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TORQUES ON A NEARLY RIGID BODY IN A RELATIVISTIC GRAVITATIONAL FIELD

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I. INTRODUCTION

In order to describe the motion of massive bodies in post-Newtonian approximations to General Relativity and other metric theories of gravity, one is led to define certain relativistic potentials. They simply add to the Newtonian potential, inducing perturbations on the Keplerian motion of a body in the gravity field of another body. Well-known examples of such perturbations are the Einstein and Lense-Thirring rotations of the orbit in inertial space.

The question is whether the same relativistic potentials also induce perturbations in the rotational motion of a nearly rigid body and, if they do, how can these perturbations be observed?

Recently [1] the author has developed an approach for an approximate description of the motion of extended bodies. The approach is based on the local hydrodynamical equations of motion for a perfect fluid in the PPN approximation [2] and, as such, is well adapted to the description of perfect fluid, weakly gravitating bodies, a class to which a large fraction of systems of astronomical interest belong.

After a review of the formalism (Section II), the main result is obtained in Section III: given a body A with spin vector $\underline{S}^A$ and traceless part of the tensor of inertia (or quadrupole moment) $I^A_{\alpha\beta}$ in the field of a body B with mass m_B, the potential which depends on the orientation of the ellipsoid of inertia of A has the structure

$$V = -\underline{\Omega}\cdot\underline{S}^A + f\,\frac{I^A_{\alpha\beta}}{R^3}\,\hat{R}_\alpha\hat{R}_\beta + g\,\frac{I^A_{\alpha\beta}}{R^3}\,\dot{\hat{R}}_\alpha\hat{R}_\beta + h\,\frac{I^A_{\alpha\beta}}{R^3}\,\dot{\hat{R}}_\alpha\dot{\hat{R}}_\beta\,, \tag{1.1}$$

where $\underline{\Omega}$ is defined exactly in (3.14); f, g, and h are scalar functions; $\underline{R}$ is the position vector of the Newtonian center-of-mass of A with respect to the one of B; $\hat{R}$ is the unit vector; and $\dot{\hat{R}} = d\hat{R}/dt$.

Thus the modification of the classical torque-inducing potential:

$$\frac{3}{2}\,\frac{G\,m_B}{R^3}\,I^A_{\alpha\beta}\,\hat{R}_\alpha\hat{R}_\beta\,, \tag{1.2}$$

basically the second term in (1.1), consists of a potential inducing a relativistic spin-precession about the direction of $\underline{\Omega}$, and of two new potentials which induce velocity-dependent torques.

In the case of the Stanford Gyro Relativity Experiment [3], it is known that the detection of the first few terms in $\underline{\Omega}$ and of the classical gravity-gradient term induced by (1.2) is within the capability of the readout system. It is natural to ask whether the velocity-dependent torques could also be measured in the experiment. The answer is negative; the reason is that both g and h depend linearly on the product of velocity of body A (the gyroscope) and the velocity of body B (the Earth), measured in any inertial frame. Because of the large mass of the Earth in comparison to that of the gyro, we can always choose an inertial frame in which the Earth is at rest, so that $g = h = 0$.

This analysis then, in particular, confirms that the only gravitational causes which may affect the rotational motion of the gyroscope are the first few terms of the relativistic spin precession and the Newtonian gravity gradient (1.2).

Throughout this report use will be made of the notion of a nearly rigid body. While in an exact relativistic context Born's concept of rigid rotation has no meaning and dynamical conditions (see [4]) must instead be used, in the approximate framework used here the Born kinematical condition can nevertheless be assumed valid in an approximate sense. Also, deviations from Born's rigid rotation will be assumed as small as terms of second post-Newtonian order.

Units such that $G = C = 1$ are used throughout. Greek indices range from 1 to 3, Latin indices from 0 to 3.

II. FORMALISM

In this section we proceed to summarize some of the definitions and the main results in References 1 and 5.

