# Improved Chebyshev Series Ephemeris Generation Capability of GTDS

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ABSTRACT

This paper describes an improved implementation of the Chebyshev ephemeris generation capability in the operational version of the Goddard Trajectory Determination System (GTDS). Preliminary results of an evaluation of this orbit propagation method for three satellites of widely different orbit eccentricities are also discussed in terms of accuracy and computing efficiency with respect to the Cowell integration method. An empirical formula is also deduced for determining an optimal fitting span which would give reasonable accuracy in the ephemeris with a reasonable consumption of computing resources.

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# SECTION 1 - INTRODUCTION

This document presents an improved implementation of the Chebyshev ephemeris generation capability in the Goddard Trajectory Determination System (GTDS). The reimplementation was necessary to resolve a System Failure Report on the operational version of GTDS and to improve the clarity of the computer program code to make it more readable and maintainable. The improved implementation employs the same Chebyshev polynomial/Picard iteration scheme as previously implemented (described in References 1 and 2) but exhibits a marked improvement in accuracy and efficiency (see Appendix B). The improved implementation fits the Chebyshev polynomial to satellite ephemeris data displaced as a function of time in accordance with the roots of the Chebyshev polynomial. This displacement is dependent on the degree of the polynomial.

The advantages of using Chebyshev polynomials as interpolating polynomials and the computational scheme in GTDS are briefly described in Section 2. Section 3 discusses general application of the improved implementation of the Chebyshev method to orbits over a wide range of eccentricity. The results are analyzed in deducing an empirical formula for determining an optimal fitting span that would consume a reasonable amount of computer resources and still provide a reasonably accurate ephemeris. A brief summary of conclusions is presented in Section 4.

Appendix A briefly discusses the properties of Chebyshev polynomials, the formulation of an interpolating polynomial consisting of a linear combination of Chebyshev polynomials of different degrees to represent acceleration, and the integration of the interpolating polynomial to generate satellite ephemerides. Appendix B contains the results of a comparison of the new and previous implementations of the Chebyshev ephemeris generation method in GTDS.

# SECTION 2 - THE USE OF CHEBYSHEV POLYNOMIALS TO GENERATE EPHEMERIDES

## 2.1 ADVANTAGES OF USING CHEBYSHEV POLYNOMIALS AS INTERPOLATING POLYNOMIALS

In principle, any function characterized by a discrete set of values can be approximated by a polynomial or a linear combination of polynomials. Such polynomials may be expressed as Chebyshev polynomials, Legendre polynomials, Laguerre polynomials, or any other polynomial form which is expedient for mathematical/computational analysis. For instance, in the case of a satellite trajectory, the positions at a series of selected times determine a polynomial consisting of a Chebyshev series within the time interval. The significant advantages of using Chebyshev polynomials to fit a satellite trajectory are that the error in the approximation is distributed evenly over the interval and that the maximum error is reduced to the minimum or near-minimum value (References 3 and 4).

Once this interpolating polynomial is established, the position of any other time within the interval can be easily interpolated. If a long ephemeris is to be stored for any reason, it is plausible to use a small amount of computer storage to store only coefficients for the interpolating polynomial instead of using a large amount of space to store the entire ephemeris. One familiar example is the Solar/Lunar/Planetary Ephemeris File (SLP File), which is stored as coefficients of Chebyshev polynomials for GTDS and other trajectory determination systems to interpolate noncentral body positions for evaluating perturbations on a satellite. Another possible application would be to store the coefficients of Chebyshev polynomials to represent the ephemeris of a Tracking and

Data Relay Satellite (TDRS) in the onboard computer of a user satellite for autonomous orbit determination.

## 2.2 COMPUTATION SCHEME IN GTDS

In order to apply the mathematical theory described in Appendix A, one must know the acceleration, $\ddot{x}(\xi)$, as a function of time to fit a Chebyshev interpolating polynomial. However, this is not the case for near-Earth spacecraft because of the nonlinearity of the perturbing forces, namely $\ddot{x}$ depends on x which is in turn determined from $\ddot{x}$. Therefore, the Picard iteration method is used in GTDS to incorporate the Chebyshev series ephemeris generation method. The computational procedure is described in the following paragraphs. For discussions related to the mathematical aspects of the method, see Reference 1.

Suppose an ephemeris is requested from $t_a$ to $t_z$ with a fitting span (or equivalent step size) of H which is equal to $(t_b - t_a)$. The entire ephemeris will consist of a series of spans which are represented by different Chebyshev interpolating polynomials. The default fitting span in GTDS is 5400 seconds. The allowable range of the degree of the Chebyshev interpolating polynomial is from 4 to 48 with a default of 36.

Within a fitting span, the roots ($\xi_k$ of the Chebyshev polynomial of the highest degree plus 1, (n + 1)) in the interpolating polynomial are first computed according to Equation (A-7). These roots are then transformed back into time, i.e., $\xi_k \rightarrow t_k$, k = 1, 2, ..., n + 1.

GTDS uses boundary conditions at the beginning of the fitting span, i.e., the position and velocity at $t_a$, to obtain positions and velocities at $t_1, t_2, \ldots, t_k, \ldots, t_{n+1}$ with a two-body central force field to start the iteration scheme. With the positions and velocities at $t_k$ available, the perturbations, $\ddot{x}(\xi_k)$ can now be estimated at these instants and the Chebyshev coefficients, $C_i$, are subsequently computed using Equation (A-11). At this point, the Chebyshev interpolating polynomial for the acceleration, Equation (A-8), is established.

The next step is to successively integrate the Chebyshev interpolating polynomial twice according to Equations (A-14) and (A-17) to obtain interpolating polynomials, $Q_{n+1}$ and $R_{n+2}$, for velocity and position, respectively, in the fitting span ($t_a$, $t_b$). The position and velocity with perturbations included at the end of the fitting span, $t_b$, or any other time can be easily interpolated. The first loop of the iterative scheme is essentially completed at this point.

In the next loop, GTDS uses positions and velocities interpolated from the interpolating polynomials, $Q_{n+1}$ and $R_{n+2}$, at the roots to estimate acceleration. After fitting the polynomial to the accelerations, it is again integrated twice to obtain polynomials for velocity and position. The position interpolated at the end of the fitting span in this loop is compared with that obtained in the previous loop.

This iterative scheme is repeated until the differences of the position components of the two successive loops at $t_b$ are less than a tolerance (default value = $10^{-6}$ kilometers). At this moment, the fitting procedure for the span ($t_a$, $t_b$) is completed.

After ephemerides are generated and the Chebyshev coefficients for velocity and position are optionally saved, the fitting span is advanced one step forward to $(t_b, t_b + H)$. This scheme is continued until all the spans are fitted.

## SECTION 3 - APPLICATIONS OF THE IMPROVED IMPLEMENTATION OF THE CHEBYSHEV EPHEMERIS GENERATION METHOD

The improved implementation of the Chebyshev ephemeris generation method is applied to satellites of different orbital eccentricities to study the behavior of the Chebyshev polynomial representation in order to find an optimal set of parameters, such as fitting span and degree of the Chebyshev polynomial, for different satellites. A series of computer runs on GTDS with the new Chebyshev implementation was obtained. The ephemerides from the Chebyshev ephemeris generation method are compared with those from the Cowell integration method in terms of accuracy and efficiency. The results are discussed separately for a near-circular orbit, an elliptical orbit, and a highly eccentric orbit in the following sections. An attempt to find an empirical formula for determining the optimal fitting span for these orbits is also discussed.

