FINITE ELEMENTS FOR CONTACT PROBLEMS
IN TWO-DIMENSIONAL ELASTODYNAMICS

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SUMMARY

This article summarizes some aspects of research in progress for developing finite element methods for contact problems. We propose a new"finite element approach" for contact problems in two-dimensional elastodynamics. Sticking, sliding and frictional contact can be taken into account. The method consists of a modification of the shape functions, in the contact region, in order to involve the nodes of the contacting body. The formulation is symmetric (both bodies are contactors and targets), in order to avoid interpenetration. Compatibility over the interfaces is satisfied. The method is applied to the impact of a block on a rigid target. The formulation can be applied to fluid-structure interaction, and to problems involving material nonlinearity. The extension to three dimensions presents additional difficulties, but it is possible.

INTRODUCTION

The approach presented in this article was developed while trying to simulate the movement of a gas bubble in a liquid. The original idea was to introduce the compatibility of the velocities over the gas-liquid interface via a constraint equation and to handle it by the Lagrange multiplier method. In a second step, the Lagrange multiplier method was replaced by a penalty method, which is easier to implement. In both cases, the constraint equation is a geometric relationship between gas and liquid velocities. No local remeshing was performed; the bubble and liquid meshes were simply superposed. This resulted in poor pressure fields along the interface. Looking for an improvement of this situation, remeshing appeared as the best but also the most cumbersome solution. Alternatively, a modification of the shape functions appeared to have the advantages of remeshing, without its inconveniences. This latter approach is described herein as it is applied to contact problems in two-dimensional elastodynamics. Frictional contact results in an exchange of momentum between the two contacting bodies, and can be realised by direct introduction of a contribution of the contactor's velocity into the target's equation of motion. This is conveniently done by means of a modification of the shape functions, as described in the next paragraphs.

The proposed approach has the advantage, as compared to the Lagrange multiplier method, of maintaining a constant size of the linear system to be solved. Compared to a penalty method, it has the advantage that we get automatic compatibility of the field variables over the interface. When the formulation is symmetric (i.e., both bodies are targets and contactors), interpenetration is totally avoided.

MODIFIED SHAPE FUNCTIONS FOR QUADRILATERAL FINITE ELEMENTS

Figure 1 shows a two-dimensional contact problem. Node C contacts element (1-2-3-4) of the target and from then on contributes to its shape functions. We start from the initial 4-nodes interpolation function

$$v = \sum_{a=1}^{4} N_a v_a \tag{1}$$

with

$$N_a = 0.25\,(1+\mathrm{sign}(\xi_a)\,\xi)\,(1+\mathrm{sign}(\eta_a)\,\eta) \tag{2}$$

where ξ, η are local coordinates, and v are the velocities.
In order to take the contribution of point C (node 5) into account, we modify the interpolation function as follows :

$$v = \sum_{a=1}^{5} N_a^{\star} v_a \tag{3}$$

Notice that from a global point of view there is no new node appearing.

Obviously, a "hat shape function" at node 5 is the most adequate for our purpose. This yields automatic compatibility of the velocities at the interface, if a symmetric formulation is used. Further, we want to account for tangential sliding with friction at the contact point. Therefore, we introduce a factor μ which allows the shape function at node 5 to vary in amplitude between 0 and 1, which will lead to a partial exchange of momentum. The resulting shape functions are (assuming C is on side $\eta = +1$) :

$$N_5^{\star} = \mu N_5 = \begin{cases} 0.5\,\mu\,(1+\eta)\left[1+(\xi-\xi_5)/(1+\xi_5)\right] & \text{if } \xi < \xi_5 \\ 0.5\,\mu\,(1+\eta)\left[1-(\xi-\xi_5)/(1-\xi_5)\right] & \text{if } \xi \geqslant \xi_5 \end{cases} \tag{4}$$

and

$$N_a^{\star} = N_a - \alpha_a N_5^{\star} \qquad a = 1 \longrightarrow 4 \tag{5}$$

with

$$\alpha_a = 0.5\,(1+\mathrm{sign}\,\xi_a\,\xi_5)$$

The local coordinates of C are $(\xi_5, 1)$. We can also assume, without restriction, that the contact point is associated with local coordinates (0, +1), N_5^{*}

then becomes[†]

$$N_5^* = 0.5\mu(1+\eta)(1-\mathrm{sign}(\xi)\cdot\xi) \tag{6}$$

This shape function is shown on figure 2. Observe that $N_3^*(\xi_5, \eta_5)$ does not vanish at node 5 when $\mu \neq 1$, but $N_a^* = 1$ is preserved.

