THE MEASUREMENT OF EARTH ROTATION ON A DEFORMABLE EARTH

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ABSTRACT

Until recently, the methods of geodetic positioning on the earth were limited to a precision of roughly one part in 10^6. At this level of precision, the earth can be regarded as a rigid body since the largest departure of the earth from rigidity is manifested in the strains of the earth tides which are of the order of one part in 10^7. Long baseline interferometry is expected to routinely provide global positioning to a precision of one part in 10^8 or better. At this level of precision, all parts of the earth's surface must be regarded as being, at least potentially, in continual motion relative to the geocenter as a result of a variety of geophysical effects.

This paper discusses the general implications of this phenomenon for the theory of the earth's rotation and focuses particular attention on the question of the measurement of the "earth's rotation vector" on a deformable earth.

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INTRODUCTION

Almost 10 years ago, a paper entitled "Geodetic Ties Between Continents by Means of Radio Telescopes" [Jones, 1969] appeared in the journal Canadian Surveyor. The author, Harold Jones of the Geodetic Survey of Canada, reported on what was probably the first geodetic application of the technique of long baseline interferometry (LBI). Jones' solution for the 2400 km baseline vector between antennas at Prince Albert, Saskatchewan, and Algonquin Park, Ontario, yielded a "correction" of 30m ± 20m to the value previously determined by ground-based survey methods.

Baseline vectors of comparable magnitude are now routinely determined by LBI to an accuracy of ±10 cm to ±100 cm. This represents an improvement of two orders of magnitude in 10 years and stands as a testimonial to the determination and skill of those who have worked to develop the technique over the past decade. So dramatic has been this development that LBI measurements of the near future, with expected accuracies of ±1 cm to ±5 cm between continents, coupled with other known geophysical effects, such as continental drift, threaten to render obsolete our fundamental definitions of the body-fixed and space-fixed coordinate frames which serve as a basis for defining positions on the earth and their motion relative to inertial space. In discussing this issue, it is useful to present a brief historical description of the evolution of these concepts, for the present situation is best understood in terms of its past history. This past history is in turn often useful in indicating the direction of the next logical step in the development.

The rotation of the earth or the earth's orientation in space is specified by the time dependent transformation between a set of body-fixed basis vectors ($\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$) conceived of as being "attached" to the earth and a set of space-fixed basis vectors ($\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$) conceived of as being "attached" to space. In general, the time dependent transformation can be represented as

$$\hat{e}_i(t) = T_{ij}(t)\,\hat{E}_j \tag{1}$$

where the transformation matrix $T_{ij}(t)$ expresses all that is known about the earth's rotation. (Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac, 1960; Mueller, 1969). Our understanding of $T_{ij}(t)$, which has progressively improved historically, can be usefully classified into six distinct eras.

Ancient Era

Since ancient times, mankind has identified two great circles on the celestial sphere. The first was the equator whose location on the celestial sphere was defined by the diurnal rotation of the earth. The second was the ecliptic whose location on the celestial sphere was defined by the annual motion of the brightest object in the sky – the sun.

According to the ancients, these two great circles were fixed on the sky, providing reference directions and a celestial coordinate frame with which to develop elaborate sky maps populated with mythical characters. The zodiac, which is referenced to the ecliptic and the equator, dates from these early times.

If these early conceptions were to be updated into modern language, they could be described by the introduction of two sets of basis vectors with a common origin at the center of the earth consisting of:

(a) a space-fixed set of basis vectors, $\hat{E}_1, \hat{E}_2, \hat{E}_3$, such that $\hat{E}_1$ is contained by the equator and the ecliptic at the ascending node, $\hat{E}_3$ is oriented toward the pole of the ecliptic, and $\hat{E}_2$ makes up a right-handed orthogonal triad; and

(b) a body-fixed set of basis vectors, $\hat{e}_1, \hat{e}_2, \hat{e}_3$, such that $\hat{e}_3$ is parallel with the earth's rotation axis, $\hat{e}_1$ is contained by the intersection of the equator and the prime meridian, and $\hat{e}_2$ makes up a right-handed orthogonal triad.

With the exception of the sun, the moon, and the five planets observable with the naked eye which were deified for their persistent anomalous behavior, the ancients described all motion by spinning the earth at a uniform rate about an axis fixed in the earth and fixed in space. The mathematical transformation between the space-fixed basis vectors and the body-fixed basis vectors could be represented by the introduction of a time dependent spin matrix $S_{ij}(t)$.

$$\hat{e}_i(t) = S_{ij}(t)\,\hat{E}_j \tag{2}$$

The quantities dS_{ij}/dt were considered to be constants since the earth's rotation was held to be uniform.

The Classical and Medieval Era (500 B.C. - 1600 A.D.)

During this period, the precision with which astronomical observations could be made steadily improved, as did the mathematical models required to make astronomical predictions of eclipses, occultations, and other periodic celestial phenomena. During this era, no truly dynamic theories of celestial motions were introduced, and the algorithms on which calculations depended were, for the most part, highly geometrical. The principal advance in the knowledge of the earth's rotation which occurred in this era was the discovery by Hipparchus, circa 150 B.C., of the precession of the equinoxes.

Following Hipparchus' discovery, previously held conceptions of the space-fixed basis vectors $\hat{E}_i$ were retained; however, it was necessary to redefine the space-fixed basis vectors as being the $\hat{E}_i$ referenced to some celestial pole and equinox of a particular epoch. The body-fixed basis vectors $\hat{e}_i$ retained their previous definitions.

In this era, the practice of realizing the body-fixed basis vectors by assigning numerical coordinates to observatories fixed on the earth was developed. A similar practice of realizing the space-fixed basis vectors by assigning numerical coordinates to stars at a particular epoch led to the development of extensive star catalogues.

With the discovery of precession, the mathematical transformation between the space-fixed basis vectors and the body-fixed basis vectors required the introduction of the time dependent precession matrix $P_{ij}(t)$.

$$\hat{e}_i(t) = P_{ij}(t) S_{jk}(t)\, \hat{E}_k \tag{3}$$

The quantities dS_{ij}/dt were still held to be constants.

The Newtonian Era (1600 A.D. - 1800 A.D.)

The Newtonian era was characterized by the rapid development of Newton's dynamical theory and its extensive application to geodynamics and problems of the earth's rotation. Around the year 1680, Newton offered an explanation of the precession of the equinoxes and predicted the oblate figure for the earth on the basis of his theory of gravitation. At about the same time, Newton and Huyghens published a predicted value for the polar flattening of the earth.

These important geophysical developments were being matched by corresponding dramatic astronomical discoveries. In 1718, the astronomer Edmund Halley announced the discovery of stellar proper motion, and the researches of the astronomer James Bradley, circa 1728, led to his discovery of both stellar aberration and astronomical nutation. This was followed in 1735 by the geodetic surveys in Finland and Peru under Pierre Bouguer of the French Academy which verified Newton's prediction of polar flattening and confirmed the theory of precession and nutation.

The discoveries of the Newtonian era placed the definition of the space-fixed coordinate frame on a firm dynamical footing. As a result of the discovery of the periodic phenomenon of nutation, it became necessary to reference the space-fixed basis vectors $\hat{E}_i$ to the mean equator and ecliptic of a particular epoch, averaging out the small periodic effects of nutation on the location of the equator.

In addition, the discoveries of the Newtonian era made it necessary to realize the space-fixed basis vectors $\hat{E}_i$ by assigning coordinates, not to the stars themselves which were no longer regarded as being fixed, but to locations on the celestial sphere known as "mean stellar positions of epoch" which were related to the actual stellar positions by:

(a) the removal of the effects of proper motion since the epoch, and

(b) the averaging out of the periodic effect of annual aberration. The relatively constant contribution to the aberration arising from the effects of the eccentricity of the earth's orbit were retained in the stellar mean position.

The body-fixed basis vectors retained their classical and medieval definition and continued to be realized by assigning coordinates to observatories fixed on the earth. The meridian through Greenwich was widely adopted as the prime geographic meridian at this time.

With the discovery of nutation, it became necessary to introduce a third time dependent matrix, the nutation matrix $N_{ij}(t)$, into the mathematical transformation between the space-fixed basis vectors $\hat{E}_i$ and the body-fixed basis vectors $\hat{e}_i$.

$$\hat{e}_i(t) = P_{ij}(t)\, N_{jk}(t)\, S_{k\ell}(t)\, \hat{E}_\ell . \tag{4}$$

The quantities dS_{ij}/dt were still held to be constant.

The Euler/Kelvin Era (1800 A.D. - 1900 A.D.)

The Euler/Kelvin era was characterized by two major developments in the theory of the earth's rotation. The first was inspired by the work of Leonard Euler who in 1765 published his famous treatise on the dynamics of rigidly rotating bodies. Euler showed that for an oblate spheroid such as the earth, the rotation axis, the angular momentum axis, and the figure axis will not coincide in general and furthermore, that in the event that they do not coincide the rotation axis cannot remain fixed within the body.

As a consequence of Euler's work, a search was begun circa 1840 for evidence of the effects of Eulerian motion on the earth's rotation. This search persisted for 50 years before achieving success, and at one time or another involved the active participation of Peters, Bessel, Maxwell, Kelvin, Newcomb, Künster, and Chandler (Munk and MacDonald, 1960).

The second major development of this era was inspired by the work of Kelvin (W. Thompson, 1863) who in 1863 was the first to introduce into theoretical geodynamics considerations of the deformability of the earth. Kelvin's seminal work on tidal deformations of the earth stimulated the growth of an entire branch of geodynamical research concerned with the general geodynamical effects of earth deformations which included the work of Darwin, Lamb, Love, and others and which is being vigorously pursued to this day.

These two developments were wed in a remarkable manner when, in 1891, the work of Künster and Chandler finally revealed the elusive Eulerian motion of the earth with a period 40 percent larger than that predicted by Euler's rigid body theory. This gross discrepancy between the theoretical and observed period of the wobble was reconciled the following year by Newcomb, who showed how the yielding of the deformable earth to the centrifugal forces of the wobble reduced its effective dynamical ellipticity and lengthened the wobble period to the observed value.

The Euler/Kelvin era ended in 1900 with the establishment of the five observatories of the International Latitude Service (ILS).

