FINITE ELEMENT ANALYSIS OF HYPERELASTIC STRUCTURES

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SUMMARY

Most hyperelastic materials are treated as incompressible or nearly incompressible in analytical approach. The use of the penalty function, to account for near incompressibility is discussed and compared to that of Lagrange multiplier. A scheme to use Lagrange multiplier, without having to treat it as unknown, is also presented.

INTRODUCTION

The finite element analysis of hyperelastic materials involves large displacements and nonlinear material description. These materials are often considered as incompressible or nearly incompressible in theoretical developments. The incompressibility condition leads to certain simplifications in the analysis for exact solutions. Such is not, however, the case in finite element applications.

In the variational formulation of the finite element problems of incompressible media, the incompressibility condition is introduced through the use of Lagrange multiplier (1,2). This multiplier is an additional unknown scalar function which can be accommodated in the discrete model by displacement shape functions or by similar methods. The procedure results in an increase in the number of unknowns.

The incompressibility condition, aside from being inconvenient in finite element analysis, is an approximation for rubberlike materials. Such approximation becomes increasingly less accurate as the percentage of carbon black increases in the rubber compound (3). The exact enforcement of incompressibility is therefore not actually required.

In this work the near incompressibility is accounted for by use of the penalty function (4) in the expression for the strain energy function. The constitutive equations are then obtained in incremental form. This is most suitable for a nonlinear finite element. The incompressibility is shown to be closely satisfied for large penalty numbers. The constitutive relations with penalty number are compared with the exact solution for the incompressible case. It is shown that for a given accuracy of the solution, the penalty number must vary in the course of finite element solution from element to element and from one increment to the next. Some numerical results are presented to illustrate these points. A simple scheme to use variable penalty number is also proposed. Finally the incremental form of constitutive equations for the case of incompressible material is derived from the strain energy function with the Lagrange multiplier. An alternative procedure is then proposed which does not require treatment of the Lagrange multiplier as unknown.

BASIC EQUATION

The strain energy function W for isotropic hyperelastic materials has the following form:

$$W = W(I_1, I_2, I_3) \tag{1}$$

Where I_1, I_2, and I_3 are the three invariants of Green-Lagrange strain tensor and are related to strain components, for the axisymmetric case, as follows:

$$I_1 = 3 + 2\,({}^t\gamma_{11} + {}^t\gamma_{22} + {}^t\gamma_{33}) \tag{2}$$

$$I_2 = 3 + 4\,({}^t\gamma_{11} + {}^t\gamma_{22} + {}^t\gamma_{33} + {}^t\gamma_{11}{}^t\gamma_{22} + {}^t\gamma_{11}{}^t\gamma_{33} + {}^t\gamma_{22}{}^t\gamma_{33} - {}^t\gamma_{12}{}^2) \tag{3}$$

$$I_3 = \left\{ (2\,{}^t\gamma_{11} + 1)\,(2\,{}^t\gamma_{22} + 1) - 4\,{}^t\gamma_{12}{}^2 \right\}(1 + 2\,{}^t\gamma_{33}) \tag{4}$$

${}^t\gamma_{11}$, ${}^t\gamma_{22}$ and ${}^t\gamma_{33}$ are the current values of strains. In this work, only the axisymmetric case is considered. The discussion, however, can be extended to the three-dimensional case without any difficulty. In the axisymmetric case, the strain invariants are related to current displacements by

$$ {}^{t}\gamma_{11} = \frac{\partial\, {}^{t}u_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial\, {}^{t}u_1}{\partial x_1} \right)^2 + \left(\frac{\partial\, {}^{t}u_2}{\partial x_1} \right)^2 \right] \tag{5} $$

$$ {}^{t}\gamma_{22} = \frac{\partial\, {}^{t}u_2}{\partial x_2} + \frac{1}{2}\left[\left(\frac{\partial\, {}^{t}u_1}{\partial x_2} \right)^2 + \left(\frac{\partial\, {}^{t}u_2}{\partial x_2} \right)^2 \right] \tag{6} $$

