NASA Contractor Report 163101

NASA-CR-163101
19810002523

A COMPUTER PROGRAM FOR CYCLIC PLASTICITY AND STRUCTURAL FATIGUE ANALYSIS

I. Kalev

November 1980

NASA

NF02056

NASA Contractor Report 163101

A COMPUTER PROGRAM FOR CYCLIC PLASTICITY AND STRUCTURAL FATIGUE ANALYSIS

I. Kalev

National Research Council
NASA Dryden Research Associate

NASA
National Aeronautics and
Space Administration

1980

N 81-11031#

A COMPUTER PROGRAM FOR CYCLIC PLASTICITY AND STRUCTURAL FATIGUE ANALYSIS

I. Kalev
National Research Council—NASA Dryden Research Associate
Dryden Flight Research Center

INTRODUCTION

This report outlines a computerized approach for the structural analysis of the time-independent cyclic plasticity response and of the metal fatigue failure process. The approach combines three main analytical components, as follows:

(a) A cyclic plasticity model which relates the material's uniaxial stress-strain behavior to the multiaxial response of any structural component.

(b) Damage accumulation criteria which indicate both the life to crack initiation and the rate of crack growth, up to complete failure, for metallic structural components that undergo local cyclic plasticity strains. The required test parameters are derived from only the fatigue life of smooth material specimens when subjected to constant uniaxial plastic strain cycles.

(c) A finite element model for the numerical solution of the structure's nonlinear static and dynamic equilibrium equations. The isoparametric finite elements of the plane-stress, plane-strain, and axisymmetric types are incorporated. These elements are adequate for the representation of the behavior of most aircraft structural components that undergo meaningful plasticity strains.

The present combined approach enables the following types of analysis:

(a) The analysis of cyclic plasticity time-independent and rate-independent structural response under any varying loading which induces either proportional or nonproportional stress variations. Basically, the analysis is related to the material's cyclic steady-state behavior; however, the material's cyclic transient behavior can also be approximated. The effect of the cyclic yield stress change is not included, and the material is assumed to be of the so-called Masing type, which characterizes the metallic alloys used in aircraft. In addition, the material is assumed to be initially isotropic. The effect of the material's cyclic anisotropy due to the Bauschinger phenomenon is incorporated.

(b) Crack initiation prediction under varying loadings. The prediction is made by employing the Coffin-Manson criterion for the multiaxial stress state.

(c) Crack growth rate prediction. This prediction is made by employing a novel damage criterion which relates crack growth rate to the inverse damage gradient along the crack path. The criterion accounts for (1) the effects of plasticity, (2) the effects of residual stresses and of multiaxial stress redistributions at the crack tip which lead to crack retardation, (3) the effects of multiple overloads and negative loads, and (4) the interaction of close cracks. The effect of possible crack closure is not directly incorporated; however, this phenomenon is approximated by including the effect of the residual compressive stresses at the crack tip, which is the main cause of crack closure. The effects of loading frequency, temperature, and other time-dependent phenomena are not incorporated.

(d) Propagated crack growth rate prediction. This prediction is based on the application of the above-mentioned damage criterion using developed damage data accumulated from several updated finite element models. No procedure for the inclusion of residual stresses in the propagated crack's wake is included. It is assumed that the effect of these residual stresses in the crack's wake is negligible because of their usually small magnitude and because of their accelerating relaxation rate. The orientation of the propagated crack is set either normal to the computed principal tensile stress or in a direction selected by the user upon consideration of the direction of the most damaged paths.

The computer program is an extension of the NONSAP program (ref. 1). It incorporates cyclic plasticity models and damage accumulation criteria and has an option for sorted output. A full listing of the program's new features is given in the appendix of this report. The two-dimensional isoparametric finite elements and the numerical solution procedures are those of the NONSAP program. As this program is an in-core solver, the size of the finite element model is limited. However, the analysis of structural components is still practical by using 130K of the computer core, as demonstrated later in this report.

APPLICATION TO DAMAGE-TOLERANT AIRCRAFT DESIGNS

The damage tolerance requirements specified in MIL-A-83444(USAF) (ref. 2) are based on the assumption that a crack already exists in each element of a new structure as a result of flaws in the material, corrosion, or manufacturing damage. The structure should sustain the growth of these assumed cracks without a total failure during its lifetime, and also still sustain a specific residual static strength. Reference 2 defines two approaches for the substantiation of a structure's damage-tolerant integrity: the fail-safe approach and the slow-crack-growth approach. The fail-safe approach assumes a smaller initial crack length and a shorter loading spectrum than the slow-crack-growth approach; however, it requires more structural accessibility for inspection and overall, as well as local, structural redundancies that are frequently impractical in aircraft structures. The slow-crack-growth approach can be applied to all structural types, and it is also simpler to implement.

The slow-crack-growth approach requires that crack growth be slow enough not to achieve an unstable size during the life of the structure. The initial crack length is assumed to be on the order of 0.25 inch (6.3 millimeters) (ref. 2), and the crack is also often assumed to be through the member's thickness. These requirements can lead, under the usual applied loading, to significant plastic strains at the crack tip. A typical loading specturm is composed of varying tension-compression components, with multiple overloads, as depicted in figure 1. Although the loading variation is not of a fully cyclic type, it still often imposes cyclic plasticity stresses, because of the material Bauschinger phenomenon.

The imposed cyclic plasticity at the crack tip and the resultant residual stresses exclude the implementation of the usual analytical methods, which are based on the stress-intensity range. The present computer program can handle these phenomena analytically, by combining the finite element method, the material's cyclic plasticity model, and the damage accumulation criterion. This analysis is essential both for ensuring the integrity of the structural components during their life and for the proper evaluation of the results of structural proof tests.

The present computer program can also be applied to cases in which the crack tip undergoes relatively small cyclic plasticity strains. This application can be carried out by idealizing the material's stress-strain uniaxial curve with both a low yield stress and a first segment's slope which differs only slightly from the material's Young's modulus. However, it should be noted that the accuracy of the present damage criterion decreases for smaller plastic strains, while the accuracy of the simpler stress-intensity range approach increases. The present damage criterion is suitable only for cyclic plasticity strains; therefore, monotonically increased plastic strains as exhibited in the static residual-strength analysis cannot be handled by the present computer program. In addition, repeated loads which do not cause reverse plasticity, but cause plastic reloading at the same unloading stress, are assumed to contribute to the cumulative plastic strain but not directly to the cumulative damage. This will be clarified later in this report.

The present computer program does not account for the beneficial effects of initial compressive stresses due to shot peening, fastener interference, cold-working, and the like. However, it should be realized that these effects are usually small because of the quick stress relaxation in the cyclic plasticity field.

The required input data for the computer program are outlined later in this report.

THEORETICAL APPROACH

Cyclic Plasticity Models

Three plasticity models are incorporated in the present computer program. They differ from each other in their definitions of the incremental translation of the yield surfaces during the hardening of the material. The three models are identical for the proportional stress state, but they lead to somewhat different results for the usual nonproportional stress state. As none of these models has yet been shown through solid experimental evidence to be superior to the others, the choice of model is left to the user.

The three plasticity models are based on classical incremental time-independent and rate-independent plastic flow theory for initially isotropic materials. Incremental plastic flow theory assumes that the plastic strain increment is much higher than the adjacent elastic strain increment, and that plastic strain increments can be computed independently on the basis of the previous loading step stresses. Therefore, small loading step sizes, specified by the user, are mandatory for solution accuracy. The material's uniaxial stress-strain curve can be idealized by a maximum of three elastoplastic piecewise linear segments in addition to the first elastic segment, as shown in figure 2(a). The reversal uniaxial segments are shifted by the program to twice the initial yield stress, and the length of the segments is magnified by a factor of two, assuming material of the Masing type. However, the user can change the idealization of the first reversal in order to represent the material's transient condition.

Each linear segment of the material's uniaxial curve is related to a yield surface in the multiaxial stress state, as shown in figure 2(b). Each yield surface is defined by the von Mises criterion and the associated plastic flow normality rule. It is allowed to translate in the stress space up to its bounding yield surface, to which it remains connected until the unloading stage. The translation rate is governed by one of the following three hardening rules (fig. 3). Prager's hardening rule physically assumes that the incremental translation is in the direction of the plastic strain increment, i.e. normal to the yield surface. In order to satisfy this rule unconditionally, the surfaces' translations in the zero stress directions are mathematically permitted. Ziegler's hardening rule assumes that the incremental translation is in the direction of the vector which connects the center point of the current yield surface to the existing stress point. Both of these hardening rules require continuous position corrections of the yield surfaces to ensure tangency among the surfaces in contact. Mroz's hardening rule is based on the inherent fulfillment of this tangency requirement.

The full mathematical expressions of the plasticity models are presented in reference 3.

It should be noted that the cyclic plasticity room temperature stress relaxation phenomenon is not included in the present plasticity models; however, this phenomenon is directly included in the present damage criteria, which are discussed next.

Life to Crack Initiation

According to the presently used criterion, crack initiation occurs after 2N reversals of cyclic loading, when the cumulative damage D equals a unit. The damage is expressed mathematically as follows (ref. 4):

$$D = \sum_{1}^{2N} \left(\frac{\int d\bar{\varepsilon}^p}{2\varepsilon_f'} \right)^{-1/c} \left(1 - \frac{3\bar{\sigma}_m}{\sigma_f} \right)^{1/n'c} \tag{1}$$

The quantity $\int d\bar{\varepsilon}^p$ denotes the integration of the equivalent plastic strain increment, $d\bar{\varepsilon}^p$, through each pair of reversals. The equivalent plastic strain increment is a positive scalar composed of the multiplication of the plastic strain increments $d\varepsilon_{ij}^p$ (i, j = 1, 2, 3 in tensor notation), and it is computed by the plasticity model as follows:

$$d\varepsilon^p = \left(d\bar{\varepsilon}_{ij}^p \, d\varepsilon_{ij}^p \right)^{1/2} \tag{2}$$

The quantity $\bar{\sigma}_m$ is the average value of the mean stresses at the two plastic unloadings which define the specific pair of reversals, or

$$\bar{\sigma}_m = 1/2 \left[\left(\sigma_{ii}/3 \right)_{\text{First unloading}} + \left(\sigma_{ii}/3 \right)_{\text{Second unloading}} \right] \tag{3}$$

The quantity $\bar{\sigma}_m$ also represents the effects of the tensile versus compressive stresses. If the reversal loading results in a symmetric stress variation, or $\sigma_{ii,max} = \sigma_{ii,min}$, then $\bar{\sigma}_m = 0$.

If the stress relaxation effect is to be included, as it should be when $\bar{\sigma}_m$ is not small, the user must define an experimental material parameter r (ref. 3) such that the relaxed $\bar{\sigma}_m$ value becomes

$$\bar{\sigma}_m = \bar{\sigma}_m' \, (2N)^{-r\left(\int d\bar{\varepsilon}^p/2\right)} \tag{4}$$

where $\bar{\sigma}_m'$ is the original average mean stress. For the numerical examples to be shown later in this report, a value of r of 277 has been adopted for aluminum alloy 7075-T6 plate.

The material parameters n', c, ε_f', and σ_f in equation (1) are defined by the user for the specific material. The parameter n' is the material's uniaxial cyclic exponent. It relates the uniaxial stress amplitude, $\Delta\sigma/2$, to the applied constant plastic strain amplitude, $\Delta\varepsilon^p/2$, in the form of $\Delta\sigma/2 = K'(\Delta\varepsilon^p/2)^{n'}$, where K' is assumed to be approximated by $\sigma_f/(\varepsilon_f')^{n'}$. The value of the exponent n' can be derived from several uniaxial plastic strain tests at the material's cyclic steady state, as indicated by figure 4(a). The parameter ε_f' is the material's cyclic ductility parameter, which is smaller than the monotonic ductility parameter, ε_f. The parameter σ_f is the material's fracture strength. The parameter c is the Coffin-Manson exponent (fig. 4(b)), which is derived from constant plastic strain amplitude tests of the material's uniaxial unnotched specimens.

The values of these material parameters depend on the specimen's surface treatment and environmental conditions. Therefore, the above-mentioned uniaxial tests have to be conducted under the same conditions as exist in the real structure.

Crack Growth Rate

The crack growth rate is approximated by the inverse damage gradient along the crack path. The cumulative damage is computed by equation (1) at two discrete points in front of the crack tip. These discrete points are defined by the two integration points of the finite element adjacent to the crack tip. Figure 5 designates these integration points as number 1 and number 2; they are located at distances of a_1 and a_2 from the crack tip, respectively. Assume that the accumulated damage at points 1 and 2 is termed D_1 and D_2, respectively. If the crack propagates by the small distance of $(a_2 - a_1)$, the damage at point 2 becomes D_1; thus, the average cumulative damage value is $1/2(D_1 + D_2)$. The crack growth rate, $\frac{da}{d(2N)}$, is approximated as follows (ref. 3):

$$\frac{da}{d(2N)} = \frac{a_2 - a_1}{\left(\frac{2}{D_1 + D_2} - \frac{1}{D_1}\right)} \tag{5}$$

where a is half the length of the existing crack. Equation (5) indicates that a complete fracture occurs when $D_2 \geq D_1$.

The finite element integration points, whose cumulative damage values are used for the crack growth rate prediction, are chosen by the user according to the predicted crack path, which is usually normal to the direction of the principal tensile stress. These integration points should be well within the material's cyclic plasticity range. This requires a reasonably small finite element to be used at the crack tip.

Damage Accumulation Technique

The damage criterion in equation (1) is applied to each pair of reversals separately, and the results are accumulated during the entire applied loading history.

Each pair of reversals is defined, as mentioned before, during two subsequent plastic unloadings made in reversal directions. The plastic unloadings in figure 6, for example, occur at points B, D, F, H, J, and L. However, the unloading at point F is not considered because the following plastic unloading, at point H, is not in the reversal direction. Therefore, the first pair-reversal is AB-CD, the second pair-reversal is EH-IJ, and so on.

For tensile loads, the present pair-reversal damage accumulation technique could lead to somewhat more conservative results than the well-known rainflow technique (ref. 2). This is because the rainflow technique refers only to closed loops; in figure 6, the plastic strains along the AB, E'F, G'H branches would not be considered, because no closing counterpart branches exist. However, the rainflow technique does consider the effect of the elastic loop FGG'.

