A SEMIANALYTICAL SATELLITE THEORY FOR WEAK TIME-DEPENDENT PERTURBATIONS

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ABSTRACT

Previously, Semianalytical Satellite Theories based upon the Generalized Method of Averaging have been developed for

- perturbations with no explicit dependence on time, and

- perturbations with a strong explicit dependence on time

While the assumption of time independence (TI) is exact only for zonal harmonics and for static atmosphere density models, the assumption has also been applied successfully to develop the averaged equations of motion for lunar-solar perturbations of satellite orbits with periods up to two days (see AIAA preprints 78-1382 and 75-9). However, recent testing of the lunar-solar short periodics produced via the TI assumption for the GPS orbital flight regime (12 hr period) indicates that the relative accuracy of these short-periodics is significantly less than the accuracy of the zonal short-periodic variations.

This paper describes the modifications of the Semianalytical Satellite Theory required to include these 'weak' time - dependent perturbations. The new formulation results in additional terms in the short-periodic variations but does not change the averaged equations of motion. Thus the m-monthly terms are still included in the averaged equations of motion. This contrasts with the usual approach for the strongly time-dependent perturbations in which the m-monthly (or m-daily, if tesseral harmonics are being considered) terms would be eliminated from the averaged equations of motion and included in the short-periodics computation.

Numerical test results for the GPS case obtained with a numerical averaging implementation of the new theory demonstrate the accuracy improvement.

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A SEMIANALYTICAL SATELLITE THEORY

FOR WEAK TIME-DEPENDENT PERTURBATIONS

Outline

- Review of Analytical Results for Time-Independent (TI) Case
- Numerical Results for Low Altitude Case w/TI Theory
- Numerical Results for High Altitude (GPS) Case w/TI Theory
- Analytical Development of Weak Time-Dependent (WTD) Theory
- Numerical Results for High Altitude Case w/WTD Theory (Zonals, Lunar-Solar, and Solar Pressure)
- Numerical Results for High Altitude Case w/WTD Theory (Zonals, Lunar-Solar, Solar Pressure, and 2x2 Tesserals)

SHORT PERIODICS

OSCULATING ELEMENT SPACE		MEAN ELEMENT SPACE
$\dot{a}_i = \varepsilon F_i(\underline{a},\lambda)$	$a_i = \bar{a}_i + \varepsilon n_{i,1}(\bar{\underline{a}},\bar{\lambda})$ ⟵	$\dot{\bar{a}}_i = \varepsilon A_{i,1}(\bar{\underline{a}})$
$\dot{\lambda} = n + \varepsilon F_6(\underline{a},\lambda)$	$\int_0^{2\pi} \varepsilon n_{i,1}(\bar{\underline{a}},\bar{\lambda})d\bar{\lambda} = 0$	$\dot{\bar{\lambda}} = \bar{n} + \varepsilon A_{6,1}(\bar{\underline{a}})$
	$(i = 1,2,3,4,5,6)$	

ASSUME

$$\dot{a}_i = X_{io} + \varepsilon\sum_{\sigma=1}^{\infty} [X_{i\sigma} \cos (\sigma\bar{\lambda}) + Z_{i\sigma} \sin (\sigma\bar{\lambda})]$$

BY USE OF THE GENERALIZED METHOD OF AVERAGING

$$X_{io} = \bar{n}\delta_{i6} + \varepsilon A_{i,1}(\bar{\underline{a}}) = \dot{\bar{a}}_i$$

$$A_{i,1}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda})d\bar{\lambda}$$

$$\bar{n}\left[\frac{\partial n_{i,1}(\bar{\underline{a}},\bar{\lambda})}{\partial\bar{\lambda}}\right] = \sum_{\sigma=1}^{\infty}\left[X_{i\sigma} \cos (\sigma\bar{\lambda}) + Z_{i\sigma} \sin (\sigma\bar{\lambda})\right]$$

DEFINE

$$C_{i\sigma} \equiv \frac{X_{i\sigma}}{\sigma\bar{n}} \qquad D_{i\sigma} = \frac{Z_{i\sigma}}{\sigma\bar{n}}$$

$$\boxed{\begin{aligned} \varepsilon n_{i,1}(\bar{\underline{a}},\bar{\lambda}) &= \sum_{\sigma=1}^{\infty}\left[\varepsilon C_{i\sigma} \sin (\sigma\bar{\lambda}) - \varepsilon D_{i\sigma} \cos (\sigma\bar{\lambda})\right] \\ \varepsilon C_{i\sigma} &= \frac{1}{\sigma\bar{n}\pi}\int_0^{2\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda}) \cos (\sigma\bar{\lambda})d\bar{\lambda} + \left(\frac{3\varepsilon D_{1\sigma}}{2\sigma\bar{a}_1}\right)\delta_{i6} \\ \varepsilon D_{i\sigma} &= \frac{1}{\sigma\bar{n}\pi}\int_0^{2\pi} \varepsilon F_i(\bar{\underline{a}},\bar{\lambda}) \sin (\sigma\bar{\lambda})d\bar{\lambda} - \left(\frac{3\varepsilon C_{1\sigma}}{2\sigma\bar{a}_1}\right)\delta_{i6} \\ i &= 1,2,3,4,5,6 \end{aligned}}$$

- -- SHORT PERIODIC COEFFICIENTS ARE FUNCTIONS OF THE FIVE SLOWLY VARYING MEAN ELEMENTS AND THEREFORE SHOULD ALSO BE SLOWLY VARYING.
- -- COUPLING OF THE FAST VARIABLE SHORT PERIODIC VARIATION WITH THE SEMIMAJOR AXIS SHORT PERIODIC VARIATION.
- -- FOR CONSERVATIVE FORCES, ANALYTICAL EXPRESSIONS ARE POSSIBLE FOR $\varepsilon C_{i\sigma}$ AND $\varepsilon D_{i\sigma}$.

LOW ALTITUDE TEST CASE

- EPOCH CONDITIONS: 1974, Oct. 21, 10 hrs, 24 min.

OSCULATING ELEMENTS

$a = 6644.586$

$e = .01$

$i = 67.98538419°$

$\Omega = 91.99738418°$

$\omega = 200.6741688°$

$M = 164.3173126°$

MEAN ELEMENTS (PCE)

$\bar{a} = 6636.3797$

$\bar{e} = .0106045$

$\bar{i} = 67.97090021°$

$\bar{\Omega} = 91.9949106°$

$\bar{\omega} = 200.21097331°$

$\bar{M} = 164.77124281°$

- S/C

$C_D = 2.0$

Area = $1.86 m^2$

Mass = 677.kg

- ATMOSPHERE

Modified Harris-Priester

$w/\bar{F}_{10.7} = 150$

- FORCE MODELS

COWELL (30 second step)

$J_2, \ldots, J_6$

and drag

SEMIANALYTICAL (1 day step)

First Order: $J_2, \ldots, J_6$ and Drag

Second Order: J_2^2 + J_2-Drag Coupling in the AOG

(IZSAK + Analytical Drag - J_2)

HIGH ALTITUDE TEST CASE

1. ASSUME A SET OF EPOCH MEAN ELEMENTS; THESE ARE 'CONSTANTS' FOR THE SEMI-ANALYTICAL THEORY

2. AT EPOCH, USE THE SHORT-PERIODIC GENERATOR TO PRODUCE OSCULATING ELEMENTS

3. CONVERT THE OSCULATING ELEMENTS TO POSITION AND VELOCITY; THESE ARE THE CONSTANTS FOR THE COWELL THEORY

4. PROPAGATE THE ORBIT USING BOTH THE SEMI-ANALYTICAL THEORY AND COWELL AND COMPARE THE RESULTING POSITION AND VELOCITY HISTORIES

TEST CASE #2 FORCE MODELS

COWELL	SEMI-ANALYTICAL
$J_2, \ldots, J_6$	$J_2, \ldots, J_6$ PLUS J_2^2
LUNAR-SOLAR	LUNAR-SOLAR (TI)
SOLAR RADIATION PRESSURE	SOLAR RADIATION PRESSURE (TI)

