$$

$$+ E^*B_1\theta'^2_{pt}\dot{\phi}']' - mge\cos\psi(v'\sin\theta - w'\cos\theta)$$

$$- mge\phi(\cos\theta\sin\psi + \beta_{pc}\sin\theta\cos\psi) = mge(\sin\theta\sin\psi$$

$$- \beta_{pc}\cos\theta\cos\psi) - m\Omega^2\beta_{pc}xe\cos\theta - m\Omega^2\sin\theta\cos\Theta\,(k^2_{m_2} - k^2_{m_1}) + M_\phi \quad (64d)$$

The assembled collection of boundary terms denoted by B is given by

$$B = B_T - B_V + B_{\delta W_D} + B_g + B_A \quad (65)$$

and the requirement of the vanishing of the individual variational components leads to the relations

$$[s_1 + d_1 - M_{x_3}(v'\cos\theta + w'\sin\theta)\sin\theta]\,\delta u\,\Big|_0^R = 0$$

$$(k_2 - T_C v' + s'_4 - s_7 + s'_8 + g_5 - T_g v' + d'_3 - T_A v')\delta v\,\Big|_0^R = 0$$

$$(s_4 + s_8 + d_3)\delta v'\,\Big|_0^R = 0$$

$$[-k_4 - T_C w' + s'_5 - s_9 + s'_{10} + g_6 - T_g w' + d'_4 - T_A w$$

$$- M_{x_3}(v'\cos\theta + w'\sin\theta)\cos\theta]\,\delta w\,\Big|_0^R = 0 \quad (66)$$

$$(s_5 + s_{10} + d_4)\delta w'\,\Big|_0^R = 0$$

$$(s_3 + s_6 + d_2 + d_5)\delta\phi\Big|_0^R = 0$$

The tension T_C and gravity force T_g appearing in equations (64) and (66) are given by

$$T_C = \int_x^R m[-(\ddot{u} - \ddot{U}_F) + e(\ddot{v}' \cos \theta + \ddot{w}' \sin \theta) + 2\Omega(\dot{v} \cos \theta$$

$$- \dot{\phi} \sin \theta)$$

$$+ \Omega^2 (x + u - U_F - ev' \cos \theta - ew' \sin \theta)$$

$$- \Omega^2 \beta_{pc} (w + e\phi \cos\theta + e \sin\theta)] \, dx \tag{67}$$

$$T_g = \int_x^R mg \cos \psi \, dx \tag{68}$$

The terms U_F and $\ddot{U}_F$ in the expressions for T_C given in equation (67) lead to third-degree nonlinear terms when T_C is substituted into equations (64) and (66) and can be discarded. Also, after substitution for T_C in these equations, only resulting terms which are consistent with the ordering scheme adopted in Appendix C should be retained. Using the results given in equations (67) and (68) in combination with the extensional equation of motion given in equation (64a) (with dampings and A_u set to zero), an alternative definition for the sum of T_C and T_g can be given as

$$T_C + T_g = EA \, [u' + k_A^2 \phi' \theta'_{pt}$$

$$- e_A (v'' + \phi w'') \cos \theta + e_A(\phi v'' - w'')\sin \theta] \tag{69}$$

The underlined terms in equation (64d) are associated with u' and they are called as tension-torsion coupling terms. These terms are known to be important in some cases (refs. 11 to 13). Also, there is a tension-bending term which is doubly underlined in the torsion equation. Some of these terms appear to be nonlinear, but they are not because of the relation given by equation (69). To simplify the solution of the equation (64), it is a customary practice in the rotor blade literature to eliminate the extensional equation. If such practice is followed in the case of wind turbine blade solutions, one should substitute for EAu' in equations (64b), (64c), and (64d) in terms of $T_C + T_g$ from equation (69). Also, after substitution only the resulting terms which are consistent with the ordering scheme adopted in Appendix C should be retained.

As stated earlier, the generalized aerodynamic forces A_u, A_v, A_w, and $A\phi$ are obtained from equations (60), (61), and (62), using equations (D16) and (D20) in combination with equations (51), (55), and (56), and retaining the terms through second degree in the dependent variables. Because of the generality of the present development, these second-degree expressions are extremely lengthy and will not be shown.

CONCLUDING REMARKS

The second-degree nonlinear aeroelastic equations of motion for a flexible, twisted, nonuniform horizontal-axis rotor blade undergoing combined flapwise bending, chordwise bending, torsion, and extension have been derived using Hamilton's principle. The equations have their basis in the geometric nonlinear theory of elasticity and are consistent with the small deformation level of approximation in which elongations and shears (and hence strains) are negligible compared to unity. A mathematical ordering scheme which is consistent with the assumption of a slender beam was adopted for the purpose of systematically discarding elastic and dynamic terms which are higher-order in the resultant equations of motion. The expressions for generalized aerodynamic forces, which account for windshear, tower shadow, and gusts, are left in general second-degree form from which one can obtain the aerodynamic forces loading to the order appropriate to any case of interest. A unique feature of this development is the consideration of the pretwist of the blades before the elastic deformations which is more realistic than the common practice in the most of published work in which the pretwist is combined with the elastic twist.

APPENDIX A

NONLINEAR CURVATURES AND COORDINATE SYSTEM TRANSFORMATION OF TWISTED ELASTIC WIND TURBINE BLADES

The nonlinear curvature expressions for twisted rotor blades were developed in reference 2 in a general manner using the geometric nonlinear theory of elasticity. For convenience, this development followed a practice which is common in helicopter blade literature, namely the combination of the pre-twist with the elastic twist. As a consequence of employing this simplification, if warping is considered, axial deformation exists in the initial configuration before any deformations are imposed. Such a situation would exist if an untwisted blade is twisted and then "frozen" to arrive at the pretwisted configuration. Since this situation does not arise in the fabrication of either wind turbine blades or helicopter blades, there is no axial deformation due to warping in the initial configuration before any deformations are imposed. In view of this situation, this Appendix will include the rotation due to pretwist with the control inputs and impose this rotation first while imposing the rotational transformations and will develop the second-degree nonlinear expressions for the curvature components and for the transformation matrix in terms of the variables u, v, w, and ϕ. The foreshortening of the blades due to bending is explicitly considered. The level of approximation used within the geometric nonlinear theory of elasticity is that designated small deformations I in reference 2 in which the elongations and shear are negligible compared to unity and the square of the first derivative of the extensional deformation on the elastic axis is negligible compared to unity and the squares of the bending slopes.

A schematic representation of the deformed and undeformed blade associated with a flap-lag-torsion transformation sequence is shown in figure

3. The coordinate axis x is tangential to the elastic axis, and the axes η and ζ are the principal axes of the section before deformation. Since the pretwist θ_{pt} is not a constant, the initial curvature of the elastic axis before deformation is

$$\bar{\omega}_{x\eta\zeta} = \bar{e}_x \frac{d\theta_{pt}}{ds} = \bar{e}_x \theta'_{pt} \tag{A1}$$

where s is the distance measured along the undeformed elastic axis.

The elastic deformations rotate the triad $x\eta\zeta$ to $x_3y_3z_3$, and the transformation matrix between the two triads can be written as

$$\begin{Bmatrix} \bar{e}_{x_3} \\ \bar{e}_{y_3} \\ \bar{e}_{z_3} \end{Bmatrix} = [T] \begin{Bmatrix} \bar{e}_{x} \\ \bar{e}_{\eta} \\ \bar{e}_{\zeta} \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 & \bar{e}_x \\ l_2 & m_2 & n_2 & \bar{e}_\eta \\ l_3 & m_3 & n_3 & \bar{e}_\zeta \end{bmatrix} \tag{A2}$$

Let the expression for the curvature vector of the deformed elastic axis be

$$\bar{\omega}_{x_3y_3z_3} = \omega_{x_3} \bar{e}_{x_3} + \omega_{y_3}\bar{e}_{y_3} + \omega_{z_3}\bar{e}_{z_3} \tag{A3}$$

The next step is to find the expressions for the components of the curvature vector in terms of the direction cosines $l_1, m_1, \ldots n_3$, and the blade initial curvature. From equation (A2) one can write

$$\bar{e}_{x_3} = l_1\bar{e}_x + m_1\bar{e}_\eta + n_1\bar{e}_\zeta \tag{A4}$$

Differentiating equation (A4) with respect to s and expressing the resulting expression along $x_3y_3z_3$ axes, one can obtain

$$\bar{e}'_{x_3} = (l_1l'_1 + m_1m'_1 + n_1n'_1)\,\bar{e}_{x_3}$$

$$+ [l'_1l_2 + m'_1m_2 + n'_1n_2 - \theta'_{pt}(n_1m_2 - m_1n_2)]\bar{e}_{y_3}$$

$$+ [l_1'l_3 + m_1'm_3 + n_1'n_3 - \theta'_{pt}(n_1m_3 - m_1n_3)]\bar{e}_{z_3} \qquad (A5)$$