The present damage accumulation technique does not account for the effect of elastic reversals, i.e. it ignores the effect of the elastic loop FGG' in figure 6. This is justified because the damage criterion (eq. (1)) employs the material's cyclic ductility strain ε_f', which is smaller than the material's monotonic ductility strain. The technique does incorporate the cyclic parameters n' and c; thus, it is assumed that the fatigue damage is due mainly to the plasticity cycles.

Finite Element Modeling and Equation Solutions

The NONSAP program's two-dimensional isoparametric elements and its solution procedures (ref. 1) are utilized. The eight-node element with undistorted shape and 3 X 3 integration points has been found to furnish a suitable representation of both the plastic strain variation and the damage gradient. The finite element adjacent to the crack tip should be small enough for the two integration points along the predicted crack path to be well within the cyclic plasticity range. In addition, the idealization should be such that the existing crack front is at the corner node, not at the mid-node, of the eight node element. Far from the crack tip and far from the stress concentration zones, the number of nodes can be reduced to four to save computer core and time.

The behavior of large plastic strains is approximated by employing the Green-Lagrange strain tensor and the second Piola-Kirchhoff stress tensor in Lagrangian coordinates. The use of this approximation is justified, since most of the fatigue failures are accompanied by only small to moderate cyclic strains around the material's yield strain.

The nonlinear equilibrium equations due to the plasticity and the large strains are solved incrementally. The size of the loading steps is variable and is set by the user, based on his numerical experience. Usually, several short trial and error runs are expected for each specific case before the largest possible step sizes are determined. The parameter which usually governs the step sizes is the material's uniaxial stress-strain slope. A smaller material slope requires a smaller step size.

The static analysis requires the construction of a new tangent stiffness matrix at each loading step. The dynamic analysis can be carried out either by employing

Newmark's implicit time-integration method or by employing the explicit central-difference method. The central-difference method is much less time consuming, but it is more prone to numerical instabilities and thus requires smaller step sizes. This method is especially attractive for cases of small material hardening, where the required time step sizes are already relatively small because of the small material slopes. The iterative NONSAP procedure for equilibrium corrections is not incorporated because of the possibility of nonconvergence at the plastic unloading steps.

PROGRAM OUTLINE

The program utilizes the NONSAP computer program's elements and solution techniques for large strains and plasticity, and for static or dynamic analysis. The new features presented here include the following:

(a) The incorporation of the cyclic plasticity models and fatigue data computations through a separate overlay (number 3.8; see appendix). The NONSAP overlay tree is shown in reference 1.

(b) Sorted output data. This is necessary because of the enormous available output data and the need to segregate the fatigue data required for the computation of the damage criteria.

Following is a brief summary of the main computation steps.

(a) The overall linear stiffness and mass matrices are constructed first. If dynamic analysis is required and Newmark's direct time integration technique is used. the overall linear effective stiffness matrix is constructed. In addition, the applied load vector is constructed. The large strain stiffnesses derived by using the Total Lagrangian procedure and the cyclic plasticity stiffnesses are updated at each loading or time step. These stiffness values are added to the linear stiffness matrix.

(b) The equilibrium equations are solved incrementally, and displacements and strains are obtained for each step. The program has an optional two-step restart capability, which is useful for problems which involve only partially different loads and for dividing a long computer run into two separate and more manageable runs.

(c) For each finite element integration point which is pre-defined by the user as an elastoplastic element, the following steps are executed at each loading or time step.

- The previous step's values of elastoplastic stiffness are recomputed.

- The plastic strain increment is computed, as is the total equivalent plastic strain.

- The stress increment is computed and the total stresses are updated. The mean stress is computed.

- The yield surface translations are computed, ensuring that the surfaces' non-intersection requirement is met.

•The elastoplastic stiffnesses are updated in four subincrements and added to the overall structural stiffnesses for the next loading step.

•Continuous checks are made for plastic unloading. If it occurs, the peak von Mises stress is kept in the memory to indicate the following reloading state.

•Plastic loading or reloading is considered when the current stress point reaches the first yield surface. The plastic reloading criterion distinguishes between reyielding at the reversed plastic region and reyielding at the same plastic region. Reyielding at the same plastic region is initiated when the accumulated elastic work during the unloading range is zero, or nearly zero. The computed accumulated damage value is for each pair of reversals; only fully reversed stress cycles are considered.

•The fatigue data for equation (1) are computed. After each pair of reversals, damage is accumulated for an indication of the life to crack initiation. The crack growth rate is computed by substituting the results of equation (1) into equation (5).

The mathematical formulations are presented in reference 3.

INPUT AND OUTPUT DATA

The input data are identical to the NONSAP specifications, with the following exceptions. The specified material model number for the cyclic plasticity analysis is NPAR(15) = 9. The number of constants per property set should be specified as NPAR(17) = 15, and the dimension of the storage array should be specified as NPAR(18) = 27. Then the material properties are specified on two input cards. The first input card contains eight parameters, in 8F10.0 format, as follows: the Young's modulus, the Poisson ratio, the yield stress, and the uniaxial slope of the first elastoplastic piecewise linear segment; the yield stress and the uniaxial slope of the second segment; and the yield stress and the uniaxial slope of the third segment. The second input card contains seven parameters, in 7F10.0 format, as follows: the yield stress and the uniaxial slope of the first, second, and third plastic reversal segments; and a seventh parameter, RULE, that indicates the required cyclic plasticity model. If RULE = 0, rigid plastic material is assumed. If RULE = 1, the well-known isotropic hardening rule is employed. If RULE = 2, 3, or 4, the kinematic hardening rule due to Prager, Ziegler, or Mroz is used, respectively.

For a material in the cyclic steady state, the specified reversed yield stresses and slopes should be identical to the values of the first reversal. Different slopes can be specified for the first and second reversals for representation of the material's transient state. In the following reversals the data specified for the second reversal are used.

The output data are printed on four tapes: TAPE6, TAPE12, TAPE13, and TAPE14. TAPE6 includes the input data and deflections. TAPE12 includes parameters for fatigue analysis. Included are the following terms:

NEL - The finite element number.
IPT - The integration point number.
LO - The number of plastic reversals. For the first plastic range LO = 1, for the second plastic reversal LO = 2, and so on.
IPEL - The current position of the equivalent von Mises stress. If IPEL = 1, 2, or 3, the stress point is on the first, second, or third piecewise linear segment, respectively.
DEPC - The cumulative equivalent plastic strain.
SMEAN - The mean stress.
FT - The equivalent von Mises stress.
SX - The maximum principal stress.
SY - The minimum principal stress.
ALPHA - The direction of the maximum principal stress relative to the element's coordinates.
DWE - Numerical stability indicator. It equals the stress increment times the elastic strain increment. The value should be positive; otherwise it indicates that a numerical instability due to too high step size has been introduced.
HP - Numerical stability indicator. It should be equal to the input slope of the specific material segment.
WP - Unloading indicator. If WP is negative, unloading occurs.
IRE - Reloading indicator. If IRE = 0, there is no reloading. If IRE = 1 or IRE = 3, fully reversal plastic reloading occurs. If IRE = 2, plastic reloading occurs at the same unloading point.
WP2 - The cumulative plastic work. Used for reference.
DEE - The current total work. Used for reference.

TAPE13 includes the computed stresses. TAPE14 includes the computed strains, surface translations, and other parameters explained in the printout shown in the appendix of this report.

The output data from TAPE12 are used for the fatigue analysis. The other data used in the fatigue analysis include the material's cyclic stress-plastic strain exponent n' and the Coffin-Manson material parameters c, ε_f', σ_f, which are defined in equation (1). Also needed is the material stress-relaxation exponent r, which is defined in equation (4). The $\int d\varepsilon^p$ value in equation (1) is calculated by subtracting the computed DEPC values at the two plastic unloading points which define the specific pair of reversals. The average of the SMEAN values at these two unloading points is calculated according to equation (3). This value should be iteratively reduced by employing equation (4) because of the assumed cyclic plasticity stress relaxation. Then equation (1) is employed for the accumulation of the pair-reversal damage. When it reaches a unit value, crack initiation is assumed. The crack growth rate is approximated using equation (5) by substituting the cumulative damage values at the two discrete points in front of the crack tip and along the predicted crack path. The crack growth path is usually predicted to be normal to the direction of the principal tensile stress, which is indicated by the ALPHA value.

APPLICATION EXAMPLES

This section describes the application of the present approach to the analysis of two structural components: a cracked panel under variable uniaxial loadings and stiffened aircraft skin panel under compressive loading.

The cracked panel is shown in figure 7(a). The magnitude of the applied loadings is such that significant plastic strains develop in front of the crack tip. Figure 7(b) depicts the finite element model, which employs plane-stress four-to-eight node isoparametric elements. The eighth node elements are solved by 3 X 3 integration points. The uniaxial cyclic material curve, idealized by three piecewise linear segments, is shown in figure 8. The material's fatigue properties are based on the constant strain amplitude test data from reference 5. The fatigue ductility parameter, ε_f', is assumed to be 0.18, while the measured monotonic ductility, ε_f, is 0.41. The fatigue strength, σ_f', is assumed to be equal to the monotonic fracture strength, σ_f, or 75.9 kg/mm^2 (108.0 ksi). The Coffin-Manson exponent c in equation (1) is estimated to be 0.52. The material uniaxial cyclic exponent, n', is 0.11.

In order to account for the stress relaxation, a value of r of 277 is assumed in equation (4). This value causes the relaxation of the existing mean stress down to 0.01 percent of its initially computed value, within two fully reversed strain cycles of $0.1\varepsilon_f'$. No experimental evidence exists for this value.

Results for fully cyclic loading and for tensile cyclic loading are shown in figures 9(a) and 9(b) and compared to test results which induce only small plasticity. These comparisons illustrate the significant crack growth retardation due to the plasticity stress redistributions and due to the residual compressive stresses developed after plastic unloading. The relative crack growth retardation is more significant for the tensile cyclic loading (fig. 9(b)) than for the fully cyclic loading (fig. 9(a)). This is because the residual compressive stresses in the latter case are followed by residual tensile stresses which diminish their beneficial effects. The computed crack displacements indicate that no crack closure occurs for the present loading conditions. The crack growth rate, $\frac{d(2a)}{dN}$, in figures 9(a) and 9(b) is depicted as a function of the stress intensity range $\Delta K = \beta \cdot \Delta\sigma_n \cdot \sqrt{a}$, where β is a geometric parameter, $\Delta\sigma_n$ is the net section stress range, and a is the half crack length. For cases of small and localized plasticity, the stress-intensity range is generally a representative parameter. However, in cases of gross plasticity, as in the present examples, ΔK loses its general validity; thus, the results shown in figures 9(a) and 9(b) are specific for the crack length used.

Figure 10 shows the effect of a tensile overload on the crack growth rate as computed by the present approach. It is apparent that this effect becomes more significant with increasing values of overload. This is in general agreement with the test data that have been reported in the literature.

The stiffened skin panel is shown in figure 11. The integral stiffeners' cross section at the spar location is changed as shown in figure 11(b). Axial loads due to overall wing bending could lead to high stress concentrations at the indicated point. These stress concentrations can usually be significantly reduced by the addition of a small area of structural reinforcement. Two cases, with different reinforcement area sizes, are analyzed. They are designated case 1 and case 2. Figure 12(a) shows the finite element model used. The applied loads are compression and vary with the stiffener's depth, as shown. The applied loading variation, shown in figure 12(b), causes local compressive yielding and high residual tensile stresses after unloading. Thus, although no tensile loads are applied, a cyclic compression-tension stress-strain field exists, causing crack initiation and propagation. The material's uniaxial stress-strain curve is idealized by three linear segments, as shown in figure 12(c). The material's fatigue properties are the same as those indicated for the cracked panel in the previous example.

Figure 13(a) shows the computed damage curves. Each curve indicates the equal damage accumulation value. As depicted, the small reinforcement area in case 2 significantly improves the life to crack initiation. Figure 13(b) shows the von Mises equivalent stress distribution for case 1. It is apparent that the stress gradient is much smoother than the damage gradient. This demonstrates the inability of stresses to predict the fatigue failure in a plastic field.

Figure 14(a) shows examples of the used cracked finite element models. The left-hand model represents the initial crack pattern, which is perpendicular to the component's free edge (and to the direction of the principal tensile stress). However, in order to maintain the element's parallelogram shape, which is an important factor for numerical accuracy, the crack's direction is changed slightly, as shown. The right-hand model in figure 14(a) represents progressive crack growth. The damage curves before the crack changes its direction are shown in figure 14(b). The damage accumulation gradient and the crack growth rate are derived from the curves shown in figures 13(a) and 14(b). The results are summarized in figure 14(c).

CONCLUDING REMARKS

This paper describes a computerized approach to the calculation of cyclic plasticity structural response, the prediction of life to crack initiation, and the prediction of crack growth rate. The method uses three analytical items: the finite element method and its associated numerical techniques for nonlinear static and dynamic analysis, the material cyclic plasticity theory, and the cumulative damage criteria.

The required input data include the loading spectrum, the material's cyclic uniaxial stress-strain curve, the material's cyclic stress-plastic strain exponent, and the Coffin-Manson low-cycle fatigue parameters. These parameters are derived from only smooth uniaxial specimens. The method also requires the material's stress relaxation exponent.

The damage criteria, and to some extent the cyclic plasticity models, are novel and without sound experimental supporting evidence. However, it is believed that in combination with engineering judgment, they can be used to obtain useful qualitative results.

The present in-core computer program is limited to small structural components. Provision for out-of-core computations would permit much broader application.

APPENDIX—PROGRAM LISTINGS

Following is a listing of the program CYCLIC for cyclic plasticity and fatigue analysis. The program includes the modifications to the NONSAP computer program (ref. 1) and the new overlay (number 3.8).