SEMI-ANALYTICAL THEORY
FOR WEAKLY TIME-DEPENDENT
PERTURBATIONS

- OSCULATING EQUATIONS

$$\frac{da_i}{dt} = \varepsilon F_i(\vec{a},\lambda,t)$$

$$\frac{d\lambda}{dt} = n + \varepsilon F_6(\vec{a},\lambda,t)$$

- ASSUMED FORMS

$$\frac{d\bar{a}_i}{dt} = \varepsilon A_i(\vec{\bar{a}},t) \qquad\qquad a_i = \bar{a}_i + \varepsilon \eta_i(\vec{\bar{a}},\bar{\lambda},t)$$

$$\frac{d\bar{\lambda}}{dt} = n(\bar{a}_1) + \varepsilon A_6(\vec{\bar{a}},t) \qquad\qquad \lambda = \bar{\lambda} + \varepsilon \eta_6(\vec{\bar{a}},\bar{\lambda},t)$$

- MATCHING EXPRESSIONS FOR da_i/dt AND $d\lambda/dt$ GIVES

$$A_i + \bar{n}\,\frac{\partial\eta_i}{\partial\bar{\lambda}} + \frac{\partial\eta_i}{\partial t} = F_i(\vec{\bar{a}},\bar{\lambda},t), \qquad i = 1,\ldots,5$$

$$A_6 + \bar{n}\,\frac{\partial\eta_6}{\partial\bar{\lambda}} + \frac{\partial\eta_6}{\partial t} = F_6(\vec{\bar{a}},\bar{\lambda},t) - \frac{3\bar{n}}{2\bar{a}}\,\eta_1(\vec{\bar{a}},\bar{\lambda},t)$$

- ASSUME:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\eta_i}{\partial t}\,d\bar{\lambda} = 0, \qquad i = 1,\ldots,6$$

PHYSICALLY, THIS TAKES THE M-MONTHLIES OUT OF THE SHORT PERIODICS

- THEN

$$A_i = \frac{1}{2\pi}\int_0^{2\pi} F_i(\vec{\bar{a}},\bar{\lambda},t)\,d\bar{\lambda}, \quad i = 1,\ldots,6$$

WTD SHORT-PERIODICS

- DEFINE

$$F_i^s(\vec{\bar{a}},\bar{\lambda},t) \equiv F_i(\vec{\bar{a}},\bar{\lambda},t) - A_i$$

- ASSUME

$$F_i^s(\vec{\bar{a}},\bar{\lambda},t) \equiv \sum_{\sigma=1}^{\infty} [X_{i\sigma}(\vec{\bar{a}},t) \cos \sigma\bar{\lambda} + Z_{i\sigma}(\vec{\bar{a}},t) \sin \sigma\bar{\lambda}]$$

$$\eta_i (\vec{\bar{a}},\bar{\lambda},t) \equiv \sum_{\sigma=1}^{\infty} \frac{1}{\sigma\bar{n}} [M_{i\sigma}(\vec{\bar{a}},t) \sin \sigma\bar{\lambda} - N_{i\sigma}(\vec{\bar{a}},t) \cos \sigma\bar{\lambda}]$$

- SUBSTITUTING INTO THE MATCHING EXPRESSIONS GIVES PDE's

$$X_{i\sigma} = M_{i\sigma} - \frac{1}{\sigma\bar{n}} \quad \frac{\partial N_{i\sigma}}{\partial t}$$

$$Z_{i\sigma} = N_{i\sigma} + \frac{1}{\sigma\bar{n}} + \frac{\partial M_{i\sigma}}{\partial t}$$

- ASSUME SOLUTION TO PDE

$$M_{i\sigma} \equiv X_{i\sigma} + \Delta^{(1)}$$

$$N_{i\sigma} \equiv Z_{i\sigma} + \Delta^{(2)}$$

- FIRST ORDER RESULT

$$\eta_i = \sum_{\sigma=1}^{N} \frac{1}{\sigma\bar{n}} \left\{ \left[C_{i,\sigma} + \frac{\partial D_{i,\sigma}}{\partial t} - \left(\frac{3\ \delta_{i,6}}{2\ \bar{a}_1\ \sigma}\right) \frac{\partial\ C_{1,\sigma}}{\partial t} \right] \sin \sigma\bar{\lambda} - \left[D_{i,\sigma} - \frac{\partial C_{i,\sigma}}{\partial t} - \left(\frac{3\ \delta_{1,6}}{2\ \bar{a}_1\ \sigma}\right) \frac{\partial\ D_{1,\sigma}}{\partial t} \right] \cos \sigma\bar{\lambda} \right\}$$

- NOTE: $C_{i,\sigma}$ AND $D_{i,\sigma}$ ARE THE COEFFICIENTS COMPUTED WITH THE TI ASSUMPTION

TEST CASE #2

- FORCE MODEL

COWELL	SEMI-ANALYTICAL
$J_2,\ldots,J_6$	$J_2,\ldots,J_6$ PLUS J_2^2
LUNAR-SOLAR	LUNAR-SOLAR (WTD)
SOLAR RADIATION PRESSURE	SOLAR RADIATION PRESSURE (WTD)

- MEAN ELEMENTS

$a = 26559.5$ km	$\Omega = 0.0°$
$e = .001$	$\omega = 0.0°$
$i = 63.0°$	$M = 0.0°$

- OSCULATING ELEMENTS

$a = 26561.56567$ km	$\Omega = 359.9999657°$
$e = .00104842$	$\omega = 359.8560915°$
$i = 63.001124°$	$M = .1436848842°$

TEST CASE #2 RESULTS

TIME (DAYS)	Δx(m)	Δy(m)	Δz(m)	RSS(m)
2	-.01	.8	1.317	1.54
4	-.22	1.574	2.704	3.14
6	-.63	2.278	4.395	4.99
8	-1.31	2.866	6.274	7.02
10	-2.14	3.601	7.801	8.85
12	-3.70	4.702	9.342	11.09
14	-5.58	5.322	10.510	13.04

TEST CASE #3

- FORCE MODEL

COWELL	SEMI-ANALYTICAL
$J_2,\ldots,J_6$	$J_2,\ldots,J_6$ PLUS J_2^2
LUNAR-SOLAR	LUNAR-SOLAR (WTD)
SOLAR RADIATION PRESSURE	SOLAR RADIATION PRESSURE (WTD)
$(C,S)_{2,1} + (C,S)_{2,2}$	$(C,S)_{2,1} + (C,S)_{2,2}$