### 3.1 NEAR CIRCULAR ORBIT (ECCENTRICITY = $10^{-3}$)

The GEOS-3 satellite was chosen for this case study. The eccentricity of the GEOS-3 orbit is 0.00098 and the semimajor axis is 7225 kilometers. The fitting spans used in this case range from P/4 to 2P, where P is the period of the satellite. For each fitting span, several runs with different degrees of Chebyshev polynomials were made. The ephemeris of every run was compared by using the GTDS Ephemeris Comparison Program with the reference ephemeris generated by the Cowell integration method with a 24-second step size using perturbations identical to those used in the Chebyshev method. The maximum differences in position vector, $|\Delta\vec{R}|_{max}$, between the two ephemerides are plotted in Figure 3-1 as a function of the degree of Chebyshev polynomials and the fitting span.

The maximum difference decreases very rapidly as the degree of Chebyshev polynomials increases. However, after reaching a critical degree of the Chebyshev polynomials, the maximum difference bottoms out and does not decrease any further.

The saturation of the maximum difference occurs at a lower degree of the Chebyshev polynomials for a shorter fitting span. This saturation level generally increases with the fitting span.

Since the step size of ·24 seconds used in the Cowell integration method in generating the reference ephemeris is relatively very small, the maximum difference in position vectors between the Chebyshev and Cowell ephemerides can be loosely regarded as the accuracy of the fit of the Chebyshev polynomials. Therefore, Figure 3-1 demonstrates one significant phenomenon: once the saturation level is reached, for a particular fitting span, adding higher degrees of the Chebyshev polynomials not only does not improve its accuracy, but decreases its efficiency. This is further evaluated by examining the computer resources, mainly CPU time, consumed by each of the computer runs.

All the runs for GEOS-3 satellite were executed on the GSFC IBM S/360-75 C1 computer. However, some of the runs were executed in the "low-speed" core of the CPU, which is

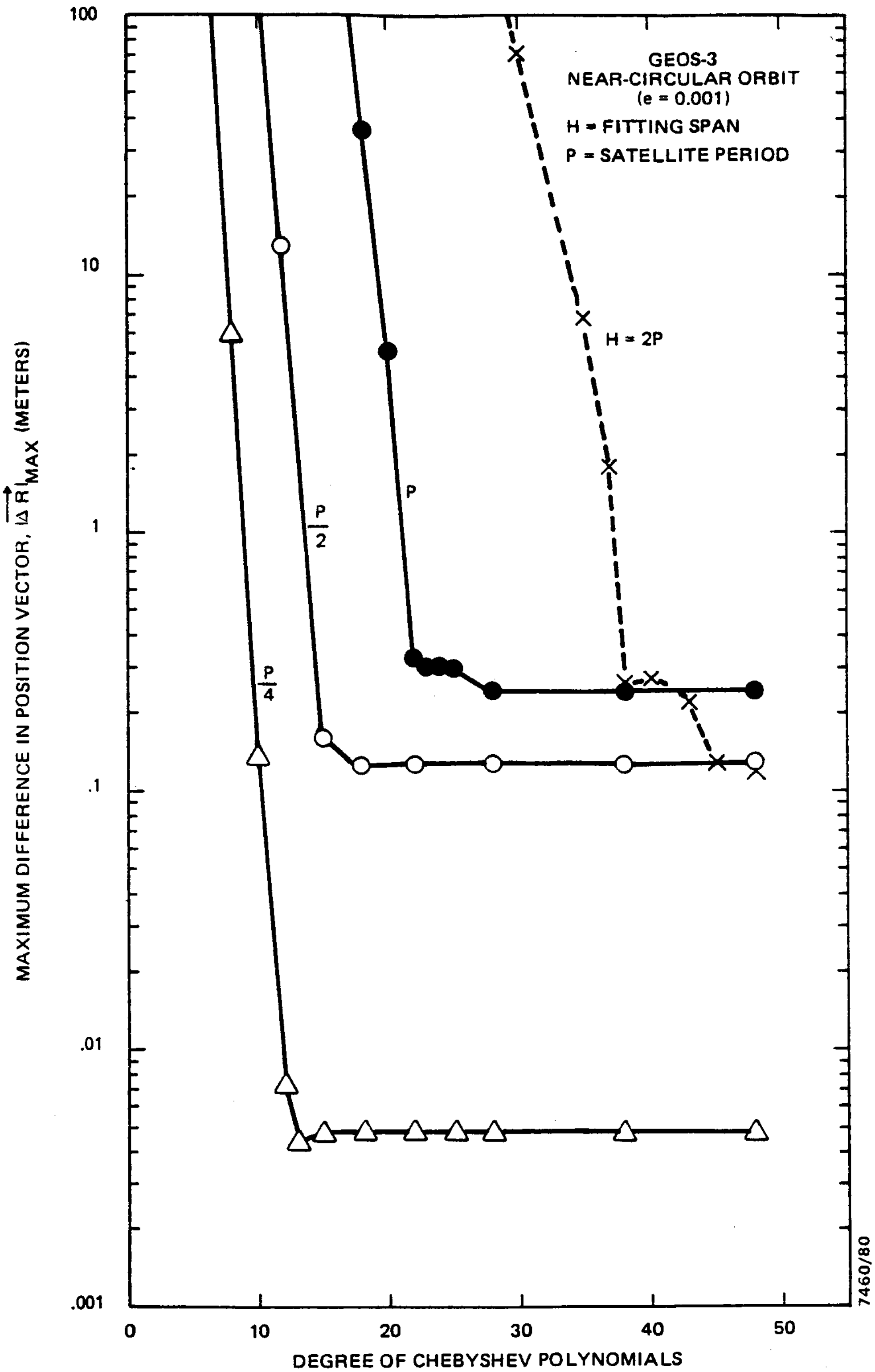


Figure 3-1. GEOS-3 Ephemeris Accuracy for the Chebyshev Polynomial Fit Compared With the Cowell Method

roughly three times slower than "high-speed" core. Also, there are a number of runs executed partly in low-speed core and partly in high-speed core. The CPU time consumed depends on the proportion of high-speed core and low-speed core used. This nonuniform CPU scale makes the comparison not so straightforward. Instead of reexecuting these runs in high-speed core, the CPU time of these runs are calibrated through force model calls, as described below.

In GTDS, the "Number of Times Forces Called For Full Model" in the statistics report is provided at the end of a run. For a numerical integration method, such as the Cowell method or the Chebyshev method, the full perturbing force, including harmonic geopotential field, noncentral body gravitational field, and nonconservative forces, is evaluated at each integration grid point according to the options specified. The number of times the full perturbing force is evaluated is proportional to the CPU time used in a run. In Figure 3-2 this number is plotted against CPU time for only those runs executed in high-speed core. Although the points plotted are somewhat scattered, there is a linear relationship between this number and the CPU time.

For comparison, the number of times the full force is evaluated (7236 times) and the CPU time (0.85 minute) are also plotted in Figure 3-2 for the reference run of Cowell method with a 24-second step size. It is interesting to note that the Chebyshev method with any reasonable accuracy is much slower than the Cowell method. Consequently, for ordinary purposes other than those that require Chebyshev coefficients, it is at least not recommended to use the Chebyshev method to generate ephemerides for a spacecraft of circular orbit at a lower altitude.