Explanatory titles and descriptions of the variables used are incorporated within the listing.

```
*IDENT CYC
*I NONSAP.3
     3TAPE12,TAPE13,TAPE14,
*D NONSAP.22
C
C                                   C Y C L I C
C
C                 CYCLIC PLASTICITY AND FATIGUE ANALYSIS PROGRAM
C
C                                     I. KALEV
C
C                                  SEPTEMBER, 1980
C
C                 THE NONSAP PROGRAM HAS BEEN MODIFIED TO INCLUDE
C                  1. CYCLIC PLASTICITY MODELS, ADDED AS OVERLAY 3.8,
C                     (*DECK CYCLIC), MATERIAL MODEL 9,
C                     FOR 2-D FINITE ELEMENTS (PLANE STRESS,
C                     PLANE STRAIN, AXISYMMETRY)
C                  2. SORTED OUTPUT DATA, AS FOLLOWS,
C                      TAPE6=OUTPUT INCLUDES DEFLECTIONS, STRESSES
C                       FOR MATERIAL MODELS 3 TO 8, AND THE INPUT DATA
C                      TAPE12 INCLUDES PAREMETERS FOR FATIGUE ANALYSIS
C                       EMPLOYING MATERIAL MODELS 1,2,9 AND
C                       NUMERICAL STABILITY CHECKS
C                      TAPE13 INCLUDES STRESSES FOR MATERIAL MODELS 1,2,9
C                      TAPE14 INCLUDES STRAINS AND OTHER COMPUTED RESULTS
C                       FOR MATERIAL MODEL 9
C
C    **    **    **    **    **    **    **    **    **    **    **    **    **    **
C
*D NONSAP.87
C .        MTOT=12000 AND NUMEST=8000 SPECIFY THE MAX. D.O.F. AND          .
C .        NONLINEAR ELEMENTS FOR THE COMPUTER CORE OF 130K                .
      MTOT=12000
*D NONSAP.88
  200 NUMEST=8000
C
*I NONSAP.460
C          OUTPUT DATA AND FORMATS
      WRITE(12,2023) KSTEP,TIME
      WRITE(12,2021)
      WRITE(12,2022)
 2023 FORMAT(/////10HTIME STEP ,I5,20X,8HAT TIME ,E10.4/)
 2021 FORMAT(2X,3HNEL,1X,3HIPT,1X,2HLO,1X,4HIPEL,9X,4HDEPC,3X,5HSMEAN,6X
     1,2HFT,6X,2HSX,6X,2HSY,2X,5HALPHA,3X,3HDWE,7X,2HHP,8X,2HWP
     2,1X,3HIRE,6X,3HWP2,6X,3HDEE)
 2022 FORMAT(2X,3H---,1X,3H---,1X,2H--,1X,4H----,9X,4H----,3X,5H-----,6X
     1,2H--,6X,2H--,6X,2H--,2X,5H-----,3X,3H---,7X,2H--,8X,2H--
     2,1X,3H---,6X,3H---,6X,3H---)
      WRITE (13,2023) KSTEP,TIME
      WRITE (14,2023) KSTEP,TIME
      WRITE (14,5051)
      WRITE (14,5052)
      WRITE (14,5053)
      WRITE(14,5054)
 5051 FORMAT (2X,3HNEL,1X,3HIPT,1X,4H  LO,1X,6H RATIO,1X,9H      YLD,1X
     1,9H      WP1,3X,9HSTRAIN(1),3X,9HSTRAIN(2),3X,9HSTRAIN(3),3X
     2,9HSTRAIN(4),1X,6HAL1(1),1X,6HAL1(2),1X,6HAL1(3),1X,6HAL1(4))
 5052 FORMAT (2X,3HNEL,1X,3HIPT,1X,4HIFEL,1X,6H  COEF,1X,9H     YMAX,1X
     1,9H      DWP,3X,9HDELEPS(1),3X,9HDELEPS(2),3X,9HDELEPS(3),3X
     2,9HDELEPS(4),1X,6HAL2(1),1X,6HAL2(2),1X,6HAL2(3),1X,6HAL2(4))
 5053 FORMAT (2X,3HNEL,1X,3HIPT,1X,4H   M,1X,6H    WE,1X,9H      WE2,1X
     1,9H       DF,3X,9H DEPSP(1),3X,9H DEPSP(2),3X,9H DEPSP(3),3X
     2,9H DEPSP(4),1X,6HAL3(1),1X,6HAL3(2),1X,6HAL3(3),1X,6HAL3(4))
 5054 FORMAT(2X,115H--- --- ---- ------ --------- ---------  ---------
     1 ---------  ---------  --------- ------ ------ ------ ------)
*I NONSAP.461
C   IN CASE PRINT OUT OF DFLECTIONS TO BE OMITTED FOR ALL LOADING STEPS
C   EXCEPT NO. 1 AND NO. 20,FOR EXAMPLE, MAKE THE FOLLOWING EFFECTIVE
C      IF (KSTEP.NE.1.OR.KSTEP.NE.20) GOTO 1001
*I NONSAP.471
```

```
  1001 CONTINUE
*D NONSAP.528
  2020 FORMAT (46H P R I N T   O U T   F O R   T I M E   S T E P ,I5,
*D TDFE.290,TDFE.294
C .       STRESSES OF MODELS 1 AND 2 ARE PRINTED ON TAPE13               .
       WRITE (13,2020) NG
       IF (ITYP2D.EQ.0) WRITE (13,2022)
       IF (ITYP2D.EQ.1) WRITE (13,2024)
       IF (ITYP2D.EQ.2) WRITE (13,2026)
       WRITE (13,2030)
*D TDFE.319
C .       STRESSES OF MODELS 1 AND 2 ARE PRINTED ON TAPE13               .
       WRITE (13,2035) N
*D TDFE.360
C .       STRESSES OF MODELS 1 AND 2 ARE PRINTED ON TAPE13               .
       WRITE (13,2040) I,STRESS,P1,P2,AG
*D TDFE.402
C .       STRESSES OF MODELS 1 AND 2 ARE PRINTED ON TAPE13               .
       WRITE (13,2040) IPT,STRESS,P1,P2,AG
*I TDFE.415
       IF (MODEL.EQ.9) GOTO 5040
*I TDFE.419
  5040 CONTINUE
C .       HEADLINES FOR STRESSES OF MODEL 9 ARE PRINTED ON TAPE13        .
       WRITE (13,2020) NG
       IF (ITYP2D.EQ.0) WRITE (13,2022)
       IF (ITYP2D.EQ.1) WRITE (13,2024)
       IF (ITYP2D.EQ.2) WRITE (13,2026)
*D TDFE.491
  2020 FORMAT(////46H S T R E S S   C A L C U L A T I O N S   F O R,3X,
*D MATRT2.74
     9 WRITE(6,2501) (PROP(I),I=1,NCON)
       WRITE (6,2061)
       RETURN
*I MATRT2.137
  2501 FORMAT(1H ,4X,42HE                                         PROP( 1)=,E14.6/
     1          ,1H ,4X,42HVNU                                       PROP( 2)=,E14.6/
     2          ,1H ,4X,42HYT1 MISES 1ST SURFACE,1 LOADING..PROP( 3)=,E14.6/
     3          ,1H ,4X,42HET1 SLOPE 1ST SURFACE,1 LOADING..PROP( 4)=,E14.6/
     4          ,1H ,4X,42HYT2 MISES 2ND SURFACE,1 LOADING..PROP( 5)=,E14.6/
     5          ,1H ,4X,42HET2 SLOPE 2ND SURFACE,1 LOADING..PROP( 6)=,E14.6/
     6          ,1H ,4X,42HYT3 MISES 3RD SURFACE,1 LOADING..PROP( 7)=,E14.6/
     7          ,1H ,4X,42HET3 SLOPE 3RD SURFACE,1 LOADING..PROP( 8)=,E14.6/
     8          ,1H ,4X,42HYC1 MISES 1ST SURFACE,RELOADING..PROP( 9)=,E14.6/
     9          ,1H ,4X,42HEC1 SLOPE 1ST SURFACE,RELOADING..PROP(10)=,E14.6/
     A          ,1H ,4X,42HYC2 MISES 2ND SURFACE,RELOADING..PROP(11)=,E14.6/
     B          ,1H ,4X,42HEC2 SLOPE 2ND SURFACE,RELOADING..PROP(12)=,E14.6/
     C          ,1H ,4X,42HYC3 MISES 3RD SURFACE,RELOADING..PROP(13)=,E14.6/
     D          ,1H ,4X,42HEC3 SLOPE 3RD SURFACE,RELOADING..PROP(14)=,E14.6/
     E          ,1H ,4X,42HRULE.............................PROP(15)=,E14.6)
  2061 FORMAT(
     F       1H ,4X,45H      IF RULE=0.    RIGID PLASTIC                /,
     G       1H ,4X,45H      IF RULE=1.    ISOTROPIC HARDENING          /,
     H       1H ,4X,45H      IF RULE=2.00 PRAGER KINEMATIC HARDENING /,
     I       1H ,4X,45H      IF RULE=3.00 ZIEGLER KINEMATIC HARDENING/,
     J       1H ,4X,45H      IF RULE=4.00 MROZ     KINEMATIC HARDENING/,
     K       1H ,4X,45HCOMBINED RULE=.XX(ISOTROPIC)+(1-.XX)KINEMATIC)
C
*I MATRT2.97
     D          29H      EQ.9, CYCLIC PLASTICITY //,
*D INITWA.63
    11 CALL OVERLAY(4HNSAP,3,10,6HRECALL)
*D INITWA.65
    12 CALL OVERLAY(4HNSAP,3,11,6HRECALL)
*D STSTN.181
    11 CALL OVERLAY(4HNSAP,3,10,6HRECALL)
*D STSTN.184
    12 CALL OVERLAY(4HNSAP,3,11,6HRECALL)
*D OVL38.2
       OVERLAY(NSAP,3,10)
*D OVL39.2
```

```
      OVERLAY(NSAP,3,11)
C
*D ELT2D9.2,ELT2D9.7
C
C
*DECK CYCLIC
C
      PROGRAM ELT2D9
C
      COMMON /EL/ IND,ICOUNT,NPAR(20),NUMEG,NEGL,NEGNL,IMASS,IDAMP,ISTAT
     1           ,NDOF,KLIN,IEIG,IMASSN,IDAMPN
      COMMON /DIMEL/ N101,N102,N103,N104,N105,N106,N107,N108,N109,N110,
     1               N111,N112,N113,N114,N120,N121,N122,N123,N124,N125
      COMMON /MATMOD/ STRESS(4),STRAIN(4),D(4,4),IPT,NEL
      COMMON A(1)
      DIMENSION IA(1)
      EQUIVALENCE (NPAR(10),NINT)
      EQUIVALENCE (A,IA)
C
C     FOR ADDRESSES N101,N102,N103,...  SEE SUBROUTINE TODMFE
C
      IF (IND.NE.0) GO TO 100
C
C     I N I T I A L I Z E   W A   W O R K I N G   A R R A Y
C
      IDW=27
      NPT=NINT*NINT
      NN=N110 + (NEL - 1)*NPT*IDW
      MATP=IA(N107 + NEL - 1)
      NM=N109 + (MATP - 1)*4
      CALL ICYCLIC (A(NN),A(NN),A(NM),NPT)
      RETURN
C
C     F I N D   S T R E S S - S T R A I N   L A W   A N D   S T R E S S
C
  100 IDW=27
      NPT=NINT*NINT
      NN=N110 + (NEL - 1)*NPT*IDW + (IPT - 1)*IDW
      MATP=IA(N107 + NEL - 1)
      NM=N109 + (MATP - 1)*4
C
      CALL CYCLIC (A(NM),A(NN),A(NN+4),A(NN+8),A(NN+12), A(NN+16)
     1, A(NN+20), A(NN+21), A(NN+22), A(NN+23),A(NN+24),A(NN+25),
     2 A(NN+26))
      RETURN
      END
C
      SUBROUTINE ICYCLIC (WA,IWA,PROP,NPT)
C
      DIMENSION WA(27,1),IWA(27,1),PROP(1)
C
C     SET INITIAL STRESSES AND STRAINS TO ZERO
C     SET INITIAL YIELD POINT TO PROP(3)
C
      DO 10 J=1,NPT
      DO 15 I=1,20
   15 WA(I,J)=0.0
      WA(21,J)=PROP(3)
      WA(22,J)=PROP(3)
      IWA(23,J)=0
      IWA(24,J)=0
      WA(25,J)=0.
      WA(26,J)=0.
   10 WA(27,J)=0.
      RETURN
      END
C
      SUBROUTINE CYCLIC (PROP,SIG,EPS,AL1,AL2,AL3,YIELD,YMAX,IPEL,LO,
     1WE2,WP2,DEPC)
C
C . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
```

```
C .     IST      NUMBER OF STRESS COMPONENTS                                    .
C .     ISR      NUMBER OF STRAIN COMPONENTS                                    .
C .     EPS      STRAINS  AT THE END OF THE PREVIOUS UPDATE                     .
C .     STRAIN   TOTAL CURRENT STRAIN                                           .
C .     DELEPS=STRAIN-EPS  TOTAL STRAIN INCREMENT                               .
C .     DEPS =(1-RATIO)/M*DELEPS                                                .
C .     DEPSP  PLASTIC STRAIN INCREMENT    PER M STEP                           .
C .       FOR PRINTING ONLY DEPSP=TOTAL OF ALL M STEPS                          .
C .     RATIC    PART OF STRAIN INCREMENT TAKEN ELASTICALLY                     .
C .     RATIC IS APPLIED IN THE ELASTIC-PLASTIC TRANSITION STEP ONLY            .
C .     DELSIG   INCREMENT IN STRESSES, ASSUMING ELASTIC BEHAVIOR               .
C .     SIG      STRESSES AT THE END OF THE PREVIOUS UPDATED STEP               .
C .     STRESS   CURRENT STRESS FOR PRINTING                                    .
C .     TAU   =SIG AT THE BEGINING OF THE STEP ,THEN UPDATED ,                  .
C .          AT THE END OF THE STEP  STRESS=TAU                                 .
C .     SMEAN  MEAN STRESS                                                      .
C .     M  NC. OF INCREMENT INTERVALS                                           .
C .      FOR ELASTIC STATE M=1, ELASTOPLASTIC STATE M=4                         .
C .          FOR TRANSITION STEP M=3 TO 15                                      .
C .     PROP(1) YOUNG S  MODULUS, E                                             .
C .     PROP(2) POISSON S  RATIO                                                .
C .     PROP(3) INITIAL YIELD STRESS IN SIMPLE TENSION                          .
C .     PROP(3),PROP(5),PROP(7)  YIELD STRESSES IN TENSION                      .
C .     PROP(4),PROP(6),PROP(8)  TANGENT MODULE  IN TENSION                     .
C .     PROP(9),PROP(11),PROP(13) YIELD STRESSES IN COMPRESSION                 .
C .     PROP(10),PROP(12),PROP(14) TANGENT MODULE  IN COMPRESSION               .
C .     PROP(15)=RULE  ,=0 RIGID PLASTIC , =1 ISOTROPIC MODEL                   .
C .                     =2.00 KINEMATIC, PRAGER S RULE                          .
C .                     =3.00            ZIEGLER S  RULE                        .
C .                     =4.00            MROZ S RULE                            .
C .                       .XX   COMBINED MODEL=                                 .
C .                       .XX(ISOTROPIC RULE)+(1-.XX)(KINAEATIC RULE).
C .          THE COMBINED MODEL IS NOT INCORPORATED IN THIS VERSION.
C .     NPAR(17)=15, NPAR(18)=IDW=24                                            .
C .     AL1,AL2,AL3  TRANSLATIONS OF THE THREE LOADING SURFACES                 .
C .     AL TRANSLATION OF THE CURRENT LOADING SURFACE                           .
C .     ALB  TRASLATION OF THE LOADING SURFACE BOUNDING THE CURRENT             .
C .       SURFACE. USED FOR MROZ S RULE ONLY                                    .
C .     ET,YY  SLOPE,YIELD STRESS OF THE CURRENT LOADING SURFACE                .