In post-Newtonian hydrodynamics for a perfect fluid it is convenient to introduce the following four independent variables and an equation of state as the basic set of quantities in a given coordinate system:

$$\begin{aligned} &\rho \quad \text{mass density} \\ &v_\alpha \quad \text{coordinate three-velocity} \\ &\P = \P(\rho) \quad \text{equation of state relating the internal specific energy density } \P \text{ to } \rho. \end{aligned} \tag{2.1}$$

In terms of the basic set (2.1), the following quantities are now introduced:

components of the metric tensor:

$$g_{ab} = g_{ab}(\rho,\ v_\alpha,\ \P) \ , \tag{2.2}$$

gravitational red-shift factor:

$$u^0 := (-\, g_{00} - g_{\alpha\beta}v_\alpha v_\beta - 2g_{0\alpha}v_\alpha)^{-\frac{1}{2}} \ , \tag{2.3}$$

coordinate four-velocity of the matter-stream:

$$v^a := (1,\ v_\alpha) \ , \tag{2.4}$$

pressure defined by the equation of state and the restriction to adiabatic processes:

$$p := \rho d\P / d\ln\rho \ , \tag{2.5}$$

energy-momentum tensor of a perfect fluid:

$$T^{ab} := \rho \left[(u^0)^2 \; (1 + \Pi + p/\rho) \; v^a v^b + (p/\rho) g^{ab} \right] . \qquad (2.6)$$

The energy momentum tensor (2.6) is entirely defined in terms of the basic set (2.1), provided that one knows how to explicitly specify—in the given coordinate system—the metric tensor in the form (2.2), or, in other words, how to solve the field equations. For a large number of systems of astrophysical interest a "slow motion" approximate representation of the type (2.2) is possible in a neighborhood of the material source and in harmonic coordinates by means of an iterative solution method of a reduced Einstein equation [1,6,7] or by means of the method originally devised by Chandrasekhar [8] or Fock [9].

In the following the metric, and correspondingly (2.3) and (2.6), will be considered known in the form (2.2) in the first post-Newtonian approximation.

Besides the Newtonian potential at the point $(\underline{x},t)$

$$U := \int_{t=\mathrm{const}} \rho' |x-x'|^{-1} \, d^3x' \; , \qquad (2.7)$$

one defines the post-Newtonian potential

$$W_\alpha := \int_{t=\mathrm{const}} \rho' v'_\alpha \, |x-x'|^{-1} \, d^3x' \; , \qquad (2.8)$$

which is a gravitational analog of the vector potential in classical electrodynamics.

Let β and γ be PPN parameters ($\beta = \gamma = 1$ in General Relativity), and

$$\rho^*: = \rho\sqrt{-g}\, u^0 = \rho(1 + \frac{v^2}{2} + 3\gamma U) \ , \quad (2.9)$$

the "conserved energy density" [2].

Given an isolated (in the sense of [10]) two-body system (A,B),[1] at any point inside the spatial support of A, say, one can uniquely decompose each potential into the sum of an "internal" and "external" part by simply restricting the domain of integration in (2.7) and (2.8) to the spatial support of body A (internal potential) and of body B (external potential). Let $U^{(A)}$, $W^{(A)}$ denote the external potentials on A, and E_k, E_p, E_I the nonrelativistic kinetic, potential, and internal energy densities, respectively, of a given body, as measured by a comoving observer.

Define for each body an inertial mass-density

$$\rho_I: = \rho^*\left(1 + \frac{E_k + E_p + E_I}{\rho^*}\right) \quad (2.10)$$

along with the corresponding inertial mass

$$m: = \int_{body} \rho_I d^3x \ , \quad (2.11)$$

and a gravitational mass-density

$$\rho_G: = \rho_I(1 + \eta E_p/\rho_I) \qquad (\eta: = 4\beta - \gamma - 3) \quad (2.12)$$

along with the corresponding gravitational mass

$$m^{(G)}: = \int_{body} \rho_G \, d^3x \ , \quad (2.13)$$

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1. The same considerations can be readily extended to any isolated system containing N > 2 bodies.

which in some metric theories of gravity may be different from the inertial mass (2.11) [11].

Let the bodies (A,B) be homogeneous and in rigid rotation, $\underline{a},\underline{b}$ the coordinates of the respective Newtonian centers of mass, and $\underline{\xi}_A,\underline{\xi}_B$ the relative positions in A,B with respect to $\underline{a},\underline{b}$, respectively.