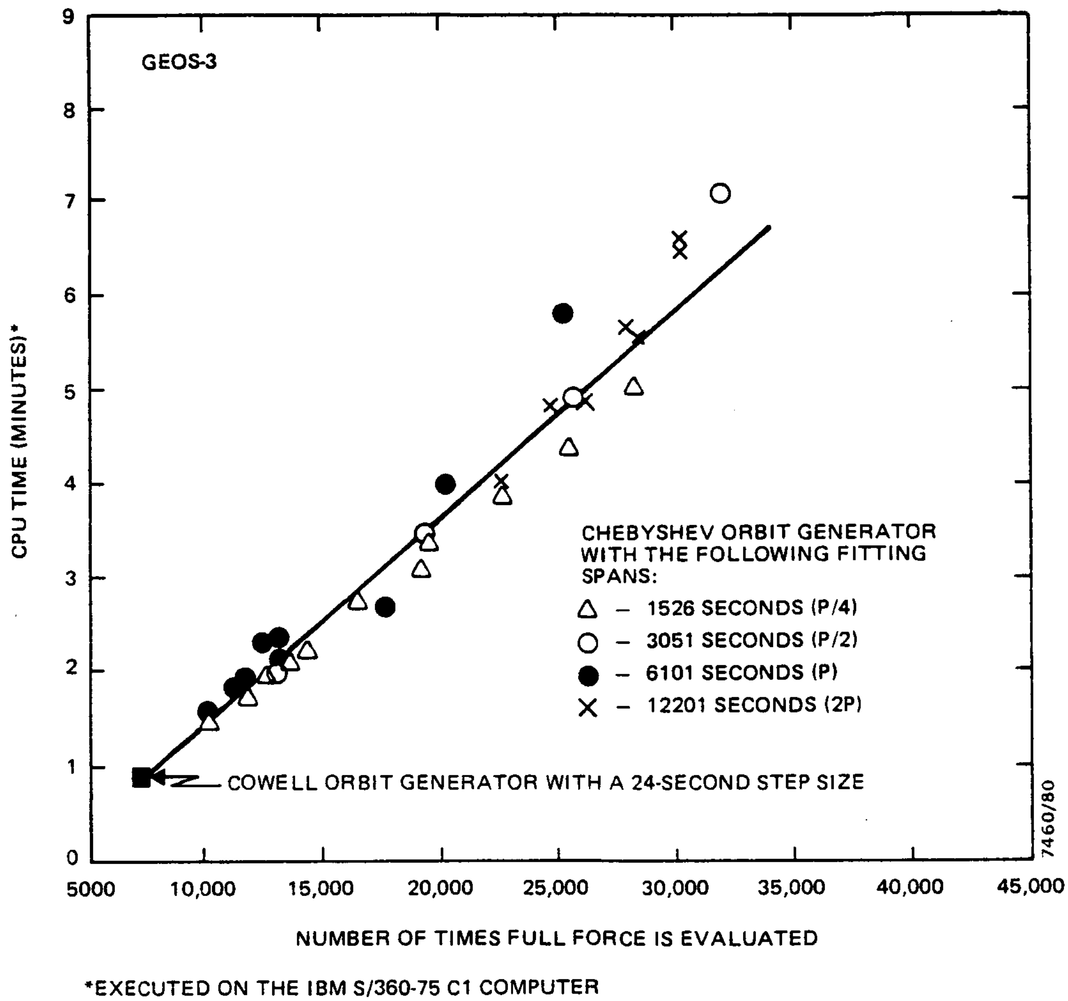


Figure 3-2. Calibration Curve of CPU Time Consumed by Computer Runs Used for This Study

After the CPU time of the runs which were executed partly in high-speed core are calibrated by using this linear relationship,[1] the CPU time of all the runs of Chebyshev method is plotted against the degree of the Chebyshev polynomials in Figure 3-3 for different fitting spans. The CPU time consumed is approximately linearly proportional to the degree of the Chebyshev polynomials. The CPU time is also plotted in Figure 3-4 against the fitting spans for degrees 18, 28, 38, and 48.

The curves in both Figures 3-3 and 3-4 give the impression that the fitting span of one satellite period would be the most desirable one to use for the Chebyshev method as far as CPU time is concerned. However, the accuracy of the fit may not be desirable for the situation. For this reason, the accuracy information is also included in Figures 3-3 and 3-4 by different shadings of the plot symbols to avoid the possibility of drawing misleading conclusions.

Since the accuracy of the Chebyshev method bottoms out at a critical degree of the polynomial (Figure 3-1) and the CPU time used increases linearly with the degrees of the polynomials, a trade-off can be performed to study the benefit or penalty of using a higher degree than is necessary.

The results of the trade-off study are presented in Figures 3-5, 3-6, 3-7, and 3-8 for fitting spans P/4, P/2, P, and 2P, respectively, by combining the results in Figures 3-1 and 3-3. The CPU time or the accuracy is normalized with respect to that of a data point

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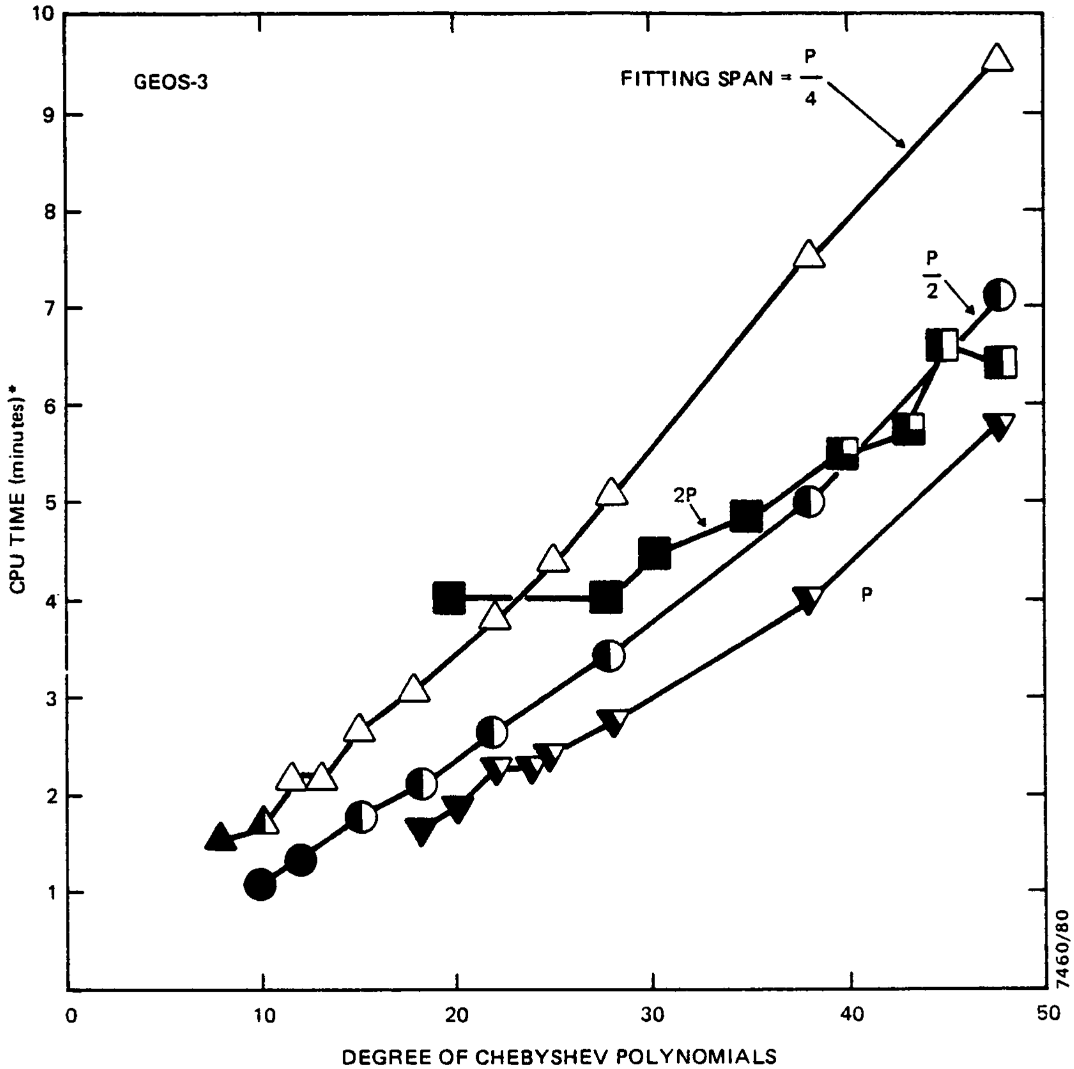
[1]This calibration curve is not necessarily valid for other experimental conditions or other satellites.