C .     IPEL= 0  ELATIC LOADING OR UNLOADING                                    .
C .     IPEL=1,2,3  PLASTIC LOADINE IN SURFACES 1,2,3                           .
C .     IP EQUALS TO IPEL FROM THE PREVIOUS LOADING INCREMENTAL                 .
C .        STEP OR FROM THE PREVIOUS SUBINCREMENTAL STEP                        .
C .     LO  NUMBER OF HALF CYCLES (REVERSALS)                                   .
C .     YIELD   PREVIOUS MISES STRESS OF THE BOUNDING SURFACE                   .
C .     YLD     CURRENT UPDATED MISES STRESS ,ALSO CRITERION FOR                .
C .             PLASTIC FLOW INTIATIVE                                          .
C .     YMAX   MISES  STRESS OF THE BOUNDING SURFACE WHEN UNLOADING             .
C .       IT IS SAVED UNTILL THE NEXT UNLOADING                                 .
C .      INITIALLY YLD=YT1 , IF WP.LT.0 YLD=YMAX FOR ISOTROPIC MODEL            .
C .       OR YLD=(YMAX-2*YC1) FOR KINEMATIC MODEL                               .
C .     WP=(TAU-AL)*DELSIG(ASSUM. ELASTIC BEHAVIOR), FOR UNLOADING              .
C .     WP1=TAU*DEPSP , WE=TAU*DEPS-WP1 , DF=(TAU-AL)*(TAU-SIG)                 .
C .     WP2=YLD*DEP CUMULATIVE PLASTIC WORK                                     .
C .     HP=YLD-YIELD/DEP , HEP=DEP/YLD                                          .
C .     WE2=ABS(TAU+SIG)/2*DELEPS  CUMULATIVE DURING ELASTIC STAGE              .
C .     DWP=(TAU-SIG)*DEPSP ,DWE=(TAU-SIG)*DELEPS-DWP                           .
C .     DEP EQUIVAL.  PLASTIC STRAIN ,DEE EQUIVALENT TOTAL STRAIN               .
C .     DEPC CUMULATIVE EQUIVALENT PLASTIC STRAIN INCREMENT, DEP CUMU.
C .     COMP=DEPSP(1+2+4) SHOULD BE ZERO                                        .
C .     COEF  FOR PERFECTLY PLASTIC , =1 P.STRESS,=0 P.STRAIN AND AXI.
C .     ITYP2D =0 AXIS, =1 P.STRAIN, =2 P.STRESS                                .
C .     IRE - INDICATOR FOR PLASTIC RELOADING.                                  .
C .        IF IRE=1   PLASTIC RELOADING WHEN WE2.GT..9*R, R=2*YC1**2/E.
C .                   INDICATES START OF FULLY PLASTICITY CYCLE                 .
C .        IF IRE=2   PLASTIC RELOADING WHEN WE2.LT.0.2*R,                      .
C .                   INDICATES START OF FLUCTUATING CYCLE                      .
C .        IF IRE=3   PLASTIC RELOADING WHEN WE2.LE..9*R AND .GE..2*R,.
C .                   INDICATES START OF FULLY PLASTICITY CYCLE, TOO            .
C .        IF IRE=0   ELASTIC OR PLASTIC LOADING AFTER THE FIRST                .
C .                   LOADING/RELOADING STEP                                    .
```

```
C .       PRINTED VALUES- WP,WP1,WE,DWP,DWE,DF,DEP,DEE ARE TOTAL OF      .
C .                       M INCREMENTS PER STEP                          .
C .                       WE2 IS CUMULATIVE FOR ELASTIC REGION ONLY      .
C .              DEPC,WP2 ARE    CUMULATIVE FOR ALL STEPS                .
C .       OUTPUT DATA ARE SORTED AS FOLLOWS-                             .
C .         TAPE6    DEFLECTIONS (IN ADDITION TO INPUT DATA)             .
C .         TAPE12   DATA FOR FATIGUE ANALYSIS AND NUMERICAL STABILITY   .
C .                  CHECK - DEPC,SMEAN,FT,SX,SY,ALPHA,DWE,HP,WP,IRE,WP2 .
C .                          ,DEE                                        .
C .         TAPE13   STRESSES                                            .
C .         TAPE14   STRAINS, SURFACES TRANSLATIONS, OTHER RESULTS       .
C . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
C
      COMMON /EL/ IND,ICOUNT,NPAR(20),NUMEG,NEGL,NEGNL,IMASS,IDAMP,ISTAT
     1            ,NDOF,KLIN,IEIG,IMASSN,IDAMPN
      COMMON /VAR/  NG,KPRI,MODEX,KSTEP,ITE,ITEMAX,IREF,IEQREF,INCMD
      COMMON /MATMOD/ STRESS(4),STRAIN(4),C(4,4),IPT,NEL
      COMMON /DISDER/ DISD(5)
      DIMENSION PROP(1),STG(1),EPS(1)
      DIMENSION TAU(4),DELSIG(4),DELEPS(4),DEPS(4),STATE(4)
      DIMENSION       AL2(1),AL3(1),DEPSP(4),AL1(1),AL(4),ALB(4)
      DIMENSION CC(4,4),CP(4,4)
      EQUIVALENCE (NPAR(3),INDNL),(NPAR(5),ITYP2D)
      DATA NGLAST/1000/, STATE/1HE,1HP,1HP,1HP/
      WP=HP=0.
      DEPSP(1)=DEPSP(2)=DEPSP(3)=DEPSP(4)=0.
      WP1=WE=DF=DWE=DWP=COMP=DEP=0.
      IRE=0
      M=1
      RATIC=0.
      SFAC=COEF=1.
      AL(1)=AL(2)=AL(3)=AL(4)=0.
      DO 101 I=1,4
      DO 101 J=1,4
  101 CC(I,J)=0.
      IF (IPT.NE.1) GO TO 110
      YT1=PROP(3)
      YT2=PROP(5)
      YT3=PROP(7)
      YC1=PROP(9)
      YC2=PROP(11)
      YC3=PROP(13)
      ET1=PROP(4)
      ET2=PROP(6)
      ET3=PROP(8)
      EC1=PROP(10)
      EC2=PROP(12)
      EC3=PROP(14)
      RULE=PROP(15)
      ET=ET1
      YY=YT1
C
      IST=4
      IF (ITYP2D.EQ.2) IST=3
      ISR=3
      IF (ITYP2D.EQ.0) ISR=4
      YM=PROP(1)
      PV=PROP(2)
      D1=PV/(PV - 1.)
      A2=YM/(1.+PV)
      B2=(1.-PV)/(1.-2.*PV)
      C2=PV/(1.-2.*PV)
      C1=A2/2.
      BM=YM/(1. - 2.*PV)/3.
      IF (ITYP2D.EQ.2) GO TO 105
C     PLANE STRAIN  /  AXISYMMETRIC
      B1=A2*C2
      A1=B1+A2
      GO TO 110
C     PLANE STRESS
  105 A1=YM/(1.-PV*PV)
```

```
      B1=A1*PV
C
  110 YLD = YIELD
C            CALCULATE INCREMENTAL STRAINS
      DO 120 I=1,ISR
  120 DELEPS(I) = STRAIN(I) - EPS(I)
      IF (ITYP2D.EQ.2) DELEPS(4)=D1*(DELEPS(1)+DELEPS(2))
      TAU(4)=0.
      DO 162 I=1,IST
  162 TAU(I)=SIG(I)
      IF (ITYP2D.EQ.2) STRAIN(4)=EPS(4)
C
      DELSIG(1) = A1*DELEPS(1) + B1*DELEPS(2)
      DELSIG(2) = B1*DELEPS(1) + A1*DELEPS(2)
      DELSIG(3) = C1*DELEPS(3)
      DELSIG(4) = 0.
      IF (ITYP2D.EQ.2) GO TO 150
      DELSIG(4) = B1 * (DELEPS(1)+DELEPS(2))
      IF (ITYP2D.EQ.1) GO TO 150
      DELSIG(1) = DELSIG(1) + B1*DELEPS(4)
      DELSIG(2) = DELSIG(2) + B1*DELEPS(4)
      DELSIG(4) = DELSIG(4) + A1*DELEPS(4)
  150 TAU(4) = 0.
      IF (IPEL.GE.1) M=4
      IF (IPEL.GE.1) GO TO 163
C
C         IF MATERIAL IN THE PLASTIC RANGE SKIP THE FOLLOWING
C         IF MATERIAL IN THE ELASTIC RANGE CALCULATE STRESSES
C
      DO 160 I=1,IST
  160 TAU(I) = SIG(I) + DELSIG(I)
C            CHECK WHETHER *TAU* STATE OF STRESS FALLS
C            OUTSIDE THE LOADING SURFACE
      WE=DWE=0.
      DO 164 I=1,4
      WE=WE+DELEPS(I)*TAU(I)
  164 DWE=DWE+DELEPS(I)*DELSIG(I)
      DO 203 I=1,4
  203 WE2=WE2+DELEPS(I)*ABS(TAU(I)+SIG(I))/2.
      SM=(TAU(1)+TAU(2)+TAU(4)-AL1(1)-AL1(2)-AL1(4))/3.
      SX=TAU(1)-AL1(1)-SM
      SY=TAU(2)-AL1(2)-SM
      SS=TAU(3)-AL1(3)
      SZ=TAU(4)-AL1(4)-SM
      FT1=1.5*(SX*SX+SY*SY+SZ*SZ+2.*SS*SS)
      FT=FT1-YLD**2
      WP=SX*DELSIG(1)+SY*DELSIG(2)+SS*DELSIG(3)*2.+SZ*DELSIG(4)
      IF (LD.GE.1.AND.RULE.GE.2.) GOTO 167
      IF (FT) 170,170,300
  167 CONTINUE
C
C         CHECK FOR PLASTICITY RELOADING
C         AVOINTING EARLY NUMERICALLY RELOADING
C          FOR FULLY CYCLIC RELOADING IRE=1 OR IRE=3
C          FOR RELOADING AT THE SAME STRESS POINT IRE=2
C
      IF (WP.LT.0.) GOTO 170
      IF ((FT1-YC1**2).LE.0.) GOTO 170
      R=2.*(YC1**2)/YM
      IF (ABS(WE2).GT.0.9*R) GOTO 190
      IF (ABS(WE2).LT.0.2*R) GOTO 191
      IRE=3
      GOTO 300
  190 IRE=1
      GOTO 300
  191 IRE=2
      GOTO 300
C
C     STATE OF STRESS WITHIN LOADING SURFACE  -  ELASTIC BEHAVIOR
C
  170 IPEL=0
```

```
      STRESS(4) = 0.
      DO 180 I=1,IST
  180 STRESS(I) = TAU(I)
      IF (ITYP2D.EQ.2) STRAIN(4)=EPS(4) + D1*(DELEPS(1) + DELEPS(2))
      DO 460 I=1,ISR
      DO 460 J=1,ISR
  460 C(I,J)=0.
      C(1,1)=A1
      C(2,1)=B1
      C(1,2)=B1
      C(2,2)=A1
      C(3,3)=C1
      IF (ITYP2D.EQ.1) GOTO 400
      IF (ITYP2D.EQ.2) GOTO 470
      C(1,4)=B1
      C(2,4)=B1
      C(4,1)=B1
      C(4,2)=B1
      C(4,4)=A1
      GOTO 400
  470 C(4,1)=B2
      C(4,2)=B2
      C(4,3)=0.
      C(4,4)=A2
      GO TO 400
C
C     STATE OF STRESS OUTSIDE LOADING SURFACE  -  PLASTIC BEHAVIOR
C     DETERMINE PART OF STRAIN TAKEN ELASTICLY
C
  300 IF (IPEL.EQ.0.AND.IPE.NE.2) LO=LO+1
      WE2=0.
      WE=DWE=0.
      SM=(SIG(1)+SIG(2)+SIG(4)-AL1(1)-AL1(2)-AL1(4))/3.
      SX=SIG(1)-SM-AL1(1)
      SY=SIG(2)-SM-AL1(2)
      SS=SIG(3)-AL1(3)
      SZ=SIG(4)-SM-AL1(4)
      DM = (DELSIG(1)+DELSIG(2)+DELSIG(4))/3.
      DX = DELSIG(1) - DM
      DY = DELSIG(2) - DM
      DS = DELSIG(3)
      DZ = DELSIG(4) - DM
      A = DX*DX + DY*DY + 2.*DS*DS + DZ*DZ
      B = SX*DX + SY*DY + 2.*SS*DS + SZ*DZ
      IF (LO.LT.1) ALD=YT1
      IF (LO.GE.1) ALD=YC1
      E = SX*SX + SY*SY + 2.*SS*SS + SZ*SZ - 2.*ALD*ALD/3.
      RATIO=0.
      IF (IPEL.GT.0) GOTO 306
      IF ((B*B-A*E).LT.0) GOTO 306
      RATIO=(-B + SQRT(B*B-A*E))/A
      IF (RATIO.GT.1.) RATIO=1.
  306 CONTINUE
      DO 350 I=1,IST
  350 TAU (I) = SIG(I) + RATIO*DELSIG(I)
      IF (ITYP2D.EQ.2) STRAIN(4)=EPS(4) + RATIO*D1*(DELEPS(1)
     1                 + DELEPS(2))
      IF (RATIO.EQ.1.) GOTO 170
C        DETERMINE NUMBER OF SUBINCREMENTS- M, AND STRAIN INTERVAL
      IF (FT.LT.0) GOTO 307
      M=20.*SQRT(FT)/YLD + 1
  307 CONTINUE
      IF (M.LT.3) M=3
      IF (M.GT.15) M=15
  163 CONTINUE
      XM = (1. - RATIO)/M
      DO 380 I=1,4
  380 DEPS(I) = XM*DELEPS(I)
C
C     .....  CALCULATION OF ELASTOPLASTIC STRESSES  .....  (START)
C .      LOOP FOR M SUBINCREMENTS. AT THE FIRST LOOP YLD=YIELD,      .
```

```
C .       TAU=SIG, AND THE COMPUTED PLASTIC STIFFNESS IS IDENTICAL
C .       TO THE ONE AT THE END OF THE PREVIOUS LOADING INCREMENTAL
C .       STEP
C
      DO 600 IM=1,M
C .       IF UNLOADING , SAVE COMPUTATIONS
      IF (WP.LT.0.AND.IM.GT.1) GOTO 600
      IF (PROP(4).EQ.0.) GOTO 941
      IP=IPEL
      IF (IPEL.LT.1) IP=1
      IPEL=3
C .       FIND THE CURRENT LOADING SURFACE WHEN  LO=1
      IF (LO.GT.1) GOTO 993
      IF (YLD.LT.YT3) IPEL=2
      IF (YLD.LT.YT2) IPEL=1
C .       IPEL CANNOT DECREASE DURING LOADING
      IF (IPEL.LT.IP) IPEL=IP
C
      IF (IPEL-2) 901,902,903
  901 ET=ET1
      YY=YT1
      GOTO 994
  902 ET=ET2
      YY=YT2
      GOTO 994
  903 ET=ET3
      YY=YT3
      GOTO 994
  993 CONTINUE
C .       FIND THE CURRENT LOADING SURFACE WHEN  LO .GT.1
      IF(IRE.EQ.2.AND.LO.EQ.1) GOTO 964
      IF(IRE.EQ.2.AND.LO.GT.1) GOTO 965
      IF (YLD.LT.ABS(YMAX-2*YC3)) IPEL=2
      IF (YLD.LT.ABS(YMAX-2*YC2)) IPEL=1
      GOTO 966
  964 IF (YLD.LT.YT3) IPEL=2
      IF (YLD.LT.YT2) IPEL=1
      GOTO 966
  965 IF (YLD.LT.YC3) IPEL=2
      IF (YLD.LT.YC2) IPEL=1
  966 CONTINUE
      IF (RULE.GT.1) GOTO 915
      IF (YLD.LT.ABS(YMAX+2.*YC2-2.*YC1)) IPEL=1
      IF (YLD.LT.ABS(YMAX+2.*YC3-2.*YC1)) IPEL=2
  915 CONTINUE
C .       IPEL CANNOT DECREASE DURING LOADING
      IF (IPEL.LT.IP)  IPEL=IP
C
      IF (IPEL-2) 911,912,913
  911 ET=EC1
      YY=YC1
      GOTO 994
  912 ET=EC2
      YY=YC2
      GOTO 994
  913 ET=EC3
      YY=YC3
  994 CONTINUE
C .
      D2=YM*ET/(YM-ET)
C
C
C         SET SURFACE TRANSLATIONS
C
      IF (LO.GT.1) GOTO 5004
      G1=YT1
      G2=YT2
      G3=YT3
      GOTO 5007
 5004 CONTINUE
      G1=YC1
      G2=YC2
```

```
      G3=YC3
 5007 CONTINUE
      IF (IPEL-2) 981,982,983
  981 DO 989 J=1,4
  989 AL(J)=AL1(J)
      DO 5020 I=1,4
 5020 ALB(I)=AL2(I)
      YRA=G2/G1
      GOTO 986
  982 DO 988 J=1,4
  988 AL(J)=AL2(J)
      DO 5021 I=1,4
 5021 ALB(I)=AL3(I)
      YRA=G3/G2
C .         TO INSURE TANGENCY OF THE SURFACES
      IF (IPEL.EQ.IP) GOTO 986
      DO 5001 J=1,4
 5001 AL(J)=TAU(J)-G2/G1*(TAU(J)-AL1(J))
      GOTO 986
  983 DO 987 J=1,4
  987 AL(J)=AL3(J)
      YRA=1.
      DO 5003 I=1,4
 5003 ALB(I)=TAU(I)
C .         TO INSURE TANGENCY OF THE SURFACES
      IF (IPEL.EQ.IP) GOTO 986
      DO 5002 J=1,4
 5002 AL(J)=TAU(J)-G3/G2*(TAU(J)-AL2(J))
  986 CONTINUE
C
C           FORMS THE ELASTO-PLASTIC MATERIAL MATRIX