In the sequel, we separate normal (n) and tangential (t) directions and therefrom the following typical possible situations.

a. $\mu_n = 1, \mu_t = 0$, corresponds to frictionless sliding in the tangential direction and sticking in the normal direction. This yields : $N_n^* = N_a$ ($a = 1 \rightarrow 5$), the standard 5-nodes interpolation function, $N_t^* = N_a$ ($a = 1 \rightarrow 4$), the standard 4-nodes interpolation function.

b. $\mu_n = 1, \mu_t = 1$ corresponds to sticking, and yields : $N_n^* = N_t^* = N_a$ ($a = 1 \rightarrow 5$).

c. $\mu_n = 1, \mu_t \in]0.1[$ this accounts for frictional sliding.

Since μ depends on orientation, we introduce a second order tensor, which we need in order to define strain rates and stresses in global coordinates.

In a local orthògonal frame tangential to the target surface we write

$$\begin{Bmatrix} v_n^T \\ v_t^T \end{Bmatrix} = \begin{bmatrix} \mu_n & 0 \\ 0 & \mu_t \end{bmatrix} \begin{Bmatrix} v_n^c \\ v_t^c \end{Bmatrix} \tag{7}$$

where the superscripts T and C stand for target and contactor, respectively, and the subscripts n and t for normal and tangential.

Equation (7) defines the contribution of node 5 (contactor) to the target velocity while the true local velocity, at C, is given by

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† $\mathrm{sign}\ (\xi) = \begin{matrix} -1 & \text{if } \xi < 0 \\ +1 & \text{if } \xi \geqslant 0 \end{matrix}$

$$\left\{ v^n \right\} = \sum_{a=1}^{5} \left[N_a^n \right] \left\{ v_a^n \right\} \tag{8}$$

where[†]

$$\left\{ v^n \right\} = \lfloor v_n^T \quad v_t^T \rfloor \tag{9}$$

$$\left\{ v_a^n \right\} = \lfloor v_{1n}^T \; v_{1t}^T \; v_{2n}^T \; v_{2t}^T \; \ldots\ldots v_{4t}^T \; v_{5n}^C \; v_{5t}^C \rfloor^T \tag{10}$$

and $\left[N_a^n \right]$ is defined as follows[††]

$$\left[N_5^n \right] = N_5 \left[\mu^n \right] \tag{11}$$

$$\left[N_a^n \right] = N_a \left[I \right] - \alpha_a \left[N_5^n \right] \tag{12}$$

$$\left[\mu^n \right] = \begin{bmatrix} \mu_n & 0 \\ 0 & \mu_t \end{bmatrix} \tag{13}$$

$$\left[I \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{14}$$

The matrix $\underset{\sim}{N}_5^n$ is diagonal when defined in a local referential, tangential to the contact surface. It does not induce a coupling of the normal and tangential components, but this would not be true in a global referential. We can, therefore, establish the stiffness in this local referential and rotate the element matrix before we assemble the elements. Alternatively we can make the derivation in a global coordinate system.

In practice, the whole effort essentially reduces to minor changes in the shape function routines.

TRANSIENT SOLUTION PROCEDURE

Search Algorithm

We need to determine at each time step the location of each node of each body in the contact zone with respect to the mesh of the other one. For that purpose a connectivity matrix is established in the input phase; this matrix lists all elements connected to each element. Assuming the time step to be small, we memorize the previous position of each node (by an element number), and search for its new position in adjacent elements. A 2-dimensional search path is shown on figure 3. Once the new position is known, we modify the shape functions of the target element as described in the previous paragraph and compute the updated stiffnesses.