The developments of the Euler/Kelvin era produced no essential modifications of the definition of the space-fixed coordinate frame spanned by the basis vectors $\hat{E}_i$. The equator of rotation and the equator of figure were now regarded as distinct and moving relative to each other. However, their relative motion was such that their mean positions respectively coincided with the equator of angular momentum, and since the basis vectors $\hat{E}_i$ were already defined in terms of the mean equator and ecliptic of epoch, the space-fixed basis vectors $\hat{E}_i$ retained to their previous Newtonian definitions and realizations.

The discoveries of the Euler/Kelvin era profoundly altered the definitions of the body-fixed coordinate frame spanned by the basis vectors $\hat{e}_i$. In the first place, since the rotation axis could no longer be regarded as a body fixed axis, it could no longer serve as a reference axis for the $\hat{e}_3$ basis vector. Consequently, $\hat{e}_3$ was redefined as being parallel to the earth's figure axis. However, as a consequence of the demonstration of the earth's tidal deformability, the figure axis was not fixed within the earth. It was necessary to finally define the $\hat{e}_3$ body-fixed basis vector as lying in the direction of the mean figure axis of the earth which is related to the instantaneous figure axis by averaging out the effects of deformations due to tides and polar motion. This averaging was actually carried out by ILS observatories over the years 1900 to 1905.

The $\hat{e}_1$ body-fixed basis vector was then defined as lying in the planes of the mean equator of figure and the meridian through Greenwich which has been adopted internationally as the prime meridian. The body-fixed basis vector $\hat{e}_2$ was defined to complete a right-handed orthogonal triad.

The body-fixed basis vectors $\hat{e}_1$ were now realized in practice by assigning coordinates, not to the observatories themselves which were no longer regarded as being fixed, but to their mean locations on the earth which were related to their actual locations by:

(a) averaging out latitude and longitude variations due to polar motion, and

(b) averaging out tidal displacements.

With the discovery of Eulerian motion, it became necessary to introduce a fourth time dependent matrix, the wobble matrix $W_{ij}(t)$, into the mathematical transformation between space-fixed basis vectors $\hat{E}_i$ and body-fixed basis vectors $\hat{e}_i$.

$$\hat{e}_i(t) = P_{ij}(t)\, N_{jk}(t)\, S_{k\ell}(t)\, W_{\ell m}(t)\, \hat{E}_m \tag{5}$$

In this stage of the development of the theory of the earth's motion, the instantaneous rotation axis is taken as an observable reference axis and the orientation of the earth (spanned by $\hat{e}_i$) in space (spanned by $\hat{E}_i$) is expressed by (see figure 1):

(a) First, orienting the reference axis in space, accomplished by the matrices $P_{ij}(t)$, $N_{ij}(t)$; and

(b) Second, orienting the earth relative to the reference axis, accomplished by the matrices $S_{ij}(t)$, $W_{ij}(t)$.

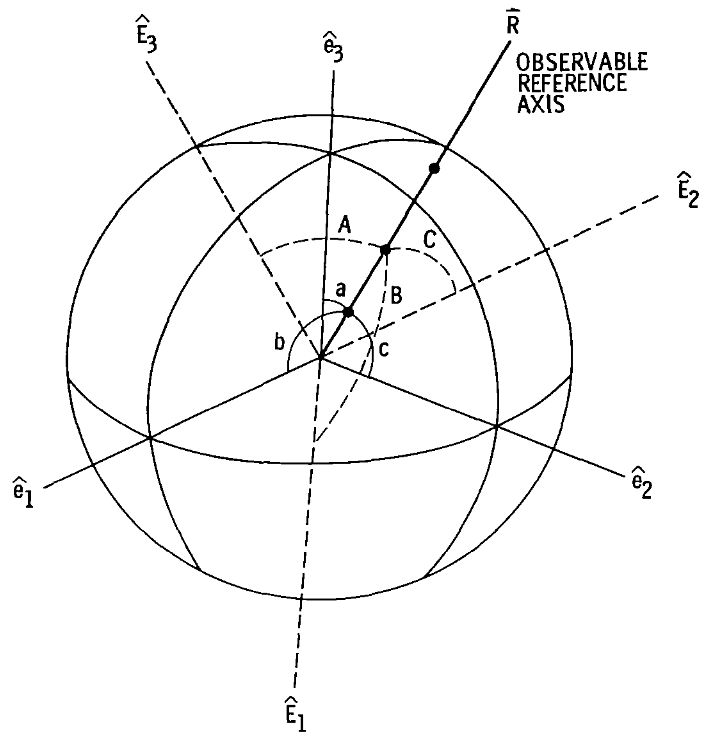

Figure 1.

The matrix $S_{ij}(t)$, which describes the orientation in azimuth of the earth about the reference instantaneous rotation axis, no longer has the property that the quantities dS_{ij}/dt are constants. This is merely a consequence of the kinematical choice of reference axis and the fact that the chosen reference axis is itself moving within the body of the earth as is explained by Woolard (1953, pp.27, 163-165). However, while they are no longer constants, the kinematical quantities dS_{ij}/dt are precisely known from the dynamics, and consequently, the elements of the matrix $S_{ij}(t)$ are precisely predictable.

The matrix $W_{ij}(t)$ which describes the location in the body-fixed frame of the instantaneous axis of rotation can be decomposed into two components:

(a) a predictable contribution $W^{\circ}_{ij}(t)$ due to the effects of the gravitational torques of the sun and moon and often referred to as the "dynamical variations of latitude," and

(b) an unpredictable part $W^{G}_{ij}(t)$ due to a variety of internal geophysical effects. The unpredictable part of $W_{ij}(t)$ is determined observationally and published in arrears.

Early Twentieth Century (1900 A.D. - 1960 A.D.)

The era extending from roughly 1900 A.D. to 1960 A.D. was characterized by a rapid and unprecedented expansion of geophysical research and international cooperation. Many important advances in instrumentation and observation techniques occurred which had bearing on the measurement of the earth's rotation. Some of these include the development of the zenith telescope (1912), the PZT (1915), the quartz crystal clock (1950), the dual rate moon camera (1954), and the atomic clock (1958).

Technological development resulted in a rapid sequence of startling geophysical discoveries in all branches of geophysics, oceans, atmosphere, and "solid" earth including the earth-moon system and "near space" as well as the two principal fields associated with the earth – the gravitational field and the electromagnetic field. The principal impact of this era on the theory of the earth's rotation came from the discovery by Stoyko in 1936 that the earth's rotation described by the matrix $S_{ij}(t)$ contained a variable component which was not predicted by the existing theory. This effect discovered by Stoyko was largely due to the seasonally varying atmospheric winds.

Throughout this era beginning with the work of the ILS and its successors, the IPMS and the BIH, the body-fixed basis vectors $\hat{e}_i$ were referenced:

(a) $\hat{e}_3$ to the mean figure axis of the earth as established by an average over the interval 1900 to 1905, and

(b) $\hat{e}_1$ to the meridian of the mean observatory and the plane of the mean figure axis of 1900 to 1905.

The "mean observatory" is a fictitious observatory occupying the meridian to which the time scales UT1 and UT2, published in the forms of UT1-UTC and UT2-UTC, are referred. In general, the meridian of the mean observatory does not contain the transit circle of the observatory at Greenwich but usually passes to the east or west of Greenwich by a few milliseconds of time depending on the particular mix of observatories and their respective weightings which have contributed to the data base from which the time scales UT1 and UT2 have been deduced. The concept of the "mean observatory" has been introduced to allow for variations in the least squares adjustment procedures by which UT1 and UT2 are obtained (Mueller, 1969).

During this era, the body-fixed basis vectors $\hat{e}_i$ were realized by assigning coordinates to mean observatory positions which have been obtained from the actual observatory positions by:

(a) averaging out latitude and longitude variations due to polar motion,

(b) averaging out tidal displacements, and

(c) allowing for the effects of the least squares adjustment procedure.

Throughout this era, the space-fixed basis vectors $\hat{E}_i$ retained their previous Euler/Kelvin era definitions and realization.

The mathematical transformation from the space-fixed basis vectors $\hat{E}_i$ to the body-fixed basis vectors $\hat{e}_i$ is still expressed as it was in the Euler/Kelvin era,

$$\hat{e}_i(t) = P_{ij}(t)\, N_{jk}(t)\, S_{k\ell}(t)\, W_{\ell m}(t)\, \hat{E}_m \tag{6}$$

except that now both $S_{ij}(t)$ and $W_{ij}(t)$ consist of a combination of predictable and unpredictable contributions. In the case of $S_{ij}(t)$, the predictable contribution constitutes the majority of it with the unpredictable portion appearing as a small perturbation. In the case of $W_{ij}(t)$, the unpredictable contribution constitutes the majority of it with the predictable portion appearing as a small perturbation.

The Present Day Era (1960 -)

The year 1960 is identified here as the epoch marking the beginning of the present-day era of the study of earth rotations, for in the few years surrounding 1960, two scientific discoveries occurred which at the time seemed entirely unrelated to each other and to the question of earth rotation but may ultimately prove to be of fundamental importance in advancing our future understanding of earth rotation.

First was the geophysical discovery of what has come to be known as "global plate tectonics." This "discovery" actually spanned about a decade of time and, like most discoveries which involve a large number of participants, was the product of, and to some extent anticipated by, earlier work. Nevertheless, this decade of discovery spanned the years 1955 to 1965 and can be regarded as being made up of two somewhat distinct intervals:

(a) 1955 to 1960 during which the palaeo-magnetic evidence of large scale polar wander also confirmed the hypothesis of continental drift, and

(b) 1960 to 1965 during which the conception of "sea floor spreading" with rigid tectonic plates in relative motion at their boundaries emerged and was confirmed by the studies of the polarity reversals of the geomagnetic field and the magnetic striping on the ocean floor at the mid-ocean ridges.

These discoveries destroyed the validity of the concept of the "mean location" of an observatory and along with it destroyed the rigorous validity of the operational procedure by which the body-fixed basis vectors $\hat{e}_i$ are realized. According to the theory of plate tectonics, the observatories are moving secularly relative to each other and relative to the bulk of the earth in quasi-random directions and at rates which vary between 1 and 10 cm/year. Under these circumstances, the concept of an observatory's mean position has no inherent meaning, for it is a function of the averaging interval and epoch chosen.