C
      HPRIME=2.*D2/3.
      BETA=1.5 /YY/YY/(1.+HPRIME/A2)
      IF (RULE     .EQ.1) BETA=BETA*YY*YY/YLD/YLD
  941 IF (RULE.EQ.0.) BETA=1.5/YT1/YT1
      BETA1=BETA
C
      IF (RULE.GE.2) GOTO 305
      DO 715 I=1,4
  715 AL(I)=0.
  305 CONTINUE
C
      SM=((TAU(1)-AL(1))+(TAU(2)-AL(2))+(TAU(4)-AL(4)))/3.
      SX=TAU(1)-AL(1)-SM
      SY=TAU(2)-AL(2)-SM
      SS=TAU(3)-AL(3)
      SZ=TAU(4)-AL(4)-SM
C           CHECK FOR UNLOADING IF WP.LT.0.
      WP=SX*DELSIG(1)+SY*DELSIG(2)+SS*DELSIG(3)*2.+SZ*DELSIG(4)
      IF (WP.LT.0.) BETA=0.
C
      C(1,1)   = A2 * (B2 - BETA*SX*SX)
      C(1,2)   = A2 * (C2 - BETA*SX*SY)
      C(2,1)=C(1,2)
      C(1,3)   = A2 * (     - BETA*SX*SS)
      C(3,1)=C(1,3)
      C(2,2)   = A2 * (B2 - BETA*SY*SY)
      C(2,3)   = A2 * (     - BETA*SY*SS)
      C(3,2)=C(2,3)
      C(3,3)   = A2 * (.5 - BETA*SS*SS)
      C(4,1)   = A2 * (C2 - BETA*SX*SZ)
      C(4,2)   = A2 * (C2 - BETA*SY*SZ)
      C(4,3)   = A2 * (     - BETA*SZ*SS)
      IF (ITYP2D.EQ.1) GOTO 5030
      C(1,4)=C(4,1)
      C(2,4)=C(4,2)
      C(3,4)=C(4,3)
      C(4,4)   = A2 * (B2 - BETA*SZ*SZ)
C
      IF (ITYP2D.EQ.0) GOTO 5030
```

```
C       PLANE STRESS / MODIFY DP MATRIX
        DO 717 I=1,3
        A=C(I,4)/C(4,4)
        DO 717 J=I,3
        C(I,J)=C(I,J) - C(4,J)*A
  717   C(J,I) = C(I,J)
        DEPS(4)=(-C(4,1)*DEPS(1)-C(4,2)*DEPS(2)-C(4,3)*DEPS(3))/C(4,4)
        IF (WP.LT.0.)   DEPS(4)=D1*(DEPS(1) + DEPS(2))
        STRAIN(4)=STRAIN(4) + DEPS(4)
 5030   CONTINUE
C
C            CALCULATE ELASTIC-PLASTIC STRESSES
C
        IF (WP.LT.0.) GOTO 193
        DO 561 I=1,4
  561   CC(I,2)=0.0
        DO 560 I=1,IST
        DO 560 J=1,ISR
        CC(I,2)=CC(I,2)+C(I,J)*DEPS(J)
  560   TAU(I) = TAU(I) + C(I,J) * DEPS(J)
  193   CONTINUE
C
C            CALCULATE PLASTIC STRAIN INCREMENT
C
        IF (WP.LT.0.) BETA=BETA1
        CP(1,1)=BETA*SX*SX
        CP(1,2)=BETA*SX*SY
        CP(1,3)=BETA*SX*SS
        CP(1,4)=BETA*SX*SZ
        CP(2,1)=CP(1,2)
        CP(2,2)=BETA*SY*SY
        CP(2,3)=BETA*SY*SS
        CP(2,4)=BETA*SY*SZ
        CP(3,1)=CP(1,3)
        CP(3,2)=CP(2,3)
        CP(3,3)=BETA*SS*SS
        CP(3,4)=BETA*SS*SZ
        CP(4,1)=CP(1,4)
        CP(4,2)=CP(2,4)
        CP(4,3)=CP(3,4)
        CP(4,4)=BETA*SZ*SZ
        DO 711 I=1,4
  711   DEPSP(I)=0.
  200   DO 123 I=1,4
        DO 123 J=1,4
  123   DEPSP(I)=DEPSP(I)+CP(I,J)*DEPS(J)
  714   CONTINUE
C
C            CALCULATE SURFACE TRANSLATIONS INCREMENTS
C
        IF (WP.LT.0.) GOTO 731
        A6=SX*CC(1,2)+SY*CC(2,2)+SS*CC(3,2)*2.+SZ*CC(4,2)
        IF (RULE.LT.2) GOTO 731
        IF(RULE-3) 732,733,734
  732   CONTINUE
C            PRAGER HARDENING RULE
        DO 124 I=1,4
  124   AL(I)=AL(I)+HPRIME*DEPSP(I)
        AL(3)=AL(3)-HPRIME*DEPSP(3)/2.
        GOTO 731
C            ZIEGLER HARDENING RULE
  733   CONTINUE
        A12=SX*(TAU(1)-AL(1))+SY*(TAU(2)-AL(2))+SS*(TAU(3)-AL(3))*2.
       1+SZ*(TAU(4)-AL(4))
        A7=A6/A12
        DO 5012 I=1,4
 5012   AL(I)=AL(I)+(TAU(I)-AL(I))*A7
        GOTO 731
  734   CONTINUE
C            MROZ HARDENING RULE
C               THE THIRD(LAST) YIELD SURFACE IS ASSUMED TO TRANSLATE
```

```
C           ACCORDING TO THE ZIEGLER S RULE, THUS YRA=1. AND ALB=TAU
      DO 5014 I=1,4
 5014 CC(I,4)=ALB(I)+YRA*(TAU(I)-AL(I))-TAU(I)
      A9=SX*CC(1,4)+SY*CC(2,4)+SS*CC(3,4)*2.+SZ*CC(4,4)
      A10=A6/A9
      AL(1)=AL(1)+A10*CC(1,4)
      AL(2)=AL(2)+A10*CC(2,4)
      AL(3)=AL(3)+A10*CC(3,4)
      AL(4)=AL(4)+A10*CC(4,4)
  731 CONTINUE
      DO 712 I=1,4
  712 CC(I,3)=CC(I,3)+DEPSP(I)
C         CALCULATE PLASTICITY PARAMETERS
      R1=TAU(1)
      R2=TAU(2)
      R3=TAU(3)
      R4=TAU(4)
      P1=DEPSP(1)
      P2=DEPSP(2)
      P3=DEPSP(3)
      P4=DEPSP(4)
      COMP=COMP+DEPSP(1)+DEPSP(2)+DEPSP(4)
      DEP=DEP+SQRT(0.667*(P1**2+P2**2+P3**2/2.+P4**2))
      IF (WP.LT.0.) DEP=0.
      WT1=     R1*DEPSP(1)+R2*DEPSP(2)+R3*DEPSP(3)+R4*DEPSP(4)
      IF (WP.LT.0.) WT1=0.
      WE= WE+ R1*DEPS(1)+R2*DEPS(2)+R3*DEPS(3)+R4*DEPS(4)-WT1
      WP1=WP1+WT1
C
C         UPDATE SURFACE TRANSLATIONS
C
      IF (WP.LT.0) GOTO 904
      IF (PROP(4).EQ.0.) GOTO 904
      A=YT1/YT2
      B=YT1/YT3
      E=YT2/YT3
      IF (LD.LE.1) GOTO 290
      A=YC1/YC2
      B=YC1/YC3
      E=YC2/YC3
  290 CONTINUE
      IF (IPEL-2) 971,972,973
  971 DO 979 J=1,4
  979 AL1(J)=AL(J)
      GOTO 976
  972 DO 978 J=1,4
      AL2(J)=AL(J)
C .       TO INSURE TANGENCY OF THE SURFACES
      AL1(J)=TAU(J)-A*(TAU(J)-AL2(J))
  978 CONTINUE
      GOTO 976
  973 DO 977 J=1,4
      AL3(J)=AL(J)
C .       TO INSURE TANGENCY OF THE SURFACES
      AL2(J)=TAU(J)-E*(TAU(J)-AL3(J))
      AL1(J)=TAU(J)-B*(TAU(J)-AL3(J))
  977 CONTINUE
  976 CONTINUE
  904 CONTINUE
C .
C .       UNLOADING
C .
      IF (WP.GE.0.) GOTO 920
      DM=(TAU(1)+TAU(2)+TAU(4))/3.
      DX=TAU(1)-DM
      DY=TAU(2)-DM
      DS=TAU(3)
      DZ=TAU(4)-DM
      YMAX=SQRT(1.5*(DX*DX+DY*DY+2*DS*DS+DZ*DZ))
      IPEL=0
C         THE ELASTIC STRAINS ARE ADJUSTED
```

```
      AM=IM
      W=M
      DO 921 I=1,IST
      DO 921 J=1,ISR
  921 TAU(I)=TAU(I)+C(I,J)*(DEPS(J)-DEPSP(J)/AM)*W
      DWE=0.
      DO 165 I=1,4
      WE=WE+TAU(I)*(DEPS(I)-DEPSP(I)/AM)*(W-IM)
  165 DWE=DWE+(TAU(I)-SIG(I))*(DEPS(I)-DEPSP(I)/AM)*W
      DO 204 I=1,4
  204 WE2=WE2+ABS(TAU(I))*(DEPS(I)-DEPSP(I)/AM)*W
      DO 962 I=1,4
  962 DEPSP(I)=0.
      WP1=DEP=0.
  920 CONTINUE
      DM = (TAU(1)+TAU(2)+TAU(4))/3.
      DX = TAU(1) - DM
      DY = TAU(2) - DM
      DS = TAU(3)
      DZ = TAU(4) - DM
C
      IF (PROP(4).EQ.0.) GO TO 580
C     STRAIN-HARDENING MATERIAL  -  UPDATE YLD
      YLD= SQRT (1.5 * (DX*DX+DY*DY+2.*DS*DS+DZ*DZ) )
      IF (WP.LT.0.) YLD=ABS(YMAX-2*YC1)
      IF (WP.LT.0.AND.RULE.EQ.1) YLD=YMAX
      GO TO 600
C
C     PERFECTLY PLASTIC MATERIAL
C
  580 FTA=.5*(DX*DX + DY*DY + DZ*DZ) + DS*DS
      FTB=(YLD*YLD)/3.
      FT=FTA - FTB
      IF (FT.EQ.0) GO TO 600
      IF (ITYP2D.EQ.2) GO TO 590
C
      COEF=-1. + SQRT(FTB/FTA)
      IF (WP.LT.0) COEF=0.
      TAU(1) = TAU(1) + COEF*DX
      TAU(2) = TAU(2) + COEF*DY
      TAU(3) = TAU(3) + COEF*DS
      TAU(4)=TAU(4) + COEF*DZ
      GO TO 600
C
  590 COEF=SQRT(FTB/FTA)
      IF (WP.LT.0) COEF=1.
      TAU(1)=TAU(1)*COEF
      TAU(2)=TAU(2)*COEF
      TAU(3)=TAU(3)*COEF
      STRAIN(4)=STRAIN(4) + (COEF - 1.)*DM/BM
C
  600 CONTINUE
C
C     .....  CALCULATION OF ELASTOPLASTIC STRESSES  .....  ( END )
C
      STRESS(4) = 0.
      DO 390 I=1,IST
  390 STRESS(I) = TAU(I)
C          FINAL STIFFNESS MATRIX
      IF (WP.LT.0.) BETA=0.
      SM=((TAU(1)-AL(1))+(TAU(2)-AL(2))+(TAU(4)-AL(4)))/3.
      SX=TAU(1)-AL(1)-SM
      SY=TAU(2)-AL(2)-SM
      SS=TAU(3)-AL(3)
      SZ=TAU(4)-AL(4)-SM
      C(1,1)  = A2 * (B2 - BETA*SX*SX)
      C(1,2)  = A2 * (C2 - BETA*SX*SY)
      C(2,1)=C(1,2)
      C(1,3)  = A2 * (    - BETA*SX*SS)
      C(3,1)=C(1,3)
      C(2,2)  = A2 * (B2 - BETA*SY*SY)
```

```
            C(2,3)   = A2 * (    - BETA*SY*SS)
            C(3,2)=C(2,3)
            C(3,3)   = A2 * (.5 - BETA*SS*SS)
            C(4,1)   = A2 * (C2 - BETA*SX*SZ)
            C(4,2)   = A2 * (C2 - BETA*SY*SZ)
            C(4,3)   = A2 * (    - BETA*SZ*SS)
            IF (ITYP2D.EQ.1) GOTO 791
            C(1,4)=C(4,1)
            C(2,4)=C(4,2)
            C(3,4)=C(4,3)
            C(4,4)   = A2 * (B2 - BETA*SZ*SZ)
      C
            IF (ITYP2D.EQ.0) GOTO 791
      C     PLANE STRESS / MODIFY DP MATRIX
            DO 792 I=1,3
            A=C(I,4)/C(4,4)
            DO 792 J=I,3
            C(I,J)=C(I,J) - C(4,J)*A
        792 C(J,I) = C(I,J)
        791 CONTINUE
      C
        400 CONTINUE
      C         CACULATE PARAMETERS
            DO 716 I=1,4
        716 DEPSP(I)=CC(I,3)
            S1=STRESS(1)-SIG(1)
            S2=STRESS(2)-SIG(2)
            S3=STRESS(3)-SIG(3)
            S4=STRESS(4)-SIG(4)
            DM=((STRESS(1)-AL(1))+STRESS(2)-AL(2) +STRESS(4)-AL(4))/3.
            DX=STRESS(1)-AL(1)-DM
            DY=STRESS(2)-AL(2)-DM
            DS=STRESS(3)-AL(3)
            DZ=STRESS(4)-AL(4)-DM
            DF=DX*S1+DY*S2+DS*S3*2.+DZ*S4
            DWP=S1*DEPSP(1)+S2*DEPSP(2)+S3*DEPSP(3)+S4*DEPSP(4)
            IF (IPEL.EQ.0) GOTO 166
            DWE=S1*DELEPS(1)+S2*DELEPS(2)+S3*DELEPS(3)+S4*DELEPS(4)-DWP
        166 CONTINUE
            P1=DELEPS(1)
            P2=DELEPS(2)
            P3=DELEPS(3)
            P4=DELEPS(4)
            DEE=SQRT(2./3.*(P1**2+P2**2+P3**2/2.+P4**2))
            DM=(STRESS(1) + STRESS(2) + STRESS(4))/3.
            DX=STRESS(1) - DM
            DY=STRESS(2) - DM
            DS=STRESS(3)
            DZ=STRESS(4) - DM
            FT=SQRT(1.5*(DX*DX+DY*DY+DZ*DZ+2.*DS*DS))
            WP2=WP2+FT*DEP
            HEP=DEP/FT
            IF (DEP.NE.0.) HP=(FT-YIELD)/DEP
            DEPC=DEPC+DEP
      C             UPDATING STRESSES, STRAINS, YIELD
            SIG(4)=EPS(4)=0.
            DO 410 I=1,IST
        410 SIG(I) = STRESS(I)
            DO 420 I=1,ISR
        420 EPS(I) = STRAIN(I)
            YIELD = YLD
            IF (ITYP2D.EQ.2) EPS(4)=STRAIN(4)
      C
            IF (KPRI.EQ.0) GO TO 700
            IF (ICOUNT.EQ.3) RETURN
            RETURN
        700 CONTINUE
      C
      C     P R I N T I N G   O F   S T R E S S E S
      C
            IF (INDNL.NE.2) GO TO 800
```

```
C
C      IN TOTAL LAGRANGIAN FORMULATION,
C      CAUCHY STRESSES ARE CALCULATED AND PRINTED
C
       CALL CAUCHY
C
   800 CONTINUE
       CALL MAXMIN (STRESS,SX,SY,SM)
       SMEAN=(STRESS(1)+STRESS(2)+STRESS(4))/3.
C
       WRITE(12,2052) NEL,IPT,LO,IPEL,DEPC,SMEAN,FT,SX,SY,SM,DWE,HP,WP
      1,IRE,WP2,DEE
C
       IF (WP.LT.0.AND.RATIO.NE.0.) GOTO 925
       GOTO 922
   925 CONTINUE
       WRITE (12,924) NEL,IPT,KSTEP
   922 CONTINUE
C
       IF (IPT.EQ.3) GOTO 1011
       IF (IPT.EQ.9) GOTO 1011
C .        THE FOLLOWING DATA IS PRINTED ON TAPE14 FOR THE ABOVE SPECIFIED
C .              INTEGRATION POINTS OF EACH ELEMENT
       GOTO 1001
  1011 CONTINUE
       WRITE (14,5055) NEL,IPT,LO,RATIO,YLD,WP1,(STRAIN(I),I=1,4)
      1,(AL1(I),I=1,4)
       WRITE (14,5055) NEL,IPT,IPEL,COEF,YMAX,DWP,(DELEPS(I),I=1,4)
      2,(AL2(I),I=1,4)
       WRITE (14,5055) NEL,IPT,M,WE,WE2,DF,(DEPSP(I),I=1,4)
      3,(AL3(I),I=1,4)
       WRITE (14,5056)
  1001 CONTINUE
C
       IF (NG.NE.NGLAST) GO TO 802
       IF (NEL.GT.NELAST) GO TO 806
       IF (IPT-1) 810,808,810
   802 NGLAST = NG
   808 WRITE (13,2003)
   806 NELAST=NEL
       WRITE (13,2004) NEL
   810 CONTINUE
       WRITE (13,2007) IPT,STATE(IPEL+1),STRESS(4),(STRESS(I),I=1,3)
      1,SX,SY,SM,FT
       RETURN
  2052 FORMAT(2X,I3,1X,I3,1X,I2,1X,I4,1X,E12.4,1X,
      14(F7.2,1X),F6.2,1X,F5.2,1X,F8.1,1X,F9.2,1X,I3,1X,2(E9.2,1X))
   924 FORMAT (10X,36HRELOADING AT THE SAME UNLOADING STEP,1X,4HNEL=,
      1I3,1X,4HIPT=,I1,1X,5HSTEP=,I4,3X,16HREDUCE STEP SIZE)
  5055 FORMAT(2X,I3,1X,I3,1X,I4,1X,F6.2,1X,E9.2,1X,E9.2,4E12.4,4F7.3)
  5056 FORMAT(2X,117H-------------------------------------------------------
      4------------------------------------------------------------)
  2003 FORMAT ( 102H ELEMENT  STRESS      STRESS-XX     STRESS-YY     STR
      1ESS-ZZ      STRESS-YZ     MAX STRESS    MIN STRESS,11X,5HYIELD /
      2 109H NUM/IPT   STATE
      3                                                  ANGLE,1X,8HFUNCTION / )
  2004 FORMAT (I4/)
  2007 FORMAT (5X,I2,2X,A1,6HLASTIC,1X,4E14.6,1X,2E14.6,1X,F6.2,1X,F8.2)
       END

7/8/9         END OF RECORD       1

6/7/8/9       END-OF-FILE
```

