A STUDY OF THE EFFECTS OF STATE TRANSITION MATRIX APPROXIMATIONS

Janet A. May
Goddard Space Flight Center

ABSTRACT

This paper investigates the effects of using an approximate state transition matrix in orbit estimation. The approximate state transition matrix results when higher order geopotential terms in the equations of motion are ignored in the formation of the variational equations. Two methods of orbit estimation were considered: the differential correction procedure (DC) and the extended Kalman filter (EKF). The system used for the study was the Research & Development version of the Goddard Trajectory Determination System (R&D GTDS). The effects of the approximation were analyzed on a number of orbits. These include orbits of various inclinations and semimajor axes. Other parameters studied include geopotential models and DC arc length.

Introduction

The state transition matrix plays an important role in orbit determination. It relates perturbations in the state at time t to perturbations in the state at epoch. Rice (4) suggests that divergence in orbit estimation methods might be linked to the use of an approximate state transition matrix. The objective of this project is to study the effects of approximating variational equations on orbit estimation methods.

We start with the equation of state:

$$\dot{X} = F(X(t); t) \tag{1}$$

which represents n-nonlinear simultaneous equations. An initial state vector $X(t_0)=X_0$ is associated with (1). The state transition matrix is described by the matrix differential equation

$$\dot{\phi} = F_x(X(t), t)\,\phi \tag{2}$$

where $\phi(t_0)=I$ and $F_X(X,t)$ is a matrix of partial derivations of $F(X,t)$ evaluated along a particular trajectory satisfying equation (1).

The force model, F(X,t), used for this study includes perturbations involving only gravitational harmonics; other perturbations, such as drag, low thrust, etc., have been ignored. Specifically, the force function looks like

$$\dot{X} = f(X(t), t) + \sum_{i=2}^{N} \sum_{K=0}^{i} J_i^K g_i^K(X, t) \tag{3}$$

with $f(X,t)$ being the point mass gravitational force caused by the central body, $\sum_{i=1}^{N}\sum_{k=0}^{i} J_i^k g_i^k(X,t)$ the perturbation due to the nonsphericity of the central body. The transcendental functions $g_i^k(X,t)$ are extremely complex for $i \geq 3$. Based on this force model, equation (2) has the form

$$\dot{\phi} = \left[f_x(X,t) + \sum_{i=2}^{M}\sum_{k=0}^{i} J_i^k g_{i_x}^k(X,t)\right]\phi\,, \quad M \leq N. \tag{4}$$

Due to the complexity of $g_i^k(X,t)$, the terms $g_{i_x}^k(X,t)$ become very cumbersome.+ The question this study addresses can now be stated as: What is the effect on orbit determination methods when M is strictly less than N even though the resulting matrix $F_X(X,t)$ is still to be evaluated along a trajectory satisfying equation (3)? The main objective for setting M<N in (4) is a reduced cost in time and space in programming and evaluating these equations.

Relationship to Orbit Estimation

Orbit estimation is the process of solving for the values of a set of parameters from the observational model which will minimize the difference between a computed and an observed trajectory. The Research Version of the Goddard Trajectory Determination System (R&D GTDS) uses two methods of orbit estimation: A classical weighted least squares estimator (differential correction procedure) and a sequential estimator (Kalman filter).

The observational model is a nonlinear regression function of the state and time:

$$y(t) = G(X,t) + n \tag{5}$$

+ Baker in Astrodynamics: Applications & Advanced Topics devotes Appendix E to "Partial Derivatives of Total Acceleration."

where n denotes random noise. The system is mx1, m being the number of observations. When least squares estimation is considered, the value of X which minimizes the weighted sum of the squares of the observational residuals is sought. The function to be minimized is called the loss function. It has the form:

$$Q(X) = [y - G(X,t)]^T W [y - G(X,t)]. \tag{6}$$

The initial estimate of the state is X_0. Equation (6) will be minimized when $\frac{\partial Q}{\partial X} = 0$. Since $\frac{\partial Q}{\partial X}$ will be nonlinear, Q(X) is first linearized by expanding G(X,t) in a truncated Taylor's series about X_0. The linearized problem is now solved and the nonlinear problem is solved recursively via a Newton-Raphson iterative scheme to give the minimum difference between computed and observed trajectories. This briefly describes the differential correction process where a "batch" of m observations are processed simultaneously. The state transition matrix is utilized in the linearization of G(X,t).