The discovery of plate tectonics injected a certain lack of rigor into the observational procedures which had developed historically for measuring earth rotation and created a vague discomfort in the minds of geophysicists who worried about such matters. However, it had little impact on the day to day operations of earth rotation measurement, because the effects being manifested by continental drift and plate tectonics were below the detectability threshold of the instruments of the instruments of the 1960's. This situation changed dramatically with the development of the long baseline interferometer.

The second event was the astrophysical discovery in 1960 of quasi-stellar objects often referred to as QSO's or quasars. By 1963, the quasars were known from their red shifts to be among the most distant objects in the universe. The long baseline interferometer was developed in 1967 specifically to study the radio emission from these objects.

It was the long baseline radio interferometer which linked the separate discoveries of sea floor spreading and quasars to the question of the earth's rotation. This was due to the fact that, in the operation of a long baseline interferometer, there was for the first time the potential to make measurements of the earth's rotation so accurately that the effects of the secular deformation rates implied by the theory of global plate tectonics could be detected. Confronted with this fact and the continued development of the long baseline interferometer as a device for making regular earth rotation measurements, it is rapidly becoming necessary to develop rigorous operational procedures and definitions which explicitly incorporate into the measurement of each rotation the secular deformability and continuing fracture of the earth's crustal structure.

Strain in Deformable Bodies: The Separation of Rotations and Deformations

We begin by considering two mass elements of a continuous deformable body which before deformation occupy points P_1 and P_2 given by position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively and separated by a vector $d\vec{x}$. The geometry of this arrangement is illustrated in figure 2. The body is then subjected to a deformation which carries the mass elements at P_1 and P_2 through displacements $\vec{u}_1$ and $\vec{u}_2$ to points Q_1 and Q_2 respectively. If the deformation field $\vec{u}(\vec{r})$ is a continuous and differentiable function of position, which is equivalent to asserting that no fracture has occurred, then for points P_1 and P_2 separated by an infinitesimal vector $d\vec{x}$, we have

$$\vec{u}_2 = \vec{u}_1 + \nabla\vec{u} \cdot d\vec{x}. \tag{7}$$

The relative displacement vector $\Delta\vec{u} = \vec{u}_2 - \vec{u}_1$ is given by

$$\Delta\vec{u} = \nabla\vec{u} \cdot d\vec{x} \tag{8}$$

where

$$\nabla\vec{u} = \frac{\partial u_i}{\partial x_j}. \tag{9}$$

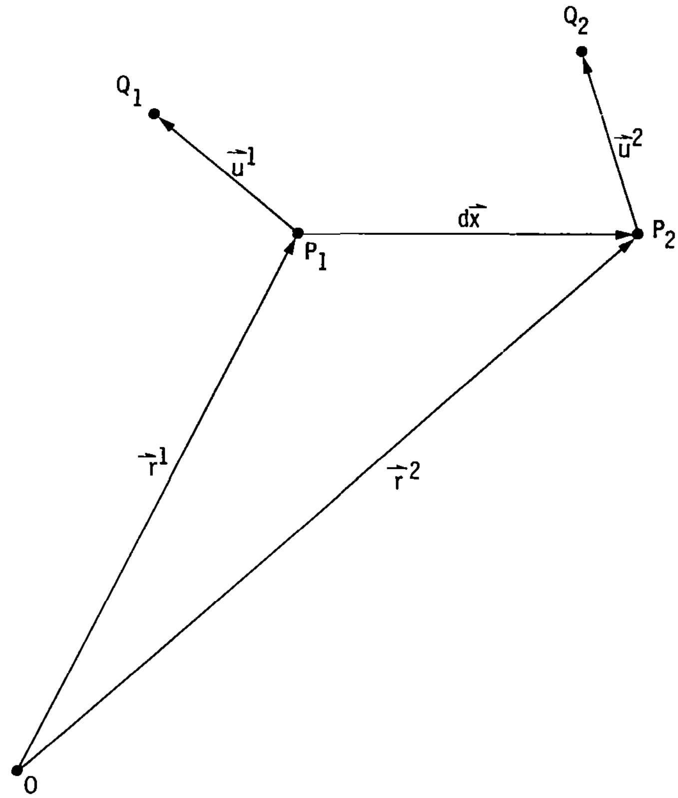

Figure 2.

The tensor $\Delta\vec{u}$ can be split up into its symmetric and antisymmetric parts

$$\nabla\vec{u} = \frac{1}{2} e_{ij} + \frac{1}{2} \Omega_{ij} \tag{10}$$

where

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{11}$$

is symmetric and where

$$\Omega_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \tag{12}$$

is antisymmetric, which allows the components of relative displacement Δu_i to be written

$$\Delta u_i = \frac{1}{2} e_{ij}\, dx_j + \frac{1}{2} \Omega_{ij}\, dx_j. \tag{13}$$

Like all antisymmetric tensors, Ω_{ij} possesses a dual tensor Ω_i given by

$$\Omega_i = \epsilon_{ijk}\, \Omega_{jk} \tag{14}$$

where ϵ_{ijk} is the alternating tensor defined by

$$\epsilon_{ijk} = \begin{cases} +1 & \text{ijk} \quad \text{cyclic } 123 \\ 0 & \text{ijk} \quad \text{not all distinct} \\ -1 & \text{ijk} \quad \text{noncyclic } 123. \end{cases}$$

It can easily be shown from the property of ϵ_{ijk} that

$$\Omega_{ij} = \epsilon_{kij}\, \Omega_k, \tag{15}$$

and, as a result, the components of relative displacement can be written

$$\Delta u_i = \frac{1}{2}\, e_{ij}\, dx_j + \frac{1}{2}\, \epsilon_{kij}\, \Omega_k\, dx_j. \tag{16}$$

By introducing the axial vector $\vec{\Omega}$ given by

$$\vec{\Omega} = \Omega_i\, e_i = \epsilon_{ijk}\, \Omega_{jk}\, e_i, \tag{17}$$

we see that equation (16) can be written in coordinate free notation as

$$\Delta\vec{u} = \frac{1}{2}\, \tilde{e} \cdot d\vec{x} - \frac{1}{2}\, \vec{\Omega} \times d\vec{x}. \tag{18}$$

We see from this result that, in general, the relative displacement of two mass elements of the medium separated by an infinitesimal vector $d\vec{x}$ can always be decomposed into the sum of a relative displacement $\Delta\vec{u}_D$,

$$\Delta\vec{u}_D = \frac{1}{2}\, \tilde{e} \cdot d\vec{x}, \tag{19}$$

arising solely from the effects of deformation (should any be present) and a relative displacement $\Delta\vec{u}_R$,

$$\Delta\vec{u}_R = -\frac{1}{2}\, \vec{\Omega} \times d\vec{x}, \tag{20}$$

arising solely from the effects of rigid body rotation (should any be present). In this way, the effects of deformational and rotational contributions to the relative displacement field are separable.

The strain tensor $\tilde{e}$ whose elements are e_{ij} has the property that if ds^2 is the squared distance between the mass elements at the points P_1 and P_2 before deformation and ds'^2 is the squared distance between the same two mass elements after deformation, then

$$\delta(ds^2) = ds'^2 - ds^2 = e_{ij}\, dx_i\, dx_j. \tag{21}$$

The rotation tensor $\tilde{\Omega}$ whose elements are Ω_{ij} has the property that the axial vector $\vec{\Omega}$ obtained from its dual gives the amount of rigid body rotation suffered by the separation vector $d\vec{x}$ as a result of the displacements.

An alternate and, in this instance, quite useful approach to the problem [Brillouin 1964, pp. 287 ff.] is to adopt the point of view that the mass elements of the medium are assigned unchanging coordinates whose numerical values are preserved under the material transformations which accompany the displacements. This requires the coordinate surfaces to be associated with the material medium and to be carried along and deformed with it as it moves. These deformed coordinate systems generally will be curvilinear.

In the usual treatment of this problem, the mass element at position $\vec{r}$ and coordinates x_i, related by

$$\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3, \tag{22}$$

gets displaced to a position $\vec{r}\,' = \vec{r} + \vec{u}(\vec{r})$ with coordinates X_i such that

$$X_i = x_i + u_i \tag{23}$$

and

$$\vec{r}\,' = X_1\hat{e}_1 + X_2\hat{e}_2 + X_3\hat{e}_3 \tag{24}$$

or equivalently

$$\vec{r}\,' = (x_1 + u_1)\hat{e}_1 + (x_2 + u_2)\hat{e}_2 + (x_3 + u_3)\hat{e}_3. \tag{25}$$

In the alternate approach, the mass element at position $\vec{r}$ and coordinates x_i gets displaced to a position $\vec{r}\,' = \vec{r} + \vec{u}(\vec{r})$ with coordinates $\overline{X}_i$, such that

$$\overline{X}_i = x_i \tag{26}$$

and

$$\vec{r}\,' = \overline{X}_1\hat{e}_1' + \overline{X}_2\hat{e}_2' + \overline{X}_3\hat{e}_3' \tag{27}$$

or equivalently

$$\vec{r}\,' = x_1\hat{e}_1' + x_2\hat{e}_2' + x_3\hat{e}_3' \tag{28}$$

where the basis vectors $\hat{e}'_1, \hat{e}'_2, \hat{e}'_3$ locally spanning the $\bar{X}$ coordinate frame are necessarily changed from the basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ locally spanning the X coordinate frame in order to accommodate the fact that $\vec{r} \neq \vec{r}.'$

Without loss of generality, we can begin by assuming that the basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ span a local Cartesian coordinate frame. The relationship between the coordinates X_i referred to the basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$, and the coordinates $\bar{X}_i$ referred to the basis vectors $\hat{e}'_1, \hat{e}'_2, \hat{e}'_3$ can be established by considering two mass elements at position $\vec{r}_1$ and $\vec{r}_2$ separated by the infinitesimal vector $d\vec{x}$, where

$$d\vec{x} = \vec{r}_2 - \vec{r}_1 . \tag{29}$$

As a result of the displacements and the motion of the mass elements, the vector $d\vec{x}$ is transformed into the vector $d\vec{X}$:

$$d\vec{X} = \vec{r}'_2 - \vec{r}'_1 = \vec{r}_2 + \vec{u}(\vec{r}_2) - \vec{r}_1 - \vec{u}(\vec{r}_1) \tag{30}$$

or

$$d\vec{X} = d\vec{x} + d\vec{u} \tag{31}$$

where

$$d\vec{u} = \vec{u}(\vec{r}_2) - \vec{u}(\vec{r}_1). \tag{32}$$

Now, in component form, equation (31) gives

$$dX_i = dx_i + du_i, \tag{33}$$

and the chain rule of differentiation gives

$$\frac{dX_i}{dx_k} = \frac{dx_i}{dx_k} + \frac{\partial u_i}{\partial x_j}\frac{dx_j}{dx_k} . \tag{34}$$

Recognizing that $dx_i/dx_j \equiv \delta_{ij}$ where δ_{ij} is the Kronecker delta, and that by the definition of equation (26) $d\bar{X}_k = dx_k$, we obtain the relationship that

$$dX_i = \left[\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right] d\bar{X}_k . \tag{35}$$

The use of equation (10) in equation (35) gives

$$dX_i = \left[\delta_{ik} + \frac{1}{2}\, e_{ik} + \frac{1}{2}\, \Omega_{ik}\right] d\bar{X}_k \tag{36}$$

which states that the X and $\overline{X}$ coordinate systems are locally transformed, one into the other, by the combination of (a) a rigid body rotation Ω_{ik} and (b) a deformation e_{ik}.

