

SOLUTIONS OF CONTACT PROBLEMS BY THE ASSUMED STRESS HYBRID MODEL

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SUMMARY

A method has been developed for contact problems which may be either frictional or frictionless and may involve extensive sliding between deformable bodies. It is based on an assumed stress hybrid approach and on an incremental variational principle for which the Euler's equations of the functional include the equilibrium and compatibility conditions at the contact surface. The tractions at an assumed contact surface are introduced as Lagrangian multipliers in the formulation. It has been concluded from the results of several example solutions that the extensive sliding contact between deformable bodies can be solved by the present method.

INTRODUCTION

The finite element method has been applied by many authors for solving solid mechanics problems which involve undetermined contact surfaces. They include the relatively simple Hertz contact problem for which there is no sliding between contact surfaces and the small displacement model can be used [1, 2]. They also include problems involving relative sliding either with friction or in frictionless conditions [3]. Existing solutions are based largely on the conventional assumed displacement finite element model.

The present paper is based on an assumed stress approach and on an incremental variational principle for which the Euler's equations of the functional include the equilibrium and compatibility conditions at the contact surface. An assumed contact surface is inserted between bodies in contact and is divided into elements. Contact tractions are independently assumed in terms of unknown values of such nodes of the contact elements. Thus, a finite element equation includes nodal displacements and nodal contact tractions as unknown. This paper is to present the variational principle and the corresponding finite element implementation for this problem.

SYMBOLS

| | |
|----------------------------------|--|
| X_i, \tilde{X} | Coordinates before deformation |
| U_i, \tilde{U} | Displacements defined within an element |
| U_i, \tilde{U} | Displacements defined along element boundaries |
| T_i, \tilde{T} | Tractions |
| $\tilde{T}_i, \tilde{\tilde{T}}$ | Contact tractions |
| μ | Coefficient of friction |
| ∂V | The whole boundary of the body |
| S_C | The contact surface |
| S_U | The portion of the boundary where displacements are prescribed |
| S_σ | The portion of the boundary where loads are applied. |
| V | Volume |
| e_{ij}, \tilde{e} | Strains |
| $\sigma_{ij}, \tilde{\sigma}$ | Stresses |
| $(\)_s$ | Quantity tangential to the contact surface |
| $(\)_n$ | Quantity normal to the contact surface |
| $(\)_{\sim}$ | Matrix |
| $(\)_{ij}, (\)_i$ | Tensor |
| $(\)^-$ | Prescribed quantity |
| $(\)^A$ | Quantity pertinent to body A |
| $(\)^B$ | Quantity pertinent to body B |
| $(\)^{A+B}$ | Quantity pertinent to bodies A and B |
| $(\)_N$ | Quantity pertinent to element N |
| $\Delta(\)$ | Incremental quantity |

GENERAL INCREMENTAL ASSUMED STRESS FORMULATION FOR CONTACT PROBLEM

The requirements for contact (the conditions of contact) are as follows:

- (1) At the point of contact between two bodies tractions exerted on each other are the same in magnitude and are opposite in directions.
- (2) The normal tractions are compressive and the tangential tractions counteract relative movement of the bodies.
- (3) There should be no gap and no penetration of material points at the place of contact.

Consider two bodies A and B shown in Fig. 1 with volumes V^A and V^B , and boundary surfaces, ∂V^A and ∂V^B which are composed of portions, S_f^A and S_f^B , and S_u^A and S_u^B . These two bodies share a contact surface, S_C through which they interact. The previously mentioned conditions of contact in incremental form are

$$(T_n^A + \Delta T_n^A) + (T_n^B + \Delta T_n^B) = 0 \quad (1)$$

$$(T_s^A + \Delta T_s^A) + (T_s^B + \Delta T_s^B) = 0 \quad (2)$$

$$(T_s^A + \Delta T_s^A) \leq \pm \mu (T_n^A + \Delta T_n^A) \quad (3)$$

$$(T_s^B + \Delta T_s^B) \leq \pm \mu (T_n^B + \Delta T_n^B) \quad (4)$$

$$(U_n^A + \Delta U_n^A + X_n^A) - (U_n^B + \Delta U_n^B + X_n^B) = 0 \quad (5)$$

$$(U_s^A + \Delta U_s^A + X_s^A) - (U_s^B + \Delta U_s^B + X_s^B) = 0 \quad (6)$$

and the signs in Eqs. 3 and 4 are chosen such that tangential tractions on this surface act to restrain the relative movement of contacting points.

