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The Unique Determination of Density From Higher-Order Potentials

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SHUTTLE PROGRAM

THE UNIQUE DETERMINATION OF DENSITY FROM HIGHER-ORDER POTENTIALS

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1.0 INTRODUCTION

There is a continuing interest in the determination of the density function of a material body. The usual approach is to attempt to recover the density function from knowledge of the total mass and the Newtonian potential (refs. 1, 2, 3, and 4). It is well known that, in general, one cannot determine the density function from knowledge of only these two items.

In this work we postulate the existence of higher-order potentials, and demonstrate that the density function can be determined from this set of potentials. Moreover, we shall show that there is a one-to-one correspondence between such potentials and the density function, provided it is continuous.

Finally, we note that this demonstration includes a method of constructing the density function from the set of potentials as well as indicating how one might approximate the density from incomplete knowledge of the potentials.

2.0 MODEL AND NOTATION

Suppose A is a material body. We model this by assuming that there is a nonnegative continuous function ρ defined on Euclidean three space denoted by E such that ρ has compact support K, where K has the same geometric shape as A and ρ is the density function for A. This notion precludes A having "point masses," but does not require that A be "in one piece;" i.e., K need not be connected.

We denote the real line by R, and the norm on E, induced by the inner product $\langle \cdot, \cdot \rangle$, by $||\cdot||$.

If r is a positive number and P is a point in E, then

$$S_r(P) = \{Q \text{ in } E: \ ||P - Q|| \leq r\}$$

and

$$\partial S_r(P) = \{Q \text{ in } E: \ ||P - Q|| = r\}$$

3.0 MATHEMATICAL ANALYSIS

Suppose K is a compact set in E, which is the support for a nonnegative continuous function ρ, which is defined on E. Let C be a point in E and r be a positive number such that K is a subset of $S_r(C)$. Let D denote the complement of $S_r(C)$.

Definition 1. The statement that M is the mass of K means M is the number

$$\int_E \rho(Q)dQ$$

We understand that the integral is a triple integral.

Definition 2. For each nonnegative integer r, V_n is the real valued function defined on D by

$$V_n(P) = \int_{S_r(C)} \frac{\rho(Q)}{||P - Q||^n} dQ \qquad \text{for } P \text{ in } D$$

Notice that for each P in D, $V_0(P) = M$, and also that $V_1(P)$ is the Newtonian potential at P due to ρ.

For each positive integer n, we shall refer to V_n as the nth-order potential and the set $\{V_n: n = 2, 3, \cdots\}$ as the higher-order potentials.

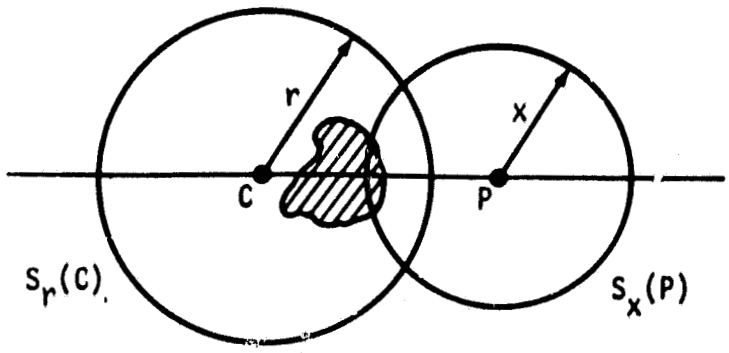
We now state the main purpose of this work.

Theorem. Suppose ρ is a nonnegative continuous function defined on E, which has compact support K. S_1 is a ball containing K, and S_2 is a ball properly containing S_1 and is concentric with S_1. There is a one-to-one correspondence between the set $V = \{ \{V_n(P)\}_{n=0}^{\infty}$. P is in the boundary of a hemisphere of $S_2\}$ and ρ; moreover, ρ can be constructed from V.

Definition 3. Suppose ρ is in D. Let m_ρ be the function defined on R by

$$m_\rho(x) = 0 \qquad \text{if } x \leq 0$$

$$m_\rho(x) = \int_{S_x(P)} \rho(Q)dQ \qquad \text{if } x > 0$$



$$m_P(x) = \int_{S_x(P)} \rho(Q)dQ \qquad \text{if} \quad x > 0$$

Figure 1.- The function m_P.