If we proceed to compute the squared distance ds^2 between the two mass elements after the displacements on both the X coordinate frame and the $\overline{X}$ coordinate frame, these two expressions must be equal since ds^2 is a scalar quantity whose value is independent of the choice of coordinates. In the X coordinate system, which has been assumed without loss of generality to be a local Cartesian coordinate frame with metric tensor $g_{ij} = \delta_{ij}$, then

$$ds_X^2 = \delta_{ij}\, dX_i\, dX_j. \tag{37}$$

The equivalent expression in the $\overline{X}$ coordinate frame is obtained from equation (36) as

$$ds_{\overline{X}}^2 = \delta_{ij}\left[\delta_{ik} + \frac{1}{2}\, e_{ik} + \frac{1}{2}\, \Omega_{ik}\right]\left[\delta_{j\ell} + \frac{1}{2}\, e_{j\ell} + \frac{1}{2}\, \Omega_{j\ell}\right] d\overline{X}_k\, d\overline{X}_\ell \tag{38}$$

which, using the antisymmetric property of Ω_{ij} reduces finally to

$$ds_{\overline{X}}^2 = (\delta_{ij} + e_{ij})\, d\overline{X}_i\, d\overline{X}_j \tag{39}$$

or

$$ds_{\overline{X}}^2 = \overline{g}_{ij}\, d\overline{X}_i\, d\overline{X}_j. \tag{40}$$

A comparison of equations (37) and (39) shows that a consequence of the theoretical development on which the displaced mass elements are allowed to retain their coordinate values is that the strain tensor e_{ij} is simply absorbed into the geometric tensor $\overline{g}_{ij}$ of the deformed $\overline{X}$ coordinate system. Whereas the basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are all orthogonal and of unit length, the basis vectors e_1', e_2', e_3' are nonorthogonal and no longer of unit length. This is illustrated in figure 3.

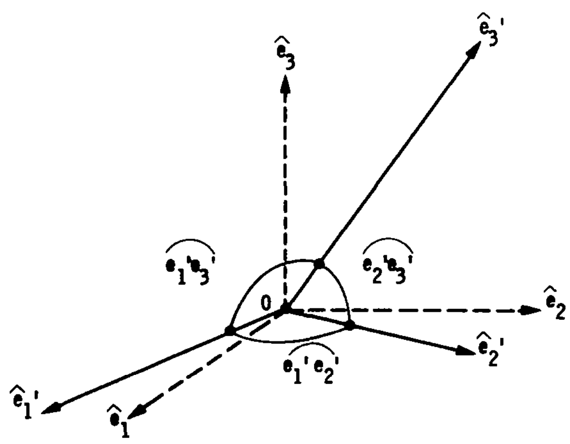

Figure 3.

The metric tensor $\bar{g}_{ij}$ of the $\bar{X}$ coordinate frame is given by definition as

$$\bar{g}_{ij} = \begin{bmatrix} |\hat{e}'_1|^2 & |\hat{e}'_1|\,|\hat{e}'_2| \cos \widehat{e'_1 e'_2} & |\hat{e}'_1|\,|\hat{e}'_3| \cos \widehat{e'_1 e'_3} \\ |\hat{e}'_2|\,|\hat{e}'_1| \cos \widehat{e'_2 e'_1} & |\hat{e}'_2|^2 & |\hat{e}'_2|\,|\hat{e}'_3| \cos \widehat{e'_2 e'_3} \\ |\hat{e}'_3|\,|\hat{e}'_1| \cos \widehat{e'_3 e'_1} & |\hat{e}'_3|\,|\hat{e}'_2| \cos \widehat{e'_3 e'_2} & |\hat{e}'_3|^2 \end{bmatrix} \tag{41}$$

So, we see from equations (39), (40), and (41) that

$$\begin{aligned} |\hat{e}'_1| &= \sqrt{1 + e_{11}} \\ |\hat{e}'_2| &= \sqrt{1 + e_{22}} \\ |\hat{e}'_3| &= \sqrt{1 + e_{33}} \end{aligned} \tag{42}$$

and that

$$\begin{aligned} \cos \widehat{e'_1 e'_2} &= \frac{e_{12}}{\sqrt{1 + e_{11}}\,\sqrt{1 + e_{22}}} \\ \cos \widehat{e'_1 e'_3} &= \frac{e_{13}}{\sqrt{1 + e_{11}}\,\sqrt{1 + e_{33}}} \\ \cos \widehat{e'_2 e'_3} &= \frac{e_{23}}{\sqrt{1 + e_{22}}\,\sqrt{1 + e_{33}}}. \end{aligned} \tag{43}$$

If the $|e_{ij}| << 1$ then these relationships reduce to

$$\begin{aligned} |\hat{e}'_1| &\cong 1 + \frac{1}{2} e_{11} \\ |\hat{e}'_2| &\cong 1 + \frac{1}{2} e_{22} \\ |\hat{e}'_3| &\cong 1 + \frac{1}{2} e_{33} \end{aligned} \tag{44}$$

and to

$$\widehat{e'_1 e'_2} \cong \frac{\pi}{2} - e_{12}$$

$$\widehat{e'_1 e'_3} \cong \frac{\pi}{2} - e_{13} \qquad (45)$$

$$\widehat{e'_2 e'_3} \cong \frac{\pi}{2} - e_{23}$$

The entire preceding analysis is based on the components of the separation vector $d\vec{x}$ and the components of the relative displacement vector $\Delta\vec{u}$. The direction of the separation vector $d\vec{x}$ is arbitrary and a reversal of the direction of $d\vec{x}$ simply reverses the sign of $\Delta\vec{u}$. The quantities in this theory have analogs in long baseline interferometry. The analog of the separation vector $d\vec{x}$ is the interferometer baseline vector $\vec{\beta}$ separating the two antennas. The analog of the relative displacement vector $\Delta\vec{u}$ is $\vec{\Delta}$ the change in the baseline vector which is observed to occur between two successive measurements of $\vec{\beta}$.

Next we shall attempt to apply these procedures to the analysis of long baseline interferometry data and to question of the measurement of earth rotation on a deformable earth. While these procedures are potentially useful in this context, they will also have certain limitations.

First, the above geometrical procedures will not give rise to tensor quantities when applied to the whole earth. Tensor quantities can only be defined in the neighborhood of a point or, for pairs of points, separated by infinitesimal separation vectors $d\vec{x}$. In long baseline interferometry, we will be concerned with relative displacement between points separated by thousands of kilometers. The analysis procedure described above will produce quantities which have many of the properties of tensors but, because they refer to pairs of points separated by a finite distance, will not possess the general transformation properties of tensors. In particular, these quantities will possess the mathematical properties of tensors only in Cartesian coordinate frames. To distinguish them from true tensors, we will refer to them as tensor analogs.

Second, the deformation field of the earth is not continuous and differentiable everywhere. Fracture is occurring on and within the earth in a complex, largely unknown pattern. This fact does not affect the validity of these procedures as applied to the earth insofar as we restrict ourselves to the discipline of kinematics or geometry. However, entering the realm of dynamics or geophysics and attempting to relate strain tensor analogs to stress tensors will be a risky procedure. Where there is fracture, there can be local strain without local stress. Without considerably more geologic knowledge, the complexity of the pattern of fracture affecting baselines of thousands of kilometers in length would make the inference of stress from strain suspect. The two possible exceptions to this might be:

(a) The case of strain tensor analogs measured on very short baselines from a few tens to a few hundred kilometers in length, in which case, the material supporting the interferometer can either be considered homogeneous and free of fracture or can be mapped in sufficient geological detail to adequately account for the effects of fracture; and

(b) The case of a mean strain tensor analog obtained by a global average of many local strain tensor analogs, in which case, the random effects of geologic faulting affecting the local measurements have been averaged out and the residual mean strain tensor analog describes that global strain field which is deforming the earth coherently as though it were the neighborhood of a point.

The Measurement of Earth Rotation on a Deformable Earth

Basic Definitions and Procedures

We presuppose a global network of interferometer baseline vectors $\vec{\beta}^k$ $k = 1, 2, 3, \ldots N$ forming a polyhedron as shown in figure 4 which will be used for the purposes of:

(a) defining the body-fixed coordinate frame, and

(b) measuring earth rotation or equivalently defining the transformation from the body-fixed coordinate frame to the space-fixed coordinate frame.