A finite element method which is based on a variational principle with relaxed continuity requirement at interelement boundaries is defined as a hybrid model [4]. Boland and Pian [5] have applied an incremental assumed stress hybrid method for large deflection analyses of thin elastic structures. The functional π_{mc} that has been derived in the reference [5] based on the Updated Lagrangian coordinate system is used as base for deriving the functional for the present problem.

The conditions of no gap and no overlapping on the place of contact are introduced into the functional by means of Lagrangian multipliers, $\tilde{T}_n + \Delta\tilde{T}_n$ and $\tilde{T}_s + \Delta\tilde{T}_s$. The functional π_{mc} becomes π_{mc}^c , i.e.

$$\begin{aligned} \pi_{mc}^c = & \sum_N \left[\int_{V_N} \left[-B(\Delta\sigma_{ij}) - \frac{1}{2}(\sigma_{ij} + \Delta\sigma_{ij}) \Delta U_{m,i} \Delta U_{m,i} \right] dV + \int_{\partial V_N^{A+B}} \Delta\tilde{T}_i \Delta\tilde{U}_i dS \right. \\ & - \int_{S_{\sigma N}^{A+B}} \Delta\tilde{T}_i \Delta\tilde{U}_i dS - \int_{S_{CN}^{A+B}} \Delta\tilde{T}_i (\Delta\tilde{U}_i^A - \Delta\tilde{U}_i^B) dS \\ & + \int_{\partial V_N^{A+B}} \tilde{T}_i \Delta\tilde{U}_i dS - \int_{S_{\sigma N}^{A+B}} \tilde{T}_i \Delta\tilde{U}_i dS - \int_{V_N^{A+B}} \Delta\sigma_{ij} \left[e_{ij} - \frac{1}{2}(U_{i,i} + U_{j,i} - U_{m,i} U_{m,i}) \right] dV \\ & \left. + \int_{S_{CN}^{A+B}} \tilde{T}_i (\Delta\tilde{U}_i^A - \Delta\tilde{U}_i^B) dS + \int_{S_{CN}^{A+B}} \Delta\tilde{T}_i \left[(\tilde{U}_i^A + X_i^A) - (\tilde{U}_i^B + X_i^B) \right] dS \right] \end{aligned}$$

(7)

The Euler's equations of π^c_{mc} are

$$e_{ij} + \Delta e_{ij} = \frac{1}{2} \left(U_{i,i} + U_{j,i} - U_{m,i} U_{m,j} + \Delta U_{i,i} + \Delta U_{j,i} + \Delta U_{m,i} \Delta U_{m,j} \right) \quad (8)$$

in V_N^{A+B}

$$T_i + \Delta T_i = \bar{T}_i + \Delta \bar{T}_i \quad \text{on } S_{\sigma N}^{A+B} \quad (9)$$

$$\left(T_i + \Delta T_i \right)^a + \left(T_i + \Delta T_i \right)^b = 0 \quad \text{on } \partial V_N^{A+B} \quad (10)$$

On the contact surface S_{C_N} , in addition to Eqs. 5 and 6

$$\left(T_n^A + \Delta T_n^A \right) - \left(\tilde{T}_n + \Delta \tilde{T}_n \right) = 0 \quad (11)$$

$$\left(T_s^A + \Delta T_s^A \right) - \left(\tilde{T}_s + \Delta \tilde{T}_s \right) = 0 \quad (12)$$

$$\left(T_n^B + \Delta T_n^B \right) + \left(\tilde{T}_n + \Delta \tilde{T}_n \right) = 0 \quad (13)$$

$$\left(T_s^B + \Delta T_s^B \right) + \left(\tilde{T}_s + \Delta \tilde{T}_s \right) = 0 \quad (14)$$

Introducing frictional constraint on $\tilde{T}_s + \Delta \tilde{T}_s$ and $\tilde{T}_n + \Delta \tilde{T}_n$, such that

$$\tilde{T}_s + \Delta \tilde{T}_s \leq \pm \mu \left(\tilde{T}_n + \Delta \tilde{T}_n \right) \quad (15)$$

and rearranging Eqs. 11, 12, 13, 14 and 15, results in Eqs. 1, 2, 3, and 4. Thus, it has been proved that the conditions of contact are Euler's equations of the functional π^c

These Euler equations are the strain displacement relations, the mechanical boundary condition, the stress equilibrium requirements along the interelement boundaries and the conditions of contact. Since they are only satisfied in an average sense within an increment they cannot be expected to satisfy these conditions in the usual sense. It is, therefore, necessary to consider a compatibility check, a stress equilibrium check, and a contact check.

