POST-BUCKLING BEHAVIOR OF A BEAM-COLUMN ON A NONLINEAR ELASTIC FOUNDATION WITH A GAP[1]

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SUMMARY

The subject of this paper is the structural behavior of an elastic beam-column placed with a gap between two nonlinearly elastic layers each resting on a rigid foundation. The beam-column is laterally supported at both ends and subjected to a uniform transverse load and axial compression. Its slenderness is such that the axial compressive force exceeds the amount that would be necessary to buckle it as a simply supported column. The elastic layers are represented by an elastic foundation with a strongly nonlinear specific reaction taken as a rapidly increasing function of the layer compression. The analytical model developed simulates the entire pattern of the deflection and stress state including layer and end support reactions, under gradually increasing axial force.

INTRODUCTION

There are many cases when a primary buckling mode occurring at the onset of buckling cannot develop freely (References 1-2) because of changing constraint or support conditions. Such is, in particular, the case of a column with lateral supports arranged with gaps, etc. In this case, the post-buckling deflection is constrained laterally and the axial force can be increased by far in excess of its first critical value. As a result, the structural behavior is characterized by a sequence of alternating gradual changes in the deformed configuration and rather abrupt jumps from one equilibrium configuration to another. A similar behavior pattern was observed (Reference 3) for a compressed plate.

The subject of this paper is an elastic beam-column placed with a gap between two nonlinearly elastic layers each resting on a rigid foundation (Figure 1). The beam-column is laterally supported at both ends and subjected

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to a uniform transverse load and axial compression. The elastic layers have a strongly nonlinear specific reaction taken as a rapidly increasing function of the layer compression. The problem consists in the analytical evaluation of the stress state and deflection under gradually increasing axial compression.

The deflection of the beam-column in consideration is small enough to justify the use of the conventional linearized expression for the curvature. However, there are two other sources of nonlinearity (nonlinearly elastic layers and the presence of a gap) which were fully accounted for. Note, that the presence of a transverse load makes the problem nonhomogeneous so that it is not a bifurcation problem.

ANALYTICAL FORMULATION AND SOLUTION METHOD

Under the above assumptions, vertical equilibrium of the beam-column requires that

$$F(Y) \cong EIY(x)^{IV} + PY(x)^{II} + Q(Y(x)) - G = 0 \tag{1}$$

Here Y is the elastic deflection of the beam-column, EI is its flexural rigidity, P is the axial compression force, Q(Y) is the foundation reaction per unit length as a function of Y, G is the distributed transverse load (assumed uniform and constant), x' is the column longitudinal axis and prime denotes differentiation with respect to x.

The elastic foundation reaction is taken in the following form:

$$Q(Y) = \begin{cases} 0 & |Y| \le c \\ (\operatorname{sgn} Y)\ k\ (|Y| - c)^n & |Y| > c \end{cases} \tag{2}$$

where k and n are given constants and c is the gap size. Thus, the foundation reaction is proportional to a power of the beam-column penetration into the elastic layer.

The self-correcting finite increment method (Reference 4) will now be applied for solving Equation (1). To this end the compression force P is given an infinitesimal increment p which results in some infinitesimal variation y(x) of deflection Y(x):

$$F(Y+y) \equiv EI(Y+y)^{IV} + (P+p)(Y+y)^{II} + Q(Y+y) - G = 0. \tag{3}$$

Specializing Q(Y) in accordance with Equation (2) yields

$$F(Y) + EIy^{IV} + Py^{II} + pY^{II} + \begin{Bmatrix} 0 \\ kn(Y-c)^{n-1}y \end{Bmatrix} - G = 0. \tag{4}$$

The operation performed is known as Frechet differentiation. It resulted in a linearized equation in unknown variation y(x) and is rigorous for infinitesimal increments only. The equation is extrapolated to small but finite increments which permits its use in a step-by-step solution procedure. A solution obtained at each step is an approximate one. Therefore, employing it as a starting point for the next step introduces some error in addition to that resulting from the next step itself. This is partially offset by a correction which consists in retaining the first term in Equation (4). For an exact Y(x), this term, according to Equation (1) would be an identical zero. Since in reality the solution obtained after the m-th step,

$$Y^m(x) = Y^{m-1}(x) + y^m(x), \tag{5}$$

is approximate, it does not turn F(Y) into zero. Retaining this term in Equation (4) compensates for the error of a current step solution thus preventing both systematic and occasional errors from passing to the next step and accumulation.