If one defines the PPN linear momentum of A

$(^{\cdot} := d/dt; \quad _{,\alpha} := \partial/\partial x^{\alpha})$

$$P_\alpha := \left(1 + \frac{\dot{a}^2}{2}\right) m_A \dot{a}_\alpha - 2(1+\gamma) \int_A \rho W_\alpha^{(A)} d^3x +$$

$$+ \frac{1}{2} \int_A \rho \int_B \rho' v'_\beta \, |x-x'|_{,\alpha\beta} \, d^3x \, d^3x' +$$

$$+ (1+2\gamma) \int_A \rho v_\alpha U^{(A)} \, d^3x \,, \tag{2.14}$$

and the PPN force on A,[2]

$$F_\alpha := \left[1 + \frac{1+2\gamma}{2}(\dot{a}^2 + \dot{b}^2)\right] \int_A \rho_G \int_B \rho'_G |x-x'|^{-1}{}_{,\alpha} \, d^3x \, d^3x' +$$

$$+ \frac{1-2\beta}{2} \int_A \rho (U^{(A)})^2{}_{,\alpha} \, d^3x +$$

$$+ (1+2\gamma) \int_A \rho \int_B \rho' (\dot{\underline{a}}\cdot\dot{\underline{\xi}}_A + \dot{\underline{b}}\cdot\dot{\underline{\xi}}'_B) |x-x'|^{-1}{}_{,\alpha} \, d^3x \, d^3x' -$$

$$- 2(1+\gamma) \int_A \rho v_\nu W^{(A)}{}_{\nu,\alpha} d^3x +$$

$$+ \frac{1}{2} \int_A \rho v_\beta \int_B \rho' v'_\gamma |x-x'|_{,\alpha\beta\gamma} \, d^3x \, d^3x', \tag{2.15}$$

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2. In the terms of post-Newtonian order the densities of gravitational and inertial mass can be replaced by ρ because the difference is negligibly small in this approximation.

then one can show [1] that the local equations of motion for the material source

$$T^{ab}{}_{;b} = 0 \qquad (2.16)$$

imply

$$\frac{d\vec{P}}{dt} = \vec{F} \qquad (2.17)$$

modulo quantities of second post-Newtonian order.

Expanding the gradient of the potentials in (2.15) into multipoles and considering only the monopole term, one can construct out of (2.17) a Lagrangian which proves to be identical to the so-called "Einstein-Infeld-Hoffmann" Lagrangian [12,13], valid for spherically symmetric field-singularities, provided one identifies the mass (2.11) or (2.13) of a perfect fluid, rigidly rotating body with the "mass" of a field-singularity.

Considering one further term in the multipole expansion of the potentials introduces a relativistic dependence of the Lagrangian on the spin of the bodies. Use of the theory of perturbation of the initial data (i.e., the orbital elements of the osculating Keplerian ellipse) leads to a secular precession of the direction of the orbital angular momentum or, equivalently, of the longitude of the ascending node of the relative orbit.

Since the spin-dependent Lagrangian has been obtained from the equation of the translational motion (2.17), it cannot be used in order to study the rotational motion of the bodies of the system. Caporali and Spyrou [5] have, however, shown that if one introduces a generalized definition of the spin which includes terms of post-Newtonian order, the equations of motion of the spin can be obtained, for each body, from the theorem of conservation of the total angular momentum of the system and the equation of motion of the orbital angular momentum, which is already known from orbital analysis. In this way they were able to show that the resulting equation of motion of the spin of each body was the same as that which could be obtained by standard techniques from the Lagrangian of the translational motion. The proof can in principle be repeated at any order of the multipole expansion. It can then be stated that at any order of the multipole expansion in (2.17) the corresponding Lagrangian contains for each body the correct dependence on the three coordinates of its center of mass and on its three Euler angles.

A Lagrangian with the same dependence on these six degrees of freedom per body was obtained by Barker and O'Connell using a quantum approach to the relativistic equations of motion of extended bodies, assumed to be described by the energy-momentum tensor of spin 1/2 particles [14].

The following section is dedicated to the study of the relativistic rotational motion of the bodies in the isolated system. The strategy will be as follows: in the multipole expansion of the relativistic potentials, the terms depending on the Euler angles of a given body will be singled out from those depending on the translational degrees of freedom to obtain the potentials entering the rotational (Euler) equations of motion of that body.