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† T as superscript of a matrix stands for transpose

†† α_a depends on the local coordinate of the fifth node

Predictor-Corrector Algorithm

We adopt here an explicit predictor-corrector algorithm, defined by the following equations, at time step (n + 1) (see ref. 1, 2 for details).

$$\underset{\sim}{M}\,\underset{\sim}{a}_{n+1} + \underset{\sim}{N}(\tilde{\underset{\sim}{d}}_{n+1}, \tilde{\underset{\sim}{v}}_{n+1}) = \underset{\sim}{F}_{n+1} \tag{15}$$

$$\tilde{\underset{\sim}{d}}_{n+1} = \underset{\sim}{d}_n + \Delta t\,\underset{\sim}{v}_n + \frac{\Delta t^2}{2}(1-2\beta)\,\underset{\sim}{a}_n \tag{16}$$

$$\tilde{\underset{\sim}{v}}_{n+1} = \underset{\sim}{v}_n + \Delta t(1-\gamma)\,\underset{\sim}{a}_n \tag{17}$$

$$\underset{\sim}{d}_{n+1} = \tilde{\underset{\sim}{d}}_{n+1} + \Delta t^2 \beta\,\underset{\sim}{a}_{n+1} \tag{18}$$

$$\underset{\sim}{v}_{n+1} = \tilde{\underset{\sim}{v}}_{n+1} + \Delta t\,\gamma\,\underset{\sim}{a}_{n+1} \tag{19}$$

$$\underset{\sim}{d}_0 = \underset{\sim}{D} \tag{20}$$

$$\underset{\sim}{v}_0 = \underset{\sim}{V} \tag{21}$$

$$\underset{\sim}{a}_0 = \underset{\sim}{M}^{-1}(\underset{\sim}{F}_0 - \underset{\sim}{N}(\underset{\sim}{d}_0, \underset{\sim}{v}_0)) \tag{22}$$

Equations (16) and (17) are predictor equations (upper tilda), (18) and (19) are corrector equations, (20) to (22) are initial conditions, and $\underset{\sim}{N}$ is a nonlinear algebraic operator†. The implementation procedure can be found in (ref. 1).

If frictional contact occurs, we need in addition a predictor equation for $\tilde{\mu}_{n+1}$. Because of lack of space, this is not developed here. For the time being we adopt

$$\tilde{\mu}_{n+1} = \mu_n \tag{23}$$

NUMERICAL RESULTS

The analysis of an impact of a rectangular block on a rigid surface is performed (see ref. 3, for comparison). Figure 4 shows the mesh. The data are

density	$\rho = 0.01$
modulus of elasticity	$E = 1{,}000$
Poisson's coefficient	$\nu = 0.3$
dimensions	$L \,.\, W = 9.9$
time step	$\Delta t = 0.002725$

† If variables are to be memorized at element integration points, as often in nonlinear problems, remember that these are moving when node 5 moves.

Newmark parameters (explicit predictor-corrector algorithm) $\gamma = 0.5 \quad \beta = 0.25$

initial velocity $v_0 = 1$

wave velocity $C_d = \{[E(1-\nu)] / [(1+\nu)(1-2\nu)\cdot\rho]\}^{0.5} = 366.9$

The time step is defined by the transit time for a dilatational wave to cross one element. The impact takes place at t = 0. Frictionless contact is assumed ($\mu_t = 0$). This is introduced via isolated nodes, as shown on figure 4a. For the purpose of testing the new formulation, both node-to-node and distinct nodal positions are tested and yield the same results.

The anticipated solution is shown on figure 4b. This exact solution has two constant zones separated by the dilatational wave front emanating from the initial impact. The circular wave front is a result of reflections off the free boundary.

During the early steps of the computation, stresses in zone II are obtained from the impulse equation applied to a one-dimensional situation ($\sigma = c\cdot\rho\cdot v_0$) Stress results shown on figure 5.a confirm the validity of the new approach. Some overshoot appears, however, in the stress results of the lowest row of elements, probably due to the absence of a discrete impact condition in the algorithm. The deformed configuration at t = 0.0218 s.is shown on figure 5b.

CONCLUDING REMARKS

A new approach to contact problems involving friction in two-dimensional elastodynamics is proposed in this article. The basic idea obviously shows some analogies with local remeshing techniques, like the one proposed in (ref. 4).

The treatment of friction via modified shape functions seems similar to the lines of thinking adopted in (ref. 5) for the treatment of shock waves.

The proposed formulation is symmetric (both bodies are contactor and target for each other) and satisfies compatibility of velocities over the contact interface (when possible), thus avoiding interpenetration. Completeness of the modified elements remains satisfied. Although no detailed comparison with different approaches has been made as yet, the following advantages can be mentioned : constant size of the system of equations (not true for local remeshing or Lagrange multiplier approach, important if an implicit solution is performed) and interface compatibility (not true in general for Lagrange multipliers or penalty methods).