The sequential estimator, or filter, handles the problem from a continuous process point of view. Rather than handling the data in batches as in differential correction, the filter processes new data immediately upon collection to yield an improved estimate of the state.

In this approach, observations from times t_0 and t_k are used to determine an estimate of the state residual from a reference trajectory $X(t_k)$ and a covariance matrix P_k. An observation from time t_{k+1} is added to this set. Values of the estimated state at t_{k+1}, $\hat{X}(t_{k+1})$, and the covariance matrix at t_{k+1}, P_{k+1}, are to be found. The filter used for this study is the Extended Kalman Filter (EKF) as programmed in R&D GTDS. The EKF corrects the reference trajectory to the most recent state estimate, which reduces the nonlinearities of the original system and is desirable in real-time solution. In the EKF, the covariance matrix is propagated via the state transition matrix.

Study Results

This study has attempted to address the question of approximate state transition matrices by initially investigating a parameter, R, formulated by Rice (4). Mr. Rice defines a single parameter to monitor the state transition matrix. He presents a statistical argument to show that the quantity $R=\left\{\sum_{i=1}^{3}\sum_{j=1}^{3}\phi_{ij}^{2}\right\}^{1/2}$, where ϕ_{ij} is an element of the transition matrix ϕ, can be interpreted as a measure of "error growth rate." Rice gives $P(t)=\phi P(0)\phi^T$ as a propagation formula for the covariance matrix, where

$$P(0) = \left(\begin{array}{c|c} \sigma^{2} I & 0 \\ \hline 0 & 0 \end{array}\right)$$

and states that the square root of the trace of P(t) is commonly used as a statistical measure of position errors. Hence,

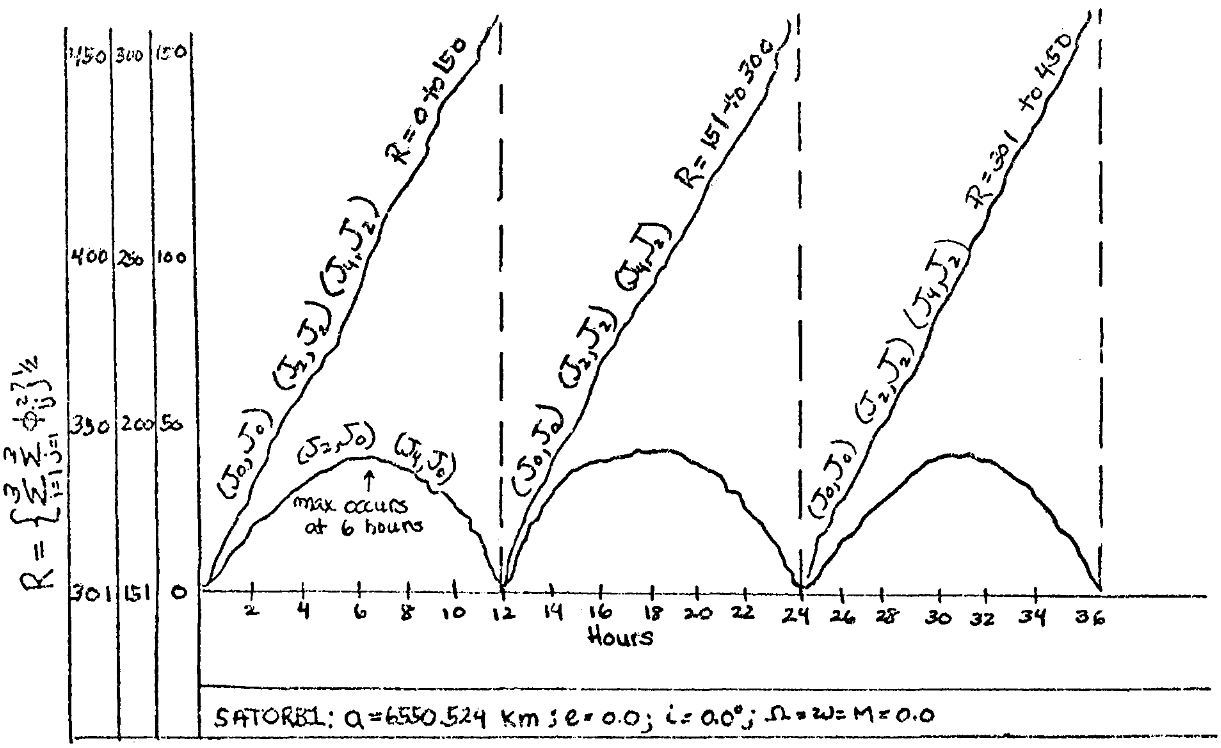
$$(T_r)^{1/2} = \left(\sum_{i=1}^{3} P_{ii}(t)\right)^{1/2} = \sigma\left(T_r(\phi_1\phi_1^T)\right)^{1/2} = \sigma R.$$