Observe that for each point P in D, m_P is a real-valued nondecreasing continuous function on R, which has a continuous derivative given by

$$m_P'(x) = 0 \qquad \text{if} \quad x \leq 0$$

$$m_P'(x) = \int_{S_r(C)\cap\partial S_x(P)} \rho(x,z)dz \qquad \text{if} \quad S_r(C) \text{ intersects } \partial S_x(P) \ (S_r(C)\cap S_x(P))$$

$$m_P'(x) = 0 \qquad \text{if} \quad x > 0 \text{ and } \partial S_x(P) \text{ does not intersect } S_r(C)$$

Definition 4. If P is a point in D, then

$$u_P = \inf.\{||P - Q||: \ Q \text{ is in } S_r(C)\}$$

$$v_P = \sup.\{||P - Q||: \ Q \text{ is in } S_r(C)\}$$

Notice that u_P and v_P are readily determined for each P in D.

Using definitions 3 and 4, observe that

$$V_n(P) = \int_{up}^{vp} \frac{1}{t^n}\, dmp(t) \qquad \text{for } n = 0, 1, 2, \ldots$$

Making a change of variables $s = 1/t$ for $up \leq t \leq vp$, we have

$$V_n(P) = \int_{up-1}^{vp-1} s^n dmp(1/s) \qquad \text{for } n = 0, 1, 2, \ldots$$

Definition 5. For each point P in D

$$Ap = vp^{-1}$$

$$Bp = up^{-1}$$

and

$$Mp(x) = M - mp(1/x) \qquad \text{for } Ap \leq x \leq Bp$$

Observe that $Mp(Ap) = 0$, and also that $Mp(Bp) = M$; hence, we may extend Mp to $(0, Bp)$ by defining $Mp(x) = 0$ for $0 \leq x \leq Ap$. Note also that $Mp'(Ap) = Mp'(1/xp)(1/Ap^2) = 0$; hence, Mp has a continuous derivative on $(0, Bp)$.

We now have

$$V_n(P) = \int_0^{Bp} x^n dMp(x) \qquad \text{for } n = 0, 1, 2, \ldots$$

Notice also that $Mp'(Bp) = Mp'(1/Bp)(1/Bp^2) = 0$.

We seek to arrange matters such that we are integrating over the interval $(0, 1)$. We have two options at this point. If we restrict our attention to only those points P outside of $S_{r+1}(C)$, then $B_P = u_P^{-1} < 1$. We then are able to extend M_P to $(0, 1)$ by setting $M_P(x) = M$, if $B_P < x \leq 1$, and to observe that M_P has a continuous derivative on $(0, 1)$, and also that

$$V_n(P) = \int_0^1 x^n dM_P(x) \text{ for } n = 0, 1, 2, \cdots$$

If we do not elect to make this restriction, then by letting $y = x/B_P$ for $0 \leq x \leq B_P$, we have

$$V_n(P) = B_P^n \int_0^1 y^n dM_P(B_P y) \text{ for } n = 0, 1, 2, \cdots$$

If we now define $\underline{V}_n(P) = B_P^{-n} V_n(P)$ for P in D and $n = 0, 1, 2, \cdots$, and define $\underline{M}_P(x) = M_P(B_P x)$ for $0 \leq x \leq 1$, then we have

$$\underline{V}_n(P) = \int_0^1 x^n d\underline{M}_P(x) \text{ for } n = 0, 1, 2, \cdots$$

Let us proceed under the assumption that $B_P < 1$, keeping in mind that M_P so defined is a nonnegative nondecreasing function that has a continuous derivative.

If P is in D, we can construct a sequence of functions $\{M_n\}_{n=1}^{\infty}$ from the number sequence $\{V_n(P)\}_{n=0}^{\infty}$ such that the function sequence $\{M_n\}_{n=1}^{\infty}$ converges uniformly to M_P on $(0, 1)$, and there is a one-to-one correspondence between M_P and $\{V_n(P)\}_{n=0}^{\infty}$. (This construction is further defined in appendix A.) Hence, we have a one-to-one correspondence between M_P' and $\{V_n(P)\}_{n=0}^{\infty}$ for each P in D.

Let $h > r + 1$ and let us restrict our attention to only those points P that are in B, the boundary of a hemisphere of $S_h(C)$. For each point P, let L_P denote the line containing P and C.

Definition 6. For each number x where $S_x(P)$ intersects $S_r(C)$, let $w_P(Q)$, for each point Q in $\partial S_x(P) \cap S_r(C)$, be the point on L_P common to $S_r(C)$ and $\partial S_x(P)$.