REFERENCES

1. Bathe, Klaus-Jürgen; Wilson, Edward L.; and Iding, Robert H.: NONSAP—A Structural Analysis Program for Static and Dynamic Response of Nonlinear Systems. Rept. No. UC SESM 74-3, Univ. Calif. Berkeley, Feb. 1974. (Available from NISEE/Computer Applications, Davis Hall, Univ. Calif., Berkley, Calif. 94720.)

2. Wood, Howard A.; and Engle, Robert M., Jr.: USAF Damage Tolerant Design Handbook: Guidelines for the Analysis and Design of Damage Tolerant Aircraft. AFFDL-TR-79-3021, Air Force Flight Dynamics Lab., Wright-Patterson AFB, March 1979.

3. Kalev, I.: Cyclic Plasticity Models and Application in Fatigue Analysis. To be published in a special volume of J. Computers and Structures for the Nonlinear Finite Element Analysis and ADINA Conf., Mass. Inst. Tech., Cambridge, Mass., June 1981.

4. Kalev, I.: Cyclic Plasticity and Failure of Structural Components. AIAA Paper 80-0693, May 1980.

5. Endo, T.; and Morrow, JoDean: Cyclic Stress-Strain and Fatigue Behavior of Representative Aircraft Metals. J. Materials, vol. 4, no. 1, March 1969, pp. 159-175.

6. Damage Tolerant Design Handbook. MCIC-HB-01, Battelle Columbus Lab., Dec. 1972.

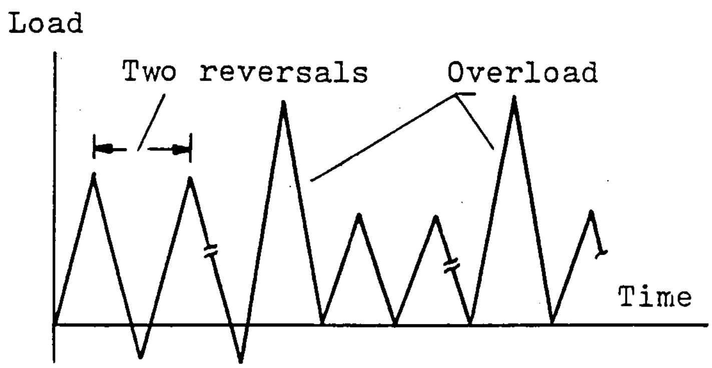
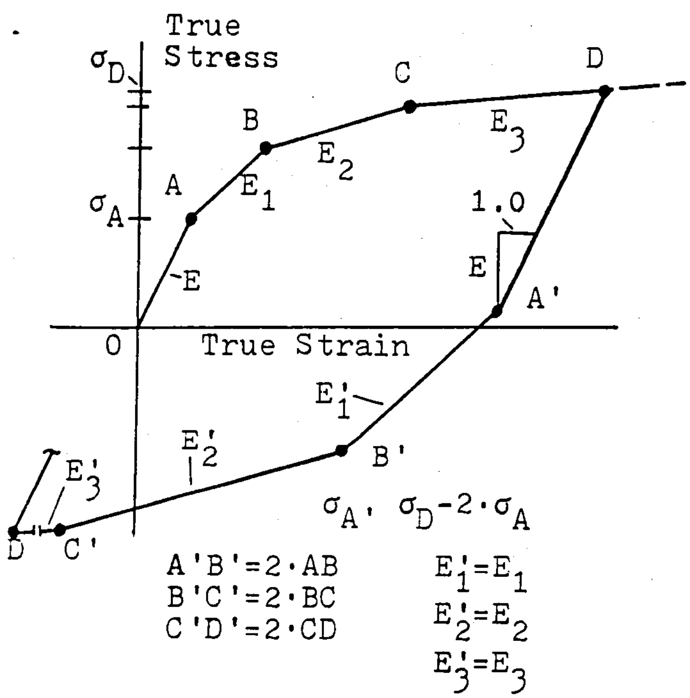
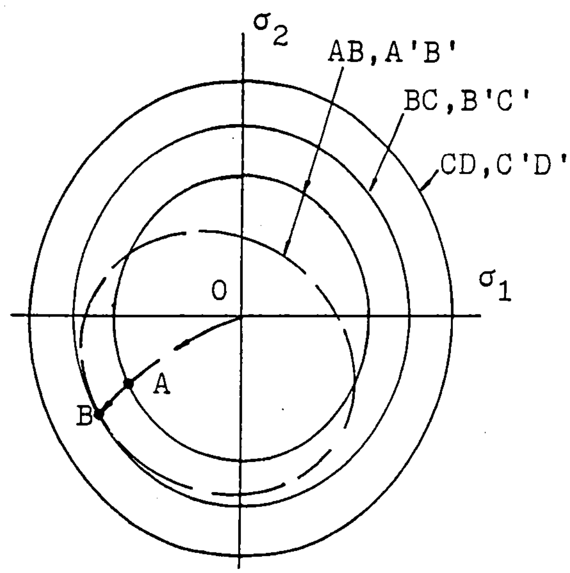


Figure 1. Typical idealization of loading spectrum.