The signature of R suggested using the GTDS estimators with several parameters to be varied. These included arc length, geopotential modeling of state and variational equations, inclination and eccentricity. Being observed were the signature of R, the convergence/divergence of the estimator, and the rate of convergence.

As a starting point, three cases discussed in Mr. Rice's paper were compared. Case one used a force model based solely on the point mass force for both the state and state transition matrices, which will be denoted (J_0,J_0). Case two included the "J_2" harmonic term in both the equations of state and the variational equations, denoted (J_2,J_2), while case three included the J_2

harmonic term in the equations of state but only the point mass model in the variational equations, (J_2, J_0). The comparison of these three cases was based on a parameter of the state transition matrix and behavior (in terms of convergence/divergence) of the two estimators in R&D GTDS discussed above.

The first orbit considered was a circular (e=0.0), equatorial ($i=0.0^o$) orbit with semi-major axis a=6550.524 km; this orbit will be referred to as SATORB1. The EPHEMERIS GENERATION (EPHEM) PROGRAM was used in the calculation of the quantity R. EPHEM is used to compute an ephemeris from a given set of initial conditions and, optionally, will compute the elements of the state transition matrix by numerically integrating the variational equations (Eq(4)). Using this option, the quantity R can be printed at any desired interval. The first results obtained printed the value of R every 5 minutes for the above-mentioned orbit with the modeling of cases 1, 2, and 3. Over 48 hours, little difference was observed between the corresponding values of



GRAPH 1

R in case 1 (J_0,J_0) and case 2 (J_2,J_2). However, the (J_2,J_0) case was vastly different. While in cases 1 and 2, R grew almost linearly with time, in case 3, R exhibited an approximately periodic behavior. Repeating the EPHEM runs with the same set of initial conditions but a different force model made the above comparisons even more striking. In this case, a 4x0 geopotential field was used in the equations of state. When generating the partial derivatives, 4x0, 2x0, and 0x0 force models were used. It is worth repeating at this time that the matrix of partial derivatives is always to be evaluated along a trajectory of the full equations of motion. Time histories of R for cases (J_4,J_4) and (J_4,J_2) were very similar to those of cases (J_2,J_2) and (J_0,J_0). Values of R for (J_4,J_0) followed the same oscillating behavior as (J_2,J_0). (See Graph 1.) These results suggest that when using a model for state with the format of eq(3), a simplified force model in the variational equations might be acceptable provided that the "J_2" geopotential term is explicitly included.

To test the effects of truncation on batch estimation, a 5-day simulated observation file was generated on tape via the GTDS DATASIM program. Range and range rate observations were made of SATORB1, where an ephemeris of SATORB1 was created with J_2 included in the state force model. The measurement standard deviations for range and range rate were 15 meters and 2 cm/sec, respectively. With a=6550.625 km, e=.00012, i=.002°, $\Omega=\omega=M=0.0$ as an initial estimate of the state, DC runs were made for cases (J_2,J_2) and (J_2,J_0) over 24 hours. In the (J_2,J_2) case, the DC procedure converged* to the correct solution in 4 iterations. The (J_2,J_0) case diverged.