It is seen that Eq. 7 already has all of these built-in checks. The 5-th and 6-th integral terms in the functional correspond to the equilibrium check and 7-th term to the compatibility check. Also, the compatibility and equilibrium checks of the contact surface (contact check) are easily identified, the equilibrium check being the 8-th integral term and the compatibility check, the 9-th term.

FINITE ELEMENT IMPLEMENTATION

Since the aim of the present work is to solve contact problems by the finite element method, expressions arising from nonlinearities, not due to contact, are excluded from equations. A technique for solving these equations, with only contact nonlinearities, will be discussed here.

Neglecting nonlinearities not due to contact, the assumed stress hybrid functional,

π_{mc}^c takes the form

$$\begin{aligned} \pi_{mc}^c = & \sum_N \left\{ \int_{V_N} \frac{1}{2} \Delta \underline{\underline{\sigma}}^T \cdot \underline{\underline{S}} \cdot \Delta \underline{\underline{\sigma}} dV - \int_{\partial V_N} \Delta \underline{\underline{T}}^T \cdot \Delta \underline{\underline{\tilde{U}}} dS + \int_{S_{\sigma N}} (\underline{\underline{\tilde{T}}} + \Delta \underline{\underline{\tilde{T}}})^T \Delta \underline{\underline{\tilde{U}}} dS \right. \\ & - \int_{S_{\sigma N}} \underline{\underline{\tilde{T}}}^T \cdot \Delta \underline{\underline{\tilde{U}}} dS - \int_{S_{CN}} (\underline{\underline{\tilde{T}}} + \Delta \underline{\underline{\tilde{T}}})^T (\Delta \underline{\underline{\tilde{U}}}^A - \Delta \underline{\underline{\tilde{U}}}^B) dS \\ & \left. - \int_{S_{CN}} (\underline{\underline{\tilde{T}}} + \Delta \underline{\underline{\tilde{T}}})^T \left[(\underline{\underline{\tilde{U}}}^A + \underline{\underline{X}}^A) - (\underline{\underline{\tilde{U}}}^B + \underline{\underline{X}}^B) \right] dS \right\} \end{aligned} \quad (16)$$

The stresses $\Delta \underline{\sigma}$ are expressed in terms of a finite number of stress parameters, $\Delta \underline{\beta}$ and the element boundary displacements interpolated in terms of the nodal displacements, \underline{q} , and $\Delta \underline{q}$. Also, coordinates are interpolated in terms of their nodal values. As a solid continuum is subdivided into elements, the contact surface is also discretized into finite number of elements referred to here as "contact elements" with "contact nodes." The contact traction $\underline{\tilde{T}} + \Delta \underline{\tilde{T}}$ is interpolated in terms of its nodal values, \underline{t} .

Thus, interpolations of them are: $\Delta \underline{\sigma} = \underline{P} \cdot \Delta \underline{\beta}$, $\Delta \underline{T} = \underline{R} \cdot \Delta \underline{\beta}$, $\underline{\tilde{U}} = \underline{L} \cdot \underline{q}$, $\Delta \underline{\tilde{U}} = \underline{L} \cdot \Delta \underline{q}$ and $\underline{\tilde{T}} + \Delta \underline{\tilde{T}} = \underline{M} \cdot \underline{t}$. Substituting these interpolations into Eq. 16, and defining the following matrices

$$\underline{H}_N = \int_{V_N} \underline{P}^T \cdot \underline{S} \cdot \underline{P} \, dV, \quad \underline{G}_N = \int_{\partial V_N} \underline{R}^T \cdot \underline{L} \, dS,$$