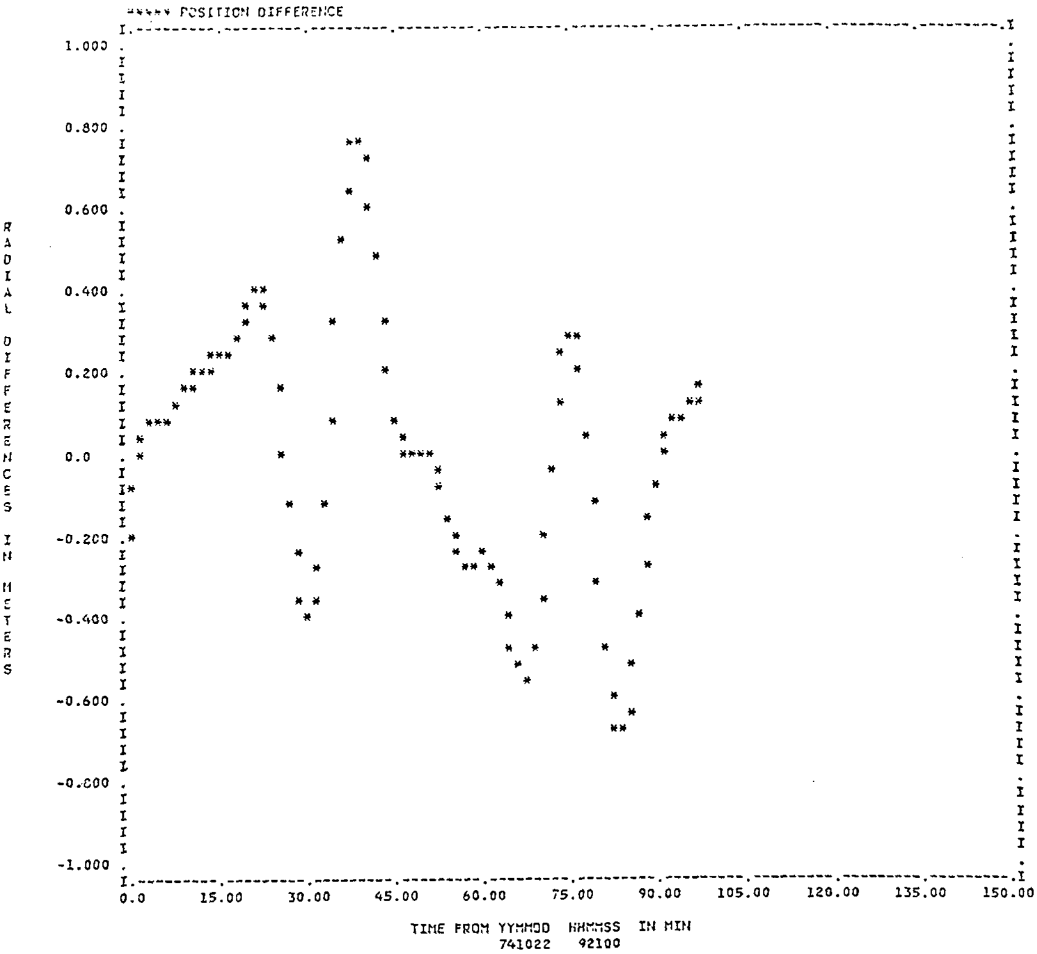
- MEAN ELEMENTS

a = 26559.5 km	Ω = 0,0°
e = .001	ω = 0.0°
i = 63.0°	M = 0.0°

- OSCULATING ELEMENTS

a = 26561.54781 km	Ω = 359.9999706°
e = .00104802	ω = 359.8538535°
i = 63.001118°	M = .1459308175°

Figure 1. Radial Difference after 23 hours from Epoch/Semianalytical minus Cowell for the Low Altitude Circular Test Case



Perturbations

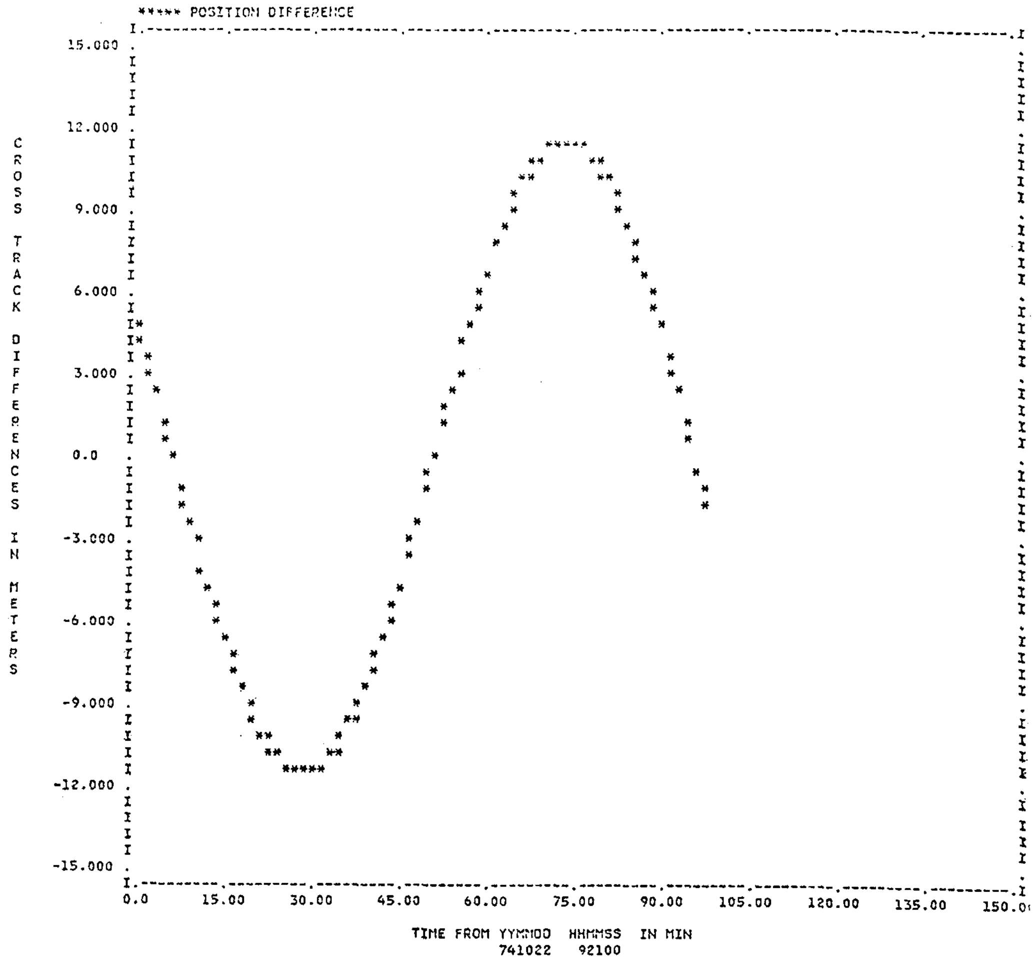
Cowell: 6x0 gravitational field any drag: 30 sec numerical integration time step

AOG: 1st order analytical expressions for the 6x0 gravitational field, Zeis' expressions for J_2^2 effects and option 7 for drag (48 pt quadrature order 1 day numerical integration time step

SPG: 6x0 gravitational field$_2$(7/48) and drag (7/48) to first order, Zeis' expressions for J_2^2 effects.

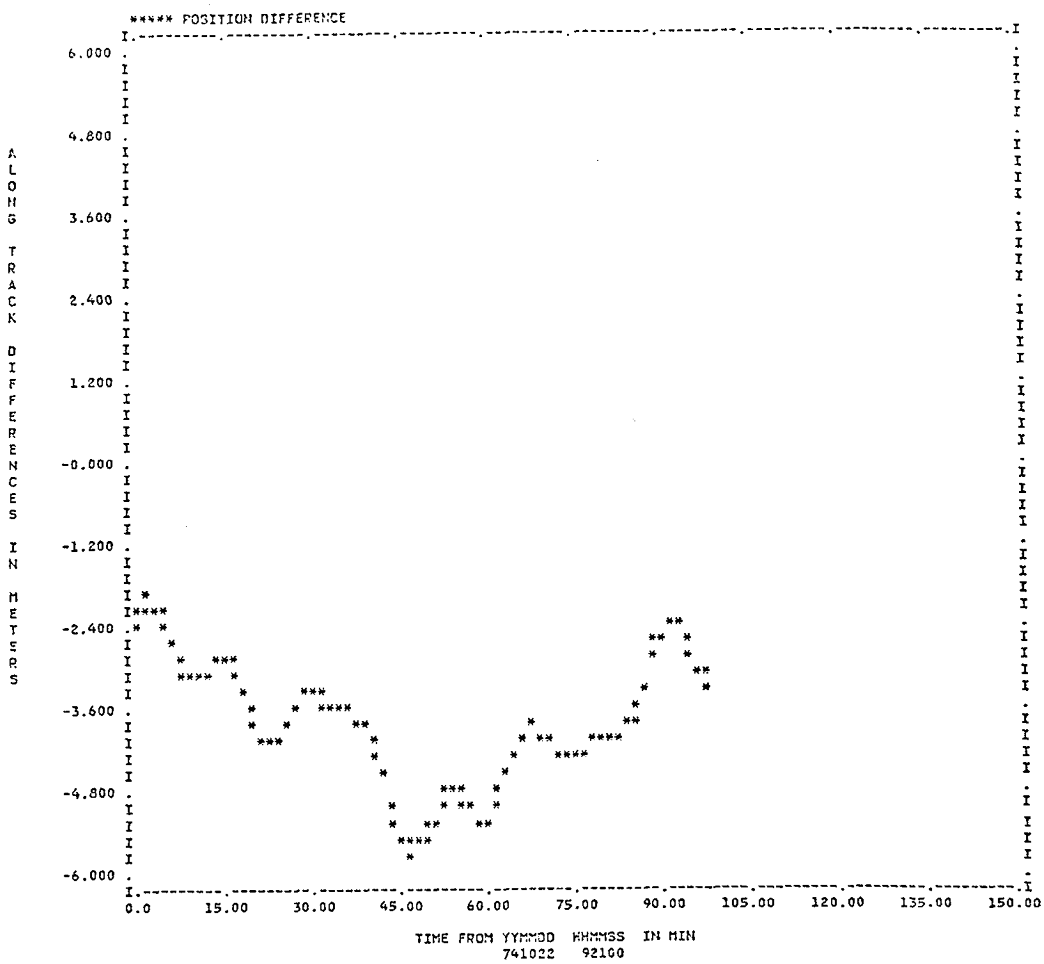
Initial Conditions: PCE

Figure 2. Cross Track Difference after 23 hours from Epoch/Semianalytical minus Cowell for the Low Altitude Circular Test Case



Same initial conditions and perturbations as in Figure 1.