These runs were repeated over a 6-hour and 12-hour arc. For the (J_2,J_0) case, 6 hours was the time at which the parameter R reached its maximum and

* The criteria for convergence of the DC are based on the iterative reduction of the RMS (square root of the mean square of the observation residuals; $RMS_i = \{\frac{1}{m}(\Delta\bar{y}_i^T w \Delta\bar{y}_i)\}^{1/2}$ where $\Delta\bar{y}_i = \bar{y} - \bar{g}(\hat{X})$ are the observation residuals, and m is the number of observations. When $RMS_{i+1} < RMS_i$, the solution is considered converging.

hence the time at which R started to decrease in value. In other words, for the initial 6 hours, R is monotonically increasing which more accurately reflects the "error growth rate" expected in the state transition matrix. The period of SATORB1 was about 90 minutes, so that a 6-hour arc covers about 4 orbits. When the DC program was used with (J_2, J_0) to correct over the 6-hour arc, it converged in 9 iterations; the 12-hour correction diverged. In other words, a short term (where 4xP, P being the period of the orbit, might be a guideline for short term) correction might be possible with a point-mass force model for the variational equations with a loss of speed in convergence.

It has been demonstrated that using an approximate state transition matrix can be a detriment to the differential correction process. A logical question at this point might be "how much, if any, of an approximation to the variational equations can be tolerated by the DC?" The behavior of the parameter R when (J_4, J_4), (J_4, J_2) and (J_4, J_0) ephemerides are compared hints that a truncation is permissible, provided J_2 is explicitly included in the variational equations. To test this hypothesis, simulated observations were made with SATORB1 elements using a 5x5 geopotential field in the equations of motion. Five DC runs were made with J_0, J_2, J_3^3, J_4^4, and J_5^5 models for the variational equations and the same initial estimate of the state as mentioned above. The DC programs converged in 4 iterations for cases (J_5^5, J_5^5), (J_5^5, J_4^4) and (J_5^5, J_3^3). Convergence was achieved in 5 iterations for the (J_5^5, J_2) case. (J_5^5, J_0) diverged. Table 1 lists RMS values for the last two iterations of these cases as an indication of how little is lost when a truncation from J_5^5 to J_2 for the variational equations is used.

Iteration #	(J_5^5, J_5^5)	(J_5^5, J_4^4)	(J_5^5, J_3^3)	(J_5^5, J_2^0)
3	1.1273619	1.0687897	1.7389176	2.1896022
4	.99626079	.99626079	.99631702	.99647488
5				.99626083

TABLE 1

The above runs were all made with a 24-hour arc. The (J_5^5, J_0) DC case did converge in 9 iterations when used for the short (6 hour) term correction. By looking at a typical term of the matrix of partial derivatives, it becomes clear that the J_2 term dominates the term $\sum_{i=2}^{M} \sum_{k=0}^{i} J_i^k g_{i_x}^k (X,t)$. From Baker, the following term is the term added to the 2-body partial derivative when forming $\frac{\partial g_x}{\partial x}$ for i=2,3 and K=0:

$$\frac{-X\mu}{r^3}\left[\frac{3}{2}\left(\frac{J_2}{r^2}\right)\left(1-5\left(\frac{z}{r}\right)^2\right)+\frac{5}{2}\left(\frac{J_3}{r^3}\right)\left(3-7\left(\frac{z}{r}\right)^2\right)\left(\frac{z}{r}\right)\right]$$

To begin with, the term J_2 is three orders of magnitude larger than J_i, $i \geq 3$. Also, J_i is divided by r^i, rapidly decreasing the relative magnitude of each J_i term for i increasing. In other words, the term J_2 will reflect the vast majority of the perturbation due to the asphericity of the earth. Hence, it is not at all surprising that the terms $\sum_{i=3}^{N} \sum_{k=0}^{i} J_i^k g_i^k$ can be truncated when forming the partial derivatives without jeopardizing convergence in the DC.

Other orbits were used to test the relationship between inclination and behavior of the DC in the (J_2, J_0) case. SATORB2 had initial elements a=6550.524 km, e=.0, i=30°, $\Omega = \omega = M = 0$. A DC program was run for a 24-hour arc with an initial estimate of the state as a=6550.624 km, e=.00012, i=30.002° and $\Omega = \omega = M = 0$. Again, the (J_2, J_2) case converged in four iterations and (J_2, J_0) diverged. Again, using SATORB2 elements with 5x5 geopotential field in the equations of motion, a (J_5^5, J_2) DC run over 24 hours will converge in six iterations. These results support the suggestion that an approximate state transition matrix might be acceptable provided the J_2 potential term is included.