Also the identity

$$\bar{e}_{x_3} = \bar{\omega}_{x_3y_3z_3} \times \bar{e}_{x_3} \qquad (A6)$$

leads to

$$\bar{e}_{x_3} = \omega_{z_3}\bar{e}_{y_3} - \omega_{y_3}\bar{e}_{z_3} \qquad (A7)$$

From the identity

$$\bar{e}_{y_3} = \omega_{x_3}\bar{e}_{z_3} - \omega_{z_3}\bar{e}_{x_3} \qquad (A8)$$

one can write

$$\omega_{x_3} = \bar{e}'_{y_3} \cdot \bar{e}_{z_3} \qquad (A8)$$

Substituting equation (A2) into (A9) and using the orthogonal property of the matrix [T], the expression for ω_{x_3} in terms of direction cosines is

$$\omega_{x_3} = l_2'l_3 + m_2'm_3 + n_2'n_3 + l_1\theta'_{pt} \qquad (A10)$$

From equations (A5) and (A7) the expressions for ω_{y_3} and ω_{z_3} are

$$\omega_{y_3} = -[l_1'l_3 + m_1'm_3 + n_1'n_3 - \theta'_{pt}(n_1m_3 - m_1n_3)] \qquad (A11)$$

$$\omega_{z_3} = [l_1'l_2 + m_1'm_2 + n_1'n_2 - \theta'_{pt}(n_1m_2 - m_1n_2)] \qquad (A12)$$

Thus for the expressions for the curvatures in terms of the direction cosines and the total section pitch of the blade have been developed. The next task is to express the direction cosines in terms of u, v, w, and ϕ. To this end, the direction cosines are first expressed in terms of the Eulerian-type angles β_1, ζ_1, and θ_1. It was shown in reference 2 that the form of the expressions for the components of the curvature depends on the order in which the rotational transformations between the deformed and undeformed coordinates are imposed. In the present development, out of the six possible rotational transformation sequences which may be employed, a flap-lag-torsion rotation sequence will be addressed. For this rotational transformation sequence, the rotations are imposed as follows:

1. A positive rotation β_1 about the negative η axis resulting in $x_1y_1z_1$.

2. A positive rotation ζ_1 about z_1 axis resulting in $x_2y_2z_2$.

3. A positive rotation θ_1 about x_2 axis resulting in $x_3y_3z_3$.

The explicit form of the transformation matrix [T] in terms of the Eulerian-type angles β_1, ζ_1, and θ_1 is

$$[T] = \begin{bmatrix} \cos\beta_1 \cos\zeta_1 & \sin\zeta_1 & \cos\zeta_1 \sin\beta_1 \\ -\sin\zeta_1 \cos\beta_1 \cos\theta_1 & \cos\zeta_1 \cos\theta_1 & \cos\beta_1 \sin\theta_1 \\ -\sin\beta_1 \sin\theta_1 & & -\sin\zeta_1 \sin\beta_1 \cos\theta_1 \\ \sin\zeta_1 \cos\beta_1 \sin\theta_1 & -\cos\zeta_1 \sin\theta_1 & \sin\zeta_1 \sin\beta_1 \sin\theta_1 \\ -\sin\beta_1 \cos\theta_1 & & +\cos\beta_1 \cos\theta_1 \end{bmatrix} \tag{A13}$$

The rotation angles ζ_1, β_1, θ_1 are to be expressed in terms of the variables u, v, w, ϕ. To this end, let $\bar{R}_0$ and $\bar{R}_1$ be the position vectors of the point 0 and 0' in figure 3, and $\Delta\bar{R}$ be the displacement of 0. Then, one can write

$$\bar{R}_1 = \bar{R}_0 + \Delta\bar{R} \tag{A14}$$

where

$$\Delta\bar{R} = (u - U_F)\,\bar{e}_X + v\bar{e}_Y + w\bar{e}_Z \tag{A15}$$

and

$$U_F = \frac{1}{2} \int_0^s \left[\left(\frac{\partial v}{\partial s}\right)^2 + \left(\frac{\partial w}{\partial s}\right)^2 \right] ds \qquad \text{(A16)}$$

Using the following relation between the unit vector triads $\bar{e}_X \bar{e}_Y \bar{e}_Z$ and $\bar{e}_x \bar{e}_\eta \bar{e}_\zeta$

$$\bar{e}_X = \bar{e}_x$$

$$\bar{e}_Y = \bar{e}_\eta \cos\theta - \bar{e}_\zeta \sin\theta \qquad \text{(A17)}$$

$$\bar{e}_Z = \bar{e}_\eta \sin\theta + \bar{e}_\zeta \cos\theta$$

the expression for $\Delta\bar{R}$ is

$$\Delta\bar{R} = (u - U_F)\bar{e}_x + (v\cos\theta + w\sin\theta)\bar{e}_\eta + (w\cos\theta - v\sin\theta)\bar{e}_\zeta \qquad \text{(A18)}$$

Differentiating equation (A18) with respect to s

$$\frac{\partial(\Delta\bar{R})}{\partial s} = \frac{\partial \bar{R}}{\partial s} + \left(\alpha_x - \frac{\alpha_y^2}{2} - \frac{\alpha_z^2}{2}\right) \bar{e}_x + \alpha_y \bar{e}_\eta + \alpha_z \bar{e}_\zeta \qquad \text{(A19)}$$

where

$$\alpha_x = u'$$

$$\alpha_y = v' \cos\theta + w' \sin\theta \qquad \text{(A20)}$$

$$\alpha_z = -v' \sin\theta + w' \cos\theta$$

$$U_F = \frac{1}{2} \int_0^s (\alpha_y^2 + \alpha_z^2)\, ds$$

Differentiating equation (A14) with respect to s and substituting equation (A19) into the resulting expression, the expression for $\partial\bar{R}_1/ds$ is

$$\frac{\partial \bar{R}_1}{\partial s} = \frac{\partial \bar{R}}{\partial s} + \left(\alpha_x - \frac{\alpha_y^2}{2} - \frac{\alpha_z^2}{2}\right) \bar{e}_x + \alpha_y \bar{e}_\eta + \alpha_z \bar{e}_\zeta \qquad \text{(A21)}$$

From calculus we have

$$\frac{\partial \bar{R}}{\partial s} = \bar{e}_x \tag{A22}$$

Substituting equation (A22) into (A21) gives

$$\frac{\partial \bar{R}_1}{\partial s} = (1 + \alpha_x - \frac{\alpha_y^2}{2} - \frac{\alpha_z^2}{2}) \bar{e}_x + \alpha_y \bar{e}_\eta + \alpha_z \bar{e}_\zeta \tag{A23}$$

The relation between the extensional component of the Green's strain tensor ε on the elastic axis and $\partial \bar{R}_1/ds$ is given by

$$\varepsilon = 1/2 \left(\frac{\partial \bar{R}_1}{\partial s} \cdot \frac{\partial \bar{R}_1}{\partial s} - 1\right) \tag{A24}$$

Substituting equation (A23) into (A24), the expression for ε reduces to

$$\varepsilon = \alpha_x + \frac{\alpha_x^2}{2} \tag{A25}$$

Invoking small strain assumption that the elongations and shears are small compared to unity, equations (A23) and (A25) reduce to

$$\frac{\partial \bar{R}_1}{\partial s} = (1 - \frac{\alpha_y^2}{2} - \frac{\alpha_z^2}{2}) \bar{e}_x + \alpha_y \bar{e}_\eta + \alpha_z \bar{e}\zeta \tag{A26}$$

If the deformed length of the element ds is ds_1, the relation between ds_1 and ds can be written as

$$ds_1 = (1 - 2\varepsilon)^{1/2} ds \tag{A28}$$

Hence,

$$\frac{\partial \bar{R}_1}{\partial s_1} = (1 + 2\varepsilon)^{-1/2} \frac{\partial \bar{R}_1}{\partial s} \tag{A29}$$

Also,

$$\frac{\partial \bar{R}_1}{\partial s_1} = \bar{e}_{x_3} \tag{A30}$$

Invoking small strain assumption and combining equations (A26), (A29), and (A30) results in

$$\bar{e}_{x_3} = \left(1 - \frac{\alpha_y^2}{2} - \frac{\alpha_z^2}{2}\right) \bar{e}_x + \alpha_y \bar{e}_\eta + \alpha_z \bar{e}_\zeta \tag{A31}$$

From equations (A2), (A13), and (A31) one can write

$$l_1 = \cos \beta_1 \cos \zeta_1 = 1 - \frac{\alpha_y^2}{2} - \frac{\alpha_z^2}{2}$$

$$m_1 = \sin \zeta_1 = \alpha_y \tag{A32}$$

$$n_1 = \cos \zeta_1 \sin \beta_1 = \alpha_z$$

Hence,

$$\sin \beta_1 = \alpha_z \qquad \cos \beta_1 = 1 - \frac{\alpha_z^2}{2}$$

$$\sin \zeta_1 = \alpha_y \qquad \cos \zeta_1 = 1 - \frac{\alpha_y^2}{2} \tag{A33}$$

