ITERATIVE METHODS BASED UPON RESIDUAL AVERAGING

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This paper concerns iterative methods for solving boundary value problems for systems of nonlinear partial differential equations. The methods involve subtracting an average of residuals from one approximation in order to arrive at a subsequent approximation.

The paper is divided into five parts. The first part gives two abstract methods in Hilbert space. The second part shows how to apply these methods to quasilinear systems to give numerical schemes for such problems. The third section contains some specific applications. The fourth part contains a discussion of some potential theoretic matters related to the iteration schemes. The final part indicates work in progress concerning extensions and improvements of the above.

1. Two abstract iterative schemes. Suppose H is a Hilbert space, H′ a closed subspace of H and P is an orthogonal projection on H whose range is a subset of H′. Suppose also that L is a strongly continuous function from H to L(H,H) so that L(U) is an orthogonal projection for each U in H. It will be seen how a variety of boundary value problems for nonlinear systems may be reduced to the problem of finding $U \in H'$ so that

$$L(U)U = 0, \quad P(U-W) = U-W \tag{1}$$

where W is a given element of H′. It will be seen that the first part of (1) represents a quasilinear system and the second part of (1) is a way of asserting that U satisfies boundary conditions described by the given element W.

For $\delta > 0$ an iterative scheme for attempting to find U satisfying (1) is

$$W_0 = W, \quad W_{n+1} = W_n - \delta PL(W_n)W_n, n = 0,1,2, \ldots . \tag{2}$$

If $\{W_n\}_{n=0}^{\infty}$ converges to $U \in H'$, then

$$P(U-W) = U-W \text{ and } PL(U)U = 0. \tag{3}$$

A solution U to (3) is called a quasisolution to the problem (1). See ref. 1 for a discussion concerning quasisolutions vs. actual solutions.

A second scheme uses a continuous iteration parameter but is otherwise similar to (2): Define $Z : (0,\infty) \to H$ so that

$$Z(0) = W, Z'(t) = -PL(Z(t))Z(t), \quad t \geq 0. \tag{4}$$

If $U = \lim_{t\to\infty} Z(t)$ exists, then U satisfies (3).

For numerical schemes one is interested in finite dimensional choices for (2) under fairly unrestrictive hypothesis on P and L. In ref. 2 it is shown that (2) always converges in the linear ($L(x)$ independent of $x \in H$) for $\delta = 1$. Similar results may be obtained for (4) by noting that (4) is a limiting case of (2) as $\delta \to 0$.

2. Quasilinear systems; use of finite differences. It is first indicated how a fairly general second order quasilinear system may be placed in a setting to which the iterative schemes (2), (4) apply. Extensive generalizations will be evident.

Suppose Ω is a bounded open subset of R^2 and each of R,S,T is a continuous real-valued function on R^3. Functions z,u,v on Ω are sought so that

$$(5) \qquad \begin{cases} R(z,u,v)u_1 + S(z,u,v)(u_2 + v_1) + T(z,u,v)v_2 = 0 \\ z_1 - u = 0 \\ z_2 - v = 0 \end{cases}$$

where $u_1 = \delta u/\delta x$, $u_2 = \delta u/\delta y$ etc.

If appropriate derivatives exist and (5) holds, then

$$(6) \qquad R(z,z_1,z_2)z_{11} + 2S(z,z_1,z_2)z_{12} + T(z,z_1,z_2)z_{22} = 0 .$$

Pick two piecewise smooth one-dimensional curves Γ and Γ' in $\overline{\Omega}$ and a function $W \in C(\overline{\Omega})$. Consider boundary conditions for (5):

$$z(p) = w(p), p \in \Gamma$$

$$(7) \qquad \left\langle \begin{pmatrix} u(p) \\ v(p) \end{pmatrix}, \begin{pmatrix} f(p) \\ g(p) \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} w_1(p) \\ w_2(p) \end{pmatrix}, \begin{pmatrix} f(p) \\ g(p) \end{pmatrix} \right\rangle, p \in \Gamma$$

where $\begin{pmatrix} f(p) \\ g(p) \end{pmatrix}$ denotes the direction normal to Γ' at p.