III. THE POTENTIAL OF THE ROTATIONAL MOTION

In this section I shall sketch the calculation which leads us to recognize the dependence of the relativistic Lagrangian corresponding to (2.17) on the Euler angles of a given body A, say (the formulae valid for B can be obtained from those of A by simply interchanging the labels). Once this dependence has been established, the Euler equations give straightforwardly the time-evolution of the spin of the body.

Let $\underline{a}, \underline{b}$ be the coordinates of the Newtonian center-of-mass of A,B, respectively. Replacing in the force (2.15) the quantity $|x - x'|^{-1}$, by its Taylor expansion in a neighborhood of $(\underline{x} = \underline{a}, \underline{x}' = \underline{b})$, one obtains a collection of terms which can be classified according to whether they are "torque-free" or "torque-inducing".

Let us first examine the torque-free terms; that is, those corresponding to a potential of the form $\underline{\Omega} \cdot \underline{S}^A$, for some $\underline{\Omega}$, $\underline{S}^A$ being the spin of A, defined below. Let

$$\underline{\xi}_A \quad \underline{x} - \underline{a}$$

be the relative position-vector in A.

In (2.15) one recognizes that the terms which have nonzero spin dependence are

$$(1 + 2\gamma)\dot{\underline{a}} \cdot \int_A \rho\, \dot{\underline{\xi}}_A \int_B \rho' \, |x-x'|^{-1}{}_{,\alpha}\, d^3x\, d^3x' \qquad (3.1)$$

and

$$-2(1 + \gamma) \int_A \rho v_\beta \int_B \rho' \, v'_\beta \, |x-x'|^{-1}{}_{,\alpha}\, d^3x\, d^3x' \, . \qquad (3.2)$$

The latter contributes through

$$-2(1+\gamma)\ \dot{\underline{b}}\cdot\int_A \rho\ \dot{\underline{\xi}}_A \int_B \rho' |x-x'|^{-1}{}_{,\alpha}\ d^3x\ d^3x' \tag{3.3}$$

and

$$-2(1+\gamma)\int_A \rho\ \ \dot{\underline{\xi}}_A\cdot \int_B \rho'\ \dot{\underline{\xi}}'_B\ |x-x'|^{-1}{}_{,\alpha}\ d^3x\ d^3x'\ . \tag{3.4}$$

Define the spin according to classical mechanics

$$S^I_\lambda\ := \epsilon_{\lambda\alpha\beta}\int_I \rho\ \xi^I_\alpha\ \dot{\xi}^I_\beta\ d^3x \qquad\qquad I = A,B.$$

and the tensors as

$$I^B_{\nu_1\ldots\nu_n} = \int_B \rho\ \xi^B_{\nu_1}\ldots\xi^B_{\nu_n}\ d^3x\ ,$$

$$P^B_{\alpha\nu_1\ldots\nu_n} = \int_B \rho\ \dot{\xi}^B_\alpha\ \xi^B_{\nu_1}\ldots\xi^B_{\nu_n}\ d^3x \qquad (n\ge 2) \tag{3.5}$$

which are symmetric in the indices $\nu_1\ldots\nu_n$.

The spin-dependent part of (3.1) or (3.3) is the gradient of

$$-\frac{1}{2}\epsilon_{\mu\alpha\gamma}\ S^A_\gamma\ (m_B\ \frac{R}{R^3}{}_\mu - \sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\ I^B_{\nu_1\ldots\nu_n}\partial^{(n+1)}{}_{\nu_1\ldots\nu_n\mu}R^{-1}) \tag{3.6}$$

contracted with $\dot{\underline{a}}$, $\dot{\underline{b}}$, respectively.