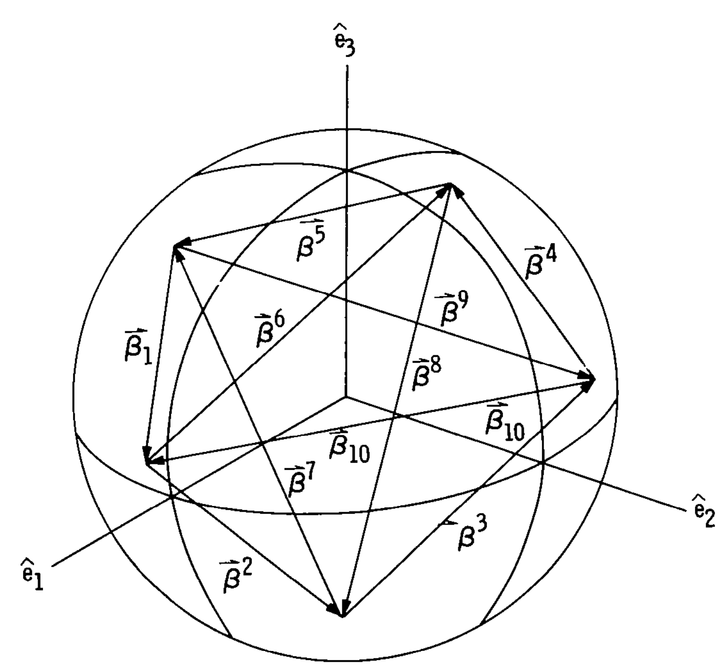

Figure 4.

These definitions must be operable in the face of the fact that as a result of the discoveries of plate tectonics, each point of the earth's surface must be regarded as possessing its own rotation axis. While points on the same tectonic plate might be expected to have nearly identical rotation axes as a consequence of Euler's theorem on rigid body motion on the surface of a sphere [McKenzie, 1972], different rotation axes will be associated with points on different tectonic plates. Furthermore, when the relative vertical motions, which occur within the plates and which can be as large as 1 cm/year, are also considered, it is clear that in general the concept of a rotation axis must be rigorously regarded as only applying locally on the earth.

Each of the time dependent interferometer baseline vectors $\vec{\beta}^k_1(t)$, k = 1, 2, 3, . . . N will possess space-fixed components $B^k_i(t)$, i = 1, 2, 3,

$$\vec{\beta}^k(t) = B^k_1(t)\,\hat{E}_1 + B^k_2(t)\,\hat{E}_2 + B^k_3(t)\,\hat{E}_3, \tag{46}$$

and body-fixed components $b^k_i(t)$, i = 1, 2, 3,

$$\vec{\beta}^k(t) = b^k_1(t)\,\hat{e}_1 + b^k_2(t)\,\hat{e}_2 + b^k_3(t)\,\hat{e}_3. \tag{47}$$

The transformation from space-fixed components to body-fixed components expresses all our knowledge of earth rotation and is currently expressed by equation (6) as

$$b^k_i(t) = W^T_{ij}(t)\,S^T_{jk}(t)\,N^T_{k\ell}(t)\,P^T_{\ell m}(t)\,B^k_m \tag{48}$$

where the superscript "k" serves merely to identify the baseline and does not obey the Einstein summation and range convention. The superscript "T" denotes a matrix transpose.

Repeated measurements of the baselines $\vec{\beta}^k(t)$ at times . . . t_{m-2} t_{m-1} t_m t_{m+1} t_{m+2}. . . yield relative displacement vectors $\vec{\Delta}^k(t_m)$ given by

$$\vec{\Delta}^k(t_m) = \vec{\beta}^k(t_m) - \vec{\beta}^k(t_{m-1}). \tag{49}$$

The relative displacement vectors $\vec{\Delta}^k(t_m)$ will possess space-fixed components $D^k_i(t)$, i = 1, 2, 3,

$$\vec{\Delta}^k(t_m) = D^k_1(t_m)\,\hat{E}_1 + D^k_2(t_m)\,\hat{E}_2 + D^k_3(t_m)\,\hat{E}_3 \tag{50}$$

and body-fixed components $d^k_i(t_m)$, i = 1, 2, 3,

$$\vec{\Delta}^k(t_m) = d^k_1(t_m)\,\hat{e}_1 + d^k_2(t_m)\,\hat{e}_2 + d^k_3(t_m)\,\hat{e}_3 \tag{51}$$

also related one to the other, according to our present knowledge of earth rotation, by equation (6)

$$d^k_i(t_m) = W^T_{ij}(t_m)\,S^T_{jk}(t_m)\,N^T_{k\ell}(t_m)\,P^T_{\ell m}(t_m)\,D^k_m(t_m). \tag{52}$$

The operation of the long baseline interferometer and the subsequent reduction of the data finally provide us with the quantities $b_i^k(t_m)$, $d_i^k(t_m)$ $i = 1, 2, 3, k = 1, 2, 3 \ldots N$ which together with the epochs t_m, $m = 1, 2, 3 \ldots$ constitute the basic data with which we must achieve our two stated objectives.

We can begin creating analogs to continuum mechanics by defining, for the k^{th} baseline, a dimensionless quantity c_{ij}^k given by

$$c_{ij}^k = \frac{d_i^k b_j^k}{b_\ell^k b_\ell^k} \tag{53}$$

which will serve as the analog of the tensor $\nabla \vec{u}$ of equation (9). The tensor analog c_{ij}^k is expanded in its symmetric and antisymmetric parts.

$$c_{ij}^k = \frac{1}{2} e_{ij}^k + \frac{1}{2} \Omega_{ij}^k \tag{54}$$

where

$$e_{ij}^k = [c_{ij}^k + c_{ji}^k] = \frac{1}{b_\ell^k b_\ell^k} [d_i^k b_j^k + d_j^k b_i^k] \tag{55}$$

is the strain tensor analog for the k^{th} interferometer baseline and where

$$\Omega_{ij}^k = [c_{ij}^k - c_{ji}^k] = \frac{1}{b_\ell^k b_\ell^k} [d_i^k b_j^k - d_j^k b_i^k] \tag{56}$$

is the rotation tensor analog for the k^{th} interferometer baseline. It can readily be shown that these definitions permit us to write

$$d_i^k = \frac{1}{2} e_{ij}^k b_j^k + \frac{1}{2} \Omega_{ij}^k b_j^k. \tag{57}$$

Equations (54) through (57) are the analogs of equations (10) through (13) for the case of continuum mechanics.

To illustrate this procedure, we consider the case of an interferometer baseline $\vec{\beta}^k$ with body-fixed components b_i^k, $i = 1, 2, 3$, given by x, y, z and a relative displacement vector $\vec{\Delta}^k$ with body-fixed components d_i^k, $i = 1, 2, 3$, given by δx, δy, δz, then:

$$c_{ij}^k = \frac{1}{x^2 + y^2 + z^2} \begin{bmatrix} x\delta x & y\delta x & z\delta x \\ x\delta y & y\delta y & z\delta y \\ x\delta z & y\delta z & z\delta z \end{bmatrix} \tag{58}$$

$$e_{ij}^k = \frac{1}{x^2 + y^2 + z^2} \begin{bmatrix} 2x\delta x & y\delta x + x\delta y & z\delta x + x\delta z \\ x\delta y + y\delta x & 2y\delta y & z\delta y + y\delta z \\ x\delta z + z\delta x & y\delta z + z\delta y & 2z\delta z \end{bmatrix} \tag{59}$$

$$\Omega_{ij}^k = \frac{1}{x^2 + y^2 + z^2} \begin{bmatrix} 0 & y\delta x - x\delta y & z\delta x - x\delta z \\ x\delta y - y\delta x & 0 & z\delta y - y\delta z \\ x\delta z - z\delta x & y\delta z - z\delta y & 0 \end{bmatrix} \tag{60}$$

To show that e_{ij}^k possesses the properties of a strain tensor insofar as the k^{th} baseline is concerned, we compute the variation in the squared distance s^{k^2} between the two observatories defining the k^{th} baseline which occurs as a result of the relative displacement. Before relative displacement

$$s^{k^2} = b_i^k \, b_i^k, \tag{61}$$

and after relative displacement

$$s^{k^2} = (b_i^k + d_i^k)\,(b_i^k + d_i^k). \tag{62}$$

The difference $\delta(s^{k^2})$ is given by

$$\delta(s^{k^2}) = (b_i^k + d_i^k)\,(b_i^k + d_i^k) - b_i^k \, b_i^k$$

or

$$\delta(s^{k^2}) = 2d_i^k \, b_i^k \left[1 + 0\left(\frac{d}{b}\right)\right] \simeq 2d_i^k \, b_i^k \tag{63}$$

neglecting terms of the order of $d/b << 1$. Equation (63) can be written

$$\delta(s^{k^2}) = \left[\frac{d_i^k b_j^k + d_j^k b_i^k}{b_\ell^k b_\ell^k}\right] b_i^k \, b_j^k = e_{ij}^k \, b_i^k \, b_j^k \tag{64}$$

which is the analog of equation (21) for the case of continuum mechanics.

A further demonstration that e_{ij}^k possesses the properties of the strain tensor for the k^{th} baseline is provided by considering the special case of an earth subjected to a uniform dilatation. In this case, the observatories at position $\vec{r}_1$ and $\vec{r}_2$ which make up the k^{th} interferometer baseline $\vec{\beta}^k$ suffer displacements $\delta\vec{r}_1$ and $\delta\vec{r}_2$ given by

$$\delta\vec{r}_1 = A\vec{r}_1$$

$$\delta\vec{r}_2 = A\vec{r}_2$$

where 3A is the magnitude of the dilatation. The relative displacement vector $\vec{\Delta}^k$ is given by

$$\vec{\Delta}^k = A(\vec{r}_2 - \vec{r}_1)$$

and will have body-fixed components d_i^k

$$d_i^k = Ab_i^k$$

where b_i^k are the body-fixed components of the k^{th} baseline vector $\vec{\beta}^k = \vec{r}_2 - \vec{r}_1$. The strain tensor analog for the k^{th} baseline in this case is

$$e_{ij}^k = \frac{A}{b_\ell^k b_\ell^k} [b_i b_j + b_j b_i] \tag{65}$$

and the trace of e_{ij}^k, $T_r(e_{ij}^k)$, in this case is

$$T_r(e_{ij}^k) = 2A. \tag{66}$$

This result should be compared to the well known result from continuum mechanics that

$$T_r(e_{ij}) = 2\theta \tag{67}$$

where θ is the dilatation.