Several more inclinations were tried: i=60°, i=90°, i=98°, i=120°. At this point, different results were achieved. The initial orbital elements used are listed in Table 2. Simulated observations were made for each orbit.

	a	e	i	Ω	ω	M
SATORB3	6550.524	0	60^0	0	0	0
SATORB4	6550.524	0	90^0	0	0	0
SATORB5	6550.524	0	98^0	0	0	0
SATORB6	6550.524	0	120^0	0	0	0

TABLE 2

For the initial estimate of the state in each DC run, the same error was added to the orbital elements: 100 meters added to the semi-major axis, eccentricity was increased to .00012, $.002^0$ added to the inclination and no error added to Ω, ω and M. Rapid convergence occured for all (J_2,J_2) DC runs. With computed observations based on an ephemeris of SATORB3, the (J_2,J_0) DC run showed a definite trend toward convergence. After 12 iterations, the current state was given as a=6550.257, e~0(-6), i=60.00003^0, $(\Omega+\omega+M)=720^0$. DC runs based on simulated observations of SATORB4, SATORB5, and SATORB6 were also converging in the (J_2,J_0) case, though at a slower rate than SATORB3. The results are summarized in Table 3.

Value of State

	No. of Iterations	a	e	i
SATORB4	20	6550.534	$.4578\times10^{-4}$	89.99754
SATORB5	27	6550.525	$.2398\times10^{-5}$	98.00005
SATORB6	24	6550.252	$.6327\times10^{-6}$	120.0

TABLE 3

Table 4 compares the RMS value for various iterations for SATORB3, 4, 5, and 6 (J_2,J_0). This serves as a monitor for the rates of convergence.

RMS Values (J_2,J_0)

Iteration #	SATORB3	SATORB4	SATORB5	SATORB6
1	2084.7695	2261.6206	1766.1576	2307.2376
6	74.458092	1175.0753	742.93831	103.57066
12	57.808147	521.06515	260.84572	112.82074

TABLE 4

As one might expect, since $\cos\frac{\pi}{3} = \left|\cos\frac{2\pi}{3}\right|$ and $\sin\frac{\pi}{3} = \sin\frac{2\pi}{3}$ the RMS values are most similar for $i=60^\circ$ and $i=120^\circ$, (SATORB3 and SATORB6). In these two cases, the first 12 iterations alternated between converging and diverging, with large decreases of RMS value in convergent iterations. (This accounts for $RMS_{12} > RMS_6$ in SATORB6.)

In order to examine the sensitivity to inclination, it is helpful to look at first order perturbations. In Methods of Orbit Determination, Escobal devotes a chapter to "Secular Perturbations," where the term secular describes variations "associated with a steady nonscillatory, continuous drift of an element from the adopted epoch value."* He represents the perturbing potential as $A \equiv \Phi - V$ where Φ is the potential due to an aspherical earth and V is the potential of a spherical earth. He segregates from A those terms which will contribute secular variations in the elements and arrives at

$$\hat{A} = K^2 n \left[\frac{3}{2}\frac{J_2}{a^2}\left(\frac{a}{r}\right)^3\left\{\frac{1}{3} - \frac{1}{2}\sin^2 i\right\}\right] \qquad (7)$$

where $K^2m=n^2a^3$. Note that this is a first order expression in J_2 and for the sake of this analysis, the J_i, $i \geq 3$, terms have been neglected. Little is lost

* Escobal, P.R., 1965, p. 362.

by neglecting J_i, $i \geq 3$ as the J_3 term is approximately 10 orders of magnitude smaller than the J_2 term and the relative magnitude of J_i and J_2 becomes even more drastic for $i > 3$. Expression (7) is then averaged over one revolution, resulting in:

$$\hat{A}^* = K^2 n \left[\frac{3}{2} \frac{J_2}{a^3} (1-e^2)^{-3/2} \left\{ \frac{1}{3} - \frac{1}{2} \sin^2 i \right\} \right] \tag{8}$$

Using this as the perturbing function due to J_2, it is easy to see that the secular effect of J_2 is eliminated in the equations of motion when $i = 54.7^\circ$ (since $1/3 - 1/2 \sin^2 54.7^\circ = 0$). In other words, at this specific inclination, the satellite, in a secular sense, perceives the earth as approximately (i.e., to first order J_2) spherical.