Define $A: R^3 \to L(R^9, R^3)$ modeled on (5) so that

$$A(r,s,t)(a,a_1,a_2,b,b_1,b_2,c,c_1,c_2) = \begin{cases} R(r,s,t)b_1 + S(r,s,t)(b_2 + c_1) + T(r,s,t)c_2 \\ a_1 - b \\ a_2 - c \\ (r,s,t),(a,a_1,a_2),(b,b_1,b_2),(c,c_1,c_2) \in R^3 . \end{cases}$$

Note that if $z,u,v \in C^{(1)}(\overline{\Omega})$ and

$$A(z(p),u(p),v(p))$$
$$(z(p),z_1(p),z_2(p),u(p),u_1(p),u_2(p),v(p),v_1(p),v_2(p)) = 0,$$

then z,u,v satisfy (5).

Denote $L_2(\Omega)^9$ by H, denote by H′ the set of all $(z,z',u,u',v,v') \in H$ where $z' = (z_1,z_2)$ etc. and all indicated derivatives are L_2 generalized derivatives (cf ref. 3). Denote by H_0' the set of all $(z,z',u,u',v,v') \in H'$ so that

$$z(p) = 0,\ p \in \Gamma$$
$$\left\langle \begin{pmatrix} u(p) \\ v(p) \end{pmatrix}, \begin{pmatrix} f(p) \\ g(p) \end{pmatrix} \right\rangle = 0,\ p \in \Gamma'$$

and denote by P the orthogonal projection of H onto H_0'.

To complete a description of how (5), (7) are carried over to (1) a description of L is required. Denote $L_2(\bar{\Omega})^3$ by K and define $C: H \to L(H,K)$ so that if $U,Z \in H$, then for almost all $p \in \Omega$,

$$(C(U)Z)(p) = (A(q)A(q)^*)^{-\frac{1}{2}} A(q)' z_1(p)$$

where $q \equiv (r,s,t)$ and r,s,t are the first, fourth and seventh elements respectively of U(p). Finally for $U \in H$, $L(U) \equiv C(U)^* C(U)$.

For w as above, define $W = (w,w_1,w_2)$. Start iteration (2) with W. Then for $n = 0,1,2,\ldots$, W_n has the property that the triple consisting of the first, fourth and seventh elements of W_n satisfy (7). Similar statements hold for the iteration (4).

It is now indicated how a finite difference scheme for (5),(7) may be constructed by defining finite dimensional spaces $\underline{H}$ and $\underline{K}$ which approximate H and K above. Suppose G_0 is a rectangular grid with even spacing δ so that $G \equiv G_0 \cap \bar{\Omega}$ has the property that if $p \in G$, then at least one of $p + \delta e_i$ is in $G, i = 1,2$, where e_1,e_2 is the standard basis for R^2. Define $\underline{K}$ to be a vector space of all real-valued functions on the grid G. For $u \in \underline{K}$, define

$$(D_i u)(p) = \begin{cases} (u(p+\delta e_i) - u(p-\delta e_i))/(2\delta) & \text{if } p+\delta e_i, p-\delta e_i \in G \\ (u(p+\delta e_i) - u(p))/\delta & \text{if } p-\delta e_i \notin G \\ (u(p) - u(p-\delta e_i))/\delta & \text{if } p+\delta e_i \notin G,\ i = 1,2,\ p \in G. \end{cases}$$

Define $\underline{H} = \underline{K}^9$. For $(z,u,v) \in \underline{K}^3$, define $D(z,u,v) = (z,D_1z,D_2z,u,D_1u,D_2u,v,D_1v,D_2v)$. Denote by $\underline{H}'$ the range of D. Define $\underline{\Gamma},\underline{\Gamma}'$ subsets of G approximating Γ and Γ' respectively. Denote by H_0' the set of all $D(z,u,v) \in \underline{H}'$ such that

$$z(p) = 0,\ p \in \underline{\Gamma}$$
$$\left\langle \begin{pmatrix} u(p) \\ v(p) \end{pmatrix}, \begin{pmatrix} f(p) \\ g(p) \end{pmatrix} \right\rangle = 0,\ p \in \underline{\Gamma}' \tag{8}$$

Denote by $\underline{P}$ the orthogonal projection of $\underline{H}$ onto H_0'. Pick $\underline{w} \in K$ approximating w above. Define $\underline{z} = \underline{w}$, $\underline{u} = D_1\underline{w}$, $\underline{v} = D_2\underline{w}$ and choose $\underline{W}_0 = (\underline{z},D_1\underline{z},D_2\underline{z},\underline{u},D_1\underline{u},D_2\underline{u},\underline{v},D_1\underline{v},D_2\underline{v})$ and

$$\underline{W}_{n+1} = \underline{W}_n - PL(\underline{W}_n)\underline{W}_n,\ n = 0,1,2,\ldots \tag{9}$$

where L is defined essentially as above. Condition (8) on $\underline{P}$ implies that boundary conditions are preserved under the iteration (9) and hence are satsified by a limit of $\{W_n\}_{n=0}^{\infty}$.