The solution of linearized Equation (4) is sought in the form of a linear combination of several approximating functions satisfying the boundary conditions of the problem:

$$y(x = \sum_{i=1}^{N} y_i \sin\frac{(2i-1)\pi x}{2\ell} \tag{6}$$

where ℓ is the beam half-length (Figure 2). As is readily seen, only symmetric configurations of the beam are taken into consideration. This was done because the particular case of interest is characterized by a relatively big transverse load, which precludes the antisymmetric configurations from occurrence at the early stages of post-buckling deformation. (The uniform transverse load would perform zero mechanical work over antisymmetric displacements).

The Galerkin method is now applied. It requires substituting the above y(x) into Equation (4), multiplying it by one of the approximating functions and integrating the product over the beam length. This results in a system of N linear algebraic equations in unknown parameters y_i with coefficients

$$a_{ii} = (D_i EI-P)D_i \frac{\ell}{2} + \int_0^{\ell} S(x) \sin^2 \frac{(2i-1)\pi x}{2\ell} dx \tag{7}$$

$$a_{ij} = \int_0^{\ell} S(x)\sin\frac{(2i-1)\pi x}{2\ell} \sin \frac{(2j-1)\pi x}{2\ell} dx$$

and free terms

$$a_{io} = -D_i Y_i P\frac{\ell}{2} + \int_0^{\ell} F(x) \sin \frac{(2i-1)\pi x}{2\ell} \tag{8}$$

where

$$D_i = [(2i-1)\pi/2\ell]^2$$

$$S(x) = kn[Y(x)-c]^{n-1}$$

$$F(x) = \sum_{i=1}^{N} (D_i EI-P)\ D_i Y_i \sin \frac{(2i-1)\pi x}{2\ell} + Q\ [Y(x)]-G \tag{9}$$

and

$$Y_i \equiv Y_i^{m-1} \tag{10}$$

Obviously, making the axial force increments smaller improves the accuracy of the solution, but increases the number of solution steps. A reasonable compromise was achieved by arranging intermediate iterations in which Equation (4) was solved without incrementing the axial force (i.e., setting $p = 0$). In these iterations, the pattern of the beam interaction with the elastic layers is refined for the fixed magnitude of the axial force. Only after some assigned level of accuracy is reached, the axial force is given its next increment.

The specificity of the problem in consideration is that more than one equilibrium configuration may correspond to a given axial force. To determine whether other equilibrium configurations exist in the vicinity of the original one, the following approach is employed. Upon achieving the convergence of the internal iterations for a fixed value of the axial force, the system is perturbed by giving the deflection some random distortion and internal iterations are performed once again. This may result in overcoming the energy barriers separating the possible equilibrium configurations and increases the likelihood of solution convergence to the most stable configuration.

The perturbation is physically meaningful: it reflects imperfections in material properties, system geometry, load application and many other factors not accounted for explicitly. The magnitude of the distortion presumably correlates with the mentioned imperfections.

A computer program implementing the above features was written and applied to the analysis of a precompressed cryogenic pipeline.