The third rotation angle θ_1 is due to torsion of the blade and hence is given by

$$\theta_1 = \phi \tag{A34}$$

The expressions for the transformation matrix [T] and for the curvature components ω_{x_3}, ω_{y_3}, ω_{z_3} are given in terms of the direction cosines in equations (A2), (A10), (A11), and (A12) and those for the direction cosines in terms of the rotation angles are given in equation (A13). The rotation angles are expressed in terms of α_x, α_y, and α_z in equations (A33) and (A34) and the expressions for α_x, α_y, and α_z are given in terms of u, v, w, and ϕ in equations (A20). Combining equations (A10), (A11), (A13), (A33), (A34), and (A20), the second-degree expressions for the curvature components and for the transformation matrix can be obtained as

$$\omega_{x_3} = \phi' - (v' \cos\theta + w' \sin\theta)(-v' \sin\theta + w' \cos\theta)' + \theta'_{pt}\left(1 - \frac{v'^2}{2} - \frac{w'^2}{2}\right) \tag{A35}$$

$$\omega_{y_3} = -w''(\cos\theta - \phi \sin\theta) + v''(\sin\theta + \phi\cos\theta) \tag{A36}$$

$$\omega_{z_3} = v''(\cos\theta - \phi\sin\theta) + w''(\sin\theta + \phi\cos\theta) \tag{A37}$$

$$[T] = \begin{bmatrix} 1 - \frac{v'^2}{2} - \frac{w'^2}{2} & v'\cos\theta + w'\sin\theta & -v'\sin\theta + w'\cos\theta \\ -v'(\cos\theta - \phi\sin\theta) & 1 - \frac{(v'\cos\theta + w'\sin\theta)^2}{2} & \phi - (v'\cos\theta + w') \\ -w'(\sin\theta + \phi\cos\theta) & -\frac{\phi^2}{2} & \cdot(-v'\sin\theta + w'\cos\theta) \\ v'(\sin\theta + \phi\cos\theta) & -\phi & 1 - \frac{(v'\sin\theta + w'\cos\phi)^2}{2} \\ -w'(\cos\theta - \phi\sin\theta) & & -\frac{\phi^2}{2} \end{bmatrix} \tag{A38}$$

APPENDIX B

NONLINEAR STRAIN-DISPLACEMENT RELATIONS

This Appendix will develop second-degree nonlinear expressions for strains. To this end, let $\bar{r}_0$ and $\bar{r}_1$ be the position vectors before and after deformation of an arbitrary mass point on the blade. These vectors can be expressed as

$$\bar{r}_0 = \bar{R}_0 + \eta \bar{e}_\eta + \zeta \bar{e}_\zeta \tag{B1}$$

$$\bar{r}_1 = \bar{R}_1(s_1) + \eta \bar{e}_{y_3} + \zeta \bar{e}_{z_3} \tag{B2}$$

where s_1 is the length measured along the deformed elastic axis of the blade. The differentials of the above vectors are given by

$$d\bar{r}_0 = ds\bar{e}_x + (d\eta - \zeta\theta'_{pt}\ ds)\ \bar{e}_\eta + (d\zeta + \eta\theta'_{pt}\ ds)\ \bar{e}_\zeta \tag{B3}$$

$$d\bar{r}_1 = (1 - \eta\omega_{z_3} + \zeta\omega_{y_3}\ \bar{e}_{x_3})\ ds_1$$

$$+ (d\eta - \zeta\omega_{x_3}\ ds_1)\ \bar{e}_{y_3} + (d\zeta + \eta\omega_{x_3}\ ds_1)\ \bar{e}_{z_3} \tag{B4}$$

where the curvature components ω_{x_3}, ω_{y_3}, w_{z_3} are defined in Appendix B. The usual practice in solid mechanics is to use the Lagrangian strain tensor which is defined by

$$d\bar{r}_1 \cdot d\bar{r}_1 - d\bar{r}_0 \cdot d\bar{r}_0 = 2\ [ds\ d\eta\ d\zeta]\ [\epsilon_{ij}] \begin{Bmatrix} ds \\ d\eta \\ d\zeta \end{Bmatrix} \tag{B5}$$

Substituting equations (B3), (B4), (A28) into equation (B5) and using the relation between the engineering strains and the components of the Lagrangian strain given by equation (5), one obtains the following expressions for the strain components

$$\gamma_{xx} = \varepsilon_{xx} = u' + (\eta^2 + \zeta^2)\left(\frac{\phi'^2}{2} + \phi'\,\theta'_{pt}\right)$$

$$- \eta[v'' \cos\theta + w'' \sin\theta + \phi\,(-v'' \sin\theta + w'' \cos\theta)]$$

$$+ \zeta\,[v'' \sin\theta - w'' \cos\theta + \phi\,(v'' \cos\theta + w'' \sin\theta)] \qquad (B6)$$

$$\gamma_{x\eta} = 2\varepsilon_{x\eta} = -\zeta\,[\phi' - (v' \cos\theta + w' \sin\theta)$$

$$\cdot(-v' \sin\theta + w' \cos\theta)' - \theta'_{pt}\left(\frac{v'^2}{2} + \frac{w'^2}{2}\right)] \qquad (B7)$$

$$\gamma_{x\zeta} = 2\varepsilon_{x\zeta} = \eta\,[\phi' - (v' \cos\theta + w' \sin\theta)$$

$$(-v' \sin\theta + w' \cos\theta)' - \theta'_{pt}\left(\frac{v'^2}{2} + \frac{w'^2}{2}\right)] \qquad (B8)$$

It should be pointed out that in arriving at the expression given in equations (B6), (B7), (B8) several higher-order terms have been discarded based either on considerations related to small deformations in which elongations and shears are small compared to unity or on considerations related to the approximations which can be made because of the slenderness of the blade as discussed in Appendix C. Retention of higher order terms in the expressions for the strain components is not a problem. However, these higher order terms in the strains lead to higher order terms in the final equations of motion. Thus, discarding these terms in the resultant strain expressions using the considerations of Appedix C simplifies the algebric manipulations.

APPENDIX C

SLENDER BEAM APPROXIMATION AND ATTENDANT ORDERING SCHEME

The simplifications of the slender beam approximation as applied to the derivation of the second-degree nonlinear equations of motion were discussed and a mathematical ordering scheme which is compatible with the assumption of a slender beam was introduced in reference 3. The same scheme has been used in the present report. In this scheme, a slender beam is systematized by introducing a parameter ε which is taken to be of the same order as the nondimensionalized bending displacements v/R and w/R. The order of the dependent variables and the geometric quantities appearing in the equations of motion of this report are as follows:

$u/R = 0(\varepsilon^2)$ $\qquad \eta/R = 0(\varepsilon)$ $\qquad \theta_c = 0(1)$

$v/R = 0(\varepsilon)$ $\qquad \zeta/R = 0(\varepsilon)$ $\qquad \theta'_{pt} = 0(\varepsilon)$

$w/R = 0(\varepsilon)$ $\qquad \beta_{pc} = 0(\varepsilon)$

$\phi = 0(\varepsilon)$ $\qquad x/R = 0(1)$

$\theta_{pt} = 0(1)$

Following this ordering scheme, the order of the elastic and inertial terms which are retained in the second-degree nonlinear coupled flap-lag-axial-torsion equations of motion of this report are given in Table A1 below.

TABLE A1. - ORDERING SCHEME

Freedom	Elastic forces	Inertial forces
Bending	ε^4	ε^2
Torsion	ε^5	ε^3
Extension	ε^3	ε^3

The rationale for this scheme was discussed in reference 3.

APPENDIX D

VELOCITY COMPONENTS EXPERIENCED BY BLADE ELEMENT

The velocity components experienced by a blade element are determined by considering the contributions from the atmospheric winds, induced velocity, and blade dynamic velocity. For a down wind rotor, the tower shadow effect will be included.