Figure 3. Along Track Difference after 23 hours from Epoch/Seminalytical minus Cowell for the Low Altitude Circular Test Case



Same perturbations and initial conditions as in Figure 1.

Figure 4 Time History of the Zonal Semimajor Axis Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $2\bar{\lambda}$ coefficients, $C_{1,2}$ and $D_{1,2}$

Same initial conditions and perturbations as in drag option 6 of AAS-79-133

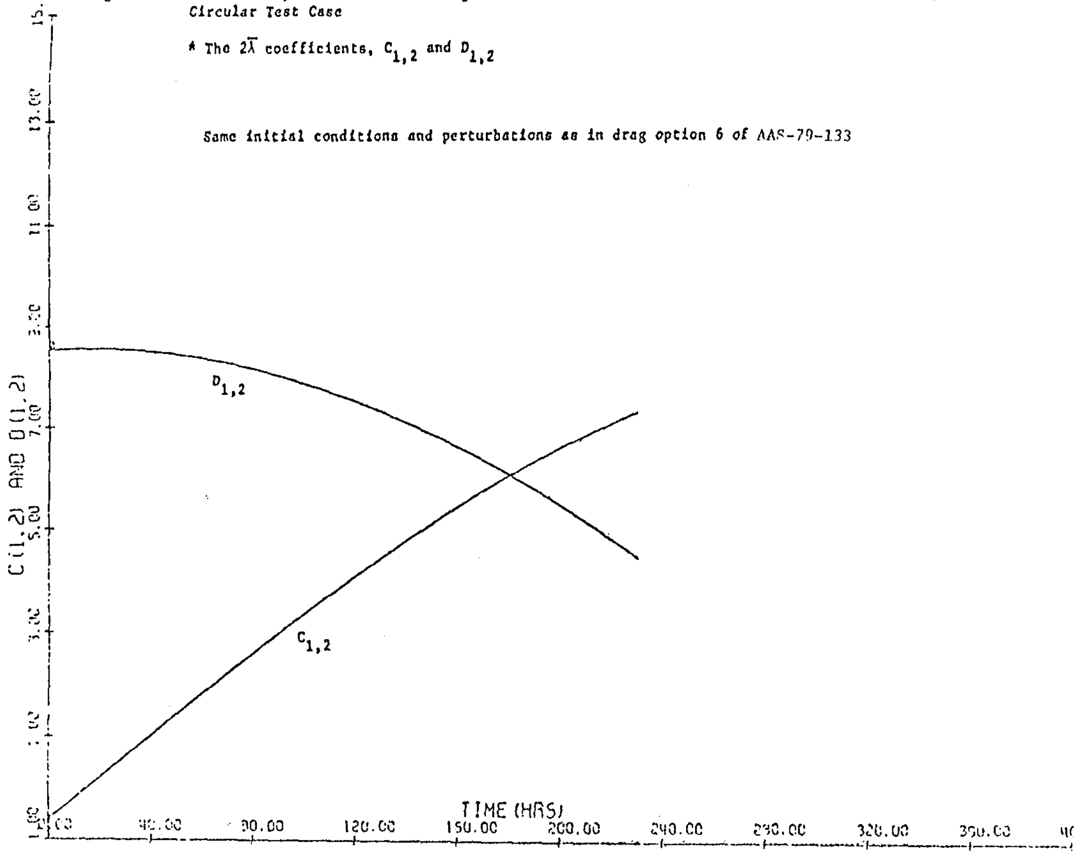


Figure 5 Time History of the Drag Semimajor Axis Short Periodic Coefficients* for the Low Altitude Circular Test Case

* The $1\bar{\lambda}$ coefficients, $C_{1,1}$ and $D_{1,1}$

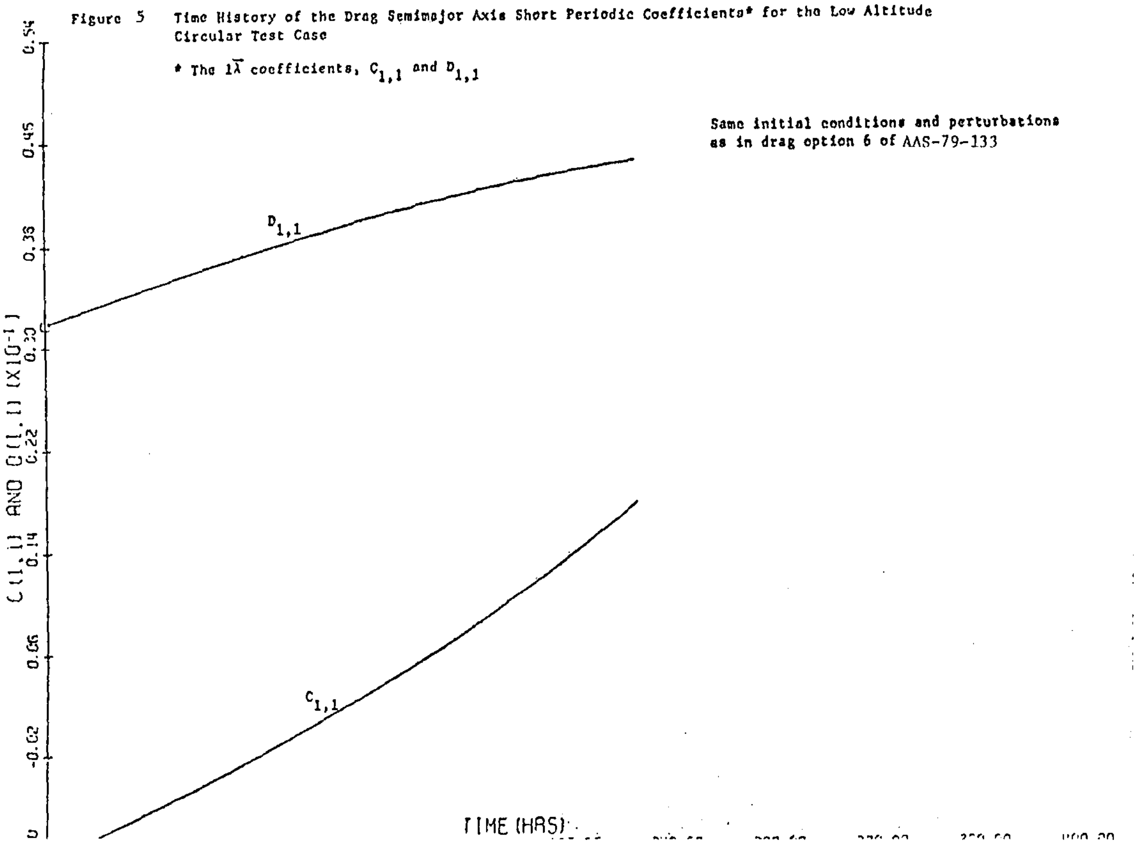
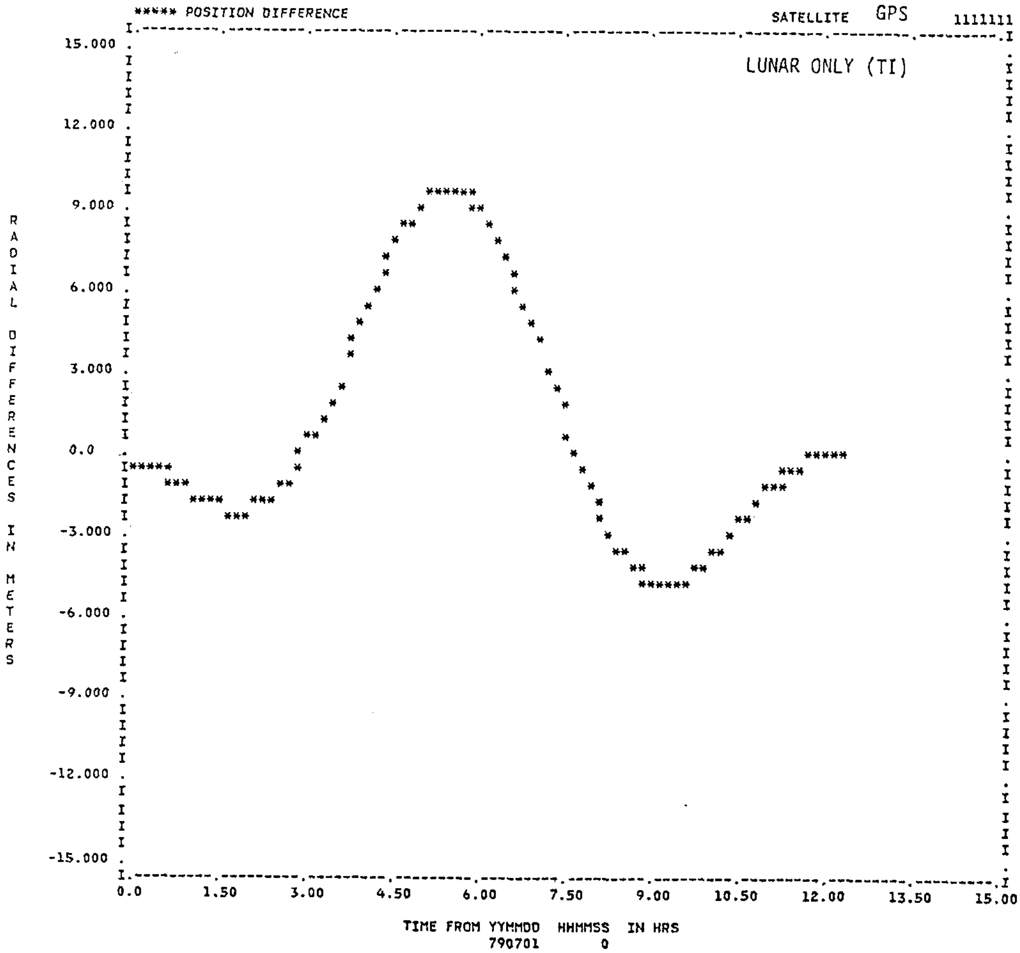