*EXECUTED ON THE IBM S/360-75 C1 COMPUTER



Figure 3-3. CPU Time Versus Degree of Chebyshev Polynomials for Various Fitting Spans for GEOS-3

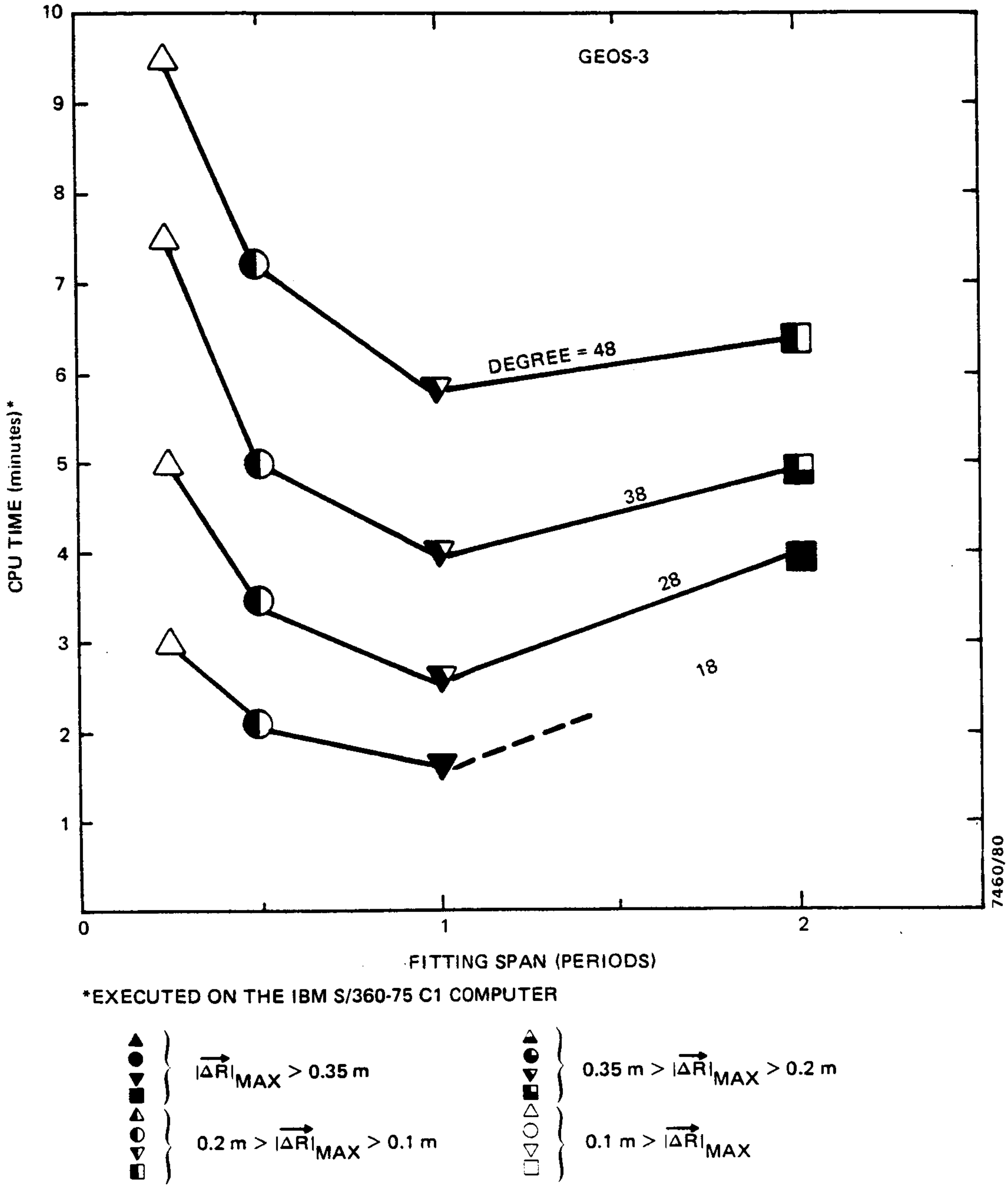



Figure 3-4. CPU Time Versus Fitting Span for Various Degrees of Chebyshev Ploynomials for GEOS-3

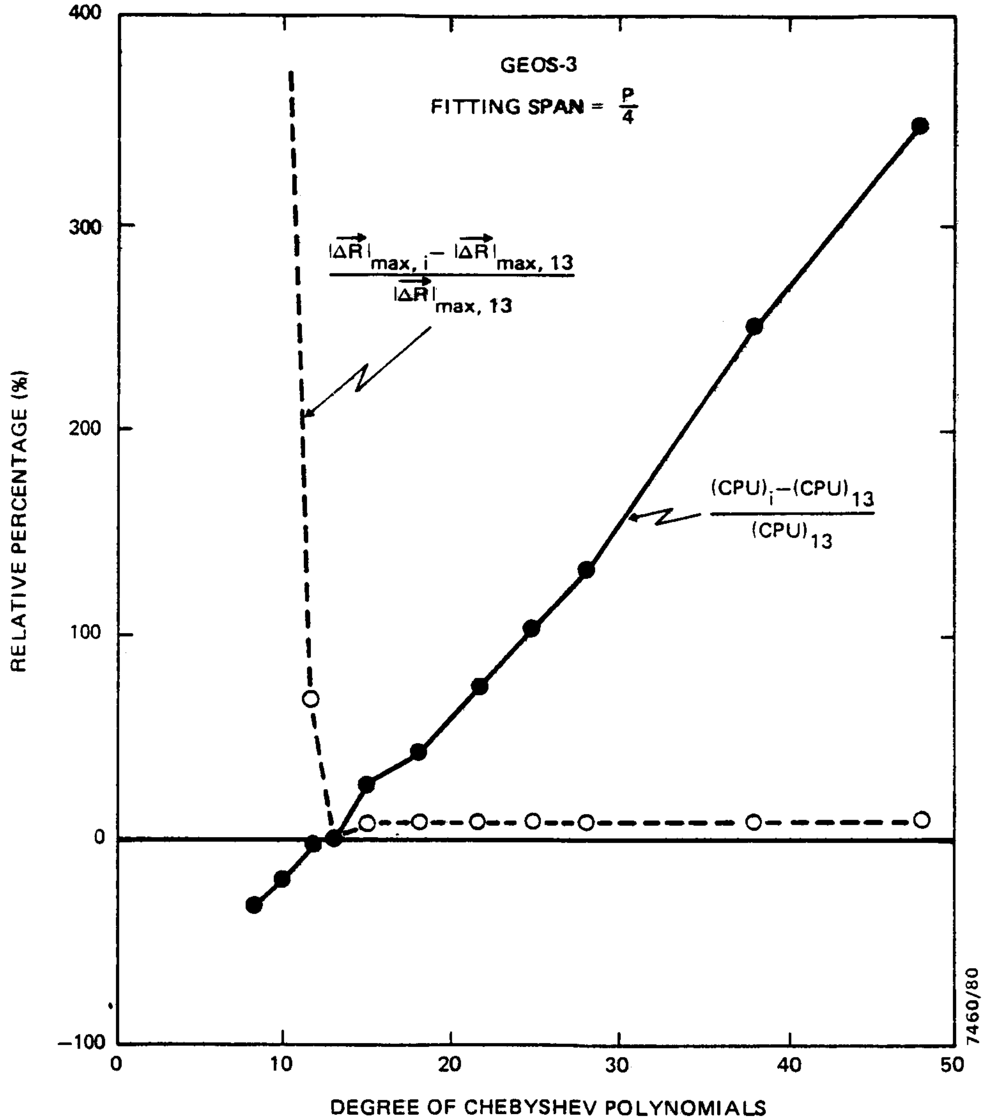



Figure 3-5. Changes in Relative Accuracy and Efficiency for a Fitting Span of P/4 for the GEOS-3 Orbit