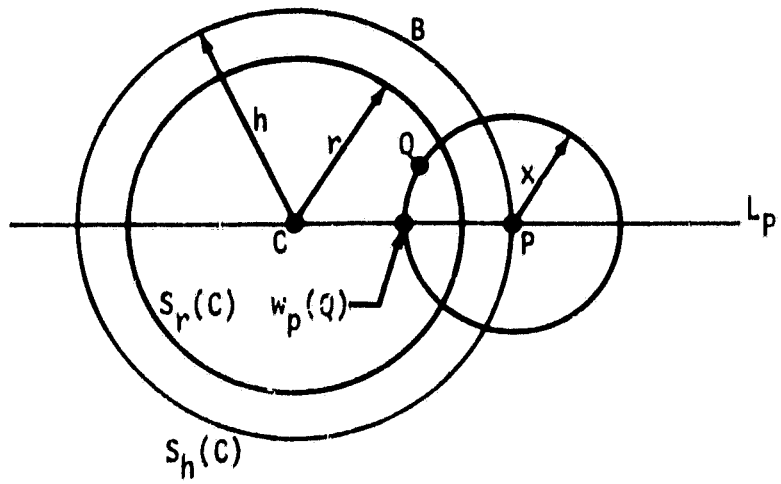


Figure 2.- The function $w_P(Q)$.

Definition 7. For each point P in B, and each number x, let

$$G(P,x) = m_P'(x)$$

Lemma. $G(P,x)$ is continuous and bounded on $B \times (-\infty, \infty)$.

The proof is given in appendix B. Notice that for P in B, $||P - C|| = h$.

Definition 8. For each point P in B, let

$$H_P(x) = m_P'(x + h) \qquad \text{for } x \text{ in } (-\infty, \infty)$$

With this definition we observe that $H_P(0) = m_P'(h)$, and also that

$$\int_{-\infty}^{\infty} Hp(x)e^{-ixd}dx$$

$$= \int_{E} \rho(Q)e^{-i\langle wp(Q),y\rangle}dQ \qquad \text{for } y \text{ in } Lp$$

where d is the number such that $y = du$ with u being a fixed unit vector at C, along Lp.

The first of the above integrals is $(2\pi)^{1/2}\hat{Hp}(d)$, where $\hat{Hp}$ is the Fourier transform of Hp.

Since Hp is zero (except on, at most, a bounded set) and Hp is continuous, it follows that $\hat{Hp}$ is a continuous $L_1(-\infty, \infty)$ function for each P in B (refs. 5 and 6). Moreover, it follows that $\hat{Hp}(d)$, when viewed as a two-place function, is bounded and continuous and is in $L_1(Bx(-\infty, \infty))$, as shown by the lemma.

For each point Y in Lp, let

$$\tilde{\rho}p(y) = (2\pi)^{-3/2} \int_{E} \rho(Q)e^{-i\langle wp(Q),Y\rangle}dQ$$

Definition 8. For each point P in B, let Tp be the transformation from E onto E, which is defined as follows. For each number x let π_x be the plane orthogonal to Lp, containing the point $D_x = (1 - x/||P - Q||)P + (x/||P - Q||)C$. If D_x is not in $S_r(C)$, then Tp is the identity map on D_x. If D_x is in $S_r(C)$, then for each point Q in $S_r(C)\eta\partial S_x(P)$, $Tp(Q)$ is the orthogonal projection of Q onto π_x. If Q is in the plane containing $\partial S_r(C)\eta\partial S_x(P)$ and Q is not in $S_r(C)$, then TpQ is the orthogonal projection of Q onto π_x.