Similar statements hold for a finite dimensional counterpart to the iteration (4).

Process (4) in this finite dimensional setting becomes a variant of the "method of lines". It specifies one equation and one 'unknown' for each point in the grid G. The 'time' parameter is iteration number, not a distinguished variable in the system of differential equations. Use of (4) then may extend the use of the 'method of lines' to a larger class of problems.

3. Applications.

Take Ω to be a bounded region in R^2. Define R,S,T in (5) so that

$$R(z,u,v) = 1 + v^2$$
$$S(z,u,v) = -uv$$
$$T(z,u,v) = 1 + u^2.$$

System (5) then is

$$(1+v^2)u_1 - uv(u_2+v_1) + (1+u^2)v^2 = 0$$
$$z_1 - u = 0$$
$$z_2 - v = 0$$

As a single second order equation this is

$$(1+z_2^2)z_{11} - 2z_1z_2z_{12} + (1+z_1^2)\, z_{22} = 0,$$

the minimal surface equation for real-valued functions on a region in R^2. Conditions are specified by

$$z(p) = f(p), p \in \partial\Omega,$$

for some given function f. The FORTRAN code listed in ref. 4 may be easily modified to deal with this equation.

If $\gamma, a_\infty, u_\infty$ are given positive numbers and R,S,T are chosen so that

$$R(z,u,v) = a_\infty^2 + ((\gamma-1)/2)(u_\infty^2-u^2-v^2) - u^2$$
$$S(z,u,v) = -uv$$
$$T(z,u,v) = a_\infty^2 + ((\gamma-1)/2(u_\infty^2-u^2-v^2) - v^2$$

then (5) reduces to the transonic flow equation used in reference 4 (and taken from reference 5). For numerical computations, boundary conditions at infinity are replaced by appropriate boundary conditions on the boundary of a large box. One also has zero normal derivative conditions on an airfoil inside the box. See references 4, 5 for details. The FORTRAN listing in reference 4 is specifically for this problem. Printouts of results for various mach numbers (u_∞/a_∞) are given there.

4. Finite dimensional potential theory. The main computational effort connected with (9) is the calculation of Px for various $x \in H$. Denote by $\underline{J}_0$ all $(z,u,v) \in K^3$ satisfying (8) and denote by π the orthogonal projection of $\underline{K}^3$ onto J_0. From ref. 2 it follows that $P = DE^{-1}\pi D^*$ where $E \equiv \pi D^*D|_{\underline{J}_0}$. Hence the main work in calculating the action of $\underline{P}$ is the solving for x (given y) in linear systems

$$Ex = y. \tag{10}$$

Now $\underline{J}_0$ is a Dirichlet space in the sense of ref. 6 and E is the corresponding Laplacian for $\underline{J}_0$. So, E^{-1} being the inverse of a Laplacian, the effect of multiplying a vector y by E^{-1} is to take a certain nonnegative weighted average of the components of y. References 2 and 4 contain descriptions of methods for solving (10).

5. Extensions and Improvements.

A promising replacement for (4) is given by

$$z(0) = wz'(t) = -(\nabla\phi)(z(t)), \ t \geq 0 \tag{11}$$

where $\sigma(x) \equiv \frac{1}{2}\|A(x)x\|^2, x \in H$, A being defined as in section 3. One has the following explicit expression for the gradient of ϕ:

$$(\nabla\phi)(x) = P[A(x)^* + C(x)^*]A(x)x, \ x \in H.$$

Then (11) becomes a steepest descent process. In a number of examples, the only critical points of ϕ seem to be solutions to (3). Furthermore, solutions z to (3) remain bounded and so converge to a solution u to (3).

Work is in progress concerning the adaptation of (2), (4) and (11) to finite element spaces rather than finite difference schemes. It is expected that methods will be developed which use finite element spaces but have little else in common with conventional finite element methods. See reference 7 for some preliminary results.

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