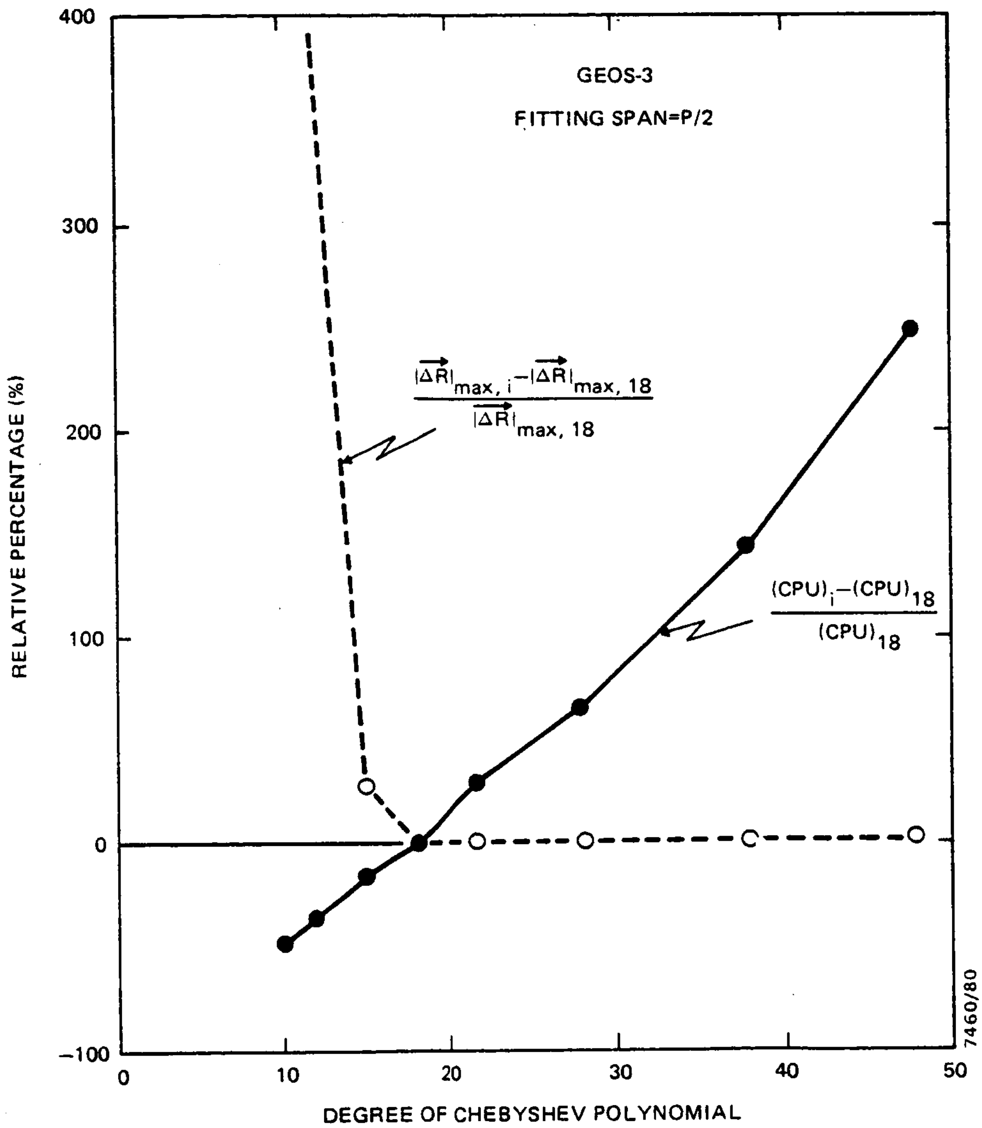



Figure 3-6. Changes in Relative Accuracy and Efficiency for a Fitting Span of P/2 for the GEOS-3 Orbit

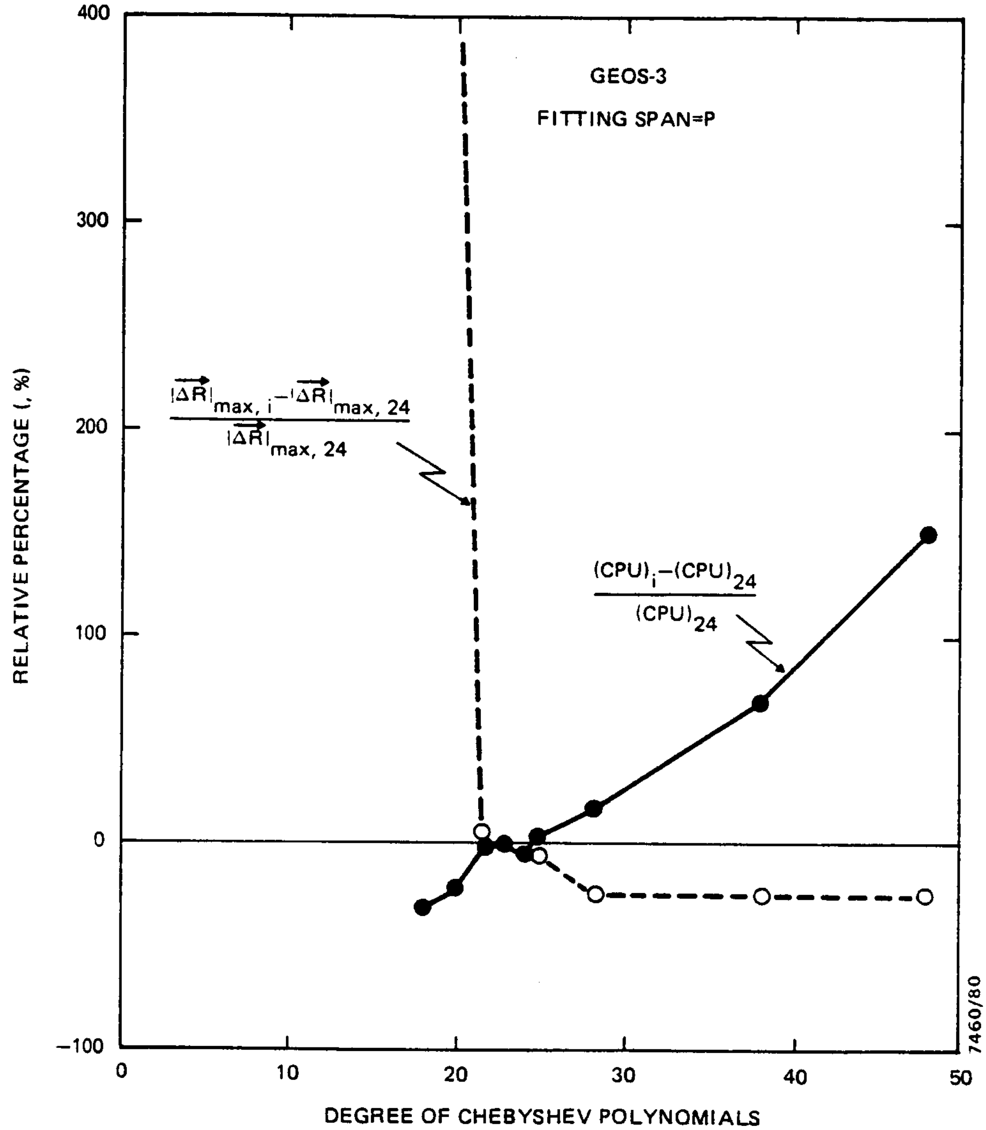



Figure 3-7. Changes in Relative Accuracy and Efficiency for a Fitting Span of P for the GEOS-3 Orbit