NUMERICAL RESULTS AND DISCUSSION

The concept of preshortening a cryogenic pipeline by compression is intended to reduce or eliminate the need for thermal expansion/contraction devices. The concept involves the compression of one pipe (inner, conveying pipe) within another (casing pipe). The pipes are separated by thermal insulation and an air gap (clearance) exists between the insulation and the inner or the outer pipe. The magnitude of the compressive force is limited by the amount that can be tolerated in the inner pipe without its local inelastic buckling as a cylindrical shell (Reference 5). Under this force, the column buckling behavior of the pipe was investigated to determine the effect of design variables such as the gap size, pipe length, insulation elastic properties, etc., upon the maximum stresses, lateral reactions, and amount of absorbed compressed length.

The following, rather typical data were used in one of the numerical examples:

- pipe length - 100 m (328 ft)
- pipe outer diameter - 46.7 cm (18 inches)
- wall thickness - 0.9525 cm (3/8 inch)
- clearance - 2.54 cm (1 inch)
- permissible stress - 320 MPa (46 ksi).

Figure 3 shows the evolution of the elastic deflection of the pipe as the compression force grows. Diagram 3a is the sagged configuration of the pipe resting on the elastic layer almost uniformly compressed. As the axial force is increased, the pipe bends and develops progressively increasing waviness (3b and c) till the end segment "snaps through" (3d). At this moment the pipe assumes another equilibrium configuration which continues its evolution in further loading (3e).

The results of numerical experiments confirmed the role of systematic perturbations applied during the analysis in order to obtain equilibrium states with lower total energy. In all cases the self-correcting finite increment method provided a rapid convergence of the computation process.

From the viewpoint of the precompression concept it was important to establish the role of the gap between the pipe and insulation. Conceivably, a wider gap could even be an advantage since it would provide more room with which to absorb the "excess" pipe length. A parametric study showed, however, an adverse effect of the gap on the relative compression of the pipe: the wider the gap, the greater the portion of the material strength spent on bending stress. Interestingly, the maximum stress (composed of the axial and bending stresses) does not grow monotonically with the compression force.

The performed study also revealed the role of the pipe length. As shown in Figure 4, the amount of compression that can be absorbed without exceeding the permissible stress increases for shorter lengths of pipe. The limiting case is the pipe length at which overall buckling does not occur at all.

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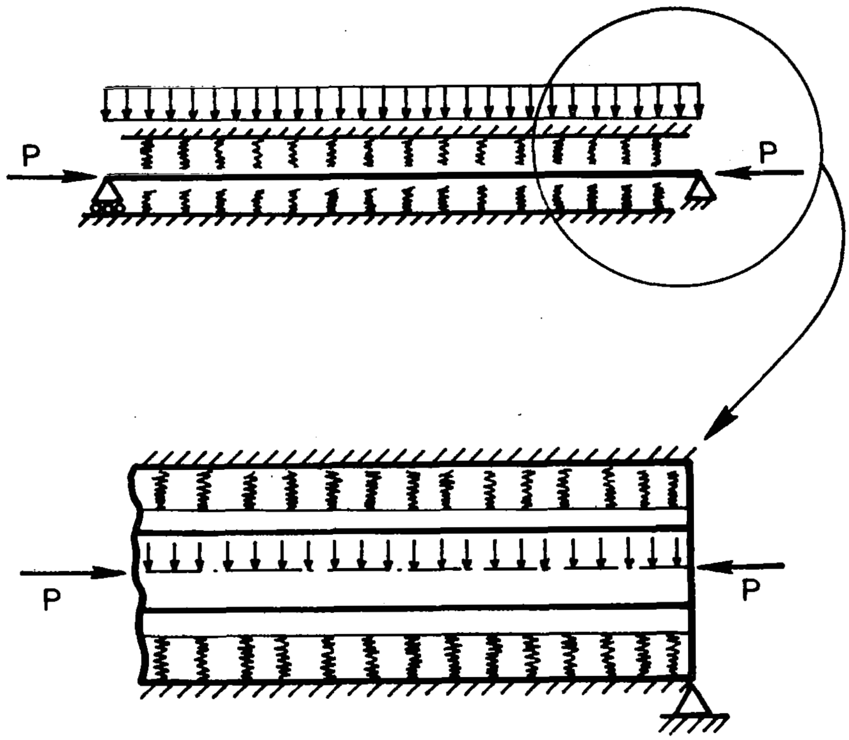

Figure 1.- Beam-column between two layers of elastic foundation with gap.