Only the traceless part of the I's, defined by Reference 15,[3]

$$I\!\!\!-^B_{\nu_1\ldots\nu_n} := \sum_{k=2}^{\mathrm{int}(n/2)}(-1)^k\frac{\binom{n}{k}\binom{n}{2k}}{\binom{2n}{2k}}\ \delta_{(\nu_1\nu_2}\ldots\delta_{\nu_{2k-1}\nu_{2k}}I^B_{\nu_{2k+1}\ldots\nu_n)\mu_1\mu_1\ldots\mu_n\mu_n} \tag{3.7}$$

---

3. int $(\frac{m}{2})$ denotes the integer part of $\frac{m}{2}$; parentheses on indices denote complete symmetrization.

is involved in (3.6). Also, introducing the operator

$$\mathcal{I}^B := \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \mathcal{I}^B_{\nu_1 \ldots \nu_n} \partial^{(n)}_{\nu_1 \ldots \nu_n} \tag{3.8}$$

and the notation

$$\langle \dot{\underline{a}}\, \mathcal{I}^B, \nabla R^{-1} \rangle_\beta := \varepsilon_{\mu\alpha\beta}\, \dot{a}_\alpha \mathcal{I}^B (\partial_\mu R^{-1}) \;, \tag{3.9}$$

the spin-dependent part of (3.1) is the gradient of

$$- \frac{1+2\gamma}{2} \left(m_B \frac{\underline{R} \times \dot{\underline{a}}}{R^3} - \langle \dot{\underline{a}}\, \mathcal{I}^B, \nabla R^{-1} \rangle \right) \cdot \underline{S}^A \tag{3.10}$$

and likewise for (3.3).

The spin-dependent part of (3.4) is given by the gradient of

$$-2(1+\gamma) \left[\frac{1}{4} S^A_\alpha S^B_\beta \partial^2_{\alpha\beta} R^{-1} - \frac{1}{2} \varepsilon_{\mu\alpha\beta} S^A_\alpha \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} P_{\alpha\nu_1 \ldots \nu_n} \partial^{(n+1)}_{\nu_1 \ldots \nu_n \mu} R^{-1} \right]. \tag{3.11}$$

Again, only the traceless part $\underline{P}$ of the P's in $(\nu_1 \ldots \nu_n)$, defined according to the same rule as in (3.7), contributes in (3.11). Defining the vector-valued operator $\underline{\mathcal{P}}$ as in (3.8)

$$\underline{\mathcal{P}}^B := \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \underline{P}^B_{\alpha\nu_1 \ldots \nu_n} \partial^{(n)}_{\nu_1 \ldots \nu_n} \tag{3.12}$$

and using the same notation as (3.9), with $\underline{\mathcal{P}}$ replacing $\dot{\underline{a}}\mathcal{I}$, the spin-dependent part of (3.6) can finally be written as the gradient of

$$- (1+\gamma) \left[\frac{1}{2} S^B_\beta \partial^2_{\alpha\beta} R^{-1} - \langle \underline{\mathcal{P}}^B, \nabla R^{-1} \rangle_\alpha \right] S^A_\alpha \;. \tag{3.13}$$

Introduce the total and reduced mass

$$m_o = m_A + m_B \qquad \mu = m_A m_B / m_o \tag{3.13}$$

and center-of-mass coordinates:

$$m_A a_\alpha = \mu R_\alpha = -m_B b_\alpha \ .$$

The torque-free potential is finally given exactly defining

$$\underline{\Omega} = -\left\{ -\left[\frac{1+2\gamma}{2}\frac{m_B}{m_A} + 1 + \gamma\right] \mu \frac{\underline{R} \times \dot{\underline{R}}}{R^3} + \frac{1+\gamma}{2}\left[\underline{S}^B - 3(\underline{S}^B \cdot \hat{\underline{R}})\hat{\underline{R}}\right] - \right.$$

$$\left. - \frac{1+2\gamma}{2}\frac{\mu}{m_A} < \dot{\underline{R}}\, I^B, \nabla R^{-1} > + (1+\gamma) < \frac{\mu}{m_B} \dot{\underline{R}}\, I^B + \underline{p}^B, \nabla R^{-1} > \right\} \tag{3.14}$$

The torque-free potential is then $V_{TF} = -\underline{\Omega} \cdot \underline{S}^A$. In an inertial frame locally comoving with A, the torque-free potential induces a precession of the spin of A about the direction of $\underline{\Omega}$:

$$\frac{d\underline{S}^A}{dt} = \underline{\Omega} \times \underline{S}^A \ .$$

The first two terms in the right-hand side of (3.14) have already been computed, though by means of different techniques [16,17]. The lowest order contributions to $\underline{\Omega}$ from the last two terms can be written defining

$$Q_\gamma := \frac{2 I^B_{\beta\gamma} - 5 I^B_{\beta\alpha} \hat{R}_\alpha \hat{R}_\gamma}{m_B R} \hat{R}_\beta \ . \tag{3.15}$$

