NONLINEAR FINITE ELEMENT ANALYSIS - AN ALTERNATIVE FORMULATION

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SUMMARY

A geometrical nonlinear analysis based on an alternative definition of strain is presented. Expressions for strain are obtained by computing the change in length of the base vectors in the curvilinear element coordinate system. The isoparametric element formulation is assumed in the global Cartesian coordinate system. The approach is based on the minimization of the strain energy, and the resulting nonlinear equations are solved by the modified Newton method. Integration of the first and second variation of the strain energy is performed numerically in the case of two- and three-dimensional elements. Application is made to a simple long cantilever beam.

INTRODUCTION

The nonlinear finite element formulation described here represents a part of the development of the BASIS finite element analysis system (ref. 1). The basic idea was to combine linear and nonlinear behaviour in order to deal more efficiently with structural analysis. Thus, nonlinear elements had to be developed which fit into an existing system without loss of general validity. The Lagrange formulation has therefore been chosen (ref. 2), expressing the displacement variables directly in the global Cartesian coordinate system, although locally, for practical purposes, a skew coordinate system may be prescribed. The advantage of this formulation is the numerical method arising from it. By adopting the modified Newton method (ref. 3, 4) a clear solution process has been chosen, therefore minimizing errors due to a wrong understanding of nonlinear behaviour. The strains can be adapted to the nonlinearity of the problem. This feature may considerably reduce the computational effort.

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A set of one-, two- and three-dimensional elements (in terms of curvilinear coordinates) can be derived from this formulation (Fig. 1). The elements presented here are based on the isoparametric approximation (ref. 3). One-dimensional elements are well suited to deal with cable and truss problems (ref. 5). Membrane and volume elements are mainly used to model elastic composite structures.

GOVERNING EQUATIONS

Consider a point P of the undeformed body (Fig. 2). The corresponding cartesian coordinate vector $\{r\}$ may be expressed as a function of the body's curvilinear coordinates Θ^α

$$\{r\} = \sum_{i=1}^{n} \{r\}_i \Psi_i(\Theta^\alpha) \tag{1}$$

where Ψ_i are the interpolation functions of the corresponding nodes. The base vectors at P are obtained by deriving $\{r\}$ with respect to the coordinates Θ^α

$$\{g\}_\alpha = \frac{\partial\{r\}}{\partial\Theta^\alpha} \tag{2}$$

For practical purposes, Eq. 2 can be rewritten

$$\{g\}_\alpha = [A]_\alpha \{s\} \tag{3}$$

$[A]_\alpha$ contains all functions $\Psi_{i,\alpha}$ and $\{s\}$ all Cartesian components of the node vectors $\{r\}_i$.

In order to define the strain energy the metric tensor at P

$$g_{\alpha\beta} = \{g\}_\alpha \cdot \{g\}_\beta \qquad \alpha = 1,3;\ \beta = 1,3 \tag{4}$$

and the infinitesimal volume

$$dV = \det(g_{\alpha\beta}) \cdot d\Theta^1 \cdot d\Theta^2 \cdot d\Theta^3 \tag{5}$$

are needed. In the deformed state the base vectors become (Fig. 1)

$$\{G\}_\alpha = \frac{\partial}{\partial\Theta} \sum_{i=1}^{n} (\{r\}_i \Psi_i + \{u\}_i \Psi_i) \tag{6}$$

Eq. 6 may be rewritten

$$\{G\}_\alpha = [A]_\alpha \cdot \{d\} + \{g\}_\alpha \tag{7}$$

where $\{d\}$ contains, similar to $\{s\}$, all Cartesian components of the node displacements. The metric tensor of the deformed body at P becomes

$$G_{\alpha\beta} = \{G\}_\alpha \cdot \{G\}_\beta \tag{8}$$

Substituting Eqs. 3 and 7 in Eq. 8,

$$G_{\alpha\beta} = \{d\}^T[D]_{\alpha\beta}\{d\} + \{d\}^T([D]_{\alpha\beta} + [D]_{\beta\alpha})\{s\} + g_{\alpha\beta} \tag{9}$$

with

$$[D]_{\alpha\beta} = [A]_\alpha^T \cdot [A]_\beta \tag{10}$$

The strains are now defined as the relative change of the base vectors:

$$\varepsilon_{\alpha\beta} = \frac{|\{G\}_\alpha + \{G\}_\beta| - |\{g\}_\alpha + \{g\}_\beta|}{|\{g\}_\alpha + \{g\}_\beta|} \tag{11}$$

or

$$\varepsilon_{\alpha\beta} = \left[\frac{G_{\alpha\alpha} + G_{\alpha\beta} + G_{\beta\alpha} + G_{\beta\beta}}{g_{\alpha\alpha} + 2g_{\alpha\beta} + g_{\beta\beta}}\right]^{\frac{1}{2}} - 1 \tag{12}$$