The resultant velocities seen by a point on the elastic axis of the blade in deformed and the undeformed coordinate systems are related according to

$$\bar{V}_{x_3 y_3 z_3} = [T]\, \bar{V}_{x\eta\zeta} \tag{D1}$$

where, from figure 6,

$$\bar{V}_{x_3 y_3 z_3} = U_R \bar{e}_{x_3} - U_T \bar{e}_{y_3} - U_P \bar{e}_{z_3} \tag{D2}$$

and
$$\bar{V}_{x\eta\zeta} = \left(\bar{V}_a - \frac{d\bar{r}_1}{dt}\right)_{x\eta\zeta} \tag{D3}$$

The aerodynamic velocity components seen by a blade element are shown in figure 7. The aerodynamic velocity consists of three components: (1) the free stream velocity; (2) the gust velocity; and (3) the induced velocity. The free stream velocity profile over a rough terrain is frequently approximated by a power-law relation with height, and is given by

$$V_m\,(x_I) = V_H \left(1 - \frac{x_I}{H}\right)^{\eta} \tag{D4}$$

where V_m and V_H are the mean velocities at x_I and turbine axis respectively and H is the height of the center of the hub from ground. This mean velocity, V_m is a function of the azimuth angle of the blade since the interference of the tower causes a reduction in the velocity when the blade is in the vicinity of the tower. In the present development, it is accounted for by introducing a blockage factor b which is less than or equal to 1. Inside the tower shadow band defined by the limits $\psi_1 < \psi < \psi_2$ the mean velocity is bV_m where V_m is defined in equation (D4). The limits ψ_1 and ψ_2 and the blockage factor b are to be obtained from either wind tunnel experiments or full scale tests. Expanding $V(x_I)$ in Taylor's series about $x_I = 0$ and retaining only up to second-order terms, equation (D4) reduces to

$$\lambda_m = \lambda_H (1 + C_1\chi^2 - C_2\chi \cos \psi + C_1\chi^2 \cos 2\psi) \qquad \text{(D5)}$$

where

$$\begin{aligned} \lambda_m &= V_m/\Omega R \\ x_I &= x \cos \beta_{pc} \cos \psi = x \cos \psi \\ C_1 &= (\eta^2 - \eta)R^2/4H^2 \\ C_2 &= \eta R/H \\ \chi &= x/R \\ \lambda_H &= V_H/\Omega R \end{aligned} \qquad \text{(D6)}$$

The gust velocities are assumed to be harmonic with a frequency ω_g and are represented as

$$\begin{aligned} \lambda_{ug} &= \frac{u_g}{\Omega R} \sin \omega_g t \\ \lambda_{vg} &= \frac{v_g}{\Omega R} \sin \omega_g t \\ \lambda_{wg} &= \frac{w_g}{\Omega R} \sin \omega_g t \end{aligned} \qquad \text{(D7)}$$

The velocity $V_m = \lambda_m \Omega R$ is inclined to the rotor axis at an angle ψ_0 as shown in figure 7. Resolving this velocity along Y_I and Z_I axis, one can write

$$\bar{V}_m = bV_m \sin\psi_0 \bar{e}_{Y_I} + bV_m \cos\psi_0 \bar{e}_{Z_I}$$

$$= \Omega R\,(b\lambda_m \sin\psi_0 \bar{e}_{Y_I} + b\lambda_m \cos\psi_0 \bar{e}_{Z_I}) \qquad \text{(D8)}$$

The induced velocity can be calculated using blade element and momentum theories and is

$$v_i = \lambda_i \Omega R \qquad \text{(D9)}$$

Now the aerodynamic velocity can be expressed as

$$\bar{V}_{X_I Y_I Z_I} = \Omega R\,[\lambda_{ug} \bar{e}_{X_I} + (b\lambda_m \sin\psi_0 + \lambda_{vg}) \bar{e}_{Y_I}$$

$$+ (b\lambda_m \cos\psi_0 + \lambda_{wg} - \lambda_i) \bar{e}_{Z_I}] \qquad \text{(D10)}$$

This velocity can be expressed in $x\eta\zeta$ system by the relation

$$\bar{V}_{x\eta\zeta} = [T_R]\, \bar{V}_{X_I Y_I Z_I} \qquad \text{(D11)}$$

where the transformation matrix can be written with the aid of the figures 1 and 2 as

$$[T_R] = \begin{bmatrix} \cos\psi & \sin\psi & \beta_{pc} \\ -\beta_{pc}\cos\psi\sin\theta & -\beta_{pc}\sin\psi\sin\theta & \sin\theta \\ -\sin\psi\cos\theta & +\cos\psi\cos\theta & \\ -\beta_{pc}\cos\psi\cos\theta & -\beta_{pc}\sin\psi\cos\theta & \cos\theta \\ +\sin\psi\sin\theta & -\cos\psi\sin\theta & \end{bmatrix} \qquad \text{(D12)}$$

The position vector of a point on the elastic axis is

$$\bar{r}_1 = (x + u - U_F)\bar{e}_x$$

$$+ (v\cos\theta + w\sin\theta)\bar{e}_\eta + (-v\sin\theta + w\cos\theta)\bar{e}_\zeta \qquad \text{(D13)}$$

The angular velocity of the coordinate system $x\eta\zeta$ is given by equation (20).

Using equation (D13) in conjunction with equation (20), the dynamic velocity of a point on the elastic axis taken with respect to the $x\eta\zeta$ axis system is given by

$$\frac{d\bar{r}_1}{dt} = (u - \dot{U}_F - \Omega v)\bar{e}_x$$
$$+ [\dot{v} \cos \theta + \dot{w} \sin \theta) - \Omega\beta_{pc} (-v \sin \theta + w \cos \theta)$$
$$+ \Omega \cos \theta (x + u - U_F)]\bar{e}_\eta$$
$$+ [-\dot{v} \sin \theta + \dot{w} \cos \theta + \Omega\beta_{pc} (v \cos \theta + w \sin \theta)$$
$$- \Omega \sin \theta (x + u - U_F)]\bar{e}_\zeta \quad \text{(D14)}$$

Combining equations (D11), (D12), and (D14) gives the total velocity seen by a point on the elastic axis as

$$\bar{V}_{x\eta\zeta} = \{\Omega R[\lambda_{ug} \cos \psi + (b\lambda_m \sin \psi_0 + \lambda_{vg})\sin \psi$$
$$+ \beta_{pc}(b\lambda_m \cos \psi_0 + \lambda_{wg} - \lambda_i)]$$
$$- \dot{u} + \dot{U}_F + \Omega v\}\bar{e}_x$$
$$+ \{\Omega R[\lambda_{ug}(-\beta_{pc} \cos \psi \cos \theta - \sin \psi \sin \theta)$$
$$+ (b\lambda_m \sin \psi_0 + \lambda_{vg})(-\beta_{pc} \sin \psi \sin \theta + \cos \psi \cos \theta)$$
$$+ (b\lambda_m \cos \psi_0 + \lambda_{wg} - \lambda_i)\sin \theta]$$
$$- (\dot{v} \cos \theta + \dot{w} \sin \theta) + \Omega\beta_{pc} (-v \sin \theta + w \cos \theta)$$
$$- \Omega \cos \theta (x + u - U_F)\}\bar{e}_\eta$$
$$+ \{\Omega R[\lambda_{ug}(-\beta_{pc} \cos \psi \cos \theta + \sin \psi \sin \theta)$$
$$+ (b\lambda_m \sin \psi_0 + \lambda_{vg})(-\beta_{pc} \sin \psi \cos \theta - \cos \psi \sin \theta)$$
$$+ (b\lambda_m \cos \psi_0 + \lambda_{wg} - \lambda_i)\cos \theta]$$
$$- (-\dot{v} \sin \theta + \dot{w} \cos \theta) - \Omega\beta_{pc} (v \cos \theta + w \sin \theta)$$
$$+ \Omega \sin \theta (x + u - U_F)\}\bar{e}_\zeta \quad \text{(D15)}$$

The tangential and perpendicular velocity components U_T and U_P are obtained from equations (A38), (D1), (D2), and (D15) and, to second-degree in the dependent variables, have the form

$$U_T = - \{-\Omega R[\lambda_{ug} \cos \psi + (b\lambda_m \sin \psi_0 + \lambda_{vg}) \sin \psi$$

$$+ \beta_{pc} (b\lambda_m \cos \psi_0 + \lambda_{wg} - \lambda_i)]$$

$$\cdot [v' (\cos \theta - \phi \sin \theta) + w' (\sin \theta + \phi \cos \theta)]$$

$$+ (-\dot{u} + \Omega v) (-v' \cos \theta - w' \sin \theta)$$

$$+ \Omega R[\lambda_{ug} (-\beta_{pc} \cos \psi \cos \theta - \sin \psi \sin \theta)$$

$$+ (b\lambda_m \sin \psi_0 + \lambda_{vg})(-\beta_{pc} \sin \psi \sin \theta + \cos \psi \cos \theta)$$

$$+ (b\lambda_m \cos \psi_0 + \lambda_{wg} - \lambda_i) \sin \theta]$$

$$\cdot [1 - \frac{(v' \cos \theta + w' \sin \theta)^2}{2} - \frac{\phi^2}{2}]$$

$$- (\dot{v} \cos \theta + \dot{w} \sin \theta) + \Omega\beta_{pc} (-v \sin \theta + w \cos \theta)$$