To show that Ω_{ij}^k possesses the properties of a rotation tensor insofar as the k^{th} baseline is concerned, we compute the rigid body rotation $\vec{\omega}$ which is imparted to an interferometer baseline vector $\vec{\beta}$ as a result of a relative displacement $\vec{\Delta}$. The geometry of this situation is illustrated in figure 5 with the body-fixed basis vectors shown for reference. The rotation vector $\vec{\omega}$ can be expressed as

$$\vec{\omega} = \Omega\hat{\omega} \tag{68}$$

where Ω is the magnitude of the angle between $\vec{\beta}$ and $\vec{\beta} + \vec{\Delta}$. Neglecting terms of the order of $\vec{\Delta}/\vec{\beta}$ where $|\vec{\Delta}| << |\vec{\beta}|$, we can express the area of the shaded triangle in two equivalent forms and equate them to obtain

$$\frac{1}{2}\Omega|\vec{\beta}|^2 = \frac{1}{2}|\vec{\beta} \times \vec{\Delta}|. \tag{69}$$

From this it follows that

$$\Omega = \frac{|\vec{\beta} \times \vec{\Delta}|}{|\vec{\beta}|^2}. \tag{70}$$

Since

$$\omega = \frac{\vec{\beta} \times \vec{\Delta}}{|\vec{\beta} \times \vec{\Delta}|}, \tag{71}$$

equations (70) and (71) give

$$\vec{\omega} = \frac{\vec{\beta} \times \vec{\Delta}}{|\vec{\beta}|^2} \tag{72}$$

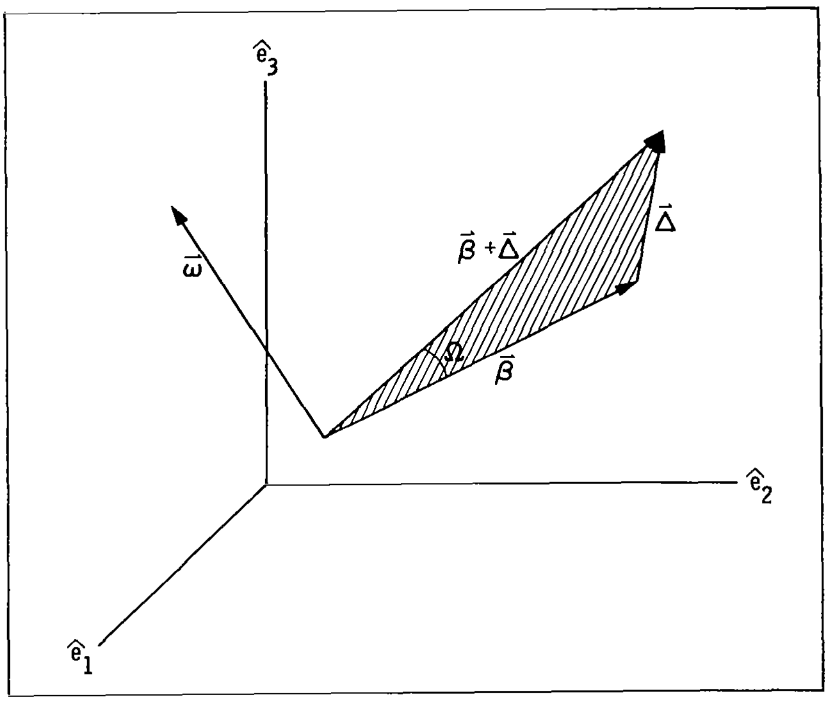

Figure 5.

In the body-fixed frame,

$$\vec{\omega} = \omega_i \hat{e}_i$$

$$\vec{\beta} = b_i \hat{e}_i$$

$$\vec{\Delta} = d_i \hat{e}_i$$

and so

$$\omega_i = \frac{\epsilon_{ijk}\, b_j\, d_k}{b_\ell\, b_\ell}\,. \tag{73}$$

The antisymmetric tensor ω_{ij} dual to the axial vector $\vec{\omega}$ is

$$\omega_{ij} = \epsilon_{kij}\, \omega_k$$

or

$$\omega_{ij} = \frac{\epsilon_{kij}\, \epsilon_{k\ell m}\, b_\ell d_m}{b_n b_n}\,. \tag{74}$$

Using the properties of the alternating tensor, this can be written

$$\omega_{ij} = (\delta_{i\ell}\, \delta_{jm} - \delta_{im}\, \delta_{j\ell})\, \frac{b_\ell d_m}{b_n b_n} \tag{75}$$

or

$$\omega_{ij} = \frac{b_i b_j - b_j d_i}{b_n b_n}\,. \tag{76}$$

A comparison of this result with the definition in equation (56) of the rotation tensor analog Ω^k_{ij} for the k^{th} baseline shows that

$$\Omega^k_{ij} = -\omega_{ij}\,. \tag{77}$$

As a consequence of this, we see that the body-fixed components Ω^k_i of the vector $\vec{\Omega}^k$ of the k^{th} interferometer baseline occurring as a result of the rotation relative displacements $\vec{\Delta}^k$ are obtained from the rotation tensor analog Ω^k_{ij} by

$$\Omega^k_i = -\epsilon_{imn}\, \Omega^k_{mn}\,. \tag{78}$$

The Body-Fixed Coordinate Frame

From observations on a global network of N interferometer baselines, we can define

$$E_{ij} = \frac{1}{\sum_{k=1}^{N} W^k} \sum_{k=1}^{N} W^k\, e^k_{ij} \tag{79}$$

and

$$\Lambda_{ij} = \frac{1}{\sum_{k=1}^{N} W^k} \sum_{k=1}^{N} W^k \, \Omega_{ij}^k \tag{80}$$

where W^k $k = 1, 2, 3, \ldots N$ are appropriate weighting factors. E_{ij} is the weighted mean global strain tensor analog and Λ_{ij} is the weighted mean global rotation tensor analog. The strain tensor analog e_{ij}^k and rotation tensor analog Ω_{ij}^k on each interferometer baseline can then be expressed as the sum of the above global means and a "local" residual ϵ_{ij}^k and ω_{ij}^k respectively specific to the k^{th} baseline.

$$e_{ij}^k = E_{ij} + \epsilon_{ij}^k \tag{81}$$

$$\Omega_{ij}^k = \Lambda_{ij} + \omega_{ij}^k \tag{82}$$

where, by definition, the weighted global means of the local residuals vanish.

$$\frac{1}{\sum_{k=1}^{N} W^k} \sum_{k=1}^{N} W^k \, \epsilon_{ij}^k = 0 \tag{83}$$

$$\frac{1}{\sum_{k=1}^{N} W^k} \sum_{k=1}^{N} W^k \, \omega_{ij}^k = 0. \tag{84}$$

If the global distribution of interferometer baselines is arranged to representatively sample the earth's crust, then E_{ij} might be expected to reveal the deformation properties of the earth as a whole. Such measurements might be expected to reveal a number of global processes which are expected to be producing coherent deformation of the whole earth.

Earth expansion or contraction would be revealed by a significant departure of $T_r(E_{ij})$ from zero. Overall earth expansion or contraction is associated with a number of outstanding problems in geophysics such as:

(a) the problem of the internal evolution of the earth and chemical phase changes possibly associated with the growth or decay of the solid inner or liquid outer cores,

(b) the problem of the effect on the earth's figure of the secular spin-down and the question of the finite strength of the mantle and the persistence or otherwise of a "fossil" equatorial bulge, and

(c) the problem of possible cosmological effects due to a non-zero value of $\dot{G}/G$ causing secular changes in the earth's size.

The off-diagonal elements of E_{ij} would be associated with global shear deformation and would reveal tidal-like effects on the earth. The eigenvectors of E_{ij} would indicate the directions of principal global stress as evidenced by crustal observations. Their evolution with time might reveal information about processes in the earth's interior. Of particular interest would be a comparison of the evolution in time of the eigenvectors E_{ij} as measured by long baseline interferometry and the evolution in time of the principal axes of inertia of the earth as evidenced by its rotational dynamics and the orbits of satellites.

For certain geophysical applications such as the study of plate tectonics, it may prove useful to introduce a material body-fixed coordinate frame $\bar{x}, \bar{y}, \bar{z}$ spanned by basis vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$ defined in the sense that for coherent global deformations E_{ij}, the body-fixed coordinate surfaces, deform along with the global motion of the material. This is easily accomplished using the observed mean global strain tensor analog E_{ij} according to the procedure described above in the section on continuum mechanics.

At some epoch t_o, the earth's internal state is adopted as an arbitrary standard with $E_{ij} = E^o_{ij} \equiv 0$ and the material body-fixed basis vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$ coinciding with the basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ defined in equation (6). The material coordinates $\bar{x}, \bar{y}, \bar{z}$ are therefore identical to the Cartesian coordinates x, y, z at the epoch. For times $t > t_o$ later than the epoch, the material body-fixed basis vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$ deform along with the mean global motion and are given by equations (44) and (45) as

$$
\begin{aligned}
|\bar{e}_1| &= 1 + \frac{1}{2} E_{11} \\
|\bar{e}_2| &= 1 + \frac{1}{2} E_{22} \\
|\bar{e}_3| &= 1 + \frac{1}{2} E_{33}
\end{aligned}
\tag{85}
$$

and

$$
\begin{aligned}
\widehat{\bar{e}_1 \bar{e}_2} &= \pi/2 - E_{12} \\
\widehat{\bar{e}_1 \bar{e}_3} &= \pi/2 - E_{13} \\
\widehat{\bar{e}_2 \bar{e}_3} &= \pi/2 - E_{23}
\end{aligned}
\tag{86}
$$

The coordinate system $\bar{x}, \bar{y}, \bar{z}$ spanned by basis vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$ might offer certain advantages in studies of plate tectonics as the coordinate velocity, or changes in the coordinates of points on the earth referenced to this coordinate frame, will be due only to "local" tectonic effects. The effects of coherent global deformation are removed automatically by deformation of the coordinate frame.

The strain tensor analog E_{ij} represents mean strain in the global network of interferometer baselines shown in figure 4. The residuals ϵ_{ij}^k given in equation (81) represent local deformations of the k^{th} baseline relative to the global network and are due to the effects of regional tectonic motion. The ϵ_{ij}^k will contain the information about regional tectonics on the earth including both inter- and intra-plate geologic processes.