Figure 6. Radial Difference (TI Theory)

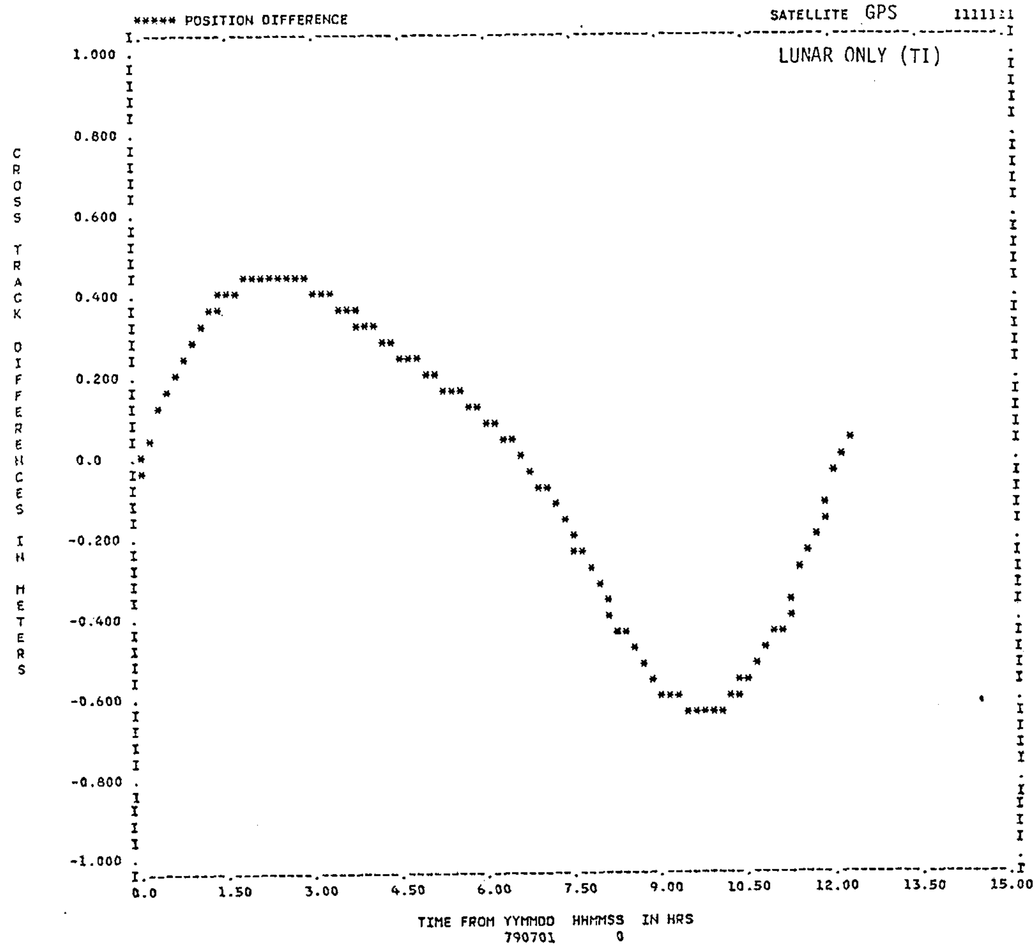


ORB1 FILE ON UNIT 24, DATA RECORDS START AT 790701 0

ORB1 FILE ON UNIT 81, DATA RECORDS START AT 790701 0

USER'S NOTES.....

Figure 7. Cross Track Difference (TI Theory)

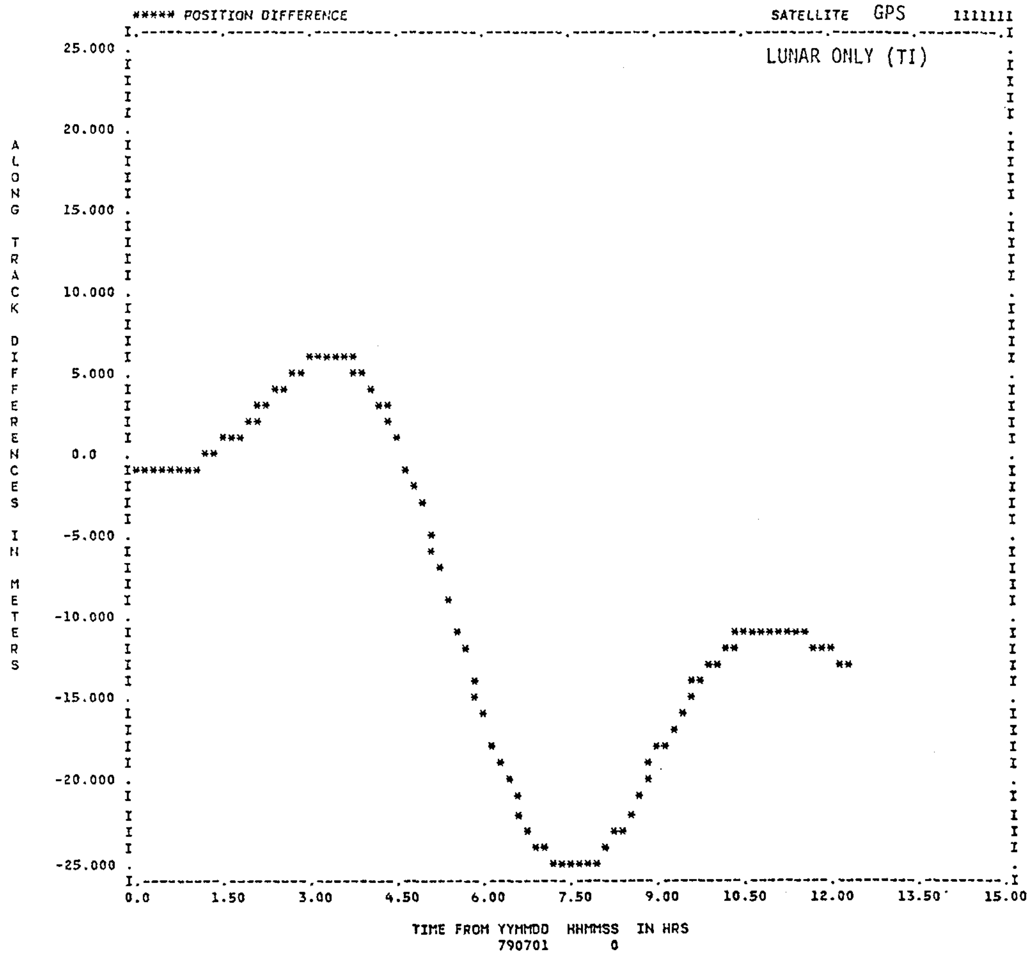


ORB1 FILE ON UNIT 24, DATA RECORDS START AT 790701 0

ORB1 FILE ON UNIT 81, DATA RECORDS START AT 790701 0

USER'S NOTES.....