(a) Material idealized uniaxial stress-strain curve at its cyclic steady state.



(b) Schematic representation of yield surfaces at initial condition and after translation of first surface (dotted line).

Figure 2. Relationship between material uniaxial curve and two-dimensional stress field.

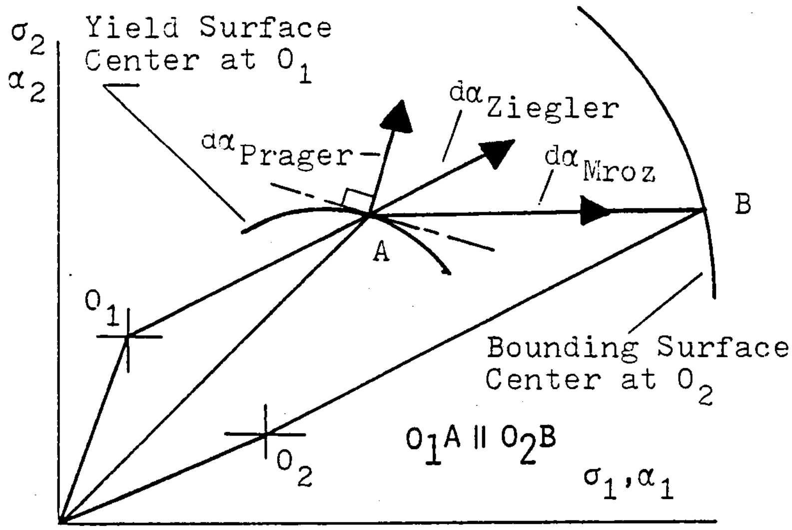
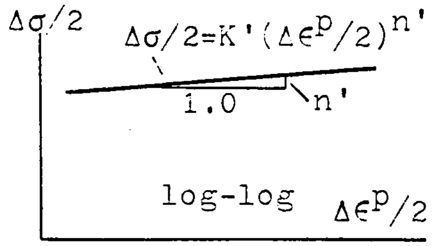
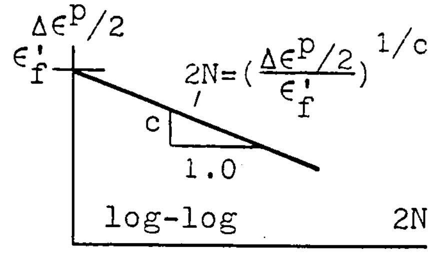


Figure 3. Incremental translations, dα, representing three hardening rules. α_1 and α_2 represent total translational components of surfaces; σ_1 and σ_2 represent stress components.



(a) Material uniaxial relationship between stress amplitude and plastic strain amplitude at cyclic steady state.



(b) Coffin-Manson low-cycle fatigue data of material uniaxial unnotched specimen.

Figure 4. Required input data for present approach.

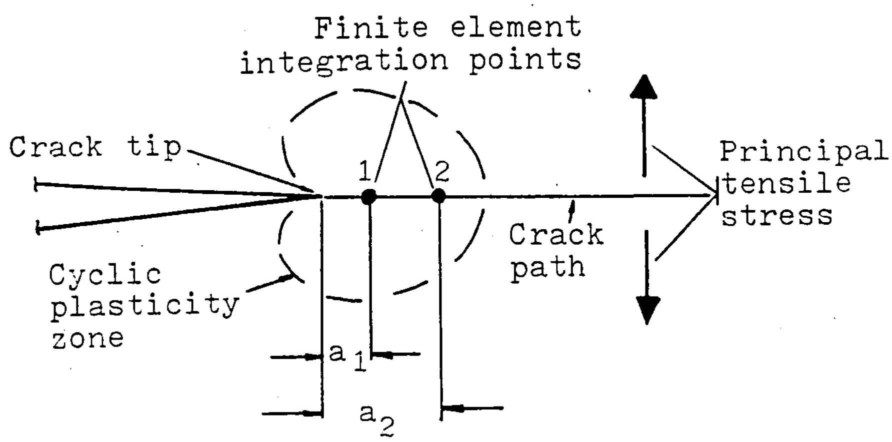


Figure 5. Location of discrete points in front of crack tip for calculation of crack growth rate.

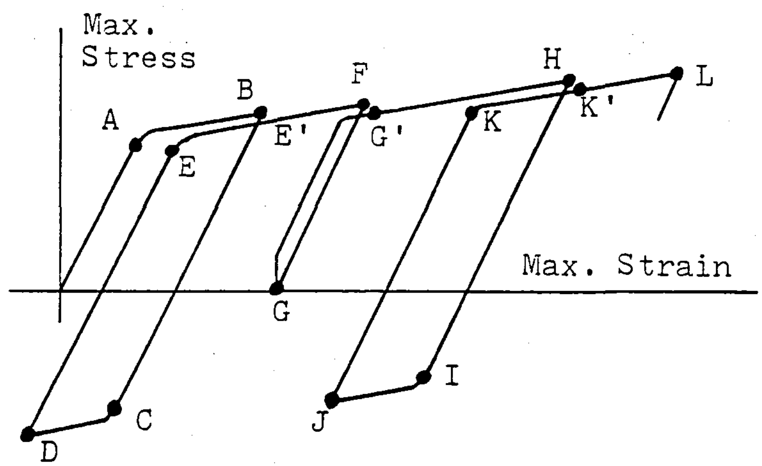
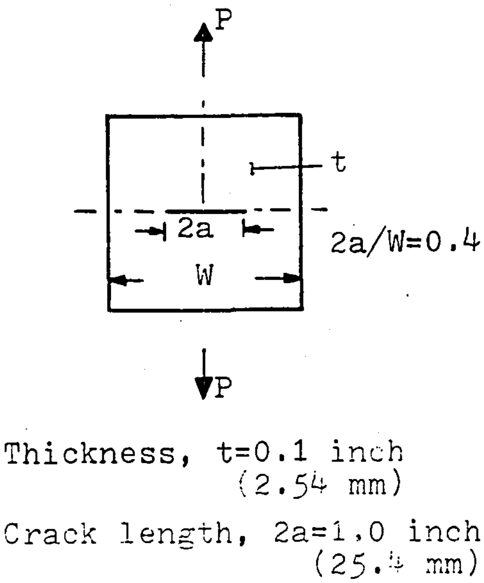
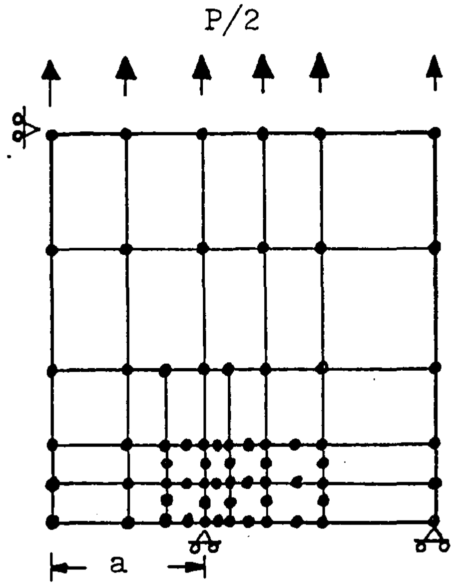


Figure 6. Pair of reversals count.



(a) Geometry of panel.



(b) Finite element model for one-quarter of panel.

Figure 7. Example of a cracked panel under uniaxial loads.

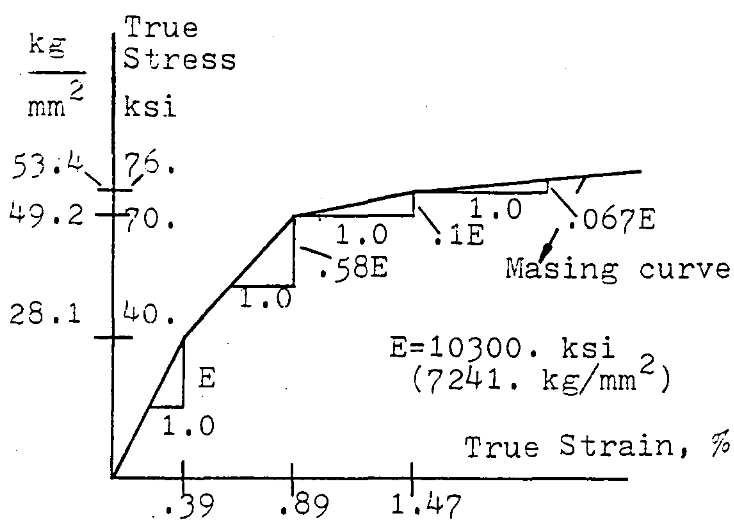
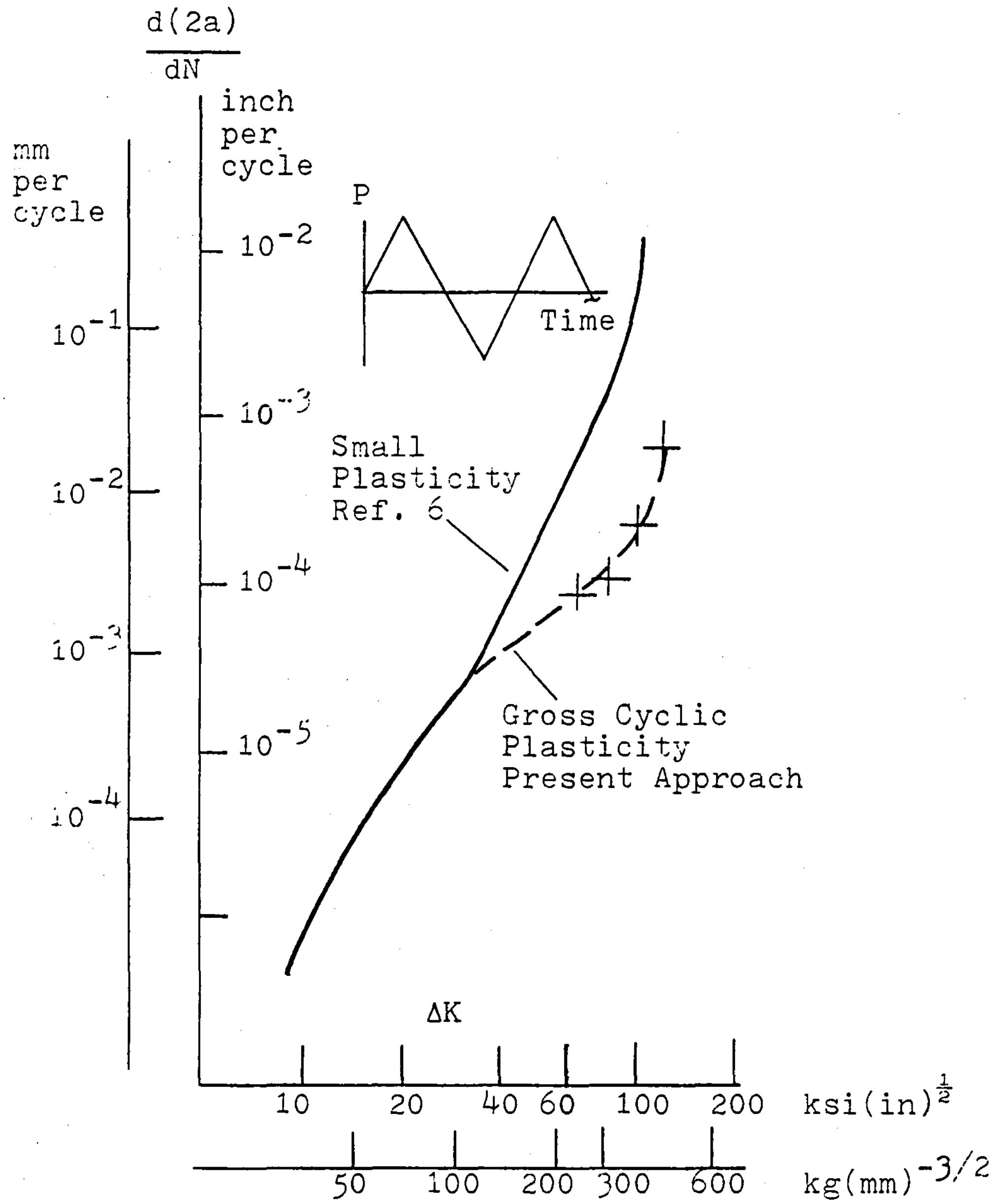
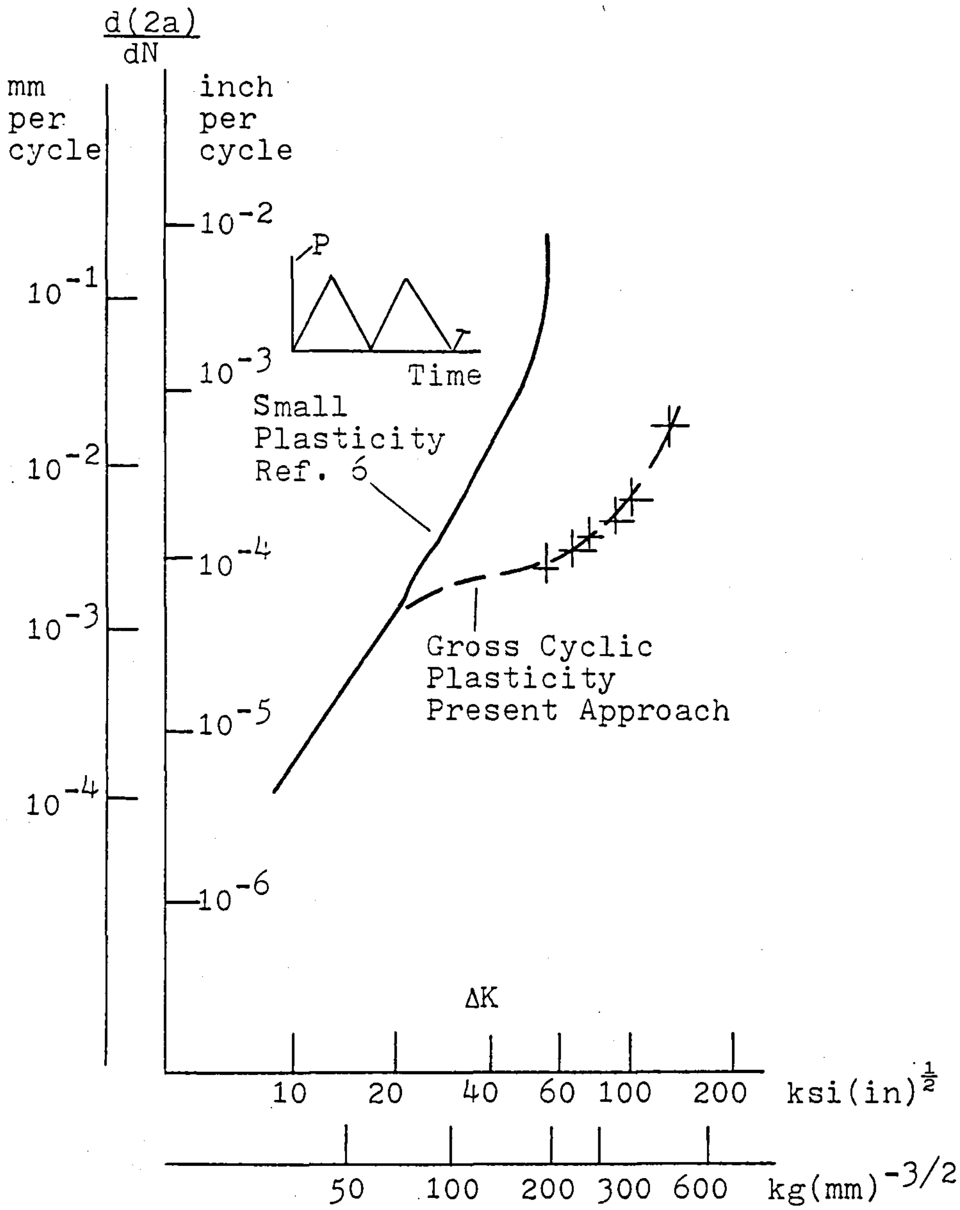


Figure 8. Idealized material curve in uniaxial cyclic steady state for cracked panel.