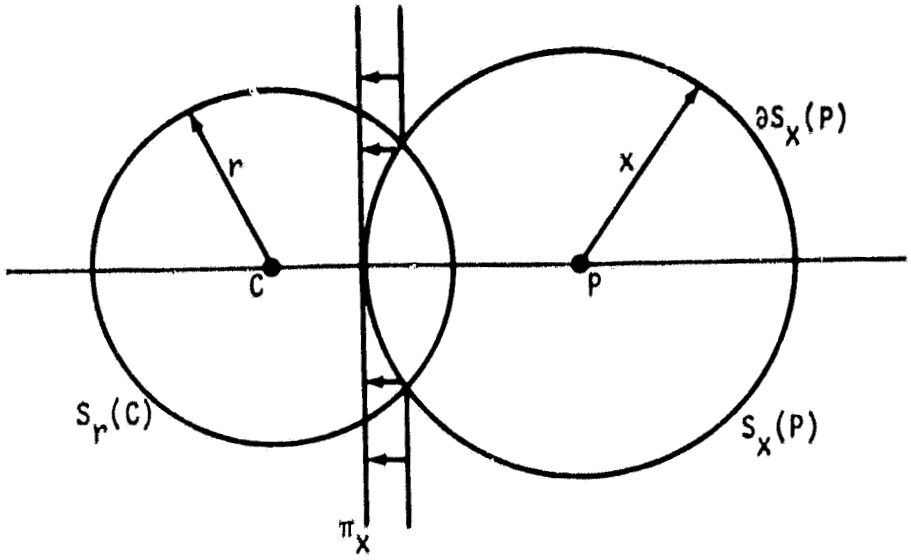


Figure 3.- The transformation Tp.

Notice that for each P in B, Tp is a reversibly continuous mapping from E onto E, which leaves each point in Lp fixed.

If Q is in $S_r(C) \cap \partial S_X(P)$ and Y is in Lp, then

$$\begin{aligned} \langle TQ, TY \rangle &= \langle TQ, Y \rangle \\ &= ||TQ||\ ||Y||\ \cos\theta \\ &= ||TQ||\ \cos\theta\ ||Y|| \\ &= ||w_P(Q)||\ ||Y|| \\ &= \langle w_P(Q),\ Y \rangle \end{aligned}$$

We can now write

$$\tilde{\rho}_P(Y) = (2\pi)^{-3/2} \int_{TE=E} \rho(TQ)e^{-i\langle TQ,Y\rangle}dTQ$$

$$= (2\pi)^{-3/2} \int_{E} \rho(Z)e^{-i\langle Z,Y\rangle}dZ$$

where $Z = TQ$ and Y is in L_P.

Hence,

$$\tilde{\rho}_P(Y) = \hat{\rho}_P(Y)$$

the Fourier transform of ρ for Y in L_P.

Therefore,

$$\hat{\rho}_P(Y) = (2\pi)^{-1} \hat{H}_P(d) \quad \text{for } Y \text{ in } L_P$$

where d is the number such that $dU = Y$.

Notice that if $Y \neq C$, there is only one point P in B such that Y is in L_P.

Definition 9. Let $\hat{H}$ be the function defined on E by

$$\hat{H}(Y) = \hat{H}_P(Y)$$

if $Y \neq C$ and Y is in L_P and $H(\hat{C}) = 0$.

Definition 10. Let $\hat{\rho}$ be the function defined on E by

$$\hat{\rho}(Y) = (2\pi)^{-1}\hat{H}(Y)$$

In review, if for each point P in B we know $\{V_n(P)\}_{n=0}^{\infty}$, then we can determine uniquely $m_p'(x)$ for x in $(-\infty, \infty)$. From this we are able to determine $H'(Y)$ for all Y in E and, hence, $\hat{\rho}$ for all Y in E.

Continuing, suppose Q is in $S_r(C)$ and Q is not C. Then there is a unique point P and B such that Q is in the line L_p and

$$\rho(Q) = (2\pi)^{-3/2} \int_E \hat{\rho}(R) e^{i\langle w_p(R), Q\rangle} dR$$

Recall that $\hat{\rho} = (2\pi)^{-1} \hat{H}$ from which we have

$$\rho(Q) = (2\pi)^{-5/2} \int_E \hat{H}(R) e^{i\langle w_p(R), Q\rangle} dR$$

Hence, for each Q in $S_r(C)$, $Q \neq C$, we can determine $\rho(Q)$ uniquely, and therefore, since ρ is assumed continuous, we have determined ρ uniquely and the argument is complete. Hence, if $\{V_n(P)\}_{n=0}^{\infty}$ is known for each point P in B, the ρ is uniquely determined under the assumptions that ρ is continuous.

4.C COMMENTS

The preceding analysis could have been carried out similarly had we elected not to assume that $B_p < 1$.

Also, the requirement that P belongs to B, the boundary of a ball containing K, is not necessary. What is essential is that for each point Q in $S_r(C)$ Q not C, there be a point P not in $S_r(C)$ such that Q is in the line L_p.