Figure 8. Along Track Difference (TI Theory)

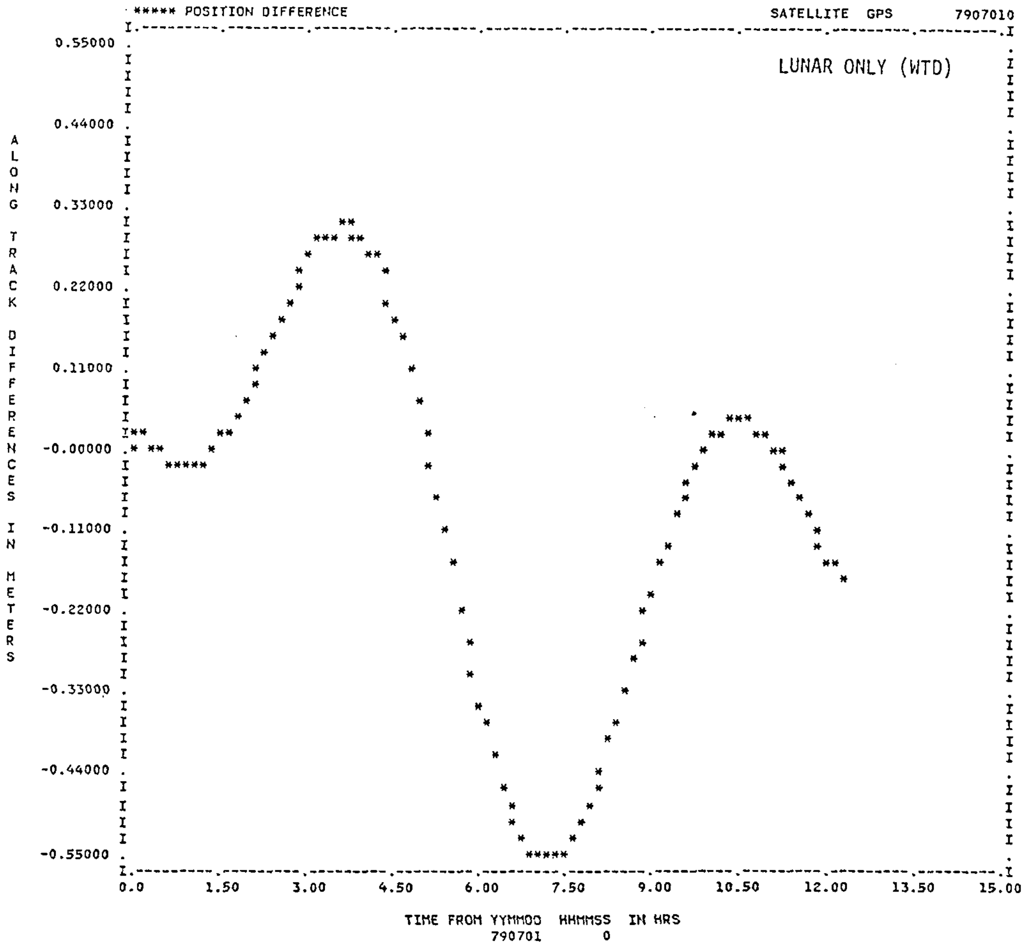


ORB1 FILE ON UNIT 24, DATA RECORDS START AT 790701 0

ORB1 FILE ON UNIT 81, DATA RECORDS START AT 790701 0

USER'S NOTES.....

Figure 9. Along Track Difference (WTD Theory)

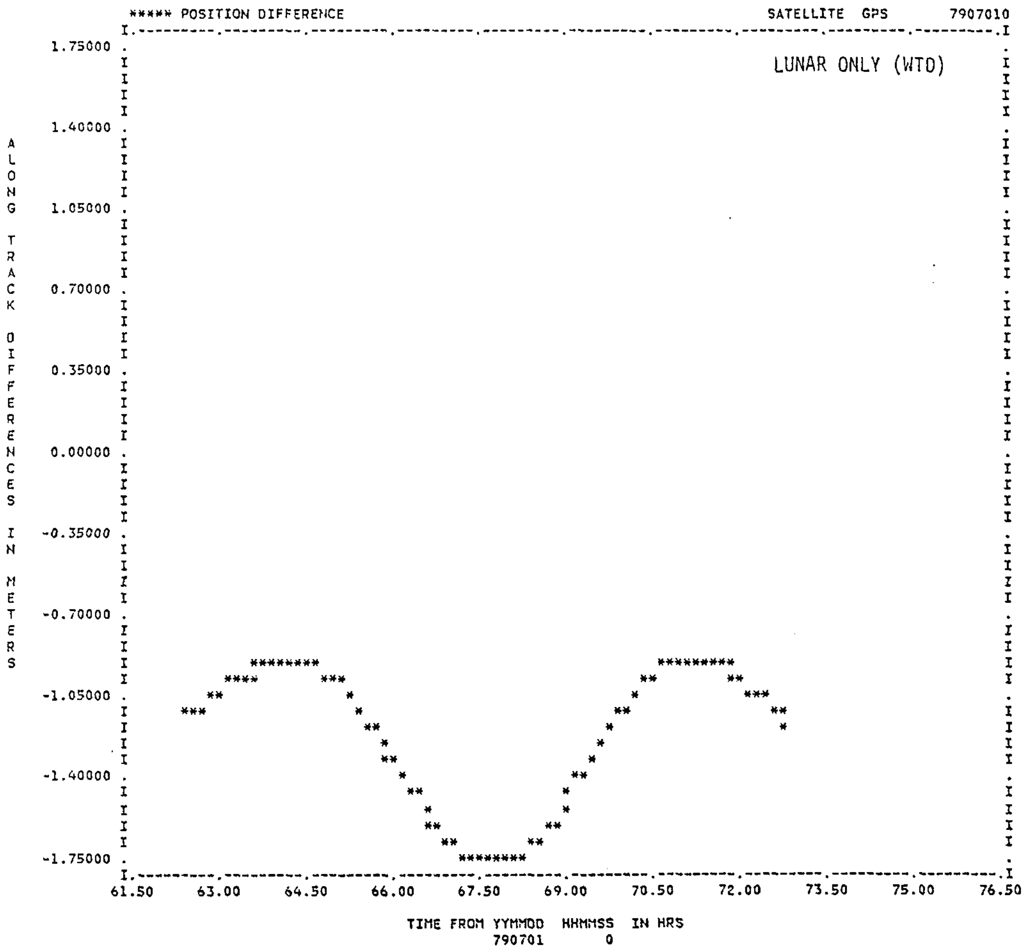


ORB1 FILE ON UNIT 24, DATA RECORDS START AT 790701 0

ORB1 FILE ON UNIT 81, DATA RECORDS START AT 790701 0

USER'S NOTES.....

Figure 10. Along Track Difference (WTD Theory)