With the aid of the above model for the perturbing function, Escobal develops the following equations representing the gradual drift of the classical elements from their adopted epoch values. Note that only Ω, ω and M experience this drift and a, e and i are taken to be constant. (It might be worthwhile to state again that this is only a first order secular perturbation theory.)

Anomalistic mean motion:

$$\bar{n} = n_0 \left[1 + \frac{3}{2} \frac{J_2 (1-e^2)^{1/2}}{p^2} \left(1 - \frac{3}{2} \sin^2 i\right) \right]$$

Mean Anomaly:

$$M = M_0 + \bar{n} (t - t_0)$$

Longitude of the ascending Node:

$$\Omega = \Omega_0 - \left(\frac{3}{2} \frac{J_2}{p^2} \cos i \right) \bar{n} (t - t_0)$$

Argument of Perigee:

$$\omega = \omega_0 + \left(\frac{3}{2} \frac{J_2}{p^2} \left[2 - \frac{5}{2} \sin^2 i \right] \right) \bar{n} (t - t_0)$$

Between these critical inclinations of 54.7° and 125.3°, the rate of the secular variations of the elements is smaller than outside of this region. This accounts for convergence in the DC procedure with (J_2, J_0) modeling for orbits with inclinations between 54° and 125°.

To conclude the differential correction section of this study, two more orbits were considered. These orbits were elliptical with a greater semi-major axis. Simulated observations were made of these orbits. The initial states are given in Table 5.

	a	e	i	Ω	ω	M
SATORB7	7278.360	.1	0	0	0	0
SATORB8	9357.89143	.3	0	0	0	0

TABLE 5

A larger value for a will decrease the effect of J_2 which is readily seen in equation (8). However, the effect of $\dot{J}_2$ is not absent from these two orbits and the graph of the parameter R with (J_2, J_0) modeling suggests that the DC procedure will have trouble converging over a 24 hour arc, which it indeed does. But the period of SATORB7 and 8 is increased to 103 minutes and 150 minutes, respectively. Because of its period, it is not surprising that SATORB8 converges in 11 iterations over a 12-hour span with the (J_2, J_0) modeling. With SATORB7, P=103 minutes so that 7 hours ($\sim$4xP) should be a reasonable time arc in the (J_2, J_0) DC. When a 6-hour arc is used, the (J_2, J_0) DC converges to SATORB7 elements in 11 iterations with rapidly decreasing RMS values. With a 12-hour arc, convergence was still not achieved after 30 iterations.

The results obtained using the FILTER as an orbit estimator are more difficult to examine than the results from the DC. As input to the FILTER program, the user supplies an initial estimate of the state along with an initial estimate of the covariance matrix.* The a priori covariance matrix contains the state standard deviation and correlations, hence points the filter in the right direction.

For this study, the observational residuals were used to monitor filter performance, with decreasing residuals within a pass and modestly larger residuals appearing after a data gap indicating convergence. The arc length used in the filter portion of this study was 18 hours.

When testing the effect of approximating the state transition matrix in the filter, only the SATORB1 orbit was used. With $i=0.0^{o}$ and $e=0.0$, this orbit is particularly susceptible to perturbations due to the earth's J_2 nonsphericity. As will be demonstrated below, the filter has an added dimension of sensitivity, that being the a priori covariance matrix. Because of this and time constraints, the use of the filter was restricted to this orbit.