$$- \Omega \cos \theta (u - U_F) - \Omega x \cos \theta [1 - \frac{(v' \cos \theta + w' \sin \theta)^2}{2} - \frac{\phi^2}{2}]$$

$$+ \Omega R [\lambda_{ug}(-\beta_{pc} \cos \psi \cos \theta + \sin \psi \sin \theta)$$

$$+ (b\lambda_m \sin \psi_0 + \lambda_{vg})(-\beta_{pc} \sin \psi \cos \theta - \cos \psi \sin \theta)$$

$$+ (b\lambda_m \cos \psi_0 + \lambda_{wg} - \lambda_i) \cos \theta]$$

$$\cdot [\phi - (v' \cos \theta + w' \sin \theta)(-v' \sin \theta + w' \cos \theta)]$$

$$+ \phi [- (-\dot{v} \sin \theta + \dot{w} \cos \theta - \Omega\beta_{pc} (v \cos \theta + w \sin \theta)$$

$$+ \Omega \sin \theta (x + u)]\} \qquad \text{(D16a)}$$

$$U_P = - \{\Omega R [\lambda_{ug} \cos \psi + (b\lambda_m \sin \psi_0 + \lambda_{vg}) \sin \psi$$

$$+ \beta_{pc} (b\lambda_m \cos \psi_0 + \lambda_{wg} - \lambda_i)]$$

$$\cdot [v' (\sin \theta + \phi \cos \theta) - w' (\cos \theta - \phi \sin \theta)]$$

$$+ (-\dot{u} + \Omega v) (v' \sin \theta - w' \cos \theta)]$$

$$- \Omega R [\lambda_{ug} (-\beta_{pc} \cos \psi \cos \theta - \sin \psi \sin \theta)$$

$$+ (b\lambda_m \sin \psi_0 + \lambda_{vg})(-\beta_{pc} \sin \psi \sin \theta + \cos \psi \cos \theta)$$

$$+ (b\lambda_m \cos \psi_0 + \lambda_{wg} - \lambda_i) \sin \theta]\phi$$

$$- [-(\dot{v} \cos \theta + \dot{w} \sin \theta) + \Omega\beta_{pc} (-v \sin \theta + w \cos \theta)- \Omega \cos\theta \, (x+u)]\phi$$

$$+ \Omega R\,[\lambda_{ug}(-\beta_{pc}\cos\psi\cos\theta + \sin\psi\sin\theta)$$

$$+ (b\lambda_m \sin\psi_0 + \lambda_{vg})(-\beta_{pc}\sin\psi\cos\theta - \cos\psi\sin\theta)$$

$$+ (b\lambda_m \cos\psi_0 + \lambda_{wg} - \lambda_i)\cos\theta]$$

$$- (-\dot{v}\sin\theta + \dot{w}\cos\theta) - \Omega\beta_{pc}(v\cos\theta + w\sin\theta) + \Omega\sin\theta\,(u - U_F)$$

$$+ \Omega\sin\theta\; x\,\left(1 - \frac{(v'\sin\theta + w'\cos\theta)^2}{2} - \frac{\phi^2}{2}\right)\} \qquad \text{(D16b)}$$

The quantity $\dot{\varepsilon}$ appearing in equations (51) and (53) is the angular velocity of the blade section about the local x_3 axis and, consistent with the present notation, can be written as $\dot{\varepsilon}_{x_3}$. It can be regarded as composed of two parts: the first part arising from the rigid-body angular velocity of the hub in space, the second part arising from the angular velocity associated with the elastic deformations. The first contribution due to Ω and is given by

$$\begin{Bmatrix} \dot{\varepsilon}_{x_3} \\ \dot{\varepsilon}_{y_3} \\ \dot{\varepsilon}_{z_3} \end{Bmatrix}_{\Omega} = [T] \begin{Bmatrix} \Omega\beta_{pc} \\ \Omega\sin\theta \\ \Omega\cos\theta \end{Bmatrix} \qquad \text{(D17)}$$

Then

$$\dot{\varepsilon}_{x_3} = \Omega\beta_{pc}\left(1 - \frac{v'^2}{2} - \frac{w'^2}{2}\right) + \Omega w' \qquad \text{(D18)}$$

The second part of the angular velocity of blade section can be obtained by replacing ϕ' by $\dot{\phi}$, $(-v'\sin\theta + w'\cos\theta)'$ by $(-v'\sin\theta + w'\cos\theta)^{\bullet}$, and θ'_{pt} by $\dot{\theta}_{pt}$ (which is zero) in the expression for the curvature ω_{x_3} given in equation (A35). Thus,

$$(\dot{\varepsilon}_{x_3})_{deformation} = \dot{\phi} - (v'\cos\theta + w'\sin\theta)(-v'\sin\theta + w'\cos\theta)^{\bullet} \qquad \text{(D19)}$$

Combining equations (D18) and (D19), the total sectional pitching velocity is

$$\varepsilon(=\dot{\varepsilon}_{x_3}) = \Omega\beta_{pc}\left(1 - \frac{v'^2}{2} - \frac{w'^2}{2}\right) + \Omega w'$$

$$+ \dot{\phi} - (v' \cos\theta + w' \sin\theta)(-\dot{v}' \sin\theta + \dot{w}' \cos\theta) \qquad \text{(D20)}$$

REFERENCES

1. Kaza, K. R. V.; and Kvaternik, R. G.: A Critical Examination of the Flap-Lag Dynamics of Helicopter Rotor Blades in Hover and Forward Flight. Paper no. 1034, American Helicopter Society 32nd Annual National Forum, Washington, D.C., May 1976.
2. Kvaternik, R. G.; and Kaza, K. R. V.: Nonlinear Curvature Expressions for Combined Flapwise Bending, Edgewise Bending, Torsion and Extension of Twisted Rotor Blades. NASA TM X-73997, December 1976.
3. Kaza, K. R. V.; and Kvaternik, R. G.: Nonlinear Aeroelastic Equations for Combined Flapwise Bending, Chordwise Bending, Torsion, and Extension of Twisted Nonuniform Rotor Blades in Forward Flight. NASA Technical Memorandum 74059, August 1977.
4. Friedmann, P.: Aeroelastic Modeling of Large Wind Turbines. Journal of American Helicopter Society, Vol. 21, No. 4, October 1976, pp. 17-27.
5. Rosen, A.; and Friedmann, P.: Nonlinear Equations of Equilibrium for Elastic Helicopter or Wind Turbine Blades Undergoing Moderate Deformation. UCLA-ENG-7718, School of Engineering and Applied Science, University of California, Los Angeles, California, June 1977, (Also as NASA CR-159478.
6. Kottapalli, S. B. R.; Friendmann, P.; and Rosen, A.: Aeroelastic Stability and Response of Horizontal Axis Wind Turbine Blades, University of California, Los Angeles, School of Engineering and Applied Science Report.
7. Warmbrodt, W.; and Friedmann, P.: Aeroelastic Response and Stability of a Coupled Rotor/Support System with Application to Large Horizontal Axis Wind Turbines. University of California, Los Angeles, School of Engineering and Applied Science Report.

8. Wendell, J.: Volume X, Wind Energy Conversion: Aeroelastic Stability of Wind Turbine Rotor Blades. Aeroelastic and Structures Laboratory, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Report ASRL TR-184-16.

9. Kaza, K. R. V.: Aeroelastic Stability of Wind Turbine Blades. Workshop on Wind Turbine Structural Dynamics, NASA Conference publication 2034 (DOE publication CONF-771148), November 15-17, 1977.

10. Greenberg, J. M.: Airfoil in Sinusoidal Motion in a Pulsating Stream. NACA TN-1326, 1974.

11. Chu, Chen.: The Effect of Initial Twist on the Torsional of Thin Prismatic Bars and Tubular Members. Proc. of the First U.S. National Congress of Applied Mechanics, June 1951.

12. Biot, M. A.: Increase of Torsional Stiffness of a Prismatic Bar Due to Tension. Journal of Applied Physics, Vol. 10, December 1939.

13. Houbolt, J. C.; and Brooks, G. W.: Differential Equations of Motion for Combined Flapwise Bending, Chordwise Bending, and Torsion of Twisted Nonuniform Rotor Blades. NACA Report 1346, 1958.