Eq. 12 can be expressed in terms of {d}

$$\varepsilon_{\alpha\beta} = \left[\frac{\{d\}^T[\bar{D}]_{\alpha\beta}(\{d\} + 2\{s\})}{\bar{g}_{\alpha\beta}} + 1\right]^{\frac{1}{2}} - 1 \tag{13}$$

with

$$[\bar{D}]_{\alpha\beta} = [D]_{\alpha\alpha} + [D]_{\alpha\beta} + [D]_{\beta\alpha} + [D]_{\beta\beta} \tag{14}$$

$$\bar{g}_{\alpha\beta} = g_{\alpha\alpha} + 2g_{\alpha\beta} + g_{\beta\beta} \tag{15}$$

Eq. 13 defines the strains $\varepsilon_{\alpha\beta}$ used to derive the numerical equations from the strain energy.

Note here that if the deformations are limited in size the root in Eq. 13 can be expressed as a series expansion according to

$$(X_{\alpha\beta} + 1)^{\frac{1}{2}} = 1 + \frac{1}{2} X_{\alpha\beta} + \ldots \tag{16}$$

Retaining only the first two terms of Eq. 16 leads to the quadratic approximation

$$\tilde{\varepsilon}_{\alpha\beta} = \frac{\{d\}^T[\bar{D}]_{\alpha\beta}(\{d\} + 2\{s\})}{2\bar{g}_{\alpha\beta}} \tag{17}$$

from which the linear solution may be obtained.

Assuming linear elastic behaviour, the variation of strain energy leads directly to the nonlinear equilibrium equations of the element.

For ease of formulation the strains $\varepsilon_{\alpha\beta}$ are rewritten in vector form, $\{\varepsilon\} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}\}$. Furthermore, as the stress-strain relations are mainly formulated in a rectangular coordinate system the strain is transformed by

$$\{\gamma\} = [T]\{\varepsilon\} \tag{18}$$

The transformation matrix [T] is constant if the scalar products of the base vectors in the undeformed and in the deformed system are the same.

The strain energy density then becomes

$$dU = \frac{1}{2}\left(\{\sigma\}^T - \{\sigma_0\}^T\right)[E]\left(\{\gamma\} - \{\gamma_0\}\right)dV \tag{19}$$

or

$$dU = \frac{1}{2}\left(\{\varepsilon\}^T[T]^T[E] - \{\sigma_0\}^T\right)[T]\{\varepsilon\}dV \tag{20}$$

Eq. 20 is then integrated over the element. For one-dimensional elements an analytical solution is possible (ref. 5), whereas for two- and three-dimensional elements numerical integration must be performed. Using Gaussian integration the strain energy becomes

$$U \cong \frac{1}{2}\sum_{i=1}^{m}\left(\{\varepsilon\}_k^T[T]_k^T[E] - \{\sigma_0\}_k^T\right)[T]_k\{\varepsilon_k\}g_k w_k \tag{21}$$

where w_k designates the weight factor at the corresponding point Θ_k^α .

The set of nonlinear equations for one element is directly obtained from the first variation of the total potential energy with respect to the global variables

$$q_i(u_j) = \frac{\partial U}{\partial u_i} = f_i \tag{22}$$

{f} denotes the vector of external loads. It is worth mentioning here that in the case of dynamic analysis the equilibrium equations can also be obtained by applying Hamilton's principle and a suitable operator for discretization in time.