$$\Delta Q_N = \int_{S_{\sigma N}} \underline{L}^T \cdot \Delta \underline{\tilde{T}} \, dS, \quad Q_N^0 = \int_{S_{\sigma N}} \underline{L}^T \cdot \underline{\tilde{T}} \, dS,$$

$$\underline{R}_{EN}^0 = \int_{S_{CN}} \underline{L}^T \cdot \underline{\tilde{T}} \, dS, \quad \underline{R}_{CN}^0 = \int_{S_{CN}} \underline{M}^T \cdot \left[(\underline{\tilde{U}}^A + \underline{X}^A) - (\underline{\tilde{U}}^B + \underline{X}^B) \right] \, dS$$

$$\underline{F}_N^A = \int_{S_{CN}} \underline{M} \cdot \underline{L}^A \, dS, \quad \underline{F}_N^B = \int_{S_{CN}} \underline{M} \cdot \underline{L}^B \, dS$$

the functional, π_{mcN}^c becomes

$$\begin{aligned} \pi_{mcN}^c = & \frac{1}{2} \Delta \underline{\beta}^T \cdot \underline{H}_N \cdot \Delta \underline{\beta} - \Delta \underline{\beta}^T \cdot \underline{G}_N \cdot \Delta \underline{q} + \Delta \underline{q}^T \cdot \Delta Q_N \\ & + \Delta \underline{q}^T \cdot Q_N^0 - \Delta \underline{q}^T \cdot \underline{R}_{EN}^0 - \left(\Delta \underline{q}^A \cdot \underline{F}_N^A - \Delta \underline{q}^B \cdot \underline{F}_N^B \right) \cdot \underline{t} - \underline{t}^T \cdot \underline{R}_{CN}^0 \end{aligned} \quad (17)$$

in which \underline{Q}_N^o and $\underline{R}_{E_N}^o$ result from stress equilibrium checks on S_{σ_n} and on the contact surface S_{C_N} , and $\underline{R}_{C_N}^o$, from the initial mismatch checks on S_{C_N} .

Equation 17 contains three unknown vectors, the incremental stress parameters $\Delta \underline{\beta}$, the incremental displacements $\Delta \underline{q}$, and the contact stresses \underline{t} . Whereas, the vector $\Delta \underline{\beta}$ are independent on the element level, $\Delta \underline{q}$ and \underline{t} are not. Thus, eliminating $\Delta \underline{\beta}$ from Eq. 17, π_{mc}^c becomes

$$\begin{aligned} \pi_{mc}^c = & -\frac{1}{2} \Delta \underline{q}^T \cdot \underline{G}_N \cdot \underline{H}_N^{-1} \cdot \underline{G}_N \cdot \Delta \underline{q} + \Delta \underline{q}^T \cdot \left(\Delta \underline{Q}_N + \underline{Q}_N^o - \underline{R}_{E_N}^o \right) \\ & - \left(\Delta \underline{q}^T \cdot \underline{F}_N^A - \Delta \underline{q}^T \cdot \underline{F}_N^B \right) \cdot \underline{t} - \underline{t}^T \cdot \underline{R}_{C_N}^o \end{aligned} \quad (18)$$

Summing up over all elements and taking the variations of the functional with respect to $\Delta \underline{q}$ and \underline{t} , results in

$$\begin{bmatrix} \underline{K} & \underline{K}_C \\ \underline{K}_C^T & \underline{0} \end{bmatrix} \begin{bmatrix} \Delta \underline{q} \\ \underline{t} \end{bmatrix} = \begin{bmatrix} \underline{Q} \\ \underline{R} \end{bmatrix} \quad (19)$$

where

$$\begin{aligned} \underline{K} &= \sum_N \underline{G}_N^T \cdot \underline{H}_N^{-1} \cdot \underline{G}_N, \quad \underline{Q} = \sum_N \left(\Delta \underline{Q}_N + \underline{Q}_N^o - \underline{R}_{E_N}^o \right) \\ \underline{K}_C &= \begin{bmatrix} \sum_N \underline{F}_N^A \\ -\sum_N \underline{F}_N^B \end{bmatrix}, \quad \underline{R} = \sum_N \underline{R}_{C_N}^o \end{aligned}$$

Equation 19 represents the total assembled finite element matrix equation.