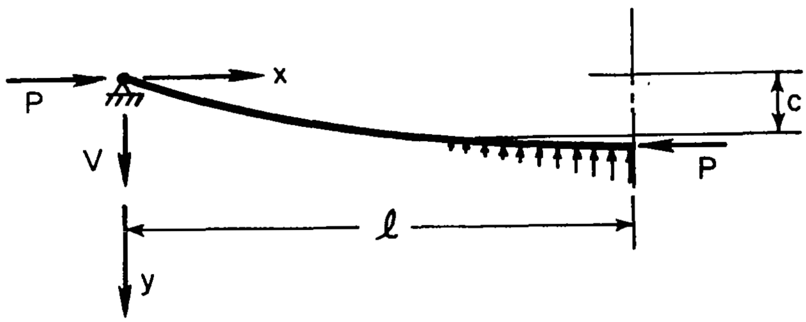

Figure 2.- Buckled column segment with partial contact with foundation.

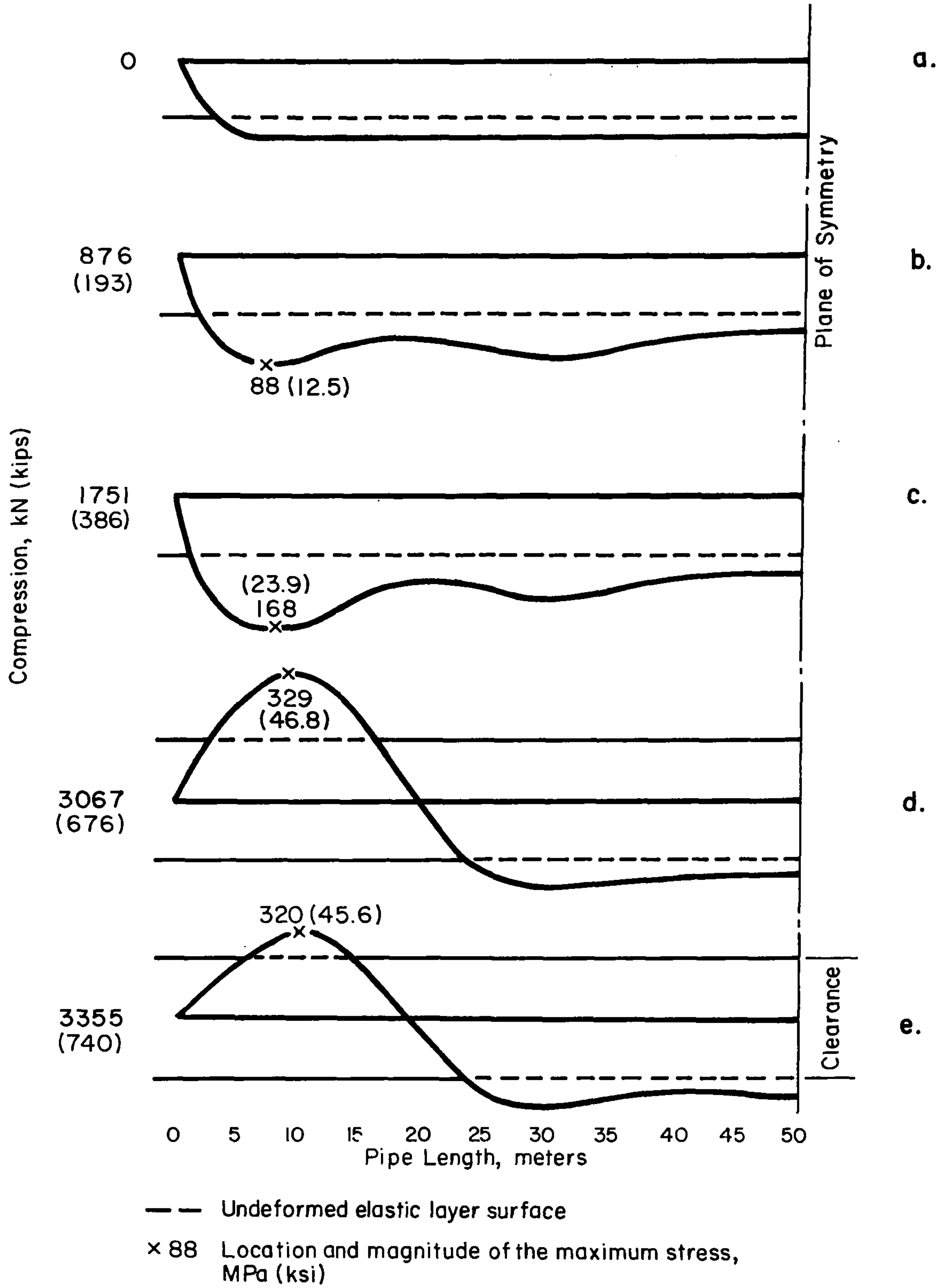

Figure 3.