After rearrangement, Eq. 22 becomes

$$\{q\} = \sum_{k=1}^{m}\left(\{\varepsilon\}_k^T\cdot[\overline{E}] - \frac{1}{2}\{\sigma_0\}_k^T\cdot[T]_k^T\right)\cdot[P]_k\cdot g_k\cdot w_k \tag{23}$$

or

$$\{q\} = \sum_{k=1}^{m}\{\overline{\sigma}\}_k^T\cdot[P]_k\cdot g_k\cdot w_k \tag{24}$$

with

$$[P]^T = [\{p\}_{\alpha\beta}^T] \qquad \alpha = 1,3;\ \beta = \alpha,3 \tag{25}$$

and

$$\{p\}_{\alpha\beta} = \frac{(X_{\alpha\beta}+1)^{-\frac{1}{2}}}{\bar{g}_{\alpha\beta}} \cdot [\bar{D}]_{\alpha\beta} \cdot (\{u\}+\{s\}) \tag{26}$$

In order to solve Eq. 22 by the modified Newton method the Jacobian of {q} is needed:

$$[K] = \left[\frac{\partial^2 U}{\partial u_i \partial u_j}\right]$$

$$= \sum_{k=1}^{m} \left[[P]_k^T \cdot [\bar{E}] \cdot [P]_k + \bar{\sigma}_{\alpha\beta}(X_{\alpha\beta}+1)^{-\frac{1}{2}} \cdot \left\{ \frac{1}{\bar{g}_{\alpha\beta}} [\bar{D}]_{\alpha\beta} - \{p\}_{\alpha\beta} \otimes \{p\}_{\alpha\beta} \right\} \right] \quad \alpha=1,3;\beta=\alpha,3 \tag{27}$$

For a set of several elements the stiffness matrices [K] are assembled to the global stiffness matrix and factored. Using the modified Newton method the n-th iteration becomes

$$\{d\}_{n+1} - \{d\}_n = \beta \cdot [K]_\ell^{-1} (\{f\} - \{q\}_n) \tag{28}$$

[K] being factored at iteration step $\ell \leqslant n$. The relaxation factor is computed using an extrapolation method for each iteration (ref. 5). Thus the final load level is reached stepwise, and for each new load step the solution is extrapolated quadratically. Convergence criteria are based on the Euclidean and maximum norms (ref. 6).

The computational procedure is essentially the same for all element types. However, for one-dimensional elements {q} and [K] can be determined analytically in terms of u_i , and the transformation [T] is not necessary. The [T] matrices for two- and three-dimensional elements are listed in the Appendix.

During the first assembly of [K] the [D], [T], [E] matrices as well as the geometry parameters $g_{\alpha\beta}$ and g are computed once for each Gaussian point and reused for further computations of [K] and {q}. For each iteration step {ε} and [P] are evaluated at each Gaussian point and the global {q} vector assembled. The global load vector contains not only external forces but also the first variation coming from linear elements. In fact, their contribution has to be evaluated only once for each load step, thus reducing the computing time.

NUMERICAL RESULTS

An application of the theory is demonstrated using results obtained by the BASIS computer program (ref. 1). A long thin cantilever beam (Fig. 3) has been idealized by 8-noded membrane elements. Its length is 1000 mm, height 10 mm and thickness 1 mm. The elastic modulus is 3000 N/mm^2 and the Poisson ratio 0.36. The beam is subjected to a load case consisting of a variable load Q at the node A and a case with constant Q and variable compression load P. Since the strains remain smaller than one in this example differences in the results are not detectable when using Eq. 17 instead of Eq. 13. The convergence criterion during iteration (Eq. 28) is based on the relative change of the displacement vector norm and has been set to 0.0001. The active load is applied stepwise. If convergence is rapid, the step is automatically increased. However, for highly nonlinear problems, it is preferable to recompute and refactor the stiffness matrix when the iteration diverges rather than to decrease the load step. The first and second variation of U has been computed using 2 by 2 Gaussian integration. Fig. 4 shows the load-displacement function at node A for transversal loading, and the same function for the divergence problem is exhibited in Fig.5. Note the good behaviour of the two-element approximation even for large nonlinearities.

To conclude, it should be mentioned that the problems currently being investigated include the influence of the integration order on numerical accuracy and convergence behaviour as well as the nonlinear creep of structures.

SYMBOLS

Vectors are symbolized by {}-brackets and matrices by []-brackets. $[\,]^T$ means transposed matrix, $[\,]^{-1}$ inverted or factored matrix, and $\otimes$ stands for dyadic product. Greek indices refer to the curvilinear coordinate system.