$$ {}^{t}\gamma_{33} = \frac{{}^{t}u_1}{x_1} + \frac{1}{2}\left(\frac{u_1}{x_1} \right)^2 \tag{7} $$

$$ {}^{t}\gamma_{12} = \frac{1}{2}\left[\frac{\partial\, {}^{t}u_1}{\partial x_2} + \frac{\partial\, {}^{t}u_2}{\partial x_1} \right] + \frac{1}{2}\left[\frac{\partial\, {}^{t}u_1}{\partial x_1}\,\frac{\partial\, {}^{t}u_1}{\partial x_2} + \frac{\partial\, {}^{t}u_2}{\partial x_1}\,\frac{\partial\, {}^{t}u_2}{\partial x_2} \right] \tag{8} $$

x_2 is the axis of symmetry and x_1x_2 is the axisymmetric plane. The current displacements parallel to x_1 and x_2 axes are denoted by ${}^{t}u_1$ and ${}^{t}u_2$ respectively.

The second Piola-Kirchoff stress tensor components ${}^{t}\tau^{ij}$ at current time, can be obtained from the following relations

$$ {}^{t}\tau^{ij} = \frac{\partial W(I_1, I_2, I_3)}{\partial\, {}^{t}\gamma_{ij}} \qquad i,j = 1,2,3 \tag{9} $$

To obtain the incremental form of constitutive equations we define

$$ \tau^{ij} = {}^{t+\Delta t}\tau^{ij} - {}^{t}\tau^{ij} \qquad i,j = 1,2,3 \tag{10} $$

$$ \gamma_{ij} = {}^{t+\Delta t}\gamma_{ij} - {}^{t}\gamma_{ij} \qquad i,j = 1,2,3 \tag{11} $$

where τ^{ij} and γ_{ij} are incremental stress and strain components respectively. The incremental stress-strain relations may now be obtained from the preceding equations, after some algebraic manipulations, as

$$\tau^{ij} = C_{ijkl}\,\gamma_{kl} \qquad i,j,\ldots= 1,2,3 \tag{12}$$

where

$$C_{ijkl} = \left(\frac{\partial^2 W}{\partial I_m\,\partial I_n}\right)\frac{\partial I_n}{\partial \gamma_{kl}}\;\frac{\partial I_m}{\partial \gamma_{ij}}$$

$$+\ \frac{\partial W}{\partial I_m}\;\frac{\partial^2 I_m}{\partial \gamma_{kl}\,\partial \gamma_{ij}} \qquad i,j\ldots= 1,2,3 \tag{13}$$

Repeated indices imply summation convention.

STRAIN ENERGY AND PENALTY FUNCTION

The strain energy function for nearly incompressible materials can be obtained by a series expansion of the strain energy function about (I_3-1) and retaining the leading terms

$$W = W_1(I_1,I_2) + W_2(I_2,I_1)\,(I_3-1)$$

$$+ W_3(I_1,I_2)\,(I_3-1)^2 + \ldots \tag{14}$$

We only consider the following special case

$$W = W_1(I_1,I_2) + H_1\,(I_3-1) + H_2\,(I_3-1)^2 \tag{15}$$

where H_1 and H_2 are constants. Many different strain energy functions have been proposed in the literature by further expansion of W_1 and retaining the leading terms. A general form covering the proposed models is

$$W = \sum_{ij=0} C_{ij} (I_1-3)^i (I_2-3)^j + H_1(I_3-1) + H_2(I_3-1)^2 \tag{16}$$

where C_{ij} are constants. The constant H_1 is not independent if the undeformed state is stress free, and should satisfy the following relation

$$H_1 = -(C_{10} + 2 C_{01}) \tag{17}$$

$$C_{00} = 0$$

We may now consider H_2 as a penalty number to handle the imcompressibility. The satisfaction of incompressibility requires H_2 to approach infinity. For practical purpose the incompressibility can be approximately satisfied by not too large values of H_2. The incompressibility can, however, be satisfied more accurately as H_2 gets larger.