If the global distribution of interferometer baselines is arranged to representatively sample the earth's crust, then the mean global rotation tensor analog Λ_{ij} and its dual axial vector $\vec{\Lambda}$ with body-fixed components Λ_i given by

$$\Lambda_i = \epsilon_{ijk} \Lambda_{ij} \tag{87}$$

represent a net mean rotation of the network of interferometer baselines shown in figure 4 relative to the body-fixed basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ of equation (6). If the global network of interferometers is to be used to define the body-fixed coordinate frame spanned by $\hat{e}_1, \hat{e}_2, \hat{e}_3$, it is a contradiction for such a coordinate frame to be rotating relative to itself. Hence, the body-fixed basis vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ must necessarily be defined in such a way that Λ_{ij} vanishes. Since Λ_{ij} can be regarded as an infinitesimal rotation residual to the transformation of equation (6), the most direct way of defining $\hat{e}_1, \hat{e}_2, \hat{e}_3$ so that Λ_{ij} vanishes is to include the effects of the rotation of Λ_{ij} directly into the transformation of equation (6) from the space-fixed to the body-fixed frame. This can be done by rewriting the transformation as

$$\hat{e}_i(t) = [\delta_{ij} - \Lambda_{ij}(t)] \, P_{jk}(t) \, N_{k\ell}(t) \, S_{\ell m}(t) \, W_{mn}(t) \, \hat{E}_n \tag{88}$$

However, this is probably not the most useful procedure to follow since the measured values of $\Lambda_{ij}(t)$ contain, in addition to the unknown effects of global deformability, the effects of errors in the models P_{ij}, N_{ij}, etc. for the known effects of precession, nutation, spin, and wobble. By observing the time dependence of the measured values of Λ_{ij}, it will be possible to decide between the alternatives of modifications to the known transformation matrices P_{ij}, N_{ij}, S_{ij}, W_{ij} and the addition of yet another transformation matrix D_{ij} representing the rotational effects of global deformability.

Historical precedent indicates that the discovery of new geophysical or astronomical phenomena has been accompanied by the addition of a new matrix to the transformation from the space-fixed frame to the body-fixed frame. However, since the models for precession, nutation, etc. are also expected to contain errors at the level of measurement anticipated by future long baseline interferometry, we should be prepared to exercise both options.

The residuals ω_{ij}^k given in equation (82) represent local rotations of the k^{th} baseline relative to the global network and, like the residuals ϵ_{ij}^k of equation (81), are due to the effects of regional tectonic motion. The ω_{ij}^k, like the ϵ_{ij}^k, will contain information about regional tectonics on the earth including both inter- and intra-plate geologic processes.

The Space-Fixed Coordinate Frame

The present set of space-fixed basis vectors $\hat{E}_1, \hat{E}_2, \hat{E}_3$ are dynamically defined by the orbit of the earth about the sun and, while they play an essential role in the formulation of the theory of the orbital dynamics of the earth, they play no essential role in the formulation of the theory of the rotational dynamics of the earth. The basis vectors $\hat{E}_1, \hat{E}_2, \hat{E}_3$ have served both purposes in the past as a matter of convenience. However, since translational (orbital) and rotational motions of the earth are dynamically independent, no great difficulty should arise if the two motions were referred to different space-fixed coordinate frames – should it prove convenient to do so. It is apparent that the application of long baseline interferometry to problems of earth rotation would be greatly facilitated by referencing the space-fixed basis vectors directly to the cosmic radio frame of the quasars. Besides being the most remote objects in the observable universe, hence expected to possess very little proper motion, the choice of the quasars to define the space-fixed frame offers some important measurement advantages.

The delay observable τ^k is a scalar quantity related to the inner product of a unit vector $\hat{s}$ in the direction of the radio source and the baseline vector $\vec{\beta}^k$

$$\tau^k = \frac{1}{c} \hat{s} \cdot \vec{\beta}^k. \tag{89}$$

The delay observable can be written in two equivalent forms

$$\tau^k = \frac{1}{c} s^i \beta_i^k \tag{90}$$

$$\tau^k = \frac{1}{c} c_i \beta^{ik} \tag{91}$$

where s^i, s_i, are the contravariant and covariant components of the unit source vector $\hat{s}$ and β^{ik}, β_i^k are the contravariant and covariant components of the k^{th} baseline vector $\vec{\beta}^k$ both referred to a set of, as yet unspecified, basis vectors the usual choice for which is $\hat{E}_1, \hat{E}_2, \hat{E}_3$ or $\hat{e}_1, \hat{e}_2, \hat{e}_3$.

If we introduce a space-fixed coordinate frame whose fundamental directions are defined by three selected quasar radio sources s(1), s(2), s(3), then it is clear that the baseline vector $\vec{\beta}^k$ has two representations given by

$$\vec{\beta}^k = \beta_1^k \hat{s}^1 + \beta_2^k \hat{s}^2 + \beta_3^k \hat{s}^3 \tag{92}$$

and

$$\vec{\beta}^k = \beta^{1k} \hat{s}_1 + \beta^{2k} \hat{s}_2 + \beta^{3k} \hat{s}_3 \tag{93}$$

where $\hat{s}^1, \hat{s}^2, \hat{s}^3$ are the contravariant quantities or so-called basis 1-forms [Misner, Throne, Wheeler, 1973, pp. 53 ff.] and $\hat{s}_1, \hat{s}_2, \hat{s}_3$ are the covariant quantities or so-called basis vectors spanning the space-fixed frame appropriate to radio sources s(1), s(2), s(3).

An analysis of the operation of a long baseline interferometer [Thomas 1972, Cannon 1978] shows that the delay observable yields the covariant components of the baseline vector as shown schematically in figure 6. The stopped fringes of the incoming wave fronts from the radio source provide a physical realization of the basis 1-forms which define the contravariant quantities $\hat{s}^1, \hat{s}^2, \hat{s}^3$. In the space-fixed coordinate frame defined by radio sources s(1), s(2), s(3), the contravariant basis vectors or 1-forms $\hat{s}^1, \hat{s}^2, \hat{s}^3$ have components

$$\begin{aligned} \hat{s}^1 &= [\,1 \quad 0 \quad 0\,] \\ \hat{s}^2 &= [\,0 \quad 1 \quad 0\,] \\ \hat{s}^3 &= [\,0 \quad 0 \quad 1\,] \end{aligned} \tag{94}$$

by definition. From equation (90), the baseline vector $\vec{\beta}^k$ has representation

$$\vec{\beta}^k = c\tau_1^k \hat{s}^1 + c\tau_2^k \hat{s}^2 + c\tau_3^k \hat{s}^3 \tag{95}$$

where the covariant space-fixed components of $\vec{\beta}^k$ given by

$$\begin{aligned} \beta_1^k &= c\tau_1^k \\ \beta_2^k &= c\tau_2^k \\ \beta_3^k &= c\tau_3^k \end{aligned} \tag{96}$$

are thus measured directly without reference to source coordinates.

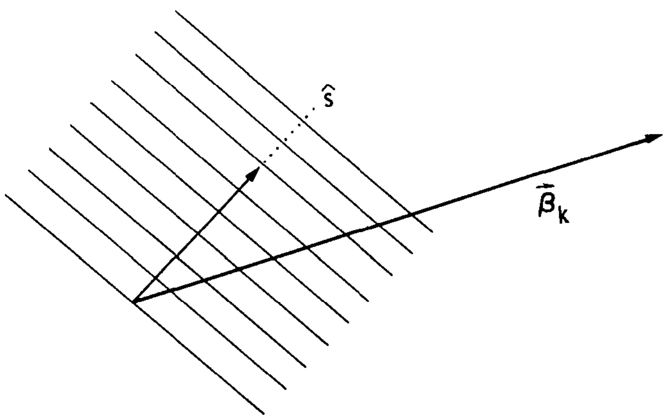

Figure 6.

It is traditional in geodesy to measure contravariant components of baseline vectors – the infinitesimal displacement vector being defined as the archetypical contravariant vector. To obtain the contravariant space-fixed components β^{ik} of the baseline vector $\vec{\beta}^k$ from the covariant space-fixed components β_i^k given by the delay observable and equations (96) requires knowledge of the metric tensor g_{ij} of the nonorthogonal space-fixed coordinate frame of the quasars s(1), s(2), s(3) shown schematically in figure 7. The contravariant components β^{ik} are given by

$$\beta^{ik} = g^{ij}\,\beta_j^k \tag{97}$$

where

$$g^{ij} = g_{ij}^{-1} \tag{98}$$

and where

$$g_{ij} = \begin{bmatrix} 1 & \cos\widehat{s(1)s(2)} & \cos\widehat{s(1)s(3)} \\ \cos\widehat{s(2)s(1)} & 1 & \cos\widehat{s(2)s(3)} \\ \cos\widehat{s(3)s(1)} & \cos\widehat{s(3)s(2)} & 1 \end{bmatrix} \tag{99}$$

This procedure does not require knowledge of absolute source coordinates but merely the relative angles or the direction cosines between the quasars s(1), s(2), s(3). This measurement can also be made very accurately by long baseline interferometry.

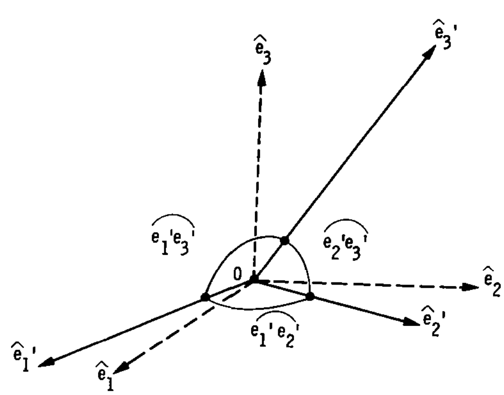

Figure 7.

We now consider the case where observations are made of a radio source s(J), J = 4, 5, 6 . . . which is not one of the three sources s(1), s(2), s(3) defining the fundamental directions in the space-fixed frame. The delay observable on the source s(J) and baseline $\vec{\beta}^k$ is $\tau^k(J)$ where

$$\tau^k(J) = \frac{1}{c}\,\hat{s}(J)\cdot\beta^k \tag{100}$$

and where $\hat{s}(J)$ is a unit vector in the direction of the radio source $\hat{s}(J)$. $\hat{s}(J)$ can be written in terms of the funamental contravariant basis vectors $\hat{s}^1, \hat{s}^2, \hat{s}^3$ as

$$\hat{s}(J) = s_1(J)\hat{s}^1 + s_2(J)\hat{s}^2 + s_3(J)\hat{s}^3 \tag{101}$$

where

$$\begin{aligned} s_1(J) &= \cos \widehat{s(1)s(J)} \\ s_2(J) &= \cos \widehat{s(2)s(J)} \\ s_3(J) &= \cos \widehat{s(3)s(J)} \end{aligned} \tag{102}$$

Equations (102) indicate that any of the quasar radio sources s(J), J = 4, 5, 6 . . . can be observed to obtain the space-fixed covariant components of the interferometer baseline providing the direction cosines of the unit vector s(J) relative to the three chosen fundamental space-fixed directions given by sources s(1), s(2), s(3) are known. The angles $\widehat{s(1)\,s(J)}$, $\widehat{s(2)\,s(J)}$, $\widehat{s(3)\,s(J)}$ can be measured accurately by long baseline interferometers. This suggests that a cataloguing procedure in terms of the direction cosines to three chosen fundamental sources instead of right ascension and declination might be usefully adopted in the near future.