$[A]_\alpha$	matrix of form functions of curvilinear coordinate Θ^α
$[D]_{\alpha\beta}$	product of form function matrices (Eq. 11)
$\{d\}$	nodal displacement vector (global Cartesian components)
$\{\varepsilon\}$	strain vector, contains components $\varepsilon_{\alpha\beta}$
$[E]$	elasticity matrix
$g_{\alpha\beta}$	metric tensor of the undeformed body
$G_{\alpha\beta}$	metric tensor of the deformed body
$\{g\}_\alpha$	base vector of the undeformed body, coordinate Θ^α

$\{G\}_\alpha$	base vector of the deformed body
g	determinant of the metric tensor $g_{\alpha\beta}$
$\{\gamma\}$	strain vector in the local Cartesian system
$\{\gamma_0\}$	initial strain vector in the local Cartesian system
$[K]$	second variation of the strain energy (stiffness matrix)
$[P]$	matrix defined in Eq. 26
Ψ_i	interpolation functions for geometry and displacements
$\{q\}$	first variation of the potential energy
$\{r\}_i$	global Cartesian coordinate vector of node i
$\{s\}$	vector containing all Cartesian node coordinates
$\{\sigma\}$	stress vector in local Cartesian system
$\{\sigma_0\}$	initial stress vector in local Cartesian system
$[T]$	transformation matrix relating $\{\varepsilon\}$ to $\{\gamma\}$ (see Appendix)
Θ^α	curvilinear element coordinates
$\{u\}_i$	global Cartesian displacement at node i
U	strain energy
w_k	weighting factor for Gaussian integration
x^1, x^2, x^3	orthogonal Cartesian coordinates
$X_{\alpha\beta}$	expression defined in Eq. 13.

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APPENDIX

The strain is always transformed to a Cartesian coordinate system defined by $\{v\} = \{r\}_2 - \{r\}_1$ for two-dimensional elements and by $\{v\}$ and $\{r\}_4$ for three-dimensional elements. For membrane elements, [T] becomes

$$[T]^{-1} = \frac{1}{2}\left[\{1 + \cos 2\phi_\alpha \text{ , } 1 - \cos 2\phi_\alpha \text{ , } 2\sin\phi_\alpha \cos\phi_\alpha\}\right] \quad \alpha = 1,3$$

$$\cos\phi_\alpha = \frac{\{v\} \cdot \{g\}_\alpha}{|\{v\}| \cdot |\{g\}_\alpha|}$$

For volume elements the six strain components are transformed similarly.

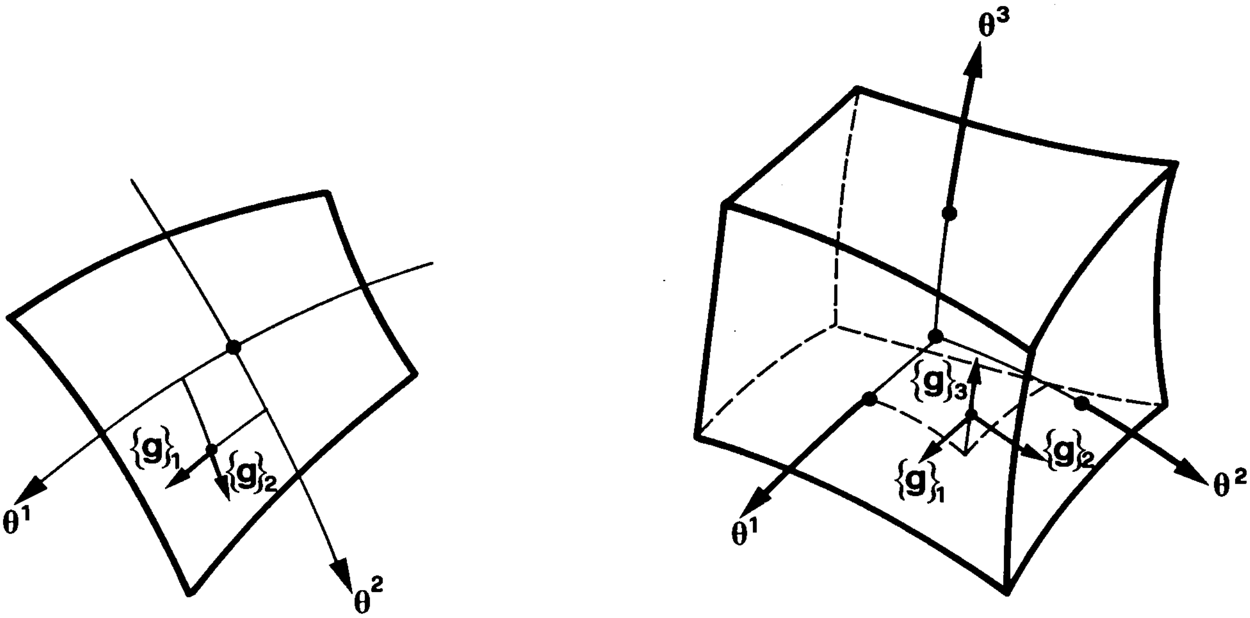

Figure 1.- Two- and three-dimensional elements.