In finite element analysis, however, the large values of H_2 can lead to computational problems due to overriding stiffness associated with H_2, as discussed in (4). A scheme to employ the variable H_2, depending on the local deviations from ideal incompressibility, can therefore improve the solution as discussed later. On the other hand we can recover the classical approach by letting H_2 be zero and treating H_1 as the unknown Lagrange multiplier

$$W = \Sigma C_{ij} (I_1-3)^i (I_2-3)^j + \lambda(I_3-1) \tag{18}$$

$$I_3 - 1 = 0$$

Unlike the penalty function, λ is then considered as an additional unknown and is equivalent to hydrostatic pressure.

ONE DIMENSIONAL STRESS-STRAIN RELATION

To compare the expressions (16) and (18) and to see the behaviour of the material with penalty number, we consider the case of one dimensional stress-strain relation. Let us consider an axisymmetric medium subject to the following uniform strain field:

$$\gamma_{11} = \bar{\gamma} = \text{constant}$$

$$\gamma_{22} = \gamma$$

$$\gamma_{33} = \bar{\gamma} \tag{19}$$

where γ and $\bar{\gamma}$ are constants. Let us further assume the simplest form of W_1 such that

$$W_1 = C_1 (I_1-3) + C_2 (I_2-3) \tag{20}$$

For strain energy function (18), the stress-strain relations are

$$\tau^{11} = 2C_1 + 4C_2 (1 + \gamma + \bar{\gamma}) + 2\lambda (1 + 2\gamma) (1 + 2\bar{\gamma})$$

$$\tau^{22} = 2C_1 + 4C_2 (1 + 2\bar{\gamma}) + 2\lambda (1 + 2\bar{\gamma})^2$$

$$\tau^{33} = 2C_1 + 4C_2 (1 + \gamma + \bar{\gamma}) + 2\lambda (1 + 2\gamma) (1 + 2\bar{\gamma})$$

$$\tau^{12} = 0 \tag{21}$$

$\bar{\gamma}$ is, however, not independent and is related to γ by

$$(1 + 2\gamma) (1 + 2\bar{\gamma})^2 - 1 = 0 \tag{22}$$

We choose λ, so that τ^{11} and τ^{33} are both zero to simulate the uniaxial loading. This choice would then lead to the following stress-strain relation

$$\tau^{22}_{inc} = \left[1 - (1 + 2\gamma)^{-3/2}\right] \left(2C_1 + 2\, C_2(1 + 2\gamma)^{-1/2}\right) \tag{23}$$

The Cauchy stress σ^{22} is then related to τ^{22} by

$$\sigma^{22} = (1 + 2\gamma)\, \tau^{22} \tag{24}$$

The one-dimensional stress-strain relation for the case of penalty number is now obtained from (16)

$$\tau^{11} = 2C_1 + 4C_2\, (1 + \gamma + \bar{\gamma}) + 2\, (1 + 2\gamma)\, (1 + 2\bar{\gamma})\, x \left[H_1 + 2H_2\, (I_3 - 1)\right]$$

$$\tau^{22} = 2C_1 + 4C_2\, (1 + 2\bar{\gamma}) + 2\, (1 + 2\bar{\gamma})^2 \left[H_1 + 2H_2\, (I_3 - 1)\right]$$

$$\tau^{33} = \tau^{11} \tag{25}$$

Following the same procedure, we arrive at the following one dimensional constitutive equation

$$\tau^{22} = \left(1 - \frac{(1 + 2\bar{\gamma})}{(1 + 2\gamma)}\right) \left(2C_1 + 2C_2\, (1 + 2\gamma)^{-1/2}\right) \tag{26}$$

where $\bar{\gamma}$ is now related to H_2 by

$$2H_2 \left[(1 + 2\bar{\gamma})^2 (1 + 2\gamma) - 1\right] =$$

$$\frac{4C_1 (\gamma + \bar{\gamma} + 2\gamma\bar{\gamma}) + 4C_2 (\gamma + \bar{\gamma} + 16\gamma\bar{\gamma})}{2 (1 + 2\gamma) (1 + 2\bar{\gamma})} \tag{27}$$