It is possible that the three chosen radio sources s(1), s(2), s(3) will exhibit proper motion relative to the cosmic space-fixed radio frame defined by the ensemble of quasars. Should this be the case it will present no great difficulty to the above procedure. The ensemble of quasars would then be chosen to define the space-fixed frame and to possess a Maxwellian velocity distribution and to exhibit no mean rotation. With this assumption, it would remain only to tie the positions of the chosen sources s(1), s(2), s(3) to the ensemble of quasars by periodic measurements of the angles $\widehat{s(1)\,s(J)}$ $\widehat{s(2)\,s(J)}$ $\widehat{s(3)\,s(J)}$, J = 4, 5, 6 . . . N where N could be of the order of 100 to correctly account for any proper motion they might exhibit.

Finally, it may perhaps be useful in the light of the general theory of relativity to retain and to distinguish between two sets of space-fixed basis vectors. The first set $\hat{s}^1, \hat{s}^2, \hat{s}^3$ described above span what could be called a cosmic inertial frame. The second set $\hat{E}_1, \hat{E}_2, \hat{E}_3$ to be defined dynamically by a relativistic ephemeris of the solar system span what could be called a local inertial frame. The theory of general relativity allows for these two inertial frames to be relatively rotating [Weinberg, 1972, pp. 239 ff].

REFERENCES

Brillouin, L., Tensors in Mechanics and Elasticity, Academic Press, 1964.

Cannon, W. H., "The Classical Analysis of the Response of a Long Baseline Radio Interferometer" Geophys. J. R. Astr. Soc., 53, 503-530, 1978.

Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac, published by the Nautical Almanac Offices of the United Kingdom and United States of America, 1961.

Jones, H. E., "Geodetic Ties Between Continents by Means of Radio Telescopes" Canadian Surv., 23, 377-388, 1969.

McKenzie, D. P., "Plate Tectonics" in The Nature of the Solid Earth, Chapt. 13, Robertson, E. C., Hays, J. F., Knopoff, L., eds. McGraw Hill, 1972.

Misner, C. W., Thorne, K. S., Wheeler, J. A., Gravitation, Freeman, 1973.

Mueller, I. I., Spherical and Practical Astronomy as Applied to Geodesy, Ungar, 1969.

Munk, W. H., MacDonald, G. J. F., The Rotation of the Earth - A Geophysical Discussion, Cambridge University Press, 1960.

Thomas, J. B., "An Analysis of Long Baseline Radio Interferometry" Deep Space Network Progress Report, J.P.L. Technical Report 32-1526, Vol. VII, 1972.

Thompson, W., "On the Rigidity of the Earth" Phil. Trans. R. Soc. London, 153, 1863.

Weinberg, S., "Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity," Wiley and Sons, 1972.

Woolard, E. W., "The Theory of the Rotation of the Earth around its Centre of Mass," Astronomical Papers Prepared for the Use of the American Ephemeris and Nautical Almanac Vol. XV, Part 1, U.S. Government Printing Office, Washington, D.C. 1953.

APPENDIX

The Definition of the Mean Global Strain Tensor E_{ij} and the Mean Global Rotation Tensor Λ_{ij}

The definition of the mean global strain tensor E_{ij} and the mean global rotation tensor Λ_{ij} as given in equations (79) and (80)

$$E_{ij} = \frac{1}{\sum_{k=1}^{N} w^k} \sum_{k=1}^{N} w^k \, e_{ij}^k$$

$$\Lambda_{ij} = \frac{1}{\sum_{k=1}^{N} w^k} \sum_{k=1}^{N} w^k \, \Omega_{ij}^k$$

can be shown to satisfy two desirable objective criteria. These are discussed below.

E_{ij} - the Mean Global Strain Tensor

The definition of E_{ij} minimizes, in a weighted least squares sense the net departure between the squared baseline length variations $\overline{\delta s}^{k^2}$ which would be predicted assuming a mean global strain E_{ij} and the individual squared baseline length variations δs^{k^2} observed by the long baseline interferometer.

$$\delta s^{k^2} = e_{ij}^k \, b_i^k \, b_j^k$$

$$\overline{\delta s}^{k^2} = E_{ij} \, b_i^k \, b_j^k$$

Net weighted squared departure between predicted $\overline{\delta s}^{k^2}$ and observed δs^{k^2} squared baseline length variations is D^2 where

$$D^2 = \sum_{k=1}^{N} w^k (\delta s^{k^2} - \overline{\delta s}^{k^2})^2$$

$$D^2 = \sum_{k=1}^{N} w^k (e^k_{ij} b^k_i b^k_j - E_{\ell m} b^k_\ell b^k_m)^2$$

$$D^2 = \sum_{k=1}^{N} w^k [(e^k_{ij} b^k_i b^k_j)^2 - 2e^k_{ij} E_{\ell m} b^k_i b^k_j b^k_\ell b^k_m + (E_{\ell m} b^k_\ell b^k_m)^2]$$

A choice of $E_{\ell m}$ which minimizes D^2 is given by

$$\frac{\partial D^2}{\partial E_{pq}} = 0$$

This gives a condition

$$\sum_{k=1}^{N} w^k \left(-2e^k_{ij} \frac{\partial E_{\ell m}}{\partial E_{pq}} b^k_i b^k_j b^k_\ell b^k_m + 2(E_{ij} b^k_i b^k_j) \frac{\partial}{\partial E_{pq}} (E_{\ell m} b^k_\ell b^k_m) \right) = 0$$

$$\sum_{k=1}^{N} w^k \left(-2e^k_{ij} \; b^k_i b^k_j b^k_\ell b^k_m \; \delta_{\ell p} \; \delta_{mq} + 2E_{ij} b^k_i b^k_j b^k_\ell b^k_m \; \delta_{\ell p} \delta_{mq} \right) = 0$$

$$\sum_{k=1}^{N} w^k \left(-2e^k_{ij} b^k_i b^k_j b^k_p b^k_q + 2E_{ij} b^k_i b^k_j b^k_p b^k_q \right) = 0$$

$$\sum_{k=1}^{N} w^k \left(-2e^k_{ij} + 2E_{ij} \right) b^k_i b^k_j b^k_p b^k_q = 0$$

Since this condition should be fulfilled independently of the particular choice of baseline net work we conclude that we shall in general require

$$\sum_{k=1}^{N} w^k \left(-2e^k_{ij} + 2E_{ij} \right) = 0$$

or

$$E_{ij} = \frac{1}{\sum_{k=1}^{N} w^k} \sum_{k=1}^{N} w^k \, e_{ij}^k \, .$$

Λ_{ij} - the Mean Global Rotation Tensor

The definition of Λ_{ij} minimizes, in a weighted least squares sense, the magnitude of the net departures between the mean global rotation vector Λ_i and the rotation vectors Λ_i^k of the individual baselines observed by the long baseline interferometer.

$$\Lambda_i = \varepsilon_{i\ell m} \, \Lambda_{\ell m}^k$$

$$\Omega_i^k = \varepsilon_{i\ell m} \, \Omega_{\ell m}^k$$

The magnitude of the net weighted squared departure between the mean global rotation vector Λ_i and the observed rotation vectors Ω_i^k of the individual baselines is θ^2

$$\theta^2 = \sum_{k=1}^{N} w^k (\Omega_i^k - \Lambda_i)(\Omega_i^k - \Lambda_i)$$

but since

$$\Omega_i^k - \Lambda_i = \varepsilon_{i\ell m}(\Omega_{\ell m}^k - \Lambda_{\ell m})$$

we have

$$\theta^2 = \sum_{k=1}^{N} w^k \, \varepsilon_{i\ell m} \, \varepsilon_{ipq} [\Omega_{\ell m}^k - \Lambda_{\ell m}][\Omega_{pq}^k - \Lambda_{pq}]$$

From the properties of the alternating tensor this can be written

$$\theta^2 = \sum_{k=1}^{N} w^k \delta_{\ell p} \, \delta_{mq} [\Omega_{\ell m}^k - \Lambda_{\ell m}][\Omega_{pq}^k - \Lambda_{pq}]$$

$$\theta^2 = \sum_{k=1}^{N} w^k [\Omega^k_{\ell m} - \Lambda_{\ell m}][\Omega^k_{\ell m} - \Lambda_{\ell m}]$$

$$\theta^2 = \sum_{k=1}^{N} w^k [\Omega^{k\,2}_{\ell m} - 2\Lambda_{\ell m}\Omega^k_{\ell m} + \Lambda_{\ell m}{}^2] \ .$$

A choice of $\Lambda_{\ell m}$ which minimizes θ^2 is given by setting

$$\frac{\partial\theta^2}{\partial\Lambda_{pq}} = 0.$$

This leads to the condition

$$\sum_{k=1}^{N} w^k [-2 \frac{\partial\Lambda_{\ell m}}{\partial\Lambda_{pq}} \Omega^k_{\ell m} + 2\Lambda_{\ell m} \frac{\partial\Lambda_{\ell m}}{\partial\Lambda_{pq}}] = 0$$

$$\sum_{k=1}^{N} w^k [-2\Omega^k_{\ell m} \delta_{\ell p}\delta_{mq} + 2\Lambda_{\ell m} \delta_{\ell p}\delta_{mq}] = 0$$

$$\sum_{k=1}^{N} w^k [-2\Omega^k_{pq} + 2\Lambda_{pq}] = 0 \ .$$

The solution to this equation is simply

$$\Lambda_{pq} = \frac{1}{\sum\limits_{k=1}^{N} w^k} \sum_{k=1}^{N} w^k \Omega^k_{pq} \ .$$