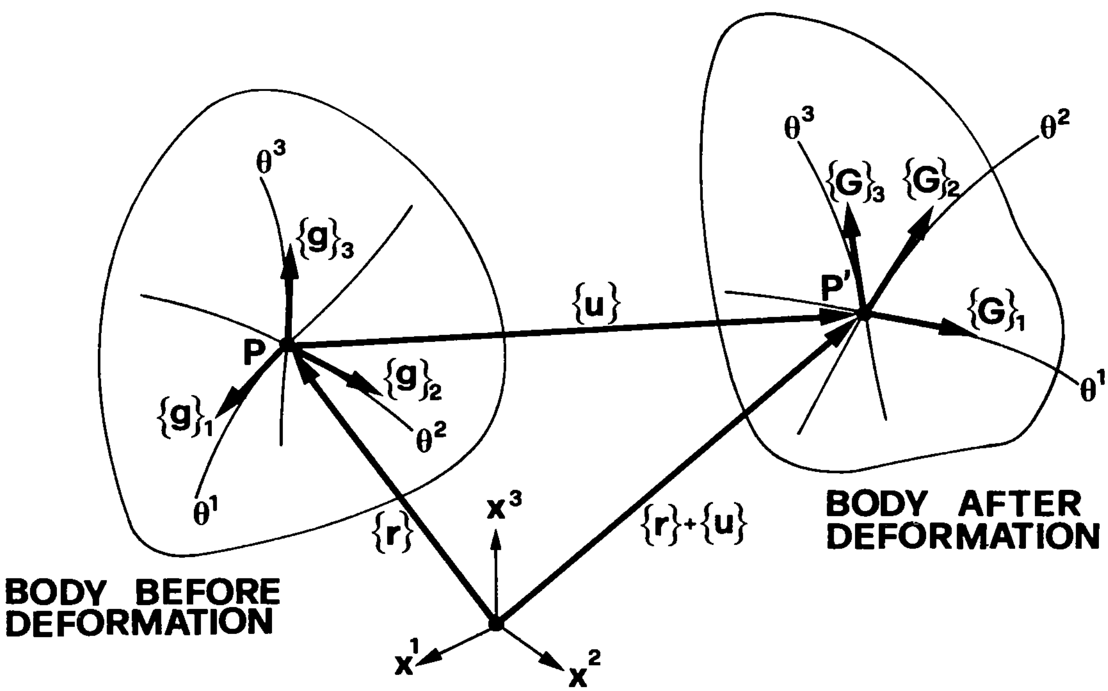

Figure 2.- Geometry of deformation.