5.0 APPROXIMATION OF DENSITY

Suppose that for each point P in B, only $\{V_n(P)\}_{n=0}^{N}$ is known. Then one can only construct, for each P in B, the first N approximates to M_p. This then leads to only the first N approximates to M_p and, hence, to H_p. One can still proceed, using only these first N approximates since each is bounded by a summable function. Notice that for P in D

$$V_n(P) = \int_0^1 x^n dM_P(x) \qquad \text{for } n = 0, 1, 2, \cdots$$

and, hence

$$V_n(P) = (1/(n+1)) \int_0^1 M_P'(x) dx^{n+1}$$

$$= (1/(n+1))M_P'(1) - (1/(n+1)) \int_0^1 x^{n+1} dM_P'(x)$$

for $n = 0, 1, 2, \cdots$

Therefore, if we let

$$D_n(P) = M_P'(1) - (n+1)\, V_n(P) \qquad \text{for } n = 0, 1, 2, \cdots$$

and P in D, we could, using the construction in appendix A, generate a sequence $\{\bar{M}_n\}_{n=0}^{\infty}$ of functions on $(0, 1)$, which will converge to M_P' on $(0, 1)$.

A second notion is that perhaps for only a finite number of points P in B is $\{V_n(P)\}_{n=0}^{\infty}$ known. Then one could proceed as in the argument except that one interpolates values for $\hat{H}_P$ for those points P in B for which $\{V_n(P)\}_{n=0}^{\infty}$ are not known; then proceed as before.

Both of these notions, as well as the convergence, rates should be addressed.

6.0 ADDITIONAL PHYSICAL NOTIONS

Suppose that for each nonnegative integer n we define

$$F_n(P) = \text{grad } V_n(P) \qquad \text{for } P \text{ in } D$$

Then $F_0(P) = 0$ and $F_1(P)$ is the usual inverse square law. Notice also that

$$F_n(P) = -n \int_{S_r(C)} \frac{\rho(Q)}{||P-Q||^{n+1}} \frac{P-Q}{||P-Q||} dQ$$

for P in D and $n = 1, 2, \cdots$

If g is a real valued analytic function on R,

$$g(x) = \sum_{n=1}^{\infty} a_n x^n$$

then

$$V_g(P) = \sum_{n=1}^{\infty} a_n V_n(P)$$

is well defined, as is

$$F_g(P) = \sum_{n=1}^{\infty} a_n F_n(P)$$

This is readily observed by noting that

$$\sum_{n=1}^{N} a_n V_n(P) = \int_{S_r(C)} \rho(Q) \sum_{n=1}^{N} a_n / ||P-Q||^n \, dQ$$

and, hence

$$V_g(P) = \int_{S_r(C)} \rho(Q) \, g(1/||P-Q||) dQ$$

for P in D.

In a similiar fashion

$$\sum_{n=1}^{N} a_n F_n(P)$$

$$= \int_{S_r(C)} \rho(Q) \sum_{n=1}^{N} a_n (-n)(1/||P-Q||)^{n+1} \frac{P-Q}{||P-Q||} dQ$$

$$= \int_{S_r(C)} \frac{\rho(Q)}{||P-Q||^2} \sum_{n=1}^{N} a_n(-n)(1/||P-Q||)^{n-1} \frac{P-Q}{||P-Q||} dQ$$

and, hence

$$F_g(P) = - \int_{S_r(C)} \frac{\rho(Q)}{||P-Q||^2} g'(1/||P-Q||) \frac{P-Q}{||P-Q||} dQ$$

Let us denote V_g as the sum potential and F_g as the sum force, each relative to g. Notice that if $g(x) = ax$, then we have the familiar Newtonian potential and force (ref. 7).

An interesting question that should be addressed follows; if

$$V(P) = \sum_{n=1}^{\infty} a_n V_n(P)$$

is well defined for all P in D, then is there an entire function g such that

$$g(x) = \sum_{n=1}^{\infty} a_n x^n$$

7.0 CONCLUDING REMARKS

We have demonstrated that if A is a material body (not necessarily having only one component), which has a continuous density function, and A is contained in a sphere S, then if one knows all the potentials (Newtonian and higher orders) for A at each point P of a sufficiently dense set B, then the density function for A can be determined uniquely.

In addition, a notion of sum potential and sum force has been introduced. There is clearly a question as to how to determine the higher-order forces and potentials from experimental data as well as to determine the coefficients of g, if such a g exists (ref. 8).