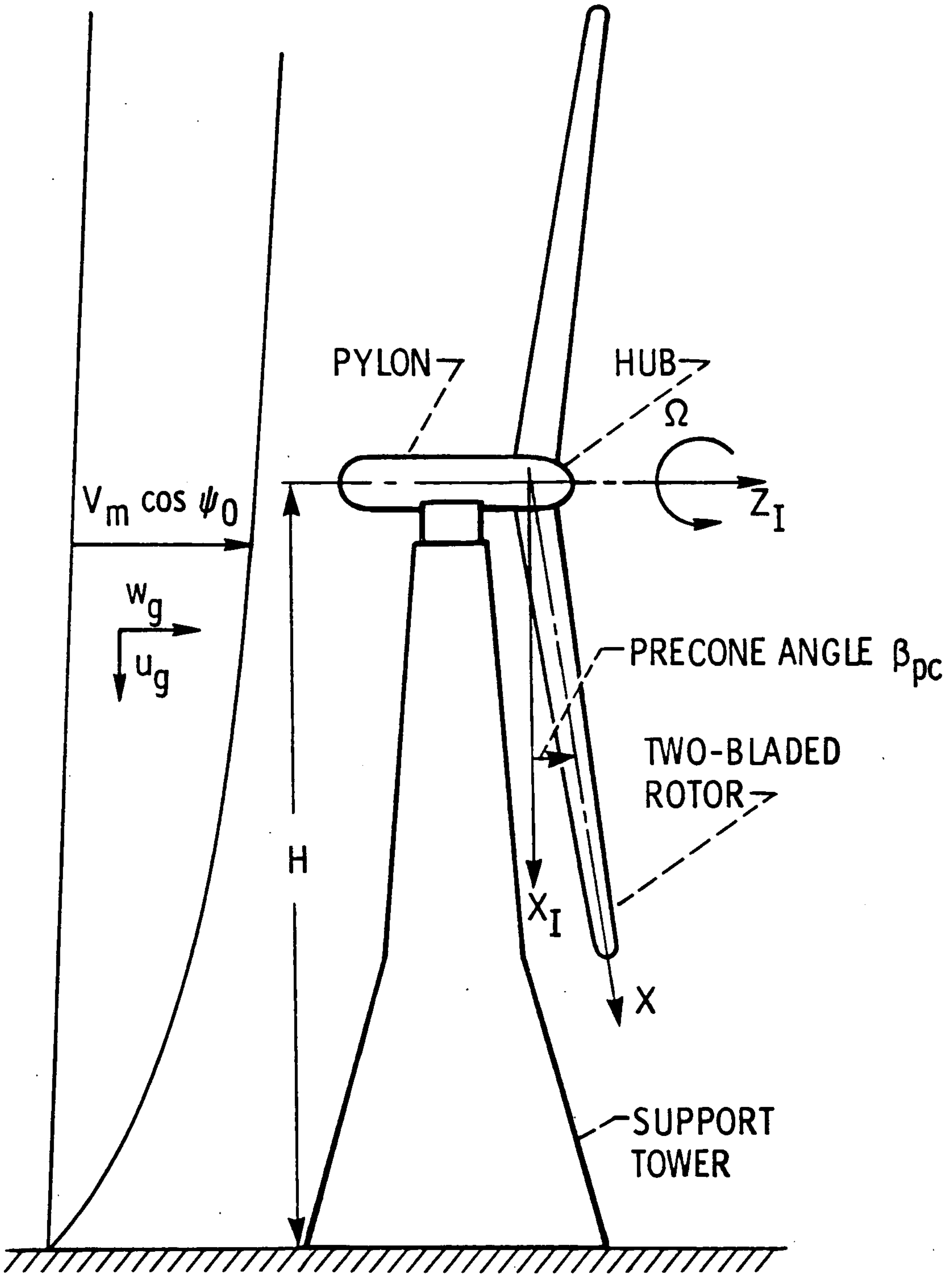


Figure 1(a). - A horizontal axis wind turbine configuration (Y_I-axis is perpendicular to the plane of the paper and the velocity components $V_m \sin\psi_o$ and v_g are parallel to Y_I-axis).

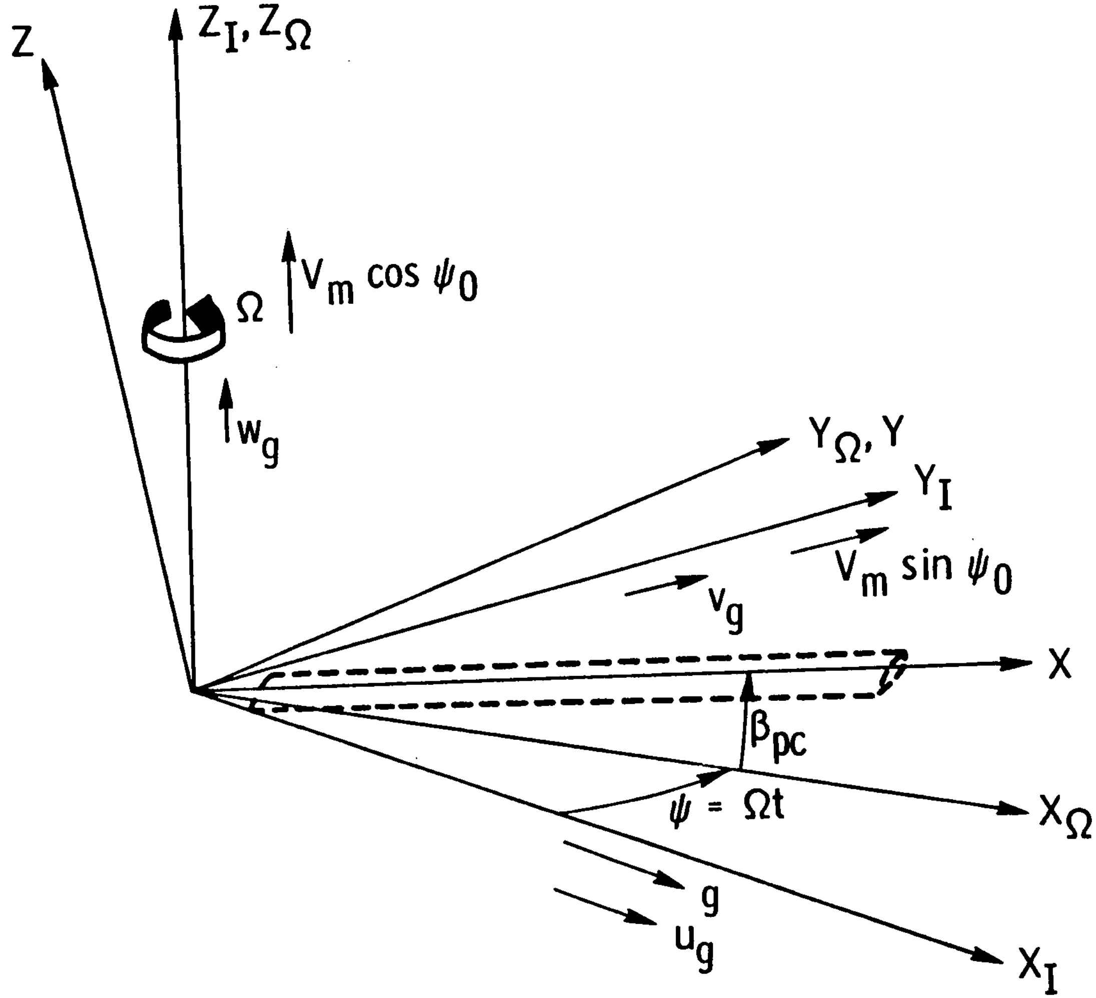


Figure 1(b). - Coordinate system of undeformed blade (section pitch angle, θ, not shown).

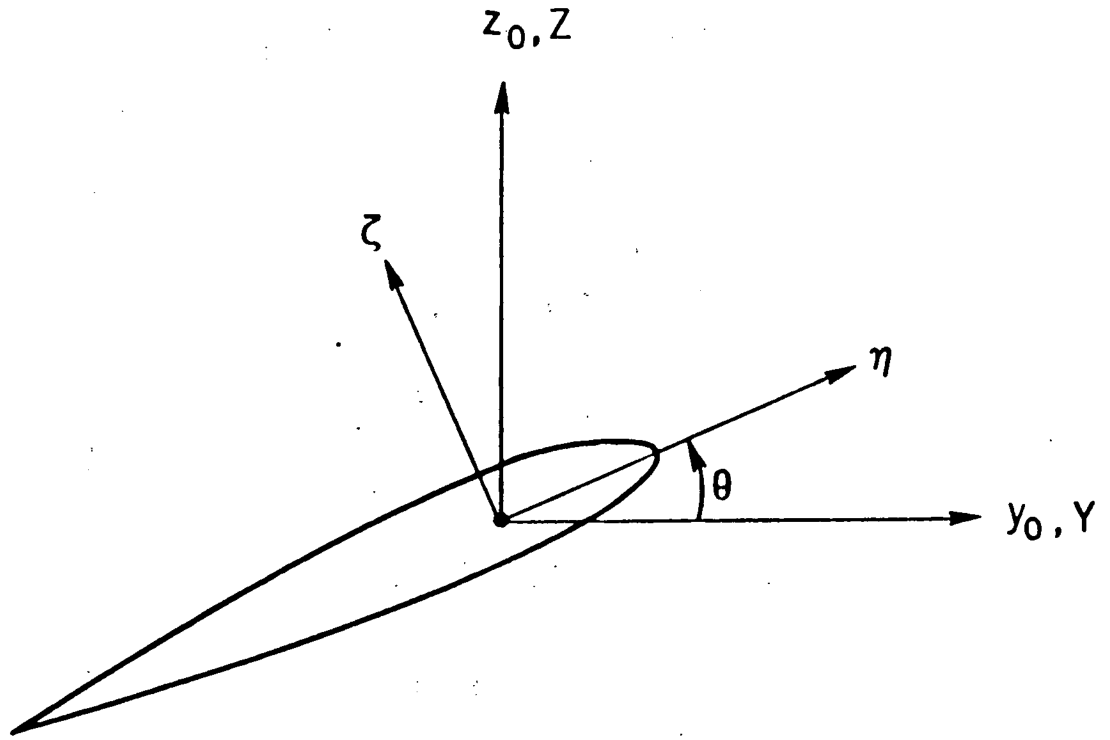


Figure 2. - Coordinate systems of blade cross section.

