THE SENSITIVITY OF A VERY LONG BASELINE INTERFEROMETER

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ABSTRACT

The theoretical sensitivity of various methods of acquiring and processing interferometer data are compared. It is shown that for a fixed digital recording capacity one bit quantization of single sideband data filtered with a rectangular bandpass and sampled at the Nyquist rate yields the optimum signal-to-noise ratio (SNR). The losses which result from imperfect bandpass, poor image rejection, approximate methods of fringe rotation, "fractional bit correction," and loss of quadrature are discussed. Also discussed is the use of the "complex delay function" as a maximum likelihood fringe estimator.

INTRODUCTION

For interferometric observations of continuum radio sources, the signal-to-noise ratio (SNR) is proportional to the square root of the bandwidth as well as depending linearly on the ratio of the geometric mean of the antenna temperatures to the geometric mean of system temperatures. The constant of proportionality depends on the method of processing. It has been shown (Rogers, 1970) the SNR with the maximum likelihood analog processing is given by:

$$SNR = A(2BT)^{1/2} \tag{1}$$

where
A = Correlation amplitude = Ta/Ts
Ta = Geometric mean of antenna temperatures (correlated portion)
Ts = Geometric mean of system temperatures
B = Bandwidth (Hz)
T = Coherent integration time (sec)

The SNR is defined in this equation to be the ratio of the magnitude of the signal vector to the rms of the component of the noise vector normal to the signal vector. Thus in the strong signal case, the rms phase noise (in radians) is (1/SNR). It was also shown (Meeks, 1976) that the best SNR is achieved with single sideband receivers and perfectly rectangular bandpass filters. The magnitude R of the noise vector has a Rayleigh distribution.

$$P(R) = R(e^{-R^2/2}) \tag{2}$$

so that the probability of the noise being falsely interpreted as a signal in a search of N independent channels of delay and delay rate is given by

$$PE = 1 - (1 - e^{-R^2/2})^N \tag{3}$$

which is plotted in figure 1. Thus, an SNR of at least six is required to detect a signal if a large search must be made in delay and rate.

SNR with Digital Data Recording

If the interferometer data has to be stored or transmitted over a link, then SNR will be limited by the recording or transmission channel capacity. The best SNR is achieved by using the largest possible bandwidth without under-sampling. This condition is achieved by using one bit (two-level sampling of infinitely clipped data) quantization of the data sampled at the Nyquist rate. Over sampling the data produces correlation between samples and increases the noise level while under-sampling reduces the signal by aliasing. In this case,