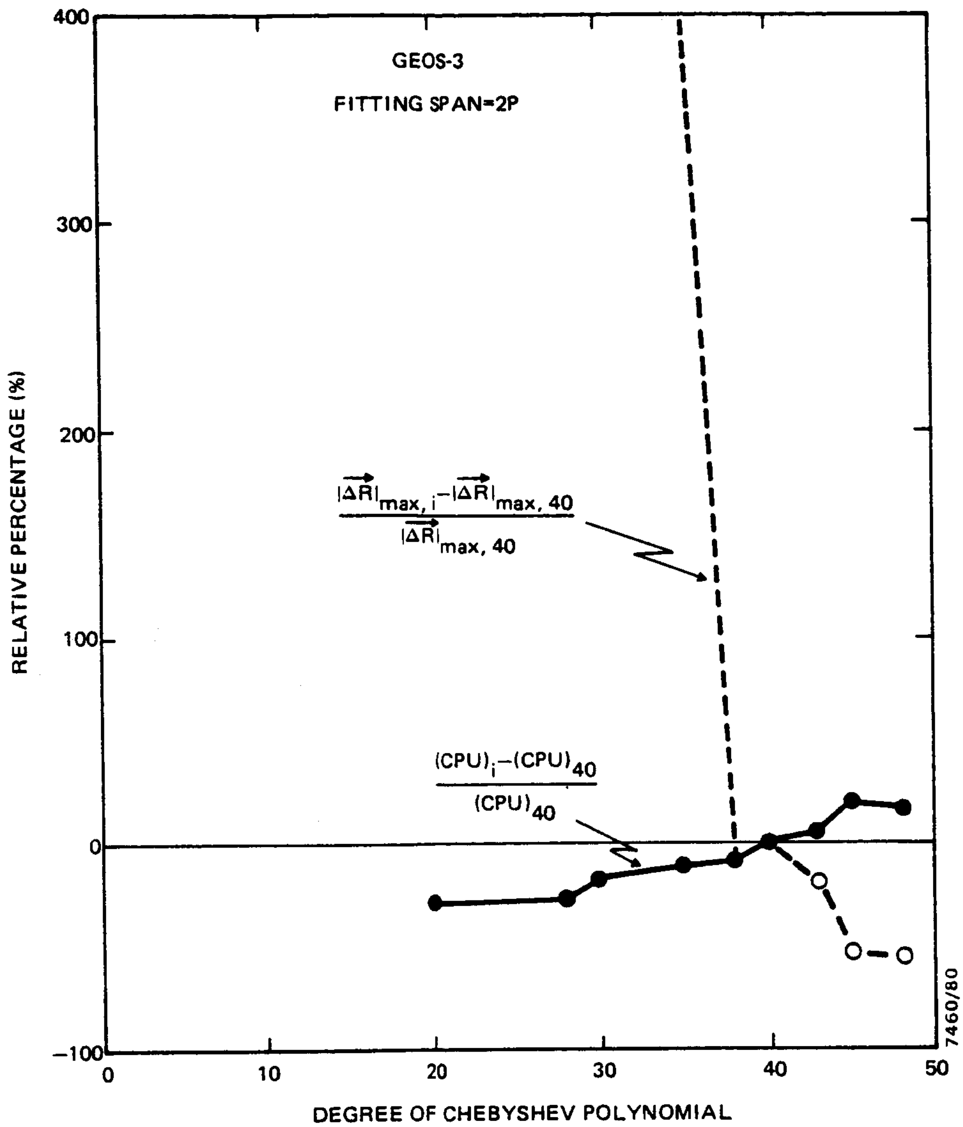



Figure 3-8. Changes in Relative Accuracy and Efficiency for a Fitting Span of 2P for the GEOS-3 Orbit

corresponding to the critical degree of the polynomial on the saturated portion of the accuracy curve. For example, the CPU time used in a computer run for fitting a Chebyshev polynomial of the ith degree over a span of P/4 is normalized with respect to the CPU time consumed for fitting 13th-degree Chebyshev polynomials over the same span, i.e.,

$$[(CPU)_i - (CPU)_{13}]/[(CPU)_{13}]$$

Likewise, the accuracy of the fit is also normalized with respect to the 13th-degree Chebyshev polynomials, i.e.,

$$(|\Delta\vec{R}|_{max,i} - |\Delta\vec{R}|_{max,13})/(|\Delta\vec{R}|_{max,13})$$

For the fitting span of P/4, CPU usage doubles without any benefit at all when the degree is increased from 13th to 25th. Actually, the accuracy has deteriorated by about 10 percent. If the degree is reduced from 13th to 12th, the CPU consumption saved is only 0.6 percent, but the penalty is a significant 70 percent decrease in accuracy. Therefore, it is very desirable to predefine the requirement for support to be accuracy-bound or CPU-bound for selecting the fitting span and the degree of the Chebyshev polynomials. An arbitrary combination of these parameters may either produce an ephemeris with accuracy so poor that it is not usable or consume more computer resources than necessary.

Another area of trade-off consideration is whether support is accuracy-bound or storage-bound. The total number of Chebyshev coefficients is directly proportional to the degree of the Chebyshev polynomials and inversely proportional to the fitting span over a predefined arc length. If these coefficients are to be saved in a limited amount of space for general applications, such as ephemeris representation on an onboard computer for satellite navigation or autonomous spacecraft, an appropriate combination of degree and fitting span must be selected for an efficient usage of the storage within a required accuracy constraint.

### 3.2 ELLIPTICAL ORBIT (ECCENTRICITY = 0.1)

The IMP-7 spacecraft, with an orbit eccentricity of 0.11 and a semimajor axis of 223,670 kilometers, was selected to represent the elliptical orbit. Three sets of computer runs were obtained for fitting spans of P/4.5, P/2, and P. The results are shown in Figure 3-9.

The behavior in the variation of accuracy with the degree of the Chebyshev polynomials is essentially the same as that shown in Figure 3-1 for a near-circular orbit. The accuracy improves very rapidly as the degree increases and then saturates after a critical degree is reached.

### 3.3 HIGHLY ECCENTRIC ORBIT (ECCENTRICITY = 0.9)

The ISEE-1 spacecraft orbit, with an eccentricity of 0.91 and a semimajor axis of 75,500 kilometers, was selected as representative of a highly eccentric orbit. With a fitting span of P/4, equivalent to 51,600 seconds, the best accuracy of the ephemeris represented by Chebyshev polynomials of the 48th degree over one satellite revolution (equivalent to a 2.4-day arc) is 228 kilometers with respect to the reference ephemeris generated by the Cowell method. An ephemeris with accuracy this poor may not be very useful.