TECHNIQUE OF ITERATIVE SOLUTION

Once the contacting bodies are adequately constrained such that the inverse of the global stiffness matrix, \underline{K}^{-1} in Eq. 19 can be calculated, it can be used throughout the iteration procedure. Thus, in locating the contact surface only \underline{K}_C needs to be recomputed in each iteration. The global stiffness matrix \underline{K} remains constant during this process. Even in the case of material and/or large deflection nonlinearities, it is possible to use a modified Newton-Raphson method; hence the global stiffness matrix \underline{K} may remain constant during this process.

For a two-dimensional problem, before each iteration, the contact surface is a line, fixed in the coordinate system, but not to the contacting bodies. Such line is assumed known in order to perform the necessary integrals. But if, before the iteration, one has assumed the location of the contact surface and the positions on it that the contacting nodes of the bodies will occupy, then it would appear that the displacement increment can be specified. This is not the case. Instead, the problem is solved for the displacement increments and if the contact surface found therefrom is not coincident with the one presupposed, a new contact surface location is calculated, and then an iteration can be followed.

It has been found that the length of the contact element which yields best results is the same as the length of the contacting side of boundary elements of the bodies. In order to facilitate programming, the nodes of the contact element are chosen to be coincident with those of one of the contacting bodies. As a result of variations in load, the place of contact changes; thus a vital part of the solution is to establish a procedure for calculating this change. A trial and error scheme is employed because it is virtually impossible to formulate a variational principle including unknown variables which locate the surface of contact.

The overall strategy for solving the contact problem is discussed here. First, an increment in the external load or prescribed displacement is applied. Second, a contact surface is assumed together with the points on it through which nodes of the bodies contact each other. Also, the types of contact (sliding or non-sliding) at each of the above-mentioned points are assumed. For the initial calculation of the first load increment, the above assumptions are made simply by inspection, and for the first iteration after each new load increment, the converged solution of the previous load step is used. Third, all the necessary matrices are calculated and assembled. At the i -th iteration of the N -th load step, incremental displacement, ${}^i\Delta U_k$ and contact tractions ${}^i(\tilde{T}_k + \Delta\tilde{T}_k)$ are solved from a finite element matrix equation. Fourth, knowing the total displacement ${}^{N-1}U_k$ at the end of the previous loading step $N-1$, the total displacements ${}^{N-1}U_k + {}^i\Delta U_k$ on the boundary, and contact tractions ${}^i(\tilde{T}_k + \Delta\tilde{T}_k)$ are checked to determine if they satisfy the conditions of contact.

If they do not satisfy these conditions, the location of the assumed contact surface is modified and the procedure repeated until they do. Next, a convergence test is made, and if the solution is not convergent, the location of the

contact surface is further modified and the solution procedure repeated.

To determine if the solution satisfies the conditions of contact, the following assurances are made:

- (1) That nodes of either body beyond the last contacting nodes from the previous iteration have not penetrated the other body.
- (2) That tractions at the contacting nodes are compressive. These normal tractions can be calculated by three different methods; (a) from the stress coefficients, (b) from the equivalent nodal forces, and (c) from the contact tractions, $\tilde{T}_n + \Delta\tilde{T}_n$. Here the last method was used.
- (3) That the relationship between normal and tangential contact tractions,

$$\left| \tilde{T}_s + \Delta\tilde{T}_s \right| \leq \mu \left| \tilde{T}_n + \Delta\tilde{T}_n \right|$$

is satisfied. Depending on which of the above checks, if any, is violated, one of the following procedures is employed to modify the assumed location of the contact surface.

- (a) If (1) is violated, the contact surface may be extended to include the points at which penetration has occurred.
- (b) If (2) is violated, the contact surface is reduced by excluding nodes at which the tractions are tensile.
- (c) If (3) is violated, sliding is allowed to occur.

After the conditions of contact are satisfied, a test for convergence can be made by calculating the following quantity:

$$R = \left| \frac{i+1 \Delta U_k - i \Delta U_k}{i+1 \Delta U_k} \right|$$

where ΔU_k is the displacement at the k-th degree of freedom. If R is less than a prescribed quantity, say 0.01, the solution is considered as converged.