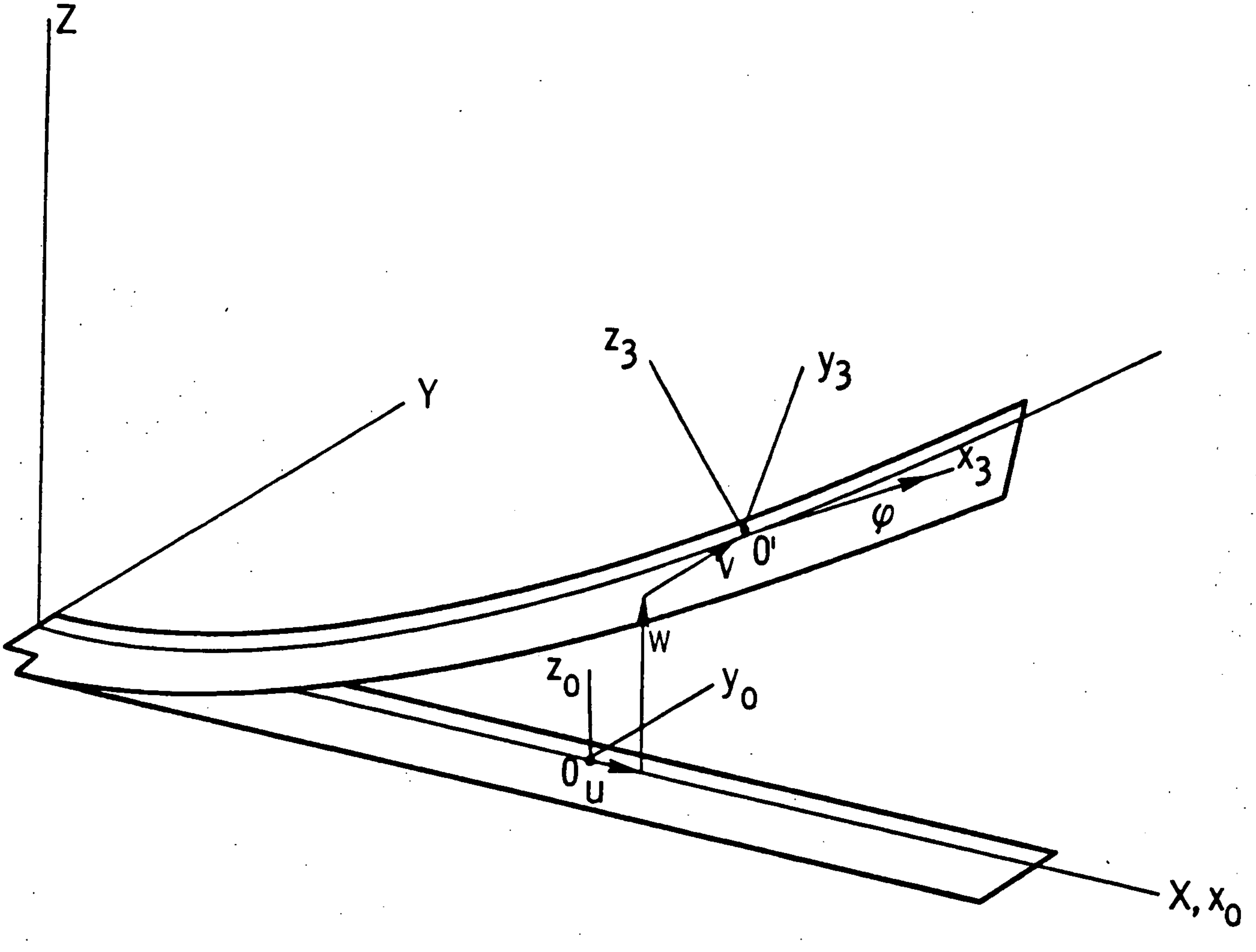


Figure 3. - Schematic representation of undeformed and deformed blade (section pitch angle, θ, not shown).

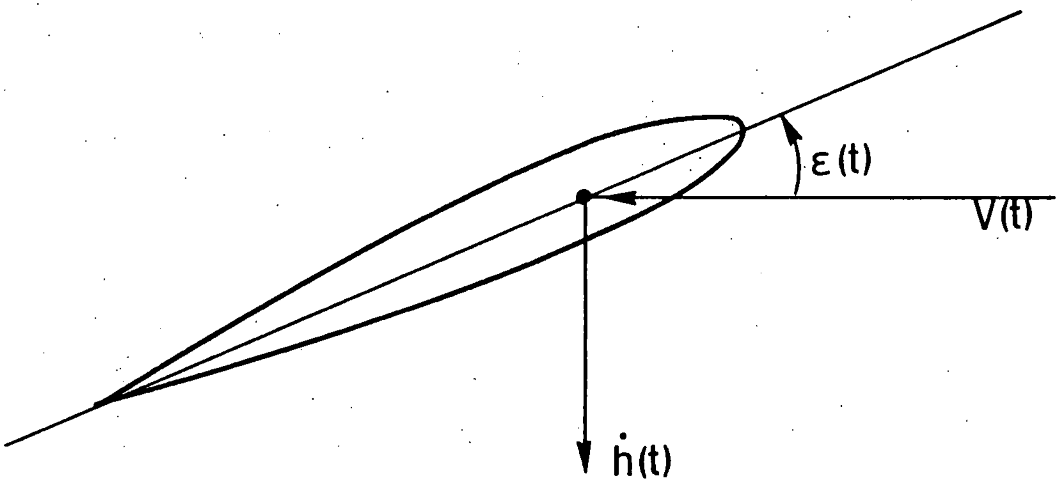


Figure 4. - Cross section of blade in general unsteady motion.