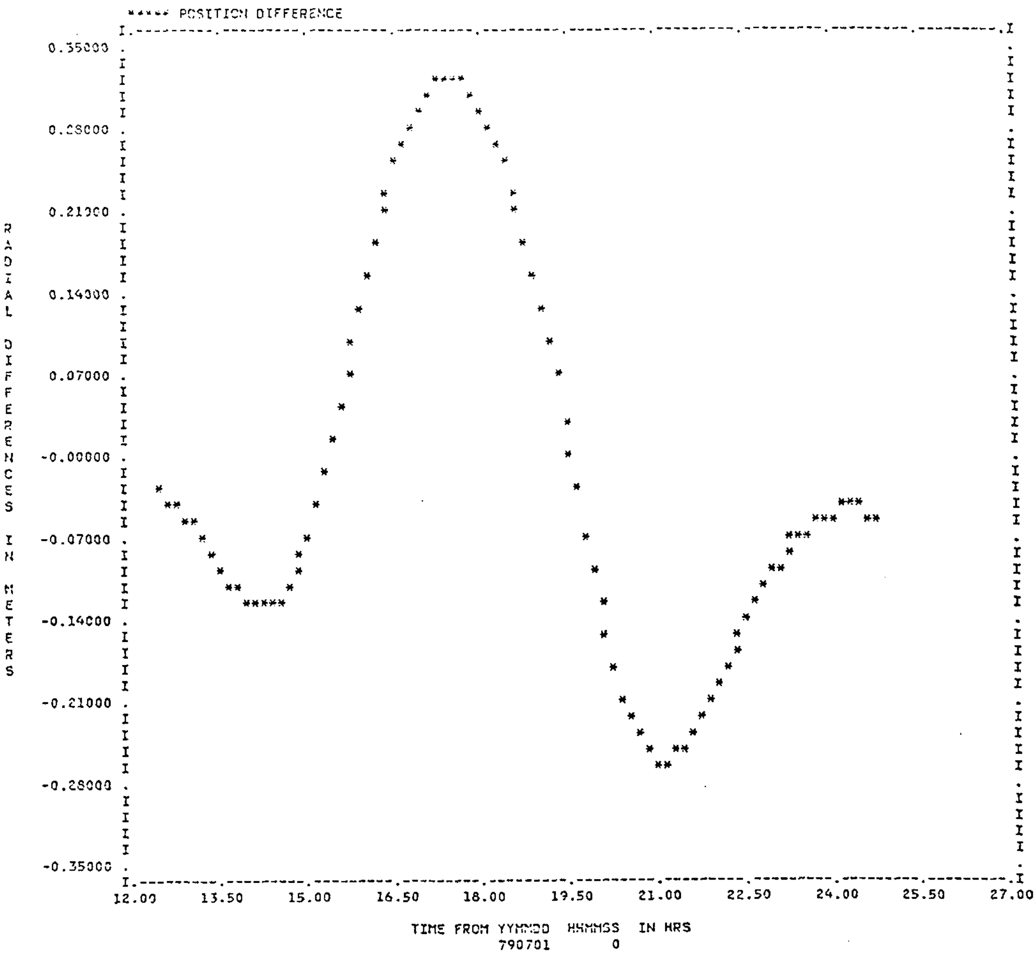


ORB1 FILE ON UNIT 24, DATA RECORDS START AT 790701 0

ORB1 FILE ON UNIT 81, DATA RECORDS START AT 790701 0

USER'S NOTES.....

Figure 11. Radial Difference/Semianalytical minus Cowell for the GPS Test Case



Perturbations

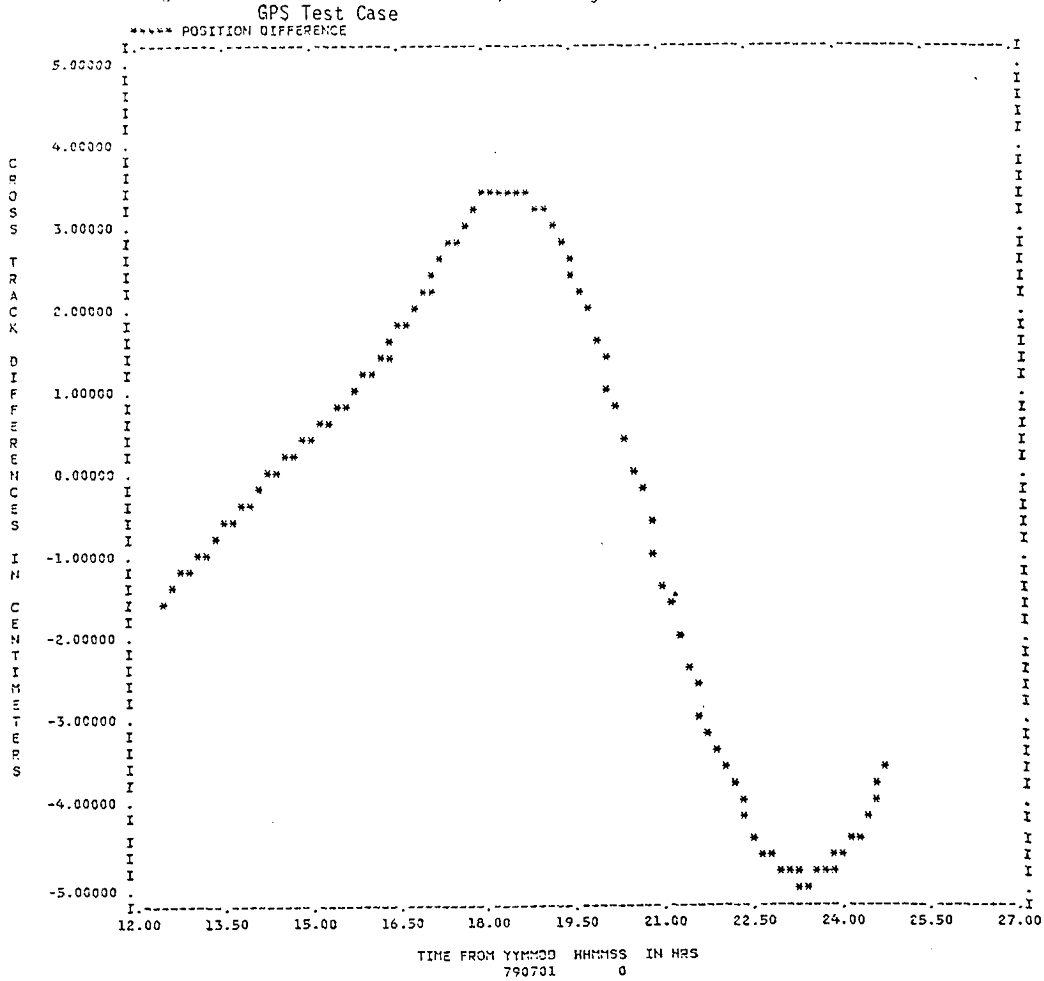
Cowell: 6x0 field, lunar-solar, solar radiation pressure: 300 sec integration time step

AOG: 1st order analytical expressions for 6x0 field and lunar-solar pt mass effects, Zeis's J_2^2 expressions, numerical solar radiation pressure effects (48 pt quadrature order): 1 day integration time step

SPG: 1st order weak time-dependent model for 6x0 field (7/48), Zeis's J_2^2 expressions

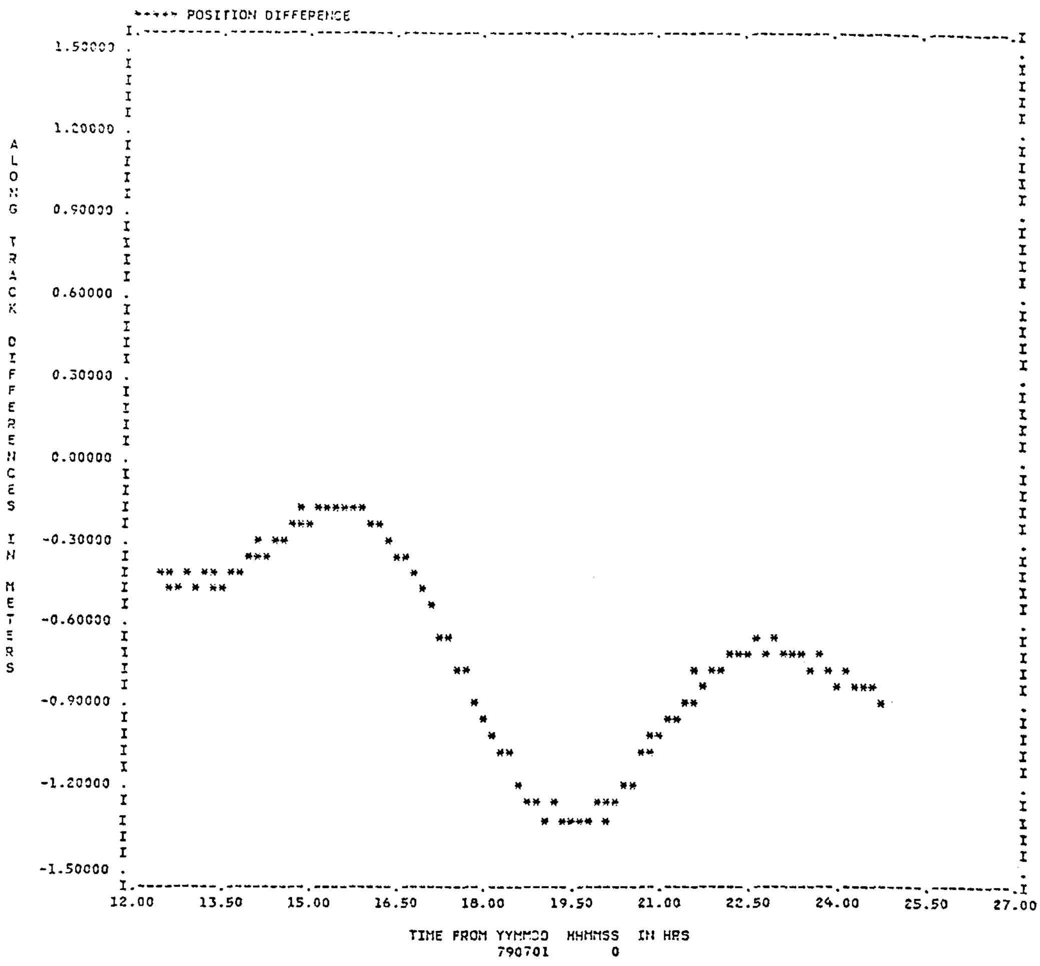
Initial Conditions: $[EPC]^{-1}$

Figure 12. Cross Track Difference/Semianalytical minus Cowell for the GPS Test Case