EXAMPLE SOLUTIONS

The finite element model and solution scheme are applied to problems of contact between a disk and a semi-infinite half-plane. The overall mesh pattern, the location of

the prescribed displacement and relevant dimensions are shown in Fig. 2-a, with the area immediately surrounding the contact surface shown in great detail in Fig. 2-b. The semi-infinite half-plane has been modeled by a finite one with overall dimensions much larger than those of the disk. The basic element used is four-node quadrilateral element derived by assuming seven β -parameters and linear displacement distribution along each edge. Five-node and six-node elements are also introduced in transition regions between coarser and finer meshes. Contact tractions along each contact element are approximated by linear interpolations.

Non-Sliding Contact

Problems are solved for the case with both applied loads and prescribed displacements at the top of the disk. In the problems, the ratio of Young's moduli are varied over a range from 1 to 10^{-4} and slightly different mesh patterns near the contact surface are used to accommodate node-to-node and node-to-internode contacts. Loads or displacements are applied by the three increments until the length of contact surface becomes about 2.4 mm. For each increment, the converged solutions are reached with three or four iterations. For these solutions, the best results for contact tractions are obtained when calculated from equivalent nodal forces and are compared excellently with the Hertz solution in all cases.

Frictionless Contact With Extensive Sliding

The half disk and the semi-infinite half-plane are also used to demonstrate the capability of this formulation to solve extensive sliding contact problems. Since the contact between the two bodies is frictionless, the solution is independent of the path; thus a Hertz solution is again available for comparison. Solutions are obtained for prescribed displacements at the top of the disk by eight increments. Stress distributions on the plane of contact, for two prescribed displacements, are plotted in Fig. 3 where zero position represents the point of initial contact. It is seen that the solution agrees almost exactly with that of the Hertz solution. It is noted that the center of symmetry of the stress distribution moves to the left as the half-disk slides in that direction.

Frictional Contact With Extensive Sliding

The same problem is again solved here with friction between the disk and the half-plane as an added consideration. No solution to this problem, analytic or otherwise, can be found; thus, results arrived at here will be justified by comparison with the results of the previous section and by showing that they satisfy the conditions of contact.

The normal tractions at the contacting nodes between the disk and the half-plane for every 4th displacement increment are shown in Fig. 3 and are compared with those of the frictionless case. Because of friction, it can be seen that the displacement of the contact surface is retarded. That the normal tractions of the plane and disk are equal in magnitude and opposite in sign is also evident in the figure. This implies that the normal tractions satisfy a condition of contact. A condition of sliding contact requires the ratio between normal and tangential components of tractions to be constant and equal to the coefficient of

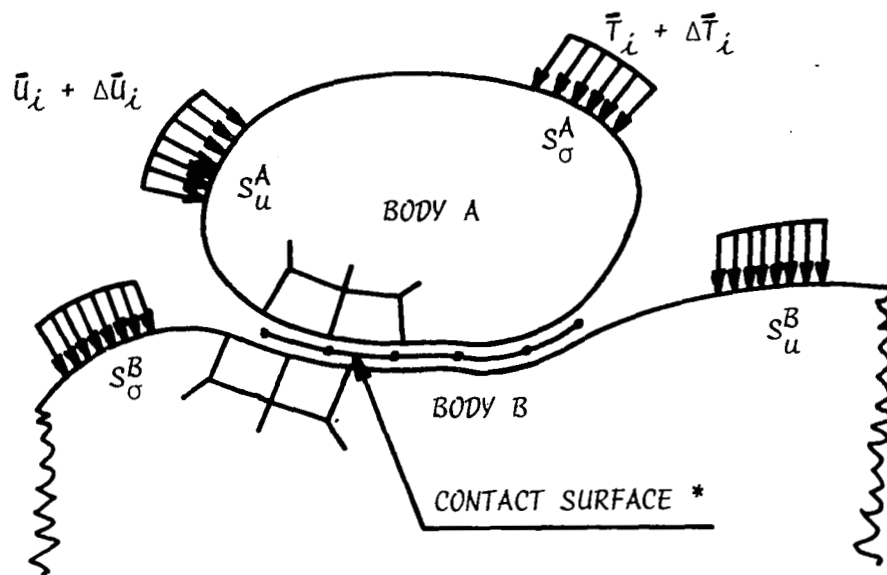
friction . They were approximately verified for all contacting nodes. Finally, the contacting surfaces of the disk and the plane are shown in Fig. 4 along with the locations of the nodes obtained in previous solutions. It can be seen that friction retards the movement of the contact surface, and as in the previous solution, through averaging over the entire contact surface, the contact condition of no separation or penetration is satisfied.