The above equation must be satisfied for all values of H_2. It can be seen that, for H_2 approaching the infinity, the above equation degenerates to

$$(1 + 2\bar{\gamma})^2 (1 + 2\gamma) - 1 = 0 \tag{28}$$

which is the expression of incompressibility condition. In this case equation (26) would become identical to equation (23).

IMCOMPRESSIBILITY AND PENALTY NUMBER

We consider the case where incompressibility is to be satisfied within some prescribed accuracy ε ; that is we require

$$(1 + 2\bar{\gamma})^2 (1 + 2\gamma) - 1 = \varepsilon \tag{29}$$

For numerical illustration, consider the case where

$$\frac{C_1}{C_2} = 4 \tag{30}$$

Combining equations (27) to (30), we arrive at the following relation

$$H\varepsilon = 6 - \frac{1}{x} - \frac{(x + 4)}{x^{1/2}} (1 + \varepsilon)^{1/2} \tag{31}$$

where

$$X = 1 + 2\gamma$$

$$H = \frac{2H_2}{C_2} \tag{32}$$

and $\varepsilon << 1$

For small values of ε , the relation (31) can be further simplified to

$$H\varepsilon = 6 - \frac{1}{x} - \frac{x + 4}{x^{1/2}} \tag{33}$$

The stress-strain relation of the equation (23) is plotted in Figure (1). The variation of $H\varepsilon$ as function of strain is plotted in Figure (2). It can be seen that as γ increases, H must also increase accordingly, to maintain the same accuracy ε on incompressibility condition. It can also be noted that higher values of H are required in compression than in tension. The relation (26) may now be written as follows:

$$\tau^{22}_{com} = \left[1 - \frac{(1 + \varepsilon)^{1/2}}{(1 + 2\gamma)^{3/2}}\right]\left(4 + (1 + 2\gamma)^{-1/2}\right) \tag{34}$$

Comparing (34) and (23), it is observed that the stress for a nearly incompressible model is always less than that of an incompressible model, for the same strain. This difference, however, depends on the ε and approaches zero as ε approached zero, or H approaches infinity. The relative error, however, is

$$e = \frac{\tau^{22}_{inc} - \tau^{22}_{com}}{\tau^{22}_{inc}} = \frac{2\varepsilon}{(1 + 2\gamma)^{3/2} - 1} \tag{35}$$

For a fixed H, ε increases with γ but the relative error in the stress is governed by (35) which is less sensitive to a variation in H . Eliminating ε between (35) and (33), we obtain

$$He = \left(6 - \frac{1}{x} - \frac{(x+4)}{x+1/2}\right) \Big/ (x^{3/2}-1) \tag{36}$$

The above equation relates the magnitude of H to level of strain for a prescribed error e. The equation (33) or (36) may serve as an approximate method of updating H, in a problem of combined stresses, for improved accuracy in incompressibility or stress calculation. Further work is, however, required to develop a more vigorious scheme in the general case.

INCREMENTAL FORM OF LAGRANGE MULTIPLIER

We now consider an alternative approach to incompressibility problems. Let us consider the following strain energy function

$$\begin{aligned} W &= W_1(I_1, I_2) + \lambda\,(I_3-1) \\ I_3 - 1 &= 0 \end{aligned} \tag{37}$$

where λ is the Lagrange multiplier. The second Piola-Kirchoff stresses can now be obtained as follow

$${}^t\tau^{ij} = \frac{\partial W}{\partial\,{}^t\gamma_{ij}} + {}^t\lambda\,\frac{\partial(I_3 - 1)}{\partial\,{}^t\gamma_{ij}} + (I_3 - 1)\,\frac{\partial\,{}^t\lambda}{\partial\,{}^t\gamma_{ij}} \tag{38}$$