The notions mentioned here should not be considered limited to density functions of material bodies. If one chooses the alternative approach (i.e., not requiring that $B_P < 1$), then one could consider questions dealing with forces or fields near to sources of such phenomena. It should be noted that the dominant terms then appear to be the higher-order potentials and forces, whereas in dealing with questions about effects far from the source the higher-order potentials and forces appear to play a far smaller role. The material in reference 8 is very interesting with regard to the above comments. It would also be interesting to consider the work in reference 9 in regard to the notions introduced here.

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APPENDIX A

GENERATING A SEQUENCE OF FUNCTIONS

We indicate here a method of generating a sequence of functions $\{M_n\}_{n=1}^{\infty}$, each of which is real valued and defined on $(0, 1)$, from a number sequence $\{C_n\}_{n=0}^{\infty}$. (Please refer to references A-1, A-2, A-3, A-4, A-5, and A-6.)

Given a number sequence $\{C_n\}_{n=0}^{\infty}$, the function sequence $\{M_n\}_{n=1}^{\infty}$ is called the associated function sequence.

Suppose n is a positive integer and x is a number in the number interval $(0, 1)$. There is a unique integer $k(n,x)$ such that

$$k(n,x)/n \leq x < (k(n,x)+1)/n$$

For each positive integer n define a step function M_n on $(0, 1)$ as follows:

$$M_n(0) = 0$$

$$M_n(x) = \sum_{t=0}^{k(n,x)} \binom{n}{t} \sum_{L=0}^{n-t} \binom{n-t}{i} (-1)^i C_{i+t}$$

for x is in $(0, 1)$.

Reference A-4 shows that if M is a real valued function on $(0, 1)$, which has a continuous derivative, and for each nonnegative integer n

$$C_n = \int_0^1 x^n dM(x)$$

then the associated function sequence $\{M_n\}_{n=1}^{\infty}$ converges uniformly to $M - M(0)$ on $(0, 1)$.

APPENDIX A REFERENCES

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APPENDIX B

ESTABLISHING THE LEMMA

The lemma is established here. For all P in B and x in $(-\infty, \infty)$

$$G(P,x) = M_P'(x)$$

If $x \leq 0$ $G(P,x) = 0$ for all P in B.

If $x > 0$ and $\partial S_x(P)$ does not intersect $S_r(C)$, then $G(P,x) = 0$.

Suppose now that $S_x(P)$ intersects $S_r(C)$

$$|G(P,x) = \left| \int_{S_r(C) \cap \partial S_x(P)} \rho(x, z)\, dz \right|$$

$$\leq |P| \left| \int_{S_r(C) \cap \partial S_x(P)} dz \right|$$

and, hence

$$|G(P,x)| < |\rho|\, \ell$$

for all x in $(-\infty, \infty)$ and P in B.

Let us now demonstrate continuity. If x is a number such that $S_r(C) \cap \partial S_x(P)$ is a singleton, then $G(P,x) = 0$.

If $\partial S_x(P)$ does not intersect $S_r(C)$, then $G(P,x) = 0$; and if $x \leq 0$, then $G(P,x) = 0$.

Suppose each of $\overline{P}$ and P is in B, and each of $\overline{x}$ and x is a positive number.

$$|G(P,x) - G(\bar{P},\bar{x})| = \int_{S_r(C)\cap\partial S_x(P)} \rho(x, z)dz$$

$$- \int_{S_r(C)\cap\partial S_{\bar{x}}(\bar{P})} \rho(\bar{x}, z)dz$$

$$\leq \int_{S_r(C)\cap\partial S_x(P)} |\rho(x, z) - \rho(\bar{x}, z)|dz$$

$$+ \int_{S_r(C)\cap\partial S_x(P)} \rho(x, z)dz - \int_{S_r(C)\cap\partial S_{\bar{x}}(P)} \rho(\bar{x}, z)dz$$

$$+ \int_{S_r(C)\cap\partial S_{\bar{x}}(P)} \rho(\bar{x}, z)dz - \int_{S_r(C)\cap\partial S_{\bar{x}}(\bar{P})} \rho(x, z)dz$$

Since ρ is continuous, we may make the first integral as small as we please by choosing $|x - \bar{x}|$ small. Since ρ is continuous and bounded, we may make the second term as small as we please by choosing $|x - \bar{x}|$ sufficiently small with P and x fixed, and by satisfying each of the above we can make the last term as small as we please by choosing $||P - \bar{P}||$ sufficiently small.

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