One obtains:

$$\underline{\Omega} = \frac{\mu}{R^3}\left(\frac{1+2\gamma}{2}\frac{m_B}{m_A} + 1+\gamma\right)(\underline{R} \times \dot{\underline{R}} - \frac{3}{2}\underline{Q} \times \dot{\underline{R}}) - \frac{1+\gamma}{2R^3}\left[\underline{S}^B - 3(\underline{S}^B \cdot \hat{\underline{R}})\hat{\underline{R}}\right] +$$

$$+ R^{-2}\, O[(\ell/R)^3] \ . \tag{3.16}$$

This expression agrees with a formula obtained by Barker and O'Connell when $m_A << m_B$ [18].

Let us consider now the torque-inducing potential; that is, the one depending on the instantaneous orientation of the ellipsoid of inertia of A w.r.t. the position of B and vice versa. An exact formula in the same fashion as (3.14) must represent the coupling of all the multipoles of A with those of B, and it is not very illuminating, in general. In the following I shall consider for the bodies the multipoles up to and including the quadrupole, neglecting in the potential terms of order $O[(\xi/R)^3]$.

It is convenient to define

$$M_{\alpha\beta} := \frac{1}{2}\left(I^{A}_{(G)\alpha\beta}\, m_B{}^{(G)} + I^{B}_{(G)\alpha\beta}\, m_A{}^{(G)} \right), \qquad (3.17)$$

where the label G indicates that the mass and the quadrupole moment correspond to the gravitational mass density (2.12).

The contribution to the torque-inducing potential coming from the first term in (2.15) is readily seen to be

$$\left[1 + \frac{1+2\gamma}{2}(\dot{a}^2 + \dot{b}^2)\right] M_{\alpha\beta}\, \partial^2_{\alpha\beta} R^{-1} + R^{-2}\, O[(\xi/R)^3]$$

modulo terms of second post-Newtonian order.

Likewise, the contribution from the second term in (2.15) is

$$(1-2\beta)\, \frac{m_B}{R} M_{\alpha\beta}\, \partial^2_{\alpha\beta} R^{-1} + R^{-2}\, O[(\xi/R)^3] ,$$

while the third integral gives no contribution, at this order.

Finally, the contributions from the fourth and fifth integral in (2.15) are, respectively,

$$-(1+\gamma)\ \underline{\dot{a}}\cdot\underline{\dot{b}}\ M_{\alpha\beta}\ \partial^2_{\alpha\beta}\ R^{-1} + R^{-2}\ O[(\xi/R)^3]$$

and

$$\frac{1}{2}\ \dot{a}_\alpha \dot{b}_\beta\ M_{\mu\nu}\ \partial^4_{\alpha\beta\mu\nu}\ R + R^{-2}\ O[(\xi/R)^3]\ .$$

Writing

$$M_{\hat{R}\nu} := M_{\mu\nu}\ \hat{R}_\mu \qquad\qquad M_{\hat{R}\hat{R}} := M_{\hat{R}\nu}\ \hat{R}_\nu\ ,$$

the torque-inducing potential is:

$$V_T = \left[1 + \frac{1+2\gamma}{2}(\dot{a}^2 + \dot{b}^2) + (1-2\beta)\frac{m_B}{R} - (1+\gamma)\ \underline{\dot{a}}\cdot\underline{\dot{b}} + \frac{1}{2}\dot{a}_\alpha \dot{b}_\beta(\delta_{\alpha\beta} - 5\hat{R}_\alpha \hat{R}_\beta)\right]$$

$$\cdot \frac{3M_{\hat{R}\hat{R}}}{R^3} - \frac{M_{\alpha\beta}}{R^3}\dot{a}_\alpha \dot{b}_\nu(\delta_{\beta\nu} - 6\ \hat{R}_\nu \hat{R}_\beta) + R^{-2}\ O[(\xi/R)^3]\ . \qquad (3.18)$$