CONCLUSIONS

- (1) An incremental variational principle and a corresponding finite element formulation have been made for contact problems based on an assumed stress hybrid method. An iterative scheme for the solution has been developed.
- (2) Successful applications of the present method for plane elasticity problems have been demonstrated for
 - (a) Non-sliding problems with node-to-node contact and with node-to-internode contact,
 - (b) Frictionless contact with extensive sliding, and
 - (c) Frictional contact with extensive sliding.
- (3) The present method should be extended to problems involving material and/or geometrical nonlinearities in addition to contact nonlinearity.

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* BODY A AND BODY B INTERACT THROUGH THE CONTACT SURFACE.

Figure 1.- Contact surface.

$$(E = 21,000 \text{ kg/mm}^2, \nu = 0.3)$$

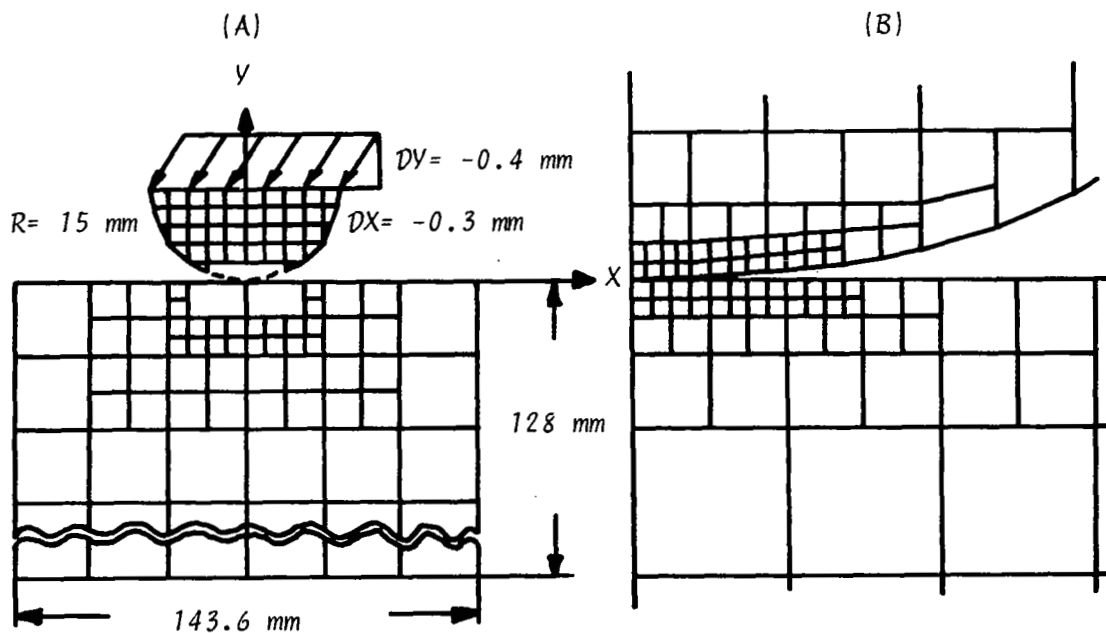


Figure 2.- Mesh pattern.