(a) Fully cyclic loading.

Figure 9. Crack growth rate results for cracked panel and comparisons with small plasticity cases.



(b) Tensile cyclic loading.

Figure 9. Concluded.

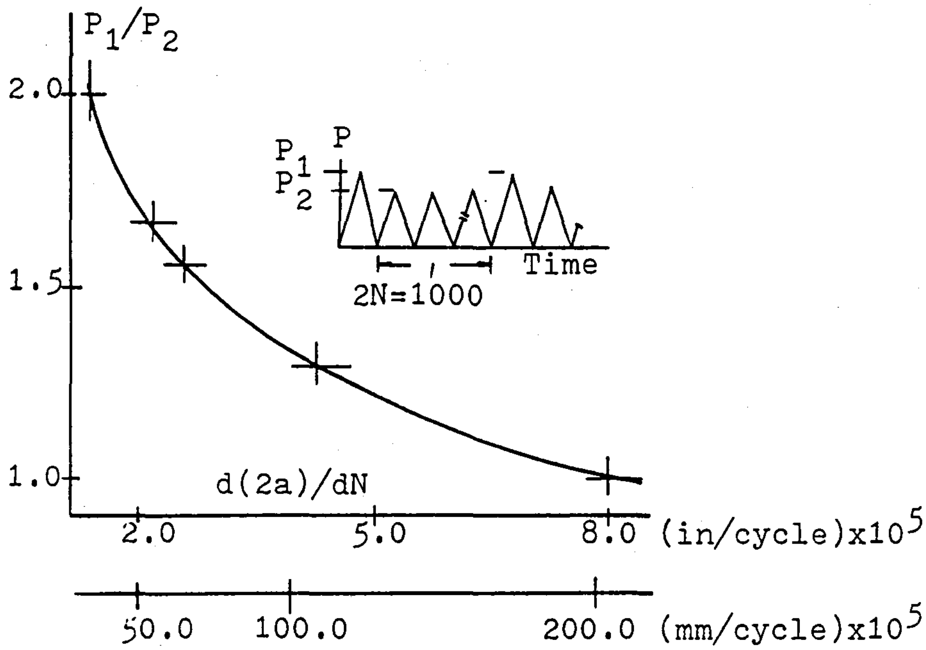
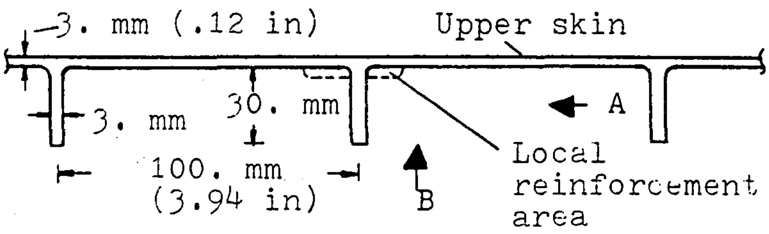
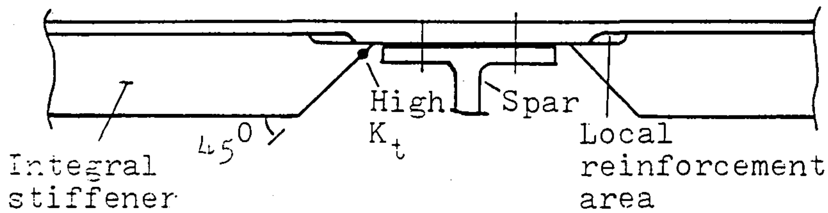


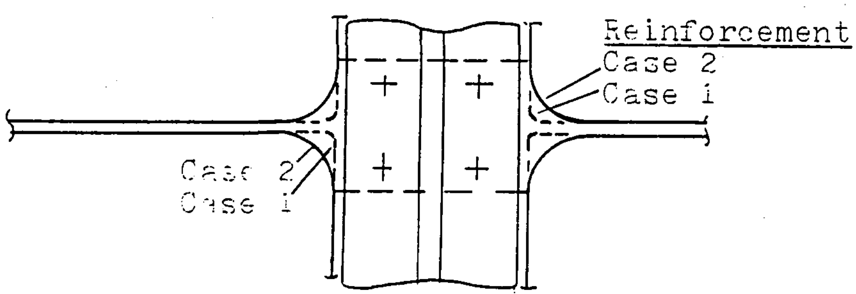
Figure 10. Numerical results for cracked panel of effect of tensile overloads on crack growth rate.



(a) Typical section.

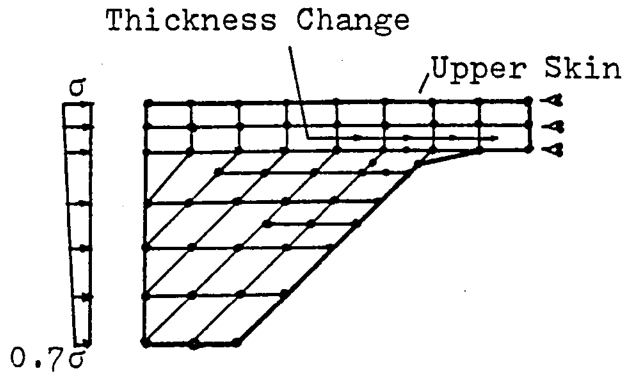


(b) View A.

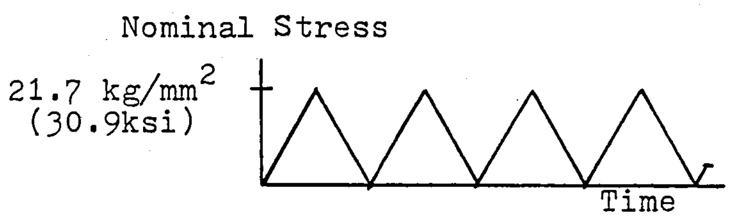


(c) View B.

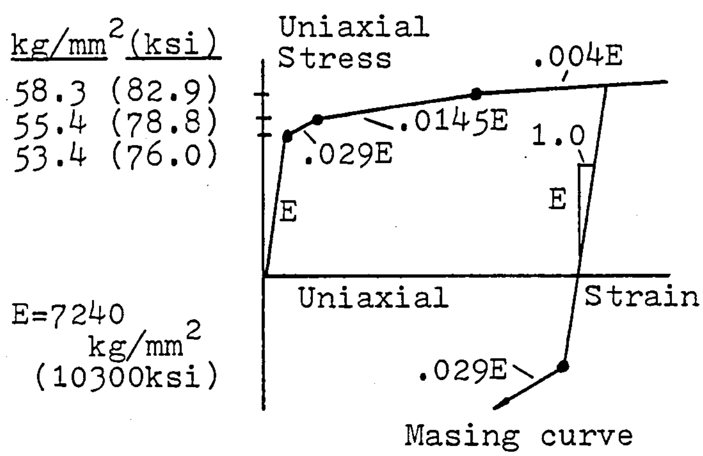
Figure 11. Details of an aircraft integral stiffened skin.



(a) Finite element model.



(b) Applied compressive loading.

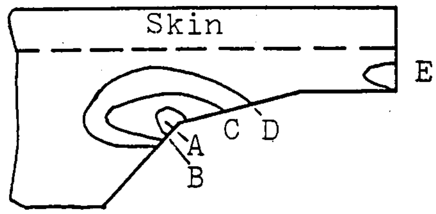


(c) Material uniaxial curve.

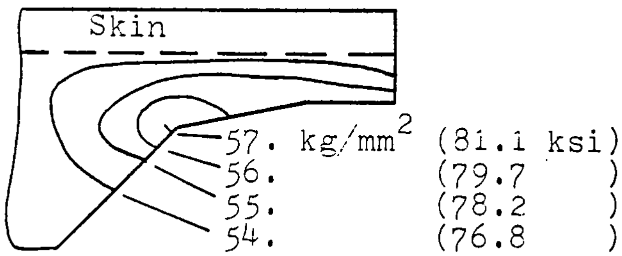
Figure 12. Idealization of stiffened skin example.

(2N) Reversals$\cdot 10^{-3}$

	Case 1	2
A	1.2	11.
B	4.4	30.
C	20.0	100.
D	60.0	500.
E	10.0	1000.

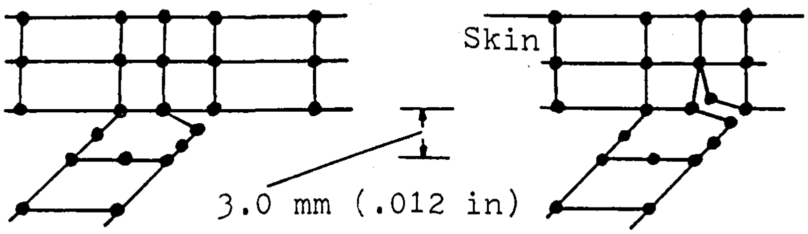


(a) Equal damage curves indicating number of reversals to crack initiation.

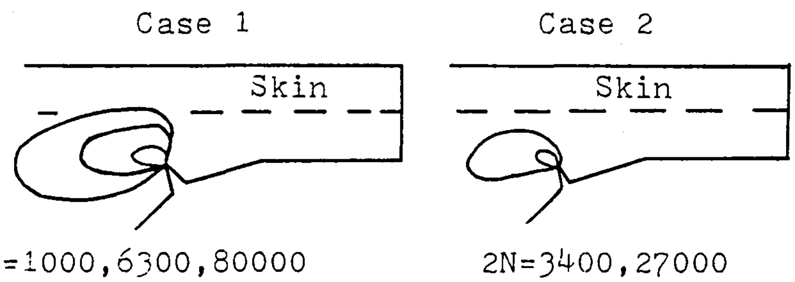


(b) Maximum von Mises stress distributions.

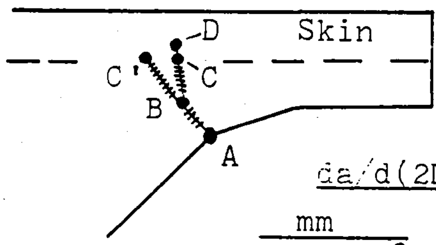
Figure 13. Results for uncracked stiffened skin.



(a) Modified finite element models due to crack growth.



(b) Distributions of equal damage curves.



da/d(2N) average

		mm	inch per two reversals
Case 1	AB	$2.4 \cdot 10^{-3}$	$9.4 \cdot 10^{-5}$
	BC	$5.0 \cdot 10^{-3}$	$2.0 \cdot 10^{-4}$
	CD	$< 10^{-6}$	$< 10^{-7}$
Case 2	AB	$2.5 \cdot 10^{-4}$	$1.0 \cdot 10^{-5}$
	BC'	$8.5 \cdot 10^{-5}$	$3.3 \cdot 10^{-6}$

(c) Crack growth rate and orientation.

Figure 14. Results for cracked stiffened skin.

1. Report No. NASA CR-163101	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle A COMPUTER PROGRAM FOR CYCLIC PLASTICITY AND STRUCTURAL FATIGUE ANALYSIS		5. Report Date November 1980
		6. Performing Organization Code
7. Author(s) I. Kalev		8. Performing Organization Report No. H-1139
9. Performing Organization Name and Address NASA Dryden Flight Research Center P.O. Box 273 Edwards, California 93523		10. Work Unit No. 506-53-64
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Research Council 2101 Constitution Avenue Washington, D.C. 20418		13. Type of Report and Period Covered Contractor Report—Topical
		14. Sponsoring Agency Code

15. Supplementary Notes

This work was performed by the author during his residence at the NASA Dryden Flight Research Center as a National Research Council Associate.
Technical Monitor: Jerald M. Jenkins Program Coordinator: Dr. Eldon E. Kordes

16. Abstract

A computerized tool for the analysis of time-independent cyclic plasticity structural response, life to crack initiation prediction, and crack growth rate prediction for metallic materials is described. Three analytical items are combined: the finite element method with its associated numerical techniques for idealization of the structural component, cyclic plasticity models for idealization of the material behavior, and damage accumulation criteria for the fatigue failure.

17. Key Words (Suggested by Author(s))		18. Distribution Statement
Computer programs	Finite element method	Unclassified—Unlimited
Crack initiation	Metal fatigue	
Crack propagation	Plastic properties	
Cyclic loads	Plasticity	
Elastoplasticity	Structural analysis	
Fatigue life		Distribution category: 05

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	42	A-02

For sale by the National Technical Information Service, Springfield, Virginia 22161

End of Document