The first offset imposed on the state was the same as that used for the DC: $\Delta a=100m, \Delta e=.00012, \Delta i=.002^{o}, \nu=\Omega=M=0.0$. With this offset, the cartesian elements at t_0 are $x=6549.8379$ km, $y=z=0.0$ km, $\dot{x}=0$ km/sec, $\dot{y}=7.801531422$ km/sec and $\dot{z}=-.0002723$ km/sec. At t_0, the true cartesian elements are $x=6550.524$ km, $y=z=0.0$ km, $\dot{x}=\dot{z}=0.0$ km/sec and $\dot{y}=7.8006548$ km/sec. Four different a priori covariance matrices were tried in the filter with this initial state estimate. All four covariance matrices were diagonal, implying there was no correlation among the errors in the state estimate. The first covariance matrix exactly reflected the errors in the state:

* The DC procedure has an a priori covariance matrix default value of infinite magnitude so that its inverse is the null matrix. In the DC procedure it is this inverse which is an additive term to the loss function (eq(6)); however, it has been omitted in eq(6) as it is the null matrix.

$\sigma_x = .68$ km, $\sigma_y = \sigma_z = .1 \times 10^{-4}$ km, $\sigma_{\dot{x}} = .1 \times 10^{-4}$ km/sec, $\sigma_{\dot{y}} = .87 \times 10^{-3}$ km/sec and $\sigma_{\dot{z}} = .27 \times 10^{-3}$ km/sec.

The results obtained for (J_2,J_2) and (J_2,J_0) were very similar, both cases converging. Table 6 below lists, for the last 6 passes, the largest residual in meters within a pass for both cases:

	Pass Number					
	19	20	21	22	23	24
(J_2,J_0) max |O-C|	23	20	29	21	19	31
(J_2,J_2) max |O-C|	31	43	22	27	21	30

TABLE 6

These results change greatly when a different a priori covariance matrix is used. With $\sigma_x = \sigma_y = .1$ km, $\sigma_z = .01$ km and $\sigma_{\dot{x}} = \sigma_{\dot{y}} = \sigma_{\dot{z}} = .1 \times 10^{-4}$ km/sec, the (J_2,J_0) case failed to converge. This covariance matrix fails to recognize the error in the $\dot{y}$ component, $\Delta\dot{y} = .00087$ km/sec, so that the actual error in $\dot{y}$ is 87 times larger than is reflected in the standard deviation associated with it: $\sigma_{\dot{y}} = .1 \times 10^{-4}$ km/sec. The largest residual in the last 6 passes is listed below. Note that the (J_2,J_2) case fares much better than (J_2,J_0) and is considered converging.

	Pass Number					
	19	20	21	22	23	24
(J_2,J_0) max |O-C|	997	1151	1184	1094	1250	982
(J_2,J_2) max |O-C|	35	72	21	58	45	81

TABLE 7

If the standard deviation of $\dot{y}$ is increased to $\sigma_{\dot{y}} = .00004$ so that $\Delta\dot{y}$ is approximately 20 times larger than is reflected in the standard deviation, the (J_2, J_0) case responds a little better. However, it is not considered converging. The residuals during the first 9 hours suggest the filter has a handle on the correct state. The next 9 hours shows increasing residuals as the filter drifts away from the possible steady state. Lastly, the standard deviation of $\dot{y}$ was increased again to $\sigma_{\dot{y}}$=.00015 or 5 times smaller than the error on $\dot{y}$. Here the (J_2, J_0) case did as well as the (J_2, J_2) case. Table 8 summarizes the results for $\sigma_{\dot{y}}$=.0004 and $\sigma_{\dot{y}}$=.00015, with $\sigma_x = \sigma_y = .1$, $\sigma_z = .01$, $\sigma_{\dot{x}} = \sigma_{\dot{z}} = .1 \times 10^{-4}$.

Pass Number

	19	20	21	22	23	24
$\sigma_{\dot{y}}$ =.00004 km/sec (J_2, J_0) max |O-C|	190	226	230	220	242	227
(J_2, J_2) max |O-C|	28	39	22	28	12	41
$\sigma_{\dot{y}}$ =.00015 km/sec (J_2, J_0) max |O-C|	32	23	31	33	29	47
(J_2, J_2) max |O-C|	28	37	23	28	14	38