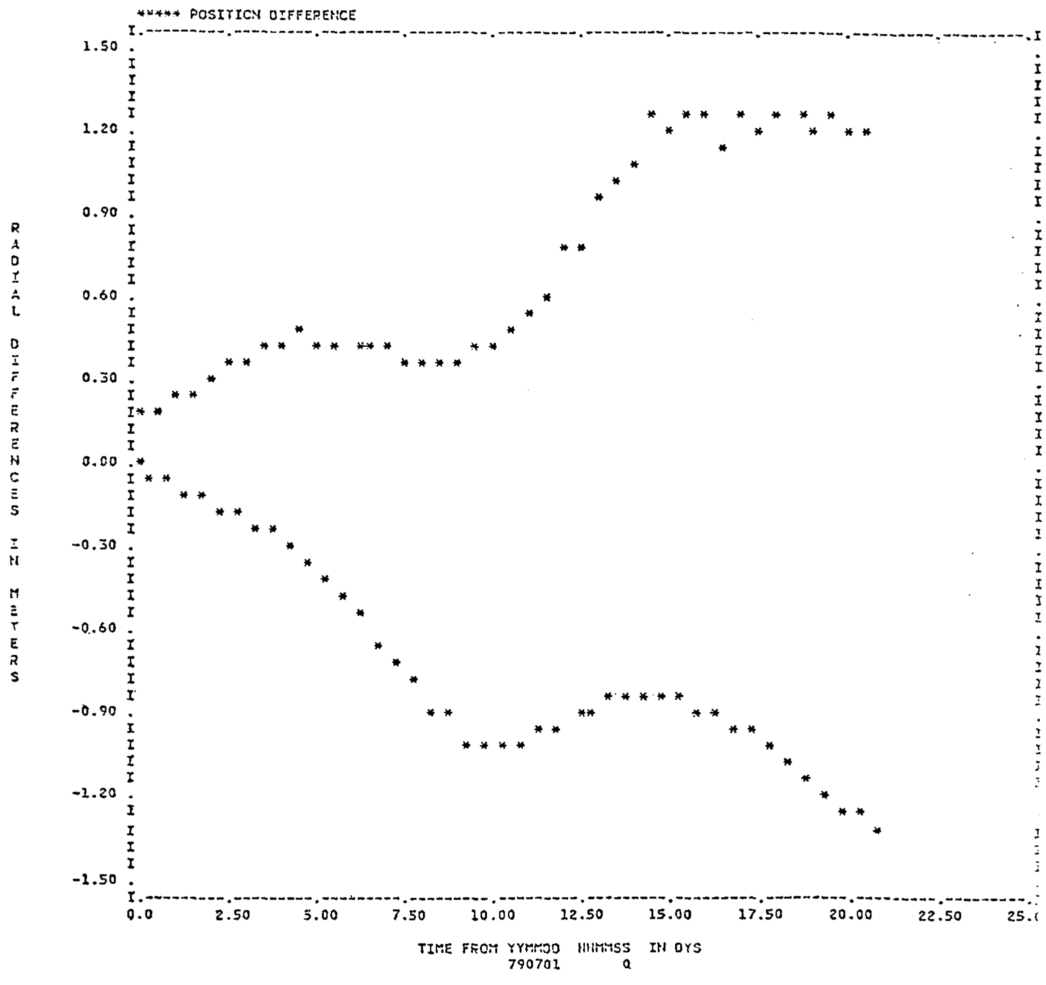
Same initial conditions and perturbations as in Figure 11.

Figure 13. Along Track Difference/Seminalytical minus Cowell for the GPS Test Case



Same initial conditions and perturbations as in Figure 11.

Figure 14. Radial Difference/Semianalytical-minus Cowell for GPS (Test Case #3)



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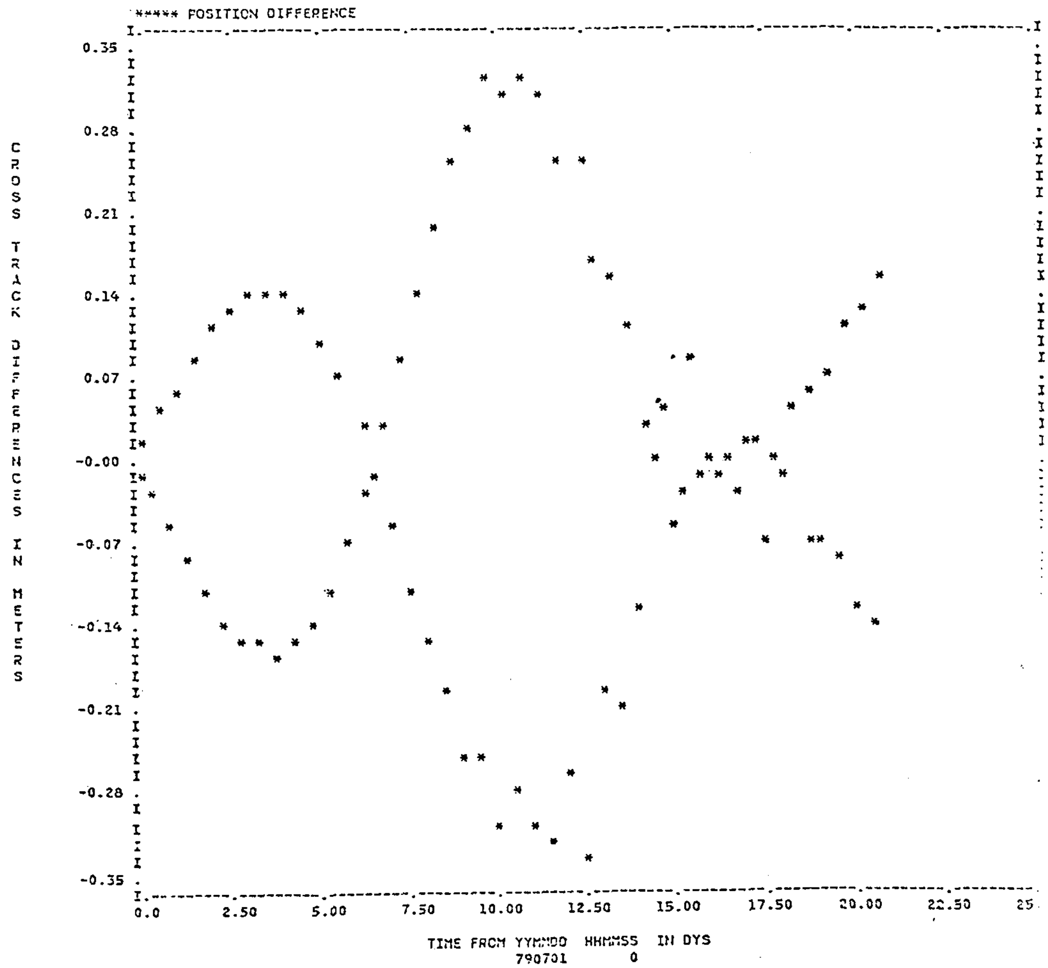
ORB1 FILE ON UNIT 81, DATA RECORDS START AT 790701 0

USER'S NOTES.....

Figure 15. Cross Tnack Difference/Semianalytical minus Cowell for GPS (Test Case #3)



ORB1 FILE ON UNIT 24, DATA RECORDS START AT 790701 0

ORB1 FILE ON UNIT 81, DATA RECORDS START AT 790701 0

Figure 16. Along Track Difference/Semianalytical minus Cowell for GPS (Test Case #3)