where ${}^t\lambda$ is the current value of λ. The incremental stress-strain relations may be obtained by a procedure similar to that used in the preceding pages. This would lead to

$$\tau^{ij} = \frac{\partial^2 W}{\partial^t\gamma_{ij}\ \partial^t\gamma_{kl}}\ \gamma_{kl} + {}^t\lambda \frac{\partial^2 (I_3 - 1)}{\partial\ {}^t\gamma_{ij}\ \partial^t\gamma_{kl}}$$

$$+ \lambda\ \frac{\partial (I_3 - 1)}{\partial^t\gamma_{ij}} \tag{39}$$

$$\lambda = {}^{t+\Delta t}\lambda - {}^t\lambda \tag{40}$$

where λ is the increment in the Lagrange multiplier between two consecutive steps in the incremental solution. The incremental form of incompressibility condition is

$$\frac{\partial I_3}{\partial^t\gamma_{ij}}\ \gamma_{ij} = 0 \tag{41}$$

multiplying both sides of (39) by γ_{ij}, and use of (41), leads to

$$\tau^{ij}\gamma_{ij} = \frac{\partial^2 W}{\partial^t\gamma_{ij}\ \partial^t\gamma_{kl}}\ \gamma_{kl}\ \gamma_{ij}$$

$$+ {}^t\lambda \left[\frac{\partial^2 (I_3 - 1)}{\partial^t\gamma_{ij}\ \partial^t\gamma_{kl}}\ \gamma_{kl}\ \gamma_{ij} \right] \tag{42}$$

or

$${}^t\lambda = \tau^{ij}\gamma_{ij} - C_{ijkl}\ \gamma_{ij}\ \gamma_{kl}$$

$$- \frac{\partial^2 (I_3 - 1)}{\partial\ {}^t\gamma_{ij}\ \partial^t\gamma_{kl}}\ \gamma_{kl}\ \gamma_{ij} \tag{43}$$

$$C_{ijkl} = \frac{\partial^2 W}{\partial^t\gamma_{ij}\ \partial^t\gamma_{kl}}$$

The following scheme may now be considered for the solution of incompressible materials without the need for introducing the unknown Lagrange multiplier. In the process of incremental solution, the last term in the right hand side of equation (39) is ignored. At the end of each increment, ${}^{t}\lambda$ can be updated from relation (43). There is, therefore, no need of treating ${}^{t}\lambda$ as an independent unknown. No numerical work, however, has been carried out with this alternative proposed scheme at this time.