$$SNR = (2/\pi)A(2BT)^{1/2} \tag{4}$$

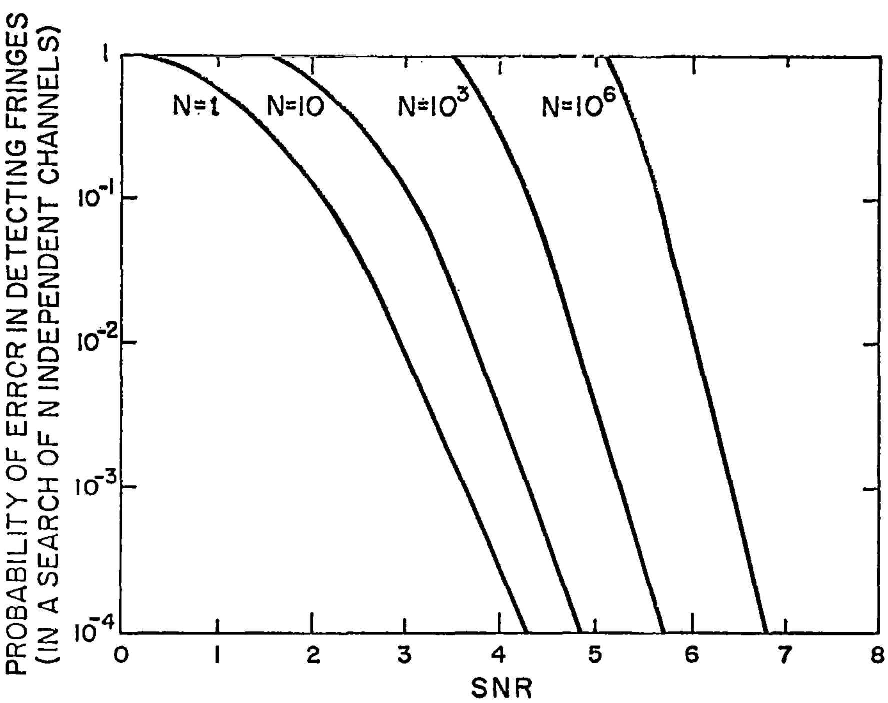

Figure 1.

where 2 BT = the total number of bits processed. The factor $(2/\pi)$ is the clipping factor which results from the one bit quantization of the signal. The signal is reduced by this factor in applying the Van Vleck correction in going from the cross-correlation $pc(\tau)$ of the clipped signals to an estimate of the true cross-correlation function $R(\tau)$ given by

$$R(\tau) = \sin((\pi/2)pc(\tau) \approx (\pi/2)pc(\tau) \text{ when } pc(\tau) << 1. \quad (5)$$

It should be emphasized that while increasing the number of levels in the quantization and oversampling improve the SNR for the spectral-line interferometry, they result in a loss of SNR in the continuum case. For example, with four-level sampling, the quantization degradation factor is only 1.135 (Bowers and Klingler, 1972) as compared with $(\pi/2) = 1.571$ for the two-level case. The spectral line SNR is thereby increased by 1.384 while the continuum SNR is reduced because the $\sqrt{2} = 1.414$ degradation, which results from having to reduce the bandwidth by a factor of two to accommodate the increased number of samples, exceeds the 1.384 gain above. Optimal three-level quantization, however, results in little change in SNR for the continuum and some improvement in the spectral line SNR.

Loss from Imperfect Bandpass

There are two loss factors which result from using a bandpass filter which is not perfectly rectangular in shape. The first is due to aliasing or foldover of noise from frequencies above the bandedge B, while the second loss factor is due to an increase in the statistical dependence of one sample upon the next. The first loss factor is given by

$$L_f = \int_0^B B(\omega)d\omega \Big/ \int_0^B (B(\omega) + B(2B - \omega))d\omega \tag{6}$$

where $B(\omega)$ is the bandpass (power response).

When two data streams x_i and y_i are cross-correlated, the noise correlation n is given by

$$n = \frac{1}{N}\sum_{i=1}^{N} x_i y_i \tag{7}$$

where N = the number of samples correlated

$$n^2 = \frac{1}{N^2}\sum_i \sum_j x_i y_i x_j y_j = \frac{1}{N}\left[1 + 2\sum_{\tau=1} pc\,(\tau)\right] \tag{8}$$

so that the second loss factor is given by

$$Lc = \left[1 + 2\sum_{\tau=1}^{N} pc\,(\tau)\right]^{-1/2} \tag{9}$$

Table 1 shows the results of numerical calculations of the loss which results from the use of Butterworth filters. The loss is reduced as the number of poles are increased, and the two loss factors total 3 percent loss for a 7-pole filter. Other filter types may result in lower loss factors but have phase characteristics that are more nonlinear.

Double Sideband

Double sideband receivers can be used for interferometry, but they result in a SNR loss of $\sqrt{2}$. At first glance, it is difficult to see why there is a reduction in SNR because one can argue that while the noise level is doubled (both sidebands folded on each other), the signal level is also doubled. Actually, the upper and lower sideband signals have opposite fringe rates. Fringes from the two

Table 1
Butterworth Low Pass Filter Losses due to Foldover and Imperfect Bandpass Shape

No. of poles	Optimum location of 3 dB point in point in percent of B	Foldover loss in percent	Loss due to imperfect bandpass in percent	Total loss in percent
2	67	7	5	12
3	79	5	3	8
4	80	3	3	6
5	88	3	2	5
6	90	3	1	4
7	91	2	1	3
9	96	2	0.3	2.3
10	96	1	0.5	1.5

sidebands can be obtained separately and then averaged, but this results in a net SNR which is a factor of $\sqrt{2}$ lower than the optimum single sideband case. Imperfect image rejection in a single sideband interferometer results in some SNR loss. The amount of this loss is given by expressions similar to that of equation (6); that is, the loss results from aliased noise. Some reduction in this loss factor can be achieved if the signals from all the images are also processed and averaged together.