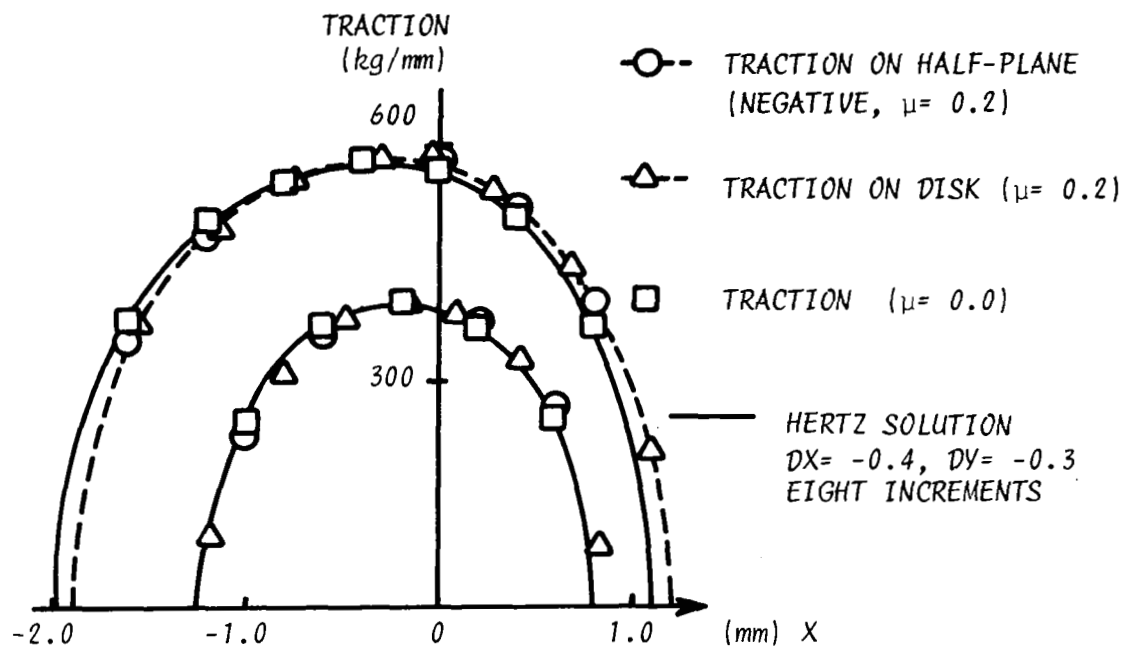


Figure 3.- Normal traction on contact surface.

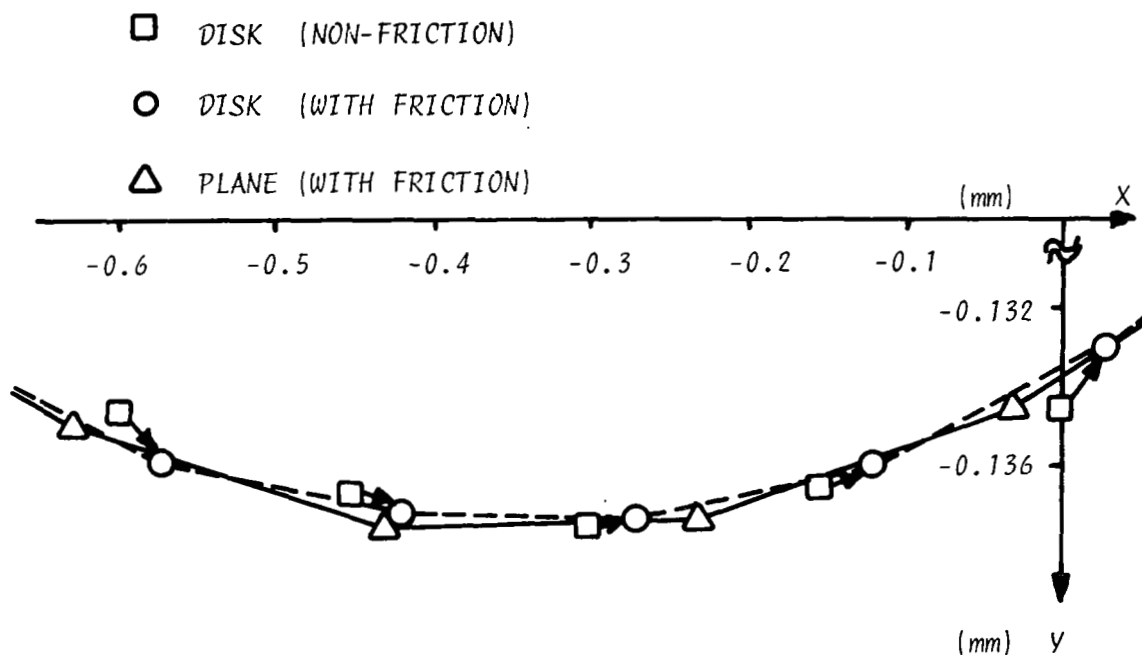


Figure 4.- Location of nodes.