The extension of the method to several contacting nodes per element is possible, but it is not trivial. The extension to contact problems in three dimensional space and inclusion problems in 2-D is possible, but at the cost of losing interface compatibility. As already mentioned, nonlinear analysis may present minor difficulties because of the fact that integration points can move.

Further research is needed on the predictor algorithm for sliding with friction and impact-release conditions have to be added.

Our main effort, at present, is directed towards testing the approach in problems involving friction.

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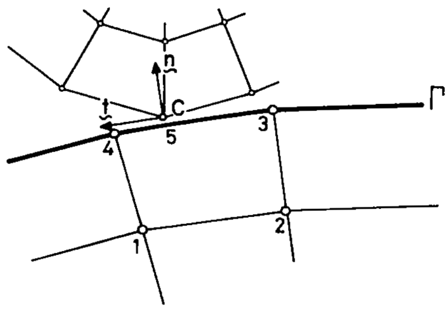


Figure 1.- Contact problem.

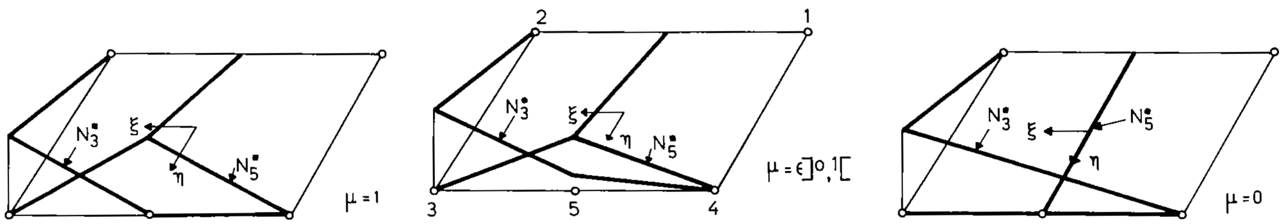


Figure 2.- Modified shape functions.

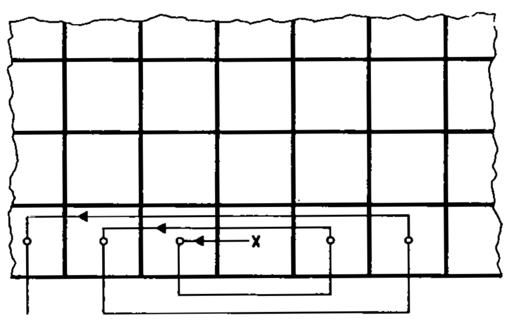
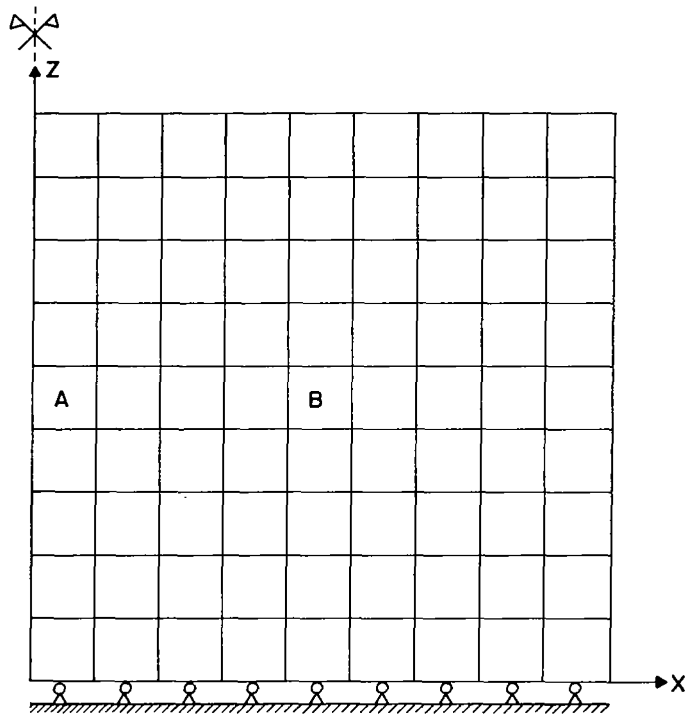
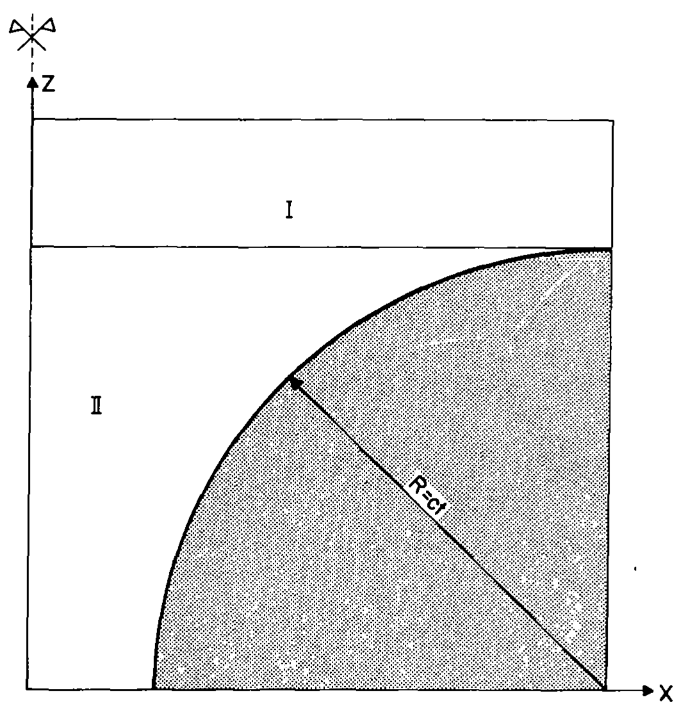