Loss of Quadrature

In order to approach the optimum SNR given by equation (4), the data must be processed in a manner which correctly extracts and coherently adds quadrature components of the interferometer. Complete lack of a quadrature channel degrades the SNR by 2. Incoherent combination of the quadrature channels degrades the SNR by $\sqrt{2}$. Any optimal processing method must completely reject fringe rate images. For example, if an interferometer is observing a radio source, fringes with the opposite fringe rate from an artificial "gedanken" radio source at the same position moving with minus twice the sidereal rate should be completely rejected.

Complex Delay Function as a Maximum Likelihood Estimate

A complex delay function $D(\tau)$ is defined as the time-reversed Fourier transform of the cross-spectral function $Sxy(\omega)$ multiplied by a window function which is unity for positive frequency and zero for negative frequency. Alternately, the delay function can be derived by convolving the cross-correlation function with a complex function whose real and imaginary parts are the noiseless cross-correlation for a fringe phase of 0 and 90 degrees respectively. The delay function is also the likelihood function which, when maximized, gives the best estimate of delay and delay rate.

Loss from Approximate Methods of Fringe Rotation

Most VLBI correlators use approximate sine and cosine functions for fringe rotation. The harmonic content in these approximations results in a small loss in signal. The loss is 4 percent for the three-level approximation used in the Mark II and III correlators.

"Fractional Bit Correction" Loss

When the delay offset between the data streams being correlated is changed, it results in a frequency dependent phase jump which can be corrected without loss by applying continuous correction to the cross-spectral function (Meeks, 1976). Alternately, the fringe rotation phase can be automatically changed by 90 degrees when the delay offset is changed. This simple procedure results in a continuous phase at midband with 45 degree jumps at each edge of the band. The phase averaged over the band is continuous but the SNR is reduced by about 3.5 percent.

Sensitivity of VLBI

(1) Analog System

$$SNR = A(2BT)^{1/2}$$

where

A = Correlation coefficient = TA/Ts
TA= Geometric mean of antenna temperatures
Ts = Geometric mean of system temperatures
B = Bandwidth recorded or transmitted (Hz)
T = Coherent integration time (sec)
$SNR \triangleq 1/(\text{R.M.S. phase noise in radians})$
$PE = 1 - (1 - e^{-SNR^2/2})^N$ – See graph.

(2) Digital System

$SNR = (2/\pi)A(2BT)^{1/2}$ 2-level sampling at Nyquist rate
2BT = Total # bits recorded per station

# Levels	Relative SNR (continuum)	Spectral SNR
2	1	1
3	1	1.271
4	0.979	1.384
2 (oversampled by 2)	0.823	1.164

(3) Loss Factors

(a)	Aliased or folded noise from failure of filter to cut-off at bandedge	2%
(b)	Imperfect shape factor of filter	1%
(c)	Approximations in fringe rotation	
	3-Level done on only one station	4%
	3-Level done on both stations	8%
(d)	"Fractional bit correction"	
	Method 1 – continuous correction	None
	Method 2 – "Autocorrection"	3.5%
(e)	Double-Sideband – If used SNR reduced by $\sqrt{2}$	
(f)	Loss of quadrature – imperfect processing	
	Complete loss — SNR reduced by 2	
	Partial loss — SNR reduced by $\sqrt{2}$	

References

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Meeks, M. L., Methods of Experimental Physics, Vol. 12C Academic Press, Chapter 5, 1976.