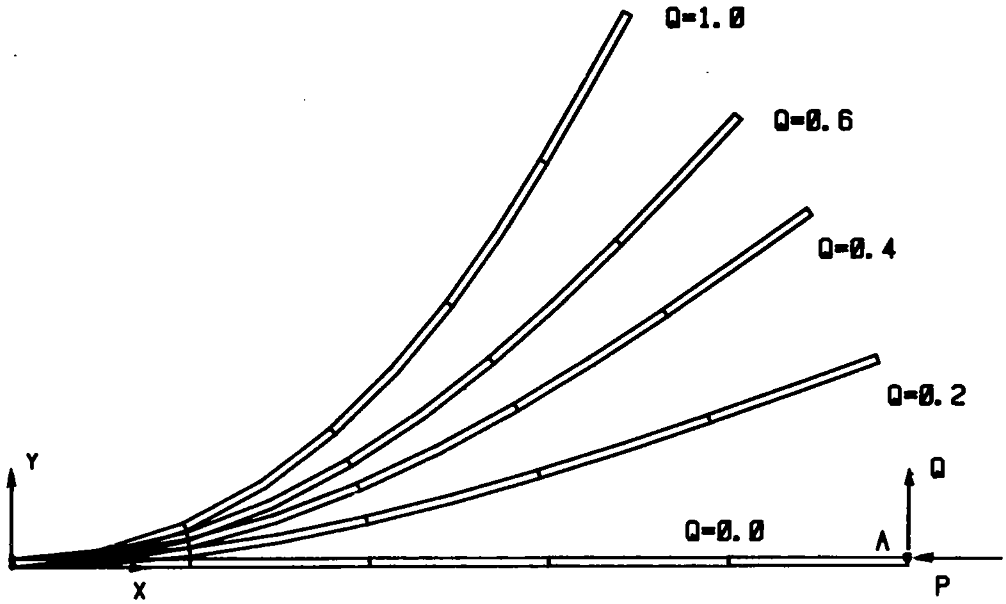

Figure 3.- Deformation of cantilever beam under transversal load Q.

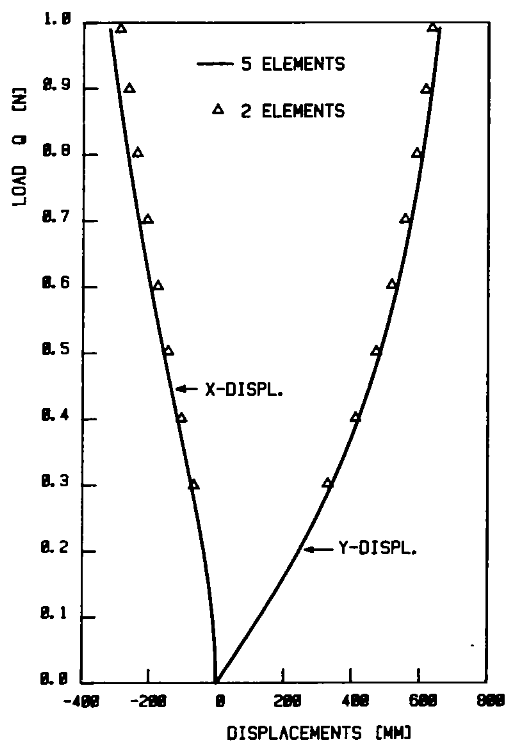

Figure 4.- Load-displacement function at node A under transversal load Q.

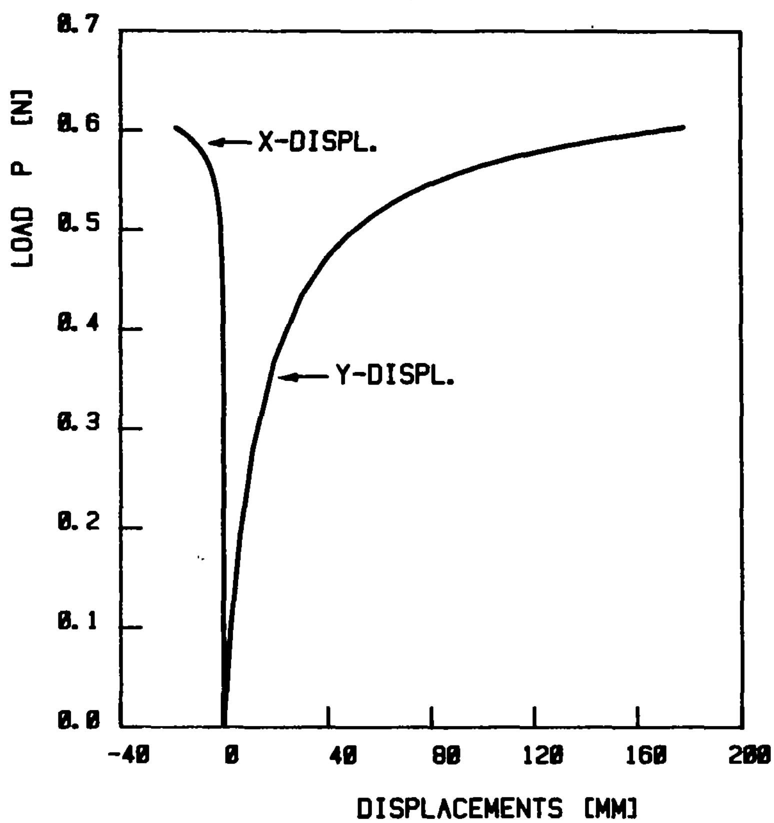

Figure 5.- Load-displacement function at node A
under axial compression P.