TABLE 8

Although these results are far from conclusive, some inferences can be drawn from them. It has been demonstrated that for this case the filter responds well with an approximate state transition matrix provided the a priori covariance matrix reflects the state errors within five standard deviations. The test cases used for this study suggest that when the error is between 20 and 90 times larger than the standard deviation, the (J_2, J_0) case fails, yet the (J_2, J_2) case is able to reach a steady state solution. This suggests that when the a priori covariance matrix is considered an accurate indication of the state error, truncated variational equations might not harm the performance of the filter. On the other hand, when the initial covariance matrix somewhat inaccurately reflects the state error, the full partial derivatives are needed to help steer the filter toward steady state. (The term "somewhat" is used here as a precaution; a totally inaccurate a priori covariance matrix can easily cause a (J_2, J_2) filter case to diverge.) This sensitivity to the initial covariance matrix makes it difficult to draw conclusions regarding the filter's performance as a function of the state transition matrix.

Conclusions

This paper has attempted to evaluate the effects of an approximate state transition matrix on the differential correction procedure and the filter procedure as used for orbit estimation. The DC results fall into four categories: the effects due to (1) extent of the approximation, (2) orbital inclination, (3) length of time arc, and (4) orbital eccentricity. Coinciding with these categories is the behavior of the parameter $R = \left\{ \sum_{i=1}^{3} \sum_{j=1}^{3} \phi_{ij}^2 \right\}^{1/2}$. R can be used to "predict" convergence/divergence in the DC and its behavior suggested these four categories as meaningful avenues to investigate. When using a force model for the state which includes the J_n^m harmonic term, it has been shown that it is a safe practice to approximate the variational equations provided that the J_2 term is included. When this approximation is made, the DC process will still converge as it would with the full variational equations with only a negligible loss of speed. A total truncation of the harmonic terms in the variational equations will, in general, cause divergence in the DC. One exception to this is orbits with inclinations between 54.7^{0} and 125.3^{0} In this range of inclination, the effect of the nonsphericity of the earth is minimized. Here (J_2,J_0) DC cases will converge but so slowly that the truncation might, in practice, be undesirable

The oscillatory signature of R when the most drastic truncation is made suggests the possibility of short term differential corrections. The maximum value of R tends to occur after four periods of the orbit. Hence an arc length of four times the orbital period becomes a reasonable guideline for short term corrections. In this arc length, convergence is achieved but speed of convergence again becomes the trade-off for the truncation. As the orbital period lengthens with greater semi-major axis, so does the time arc over which the (J_2,J_0) case converges in the DC procedure.

With the filter as an estimator, less conclusive results are found. The effect of a truncation in the variational equations on filter performance was highly correlated to the initial covariance matrix. When the a priori covariance matrix was a good indication (within 5 standard deviations) of the actual error imposed on the state, little difference was seen in the convergent behavior of the filter for the (J_2,J_2) and (J_2,J_0) cases. However, when the accuracy of the a priori covariance matrix is relaxed, the (J_2,J_0) case showed divergence in the cases tested. Furthermore, when the accuracy of the a priori covariance matrix is completely lost, neither the (J_2,J_0) nor the (J_2,J_2) case will converge.

The results obtained in the filter section of this paper leave another set of questions open. It was assumed that using the low altitude, circular equatorial orbit where the perturbations due to the nonsphericity of the earth are most pronounced would be a good orbit to test the effects of truncated variational equations in the filter. This appears to be a valid assumption in view of the analysis mentioned above based on equations (7) and (8). However, other sets of orbital elements could be tested in order to help determine the limits of accuracy needed in the a priori covariance matrix when using an approximate state transition matrix. Also, this study was restricted to variations in $\sigma_{\dot{\eta}}$. Certainly many other variations could be tested, although this starts to drift away from the original intent of the study. Also, is there a level of state noise which might be used to help compensate for the use of an approximate state transition matrix?

In general, this study could be expanded to nonpotential accelerations such as drag and solar radiation pressure. What, then, would be the effect of including a nonpotential acceleration in the equations of motion but excluding it in the variational equations?

Lastly, the stability properties of the state transition matrix is a question of interest. Does the solution to the variational equations exhibit one type of stability for the (J_2, J_2) modeling which is different from (J_2, J_0) modeling?

Although many questions remain open, it is hoped that this study sheds some light on the appropriateness of state transition matrix approximations.

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