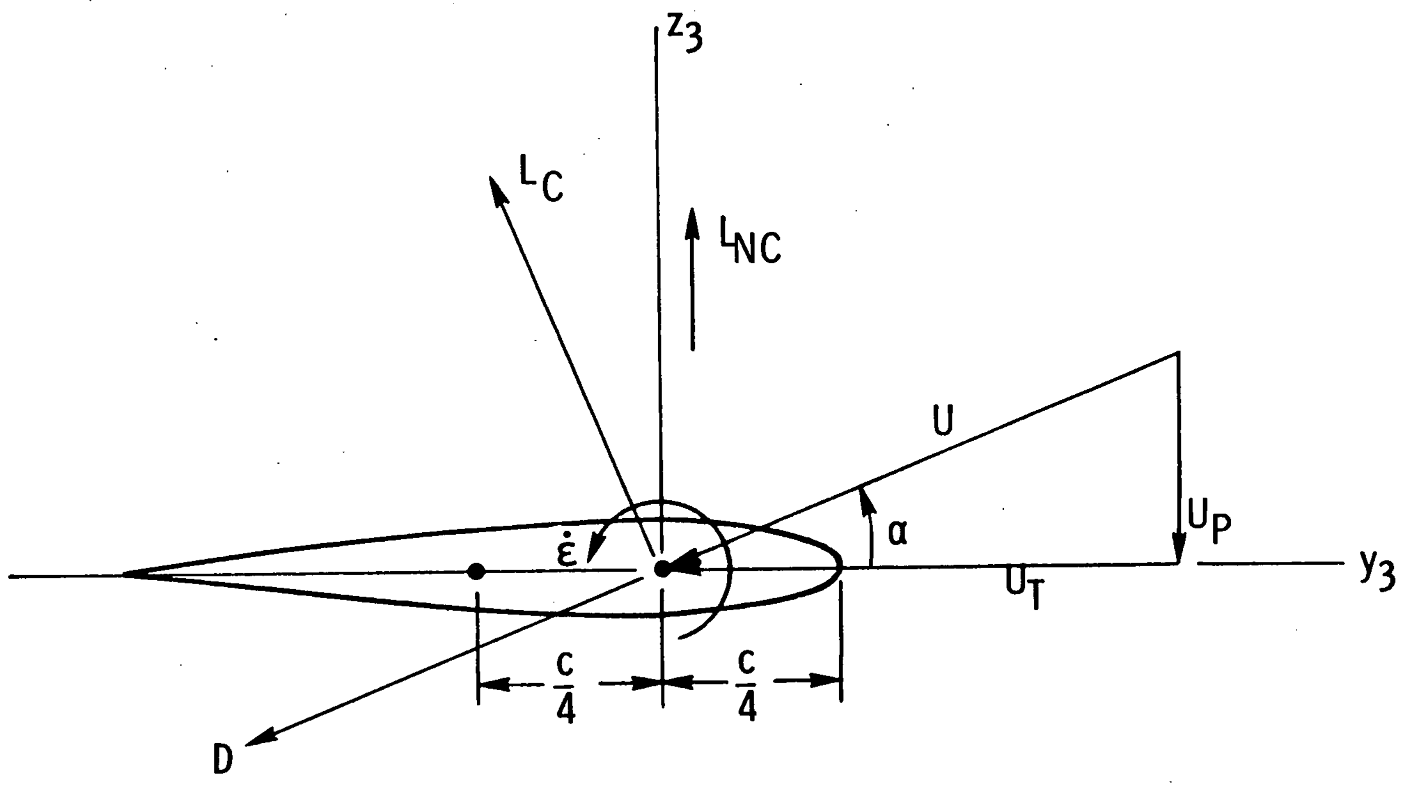


Figure 5. - Blade section inflow geometry and aerodynamic force components.

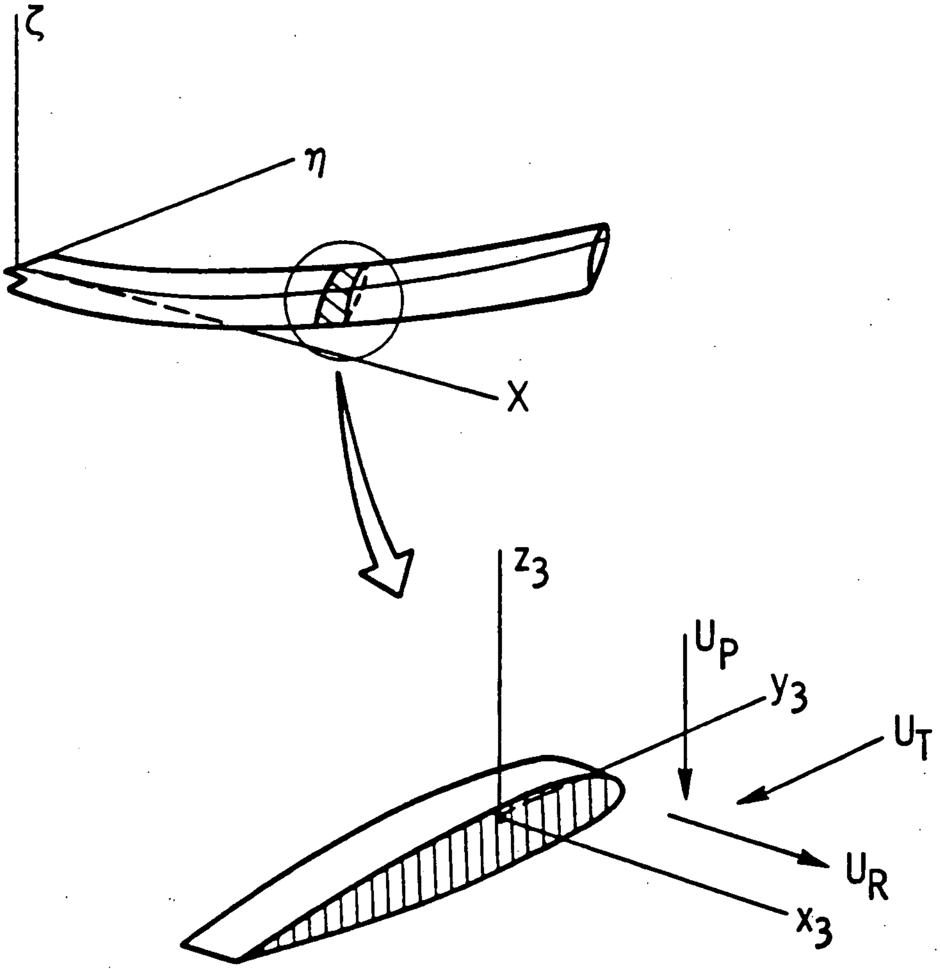
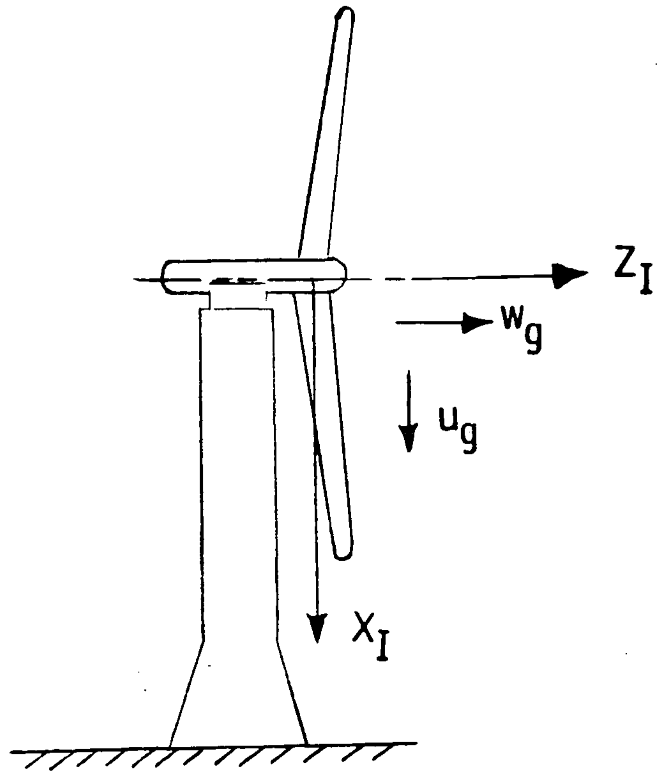
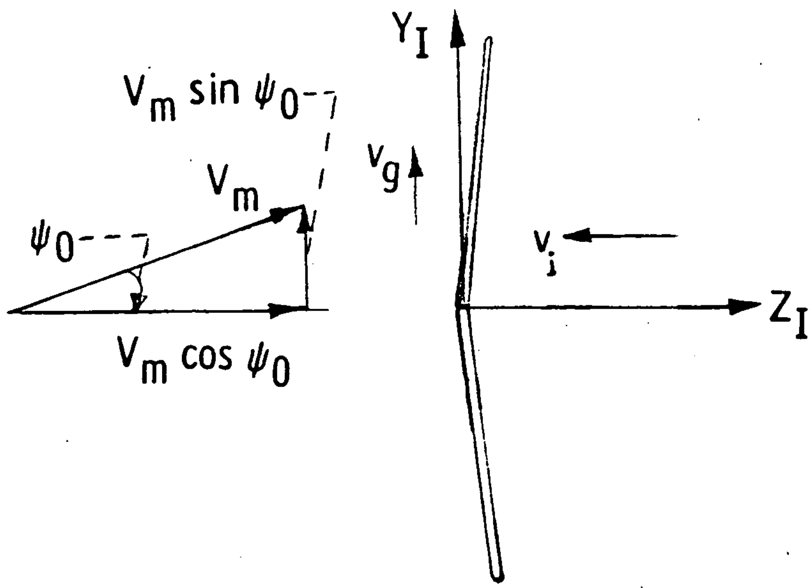


Figure 6. - Relative velocity components at blade cross section.



(a) Side view



(b) Top view

Figure 7.- Aerodynamic velocity components.

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16. Abstract
The second-degree nonlinear equations of motion for a flexible, twisted, nonuniform, horizontal-axis wind turbine blade are developed using Hamilton's principle. The derivation of the equations has its basis in the geometric nonlinear theory of elasticity, and the final equations are consistent with the small deformation approximation in which the elongations and shears are negligible compared to unity and the square of the derivative of the extensional deformation of the elastic axis is negligible compared to the squares of the bending slopes. A mathematical ordering scheme which is consistent with the assumption of a slender beam is used to discard some higher-order elastic and inertial terms in the second-degree nonlinear equations. The blade aerodynamic loading which is employed accounts for both wind shear and tower shadow and is obtained from strip theory based on a quasi-steady approximation of two-dimensional, incompressible, unsteady, airfoil theory. The resulting equations have periodic coefficients and are suitable for determining the aeroelastic stability and response of large horizontal-axis wind turbine blades.

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