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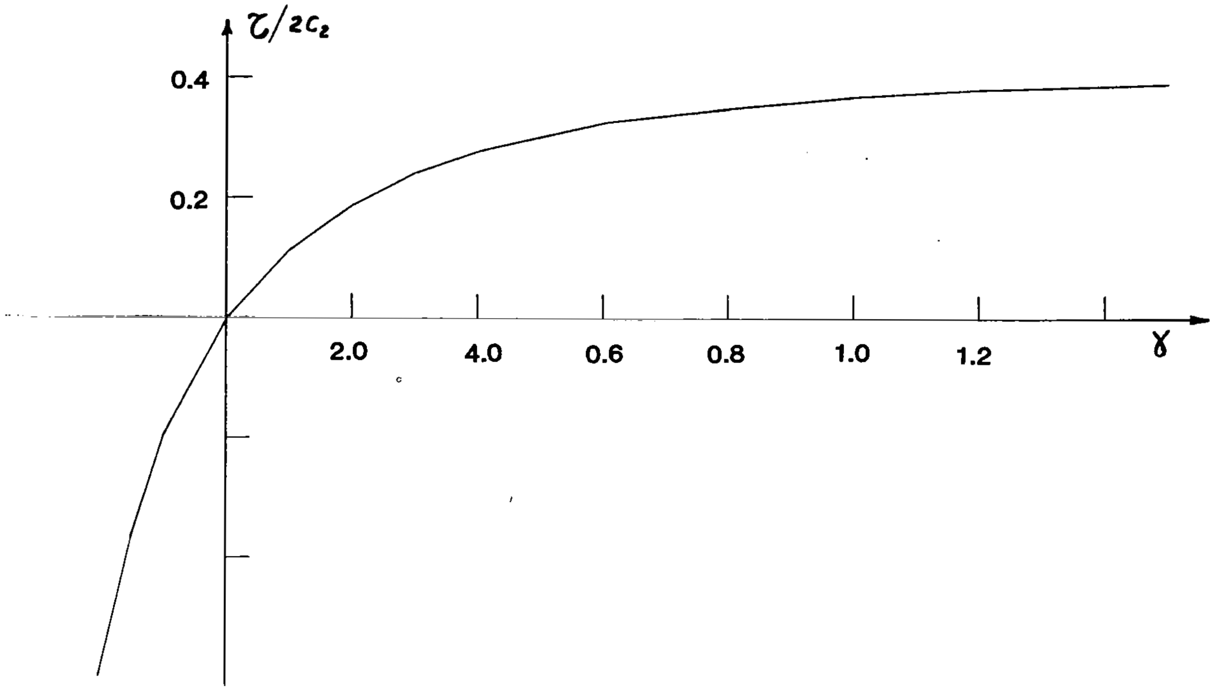

Figure 1.- One-dimensional stress-strain relation.

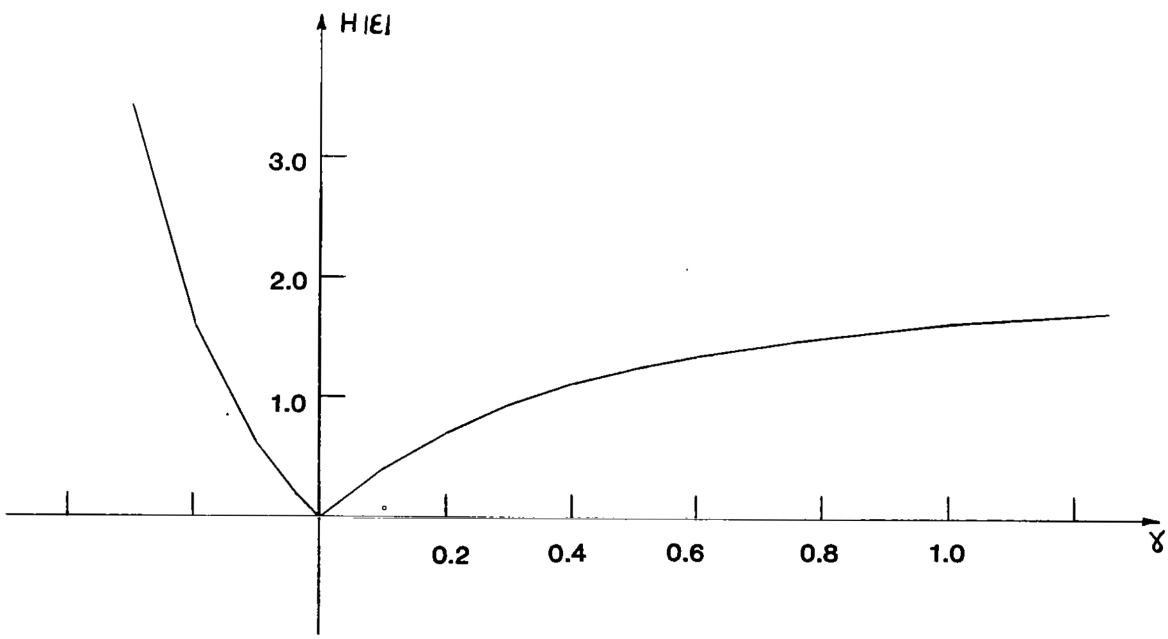

Figure 2.- Variation of $H\epsilon$ with strain.