- Development of elastic deflected shape with increasing axial load.

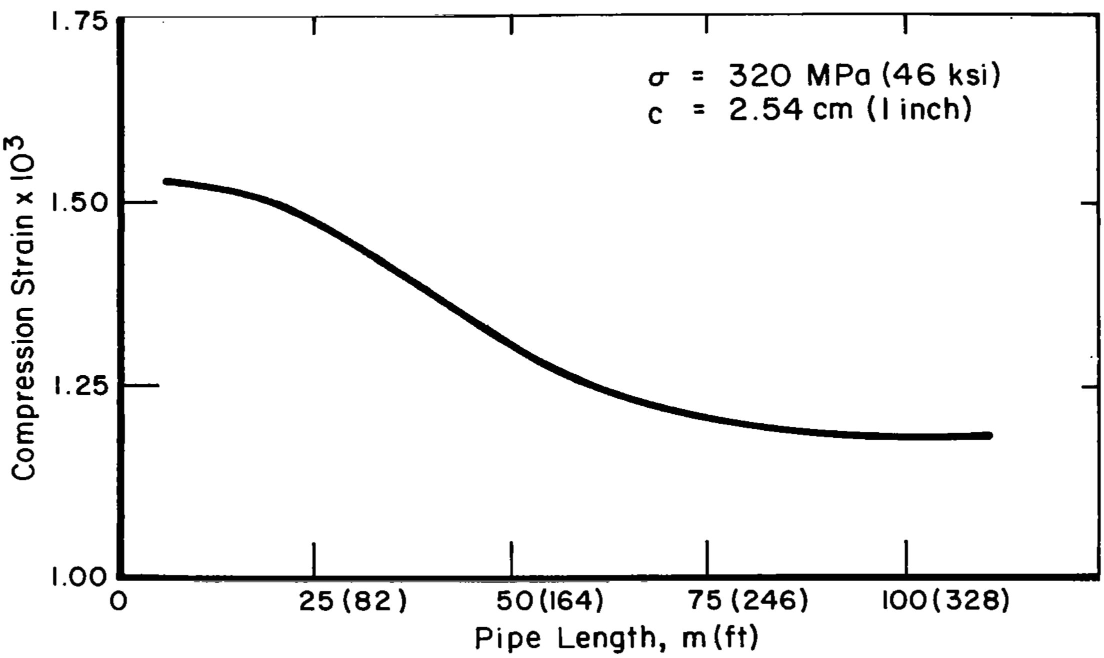

Figure 4.- Amount of compression that can be absorbed without exceeding the allowable stress versus pipe length.