Again, making use of the center-of-mass coordinates, we note that

$$\dot{R}_\alpha = \dot{R}\ \hat{R}_\alpha + R\ \dot{\hat{R}}_\alpha$$

so that

$$\mathcal{I}_{\alpha\beta}\ \dot{R}_\alpha \hat{R}_\beta = \dot{R}\ \mathcal{I}_{\hat{R}\hat{R}} + R\ \mathcal{I}_{\dot{\hat{R}}\hat{R}} \qquad (3.19)$$

and

$$\mathcal{I}_{\alpha\beta}\ \dot{R}_\alpha \dot{R}_\beta = \dot{R}^2\ \mathcal{I}_{\hat{R}\hat{R}} + 2\ \dot{R}\ R\ \mathcal{I}_{\dot{\hat{R}}\hat{R}} + R^2\ \mathcal{I}_{\dot{\hat{R}}\dot{\hat{R}}}\ . \qquad (3.20)$$

The first term in the right-hand side of (3.19) and (3.20) contributes to the first term in the right-hand side of (3.18). By contrast, $\Phi_{\dot{\hat{R}}\hat{R}}$ and $\Phi_{\dot{\hat{R}}\dot{\hat{R}}}$ introduce in the torque-inducing potential a nonclassical dependence of the orientation of the ellipsoid of inertia of one body with respect to the instantaneous position and velocity of the other body.

If the body A has a mass much smaller than that of B, the last term in (3.18) tends to zero and can be neglected. In fact, it will always be possible to choose an inertial system in which $\dot{\underline{b}} = 0$. Accordingly, the velocity-dependent, torque-inducing potentials will not be involved in the equation of the rotational motion of A.

IV. CONCLUSION

We have proved that in the post-Newtonian approximation the potential governing the rotational motion of an extended body of mass m_A, spin $\underline{S}^A$ and quadrupole moment $I^{(G)}_{\alpha\beta}$ in the relativistic field of a mass $m_B^{(G)}$ has the structure

$$V = - \underline{\Omega}\cdot\underline{S}^A + \frac{m_B^{(G)}}{R^3} I^{(G)}_{\alpha\beta}\ \dot{R}_\alpha \dot{R}_\nu\ (\delta_{\nu\beta} - 6\ \hat{R}_\nu \hat{R}_\beta)\ \mu/m_o\ +$$

$$+ [1 + F(\beta,\ \gamma,\ \text{orbital parameters})]\ 3I^{(G)}_{\hat{R}\hat{R}}\ m_B^{(G)}/2\ R^3\ . \quad (4.1)$$

The potential (4.1) is then the post-Newtonian generalization of (1.2).

An exact expression for $\underline{\Omega}$ has been derived in Section III in terms of the differential operators (3.8), (3.12).

The second term in (4.1) is responsible for a nonclassical dependence of the orientation of the ellipsoid of inertia of one body with respect to the instantaneous position and velocity of the other. When the body A has a much smaller mass than that of B, the effect of this term on the rotational motion of the body tends to zero.

The fact that the nodal line of an orbiting body, defined by the intersection of the equatorial plane and orbital plane, rotates on the orbital plane is confirmed by the third term in (4.1). Only the frequency of rotation needs to be redefined. In practice, this consists of multiplying the classical frequency by the sum of 1 and the numerically small function, of order $(V/C)^2$ contained in the parentheses.

It is worth noting that the framework developed in Reference 1 leads to the association with each body of not only a gravitational mass but also of a countable set of infinitely many gravitational multipole momenta. This fact is a consequence of the existence of a gravitational mass density (2.12) which may differ from the inertial one (2.10) in some metric theories of gravitation.

Observable consequences on the relativistic rotational motion of a small, nearly rigid body can be induced only by the torque-free term $-\underline{\Omega}\cdot\underline{S}$ under the form of a precession of the spin $\underline{S}$ about the direction of $\underline{\Omega}$. The feasibility studies of the Stanford Gyro Relativity Experiment (see, e.g., [3]) have pointed out that this precession can be observed as a variation in the orientation of the London magnetic moment of a spinning quartz ball in geodetic motion about the Earth. Careful evaluation of the attainable level of accuracy has shown the feasibility of a measure of the leading part (3.16) of the vector $\underline{\Omega}$.

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APPROVAL

TORQUES ON A NEARLY RIGID BODY IN A RELATIVISTIC GRAVITATIONAL FIELD

By Alessandro Caporali

The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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