Figure 3.- Search path.

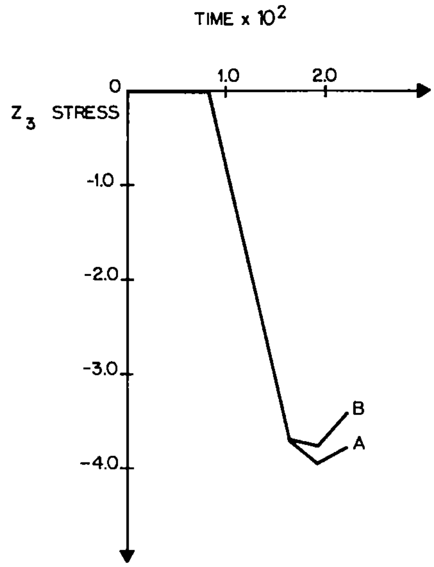


(a) Finite element mesh.

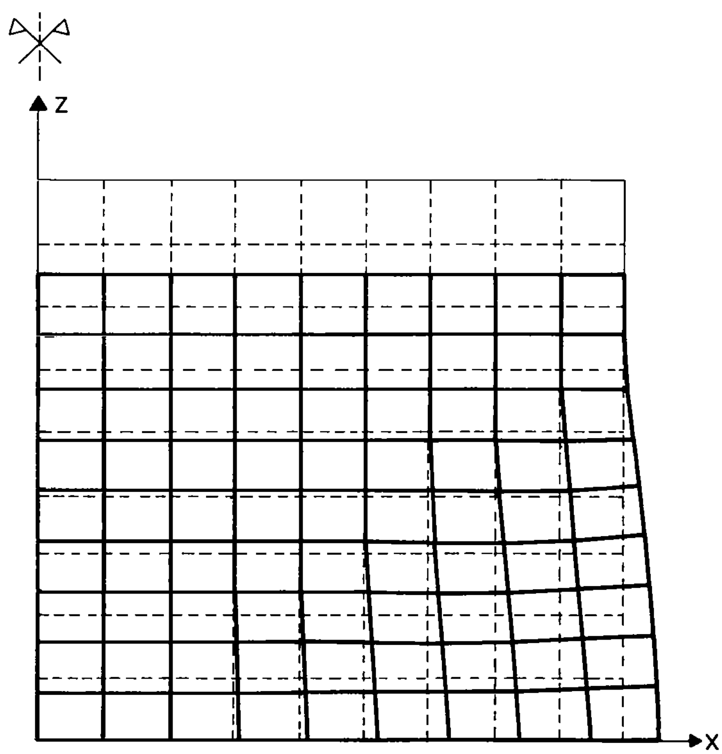


(b) Wave front diagram.

Figure 4.- Impact of rectangular block on a rigid surface.



(a) Stress results.



(b) Deformed mesh.

Figure 5.- Stress results and resulting deformed mesh.