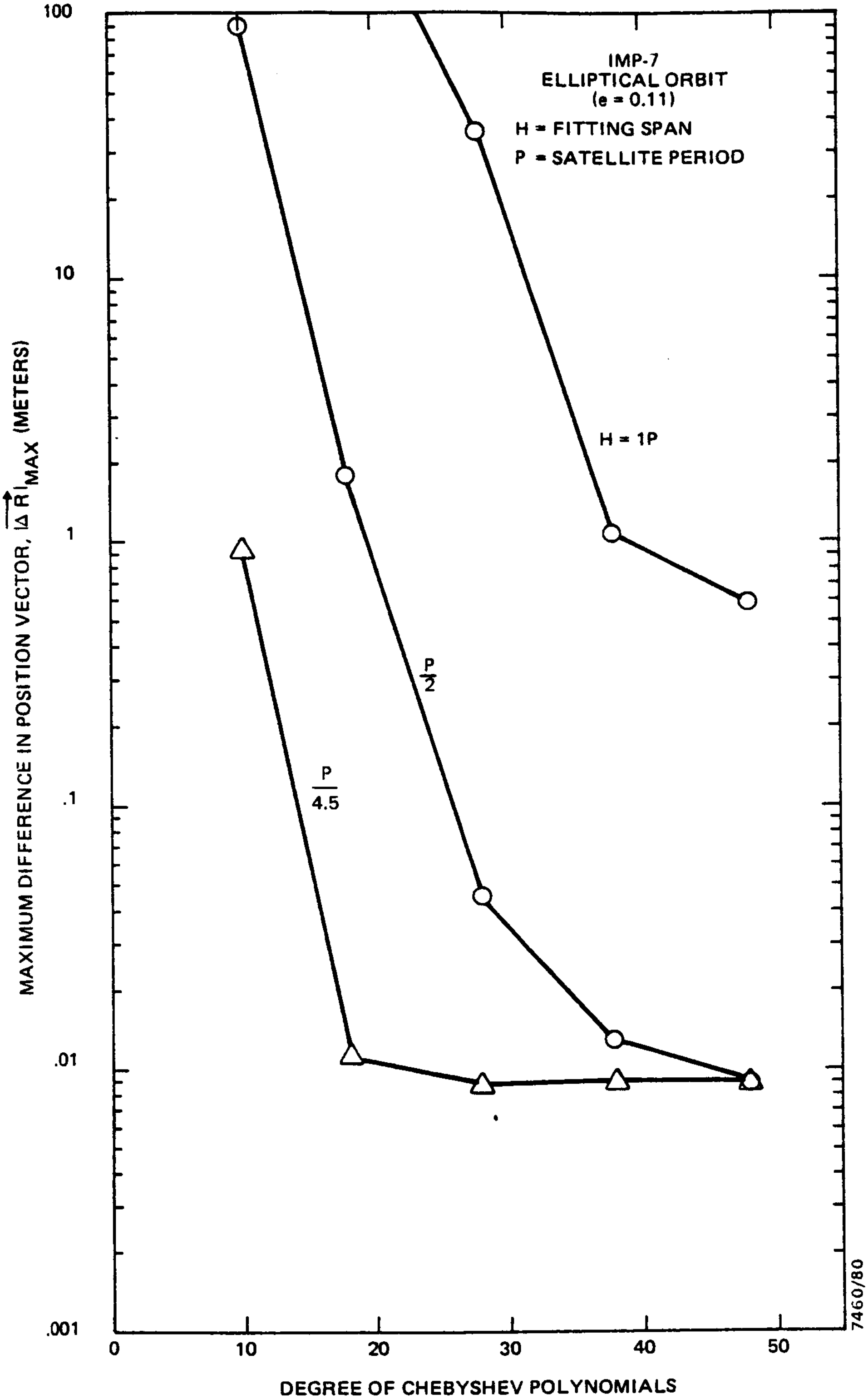



Figure 3-9. IMP-7 Ephemeris Accuracy for the Chebyshev Polynomial Fit Compared With the Cowell Method

Further tests were conducted with drastically reduced fitting spans of P/40 and P/80. The results are presented in Figure 3-10. The accuracy obtained was comparable to that shown in Sections 3.1 and 3.2. As in the cases of circular orbit and elliptical orbit, the accuracy curves show the similar behavior in the variation of accuracy with the degree of the Chebyshev polynomials.

## 3.4 AN EMPIRICAL FORMULA TO DETERMINE THE OPTIMAL FITTING SPAN

Ideally, in applying the Chebyshev method, one would like to obtain the highest accuracy with a minimum amount of CPU time for the lowest possible degree and the longest possible fitting span. However, so straightforward an application is not possible because those factors compete with each other in a rather complicated fashion as demonstrated in Sections 3.1, 3.2, and 3.3. An attempt was made to find an empirical formula for determining an optimal fitting span in terms of satellite period.

From Figures 3-1, 3-9, and 3-10, it is obvious that a longer fitting span requires a higher degree for the Chebyshev polynomials in order to achieve acceptable fitting accuracy, i.e., the fitting span should be proportional to the degree of the Chebyshev polynomials. Furthermore, the fitting span must be substantially smaller for a highly eccentric orbit than for a circular orbit. From these arguments, a very crude empirical formula results:

$$H = C \cdot DP\,(1 - e)^2 \qquad (3\text{-}1)$$

where H = the fitting span of Chebyshev polynomials in terms of satellite period

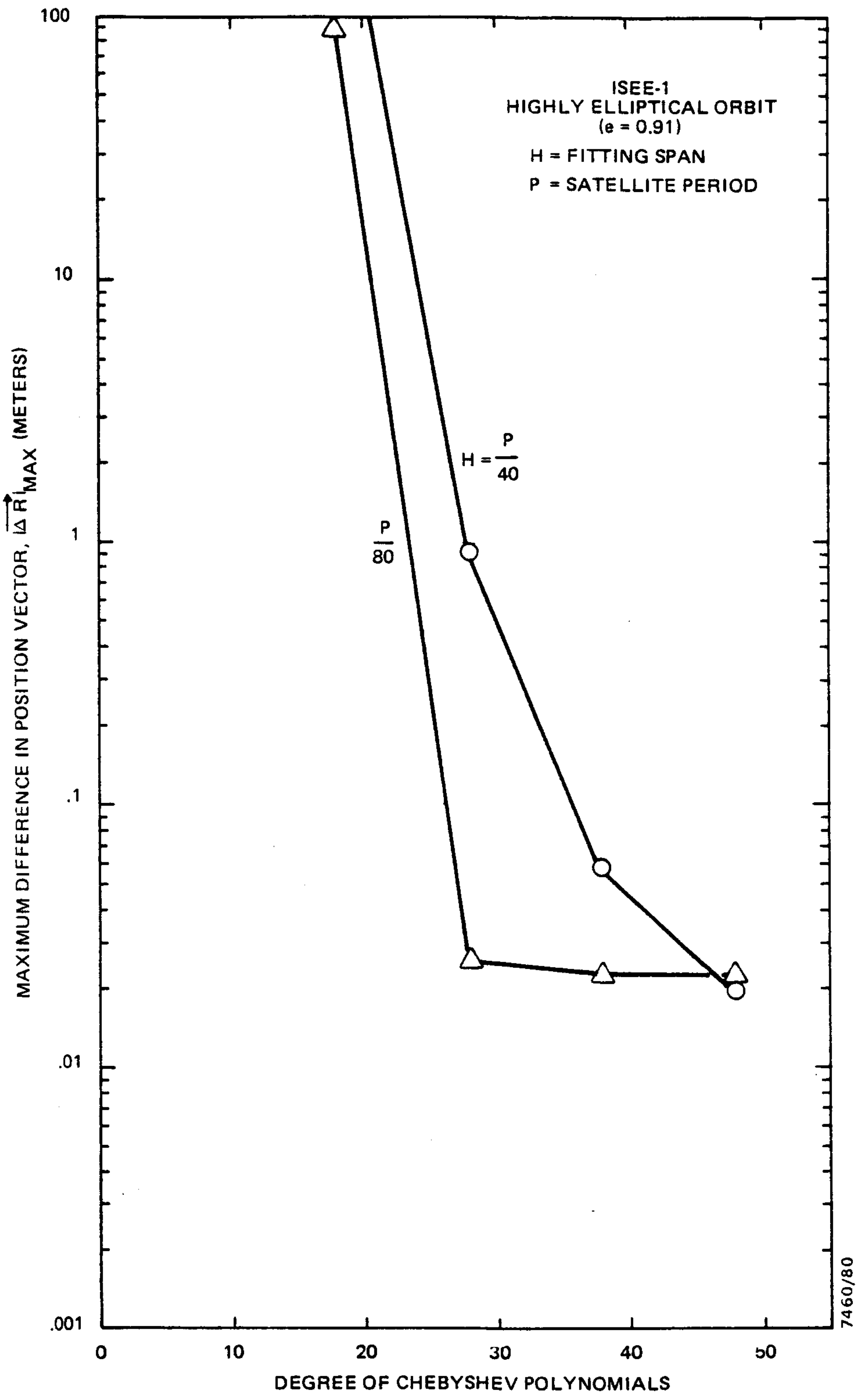



Figure 3-10. ISEE-1 Ephemeris Accuracy for the Chebyshev Polynomial Fit Compared With the Cowell Method

D = the degree of the Chebyshev polynomials

P = the satellite period

e = the eccentricity of the orbit

C = an empirical constant

The value of the constant, C, depends on the degree of the Chebyshev polynomials,

$$C = \frac{1}{40} \qquad D \leq 20$$

$$C = \frac{1}{20} \qquad D > 20$$

To demonstrate the validity of this empirical formula, the following examples are given and results are shown in Table 3-1.

To represent the GEOS-3 ephemeris ($e \simeq 0$) with Chebyshev polynomials of the 10th degree, the fitting span computed using Equation (3-1) is 2P, which gives an accuracy of 0.27 meter over 28 periods (2 days). If Chebyshev polynomials of the 20th degree are desired for GEOS-3, the fitting span given by the empirical formula is P/2, which gives an accuracy of 0.13 meter.

For IMP-7 ($e \simeq 0.1$), the fitting span computed from the empirical formula for a 40th-degree Chebyshev polynomial is roughly 3P/2 with an accuracy of 20 meters. For a 20th-degree Chebyshev polynomial, the fitting span would be P/2.5, giving an accuracy better than 0.2 meter.

In the case of ISEE-1 ($e \simeq 0.9$), fitting a 48th-degree Chebyshev polynomial requires a fitting span of P/40 to achieve 0.02 meter accuracy. For a 30th-degree Chebyshev polynomial, a fitting span of P/70 gives an accuracy of 0.03 meter.

Table 3-1. Ephemeris Accuracies for Fitting Spans Determined Using the Empirical Formula

| CATEGORY | NEAR CIRCULAR ORBIT | | | | ELLIPTICAL ORBIT | | | | HIGHLY ECCENTRIC ORBIT | |
|---|---|---|---|---|---|---|---|---|---|---|
| SATELLITE | GEOS-3 | | TDRS | | IMP-7 | | SATELLITE-X | | ISEE-1 | |
| ECCENTRICITY, e | $10^{-3}$ | | $10^{-3}$ | | 0.1 | | 0.5 | | 0.9 | |
| DEGREE OF CHEBYSHEV POLYNOMIALS, D | 20 | 40 | 20 | 40 | 20 | 40 | 20 | 40 | 30 | 48 |
| FITTING SPAN DETERMINED BY USING EMPIRICAL FORMULA, H | $\frac{P}{2}$ | 2P | $\frac{P}{2}$ | 2P | $\frac{P}{2.5}$ | $\frac{3P}{2}$ | $\frac{P}{8}$ | $\frac{P}{2}$ | $\frac{P}{70}$ | $\frac{P}{40}$ |
| ACCURACY OF EPHEMERIS* (METERS) | 0.13 | 0.27 | 0.14 | 1.6 | 0.18 | 20.0 | 0.55 | 0.62 | 0.03 | 0.02 |
| EPHEMERIS ARC LENGTH, DAYS (REVOLUTIONS) | 2 (28) | 2 (28) | 31 (31) | 31 (31) | 13.6 (1) | 13.6 (1) | 2 (11.5) | 2 (11.5) | 2.4 (1) | 2.4 (1) |

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*ACCURACY IS MEASURED WITH RESPECT TO COWELL METHOD.

The empirical formula is applied to the Tracking and Data Relay Satellite (TDRS) and the results are also included in Table 3-1. The TDRS is to be a geosynchronous satellite with an eccentricity of nearly zero and a semimajor axis of 42,000 kilometers. The fitting span computed from the empirical formula for a 20th-degree Chebyshev polynomial is P/2 which gives an accuracy of 0.14 meter for the ephemeris over a 31-day arc of 31 revolutions. If a 40th-degree Chebyshev polynomial is chosen, the computed fitting span is 2P which gives an accuracy of 0.16 meter over the same arc length.

To further verify the validity of the empirical formula, a nonexistent Satellite-X with an eccentricity of 0.5 and a semimajor axis of 13,200 kilometers was tested. The perigee height is 6,600 kilometers, about 200 kilometers above the surface of the Earth, and the apogee height is 19,800 kilometers. The relative importance of all perturbing forces, such as a higher-order harmonic geopotential field, atmospheric drag, and solar radiation pressure, exerted on Satellite-X varies at different positions on the orbit causing the magnitude of the trajectory variation to differ along the orbit. Near the perigee, a shorter fitting span and a higher degree of Chebyshev polynomials may be needed to meet required accuracy criteria because of the effects of a higher-order harmonic geopotential field and the atmospheric drag. Near the apogee, a medium fitting span and medium degrees of the Chebyshev polynomials may be required because of the large curvature of the trajectory in combination with the trajectory variation due mainly to the solar radiation pressure. While in the vicinities of 90 degrees and 270 degrees of anomaly of the orbit, the trajectory is rather linear and a lower degree and a longer fitting span may be sufficient. It is not possible, however, to apply several different fitting spans and degrees over each revolution in a single computer run setup with the current GTDS, which only allows a uniform fitting span and a single choice of degree for the Chebychev polynomials.

Two test runs were made with fitting spans of P/8 and P/2 computed by using the empirical formula, Equation (3-1), for Chebyshev polynomials of the 20th and 40th degrees, respectively. The accuracy for a P/8 fitting span with a 20th-degree Chebyshev polynomial is 0.55 meter, and 0.62 meter for a P/2 fitting span with a 40th-degree polynomial over a two-day arc of 11.5 revolutions.

With the exception of the case of the 40th-degree polynomial with a 3P/2 fitting for IMP-7, all the cases seem to favorably support the validity of the empirical formula. However, the formula still should be used with extreme caution, perhaps only as a rough guideline to establish a preliminary set of parameters for the Chebyshev ephemeris generation method.

# SECTION 4 - CONCLUSIONS

The Chebyshev ephemeris generation method is reimplemented in the operational version of GTDS. The conclusions from the testing results for this new implementation are summarized below.

- The new implementation is more efficient and produces more accurate ephemerides.
- The accuracy of the ephemeris generated by the Chebyshev method increases with the degree of the Chebyshev polynomials very rapidly but bottoms out after a critical degree is reached.
- The accuracy is generally better for smaller fitting spans.
- The efficiency of the Chebyshev method is mainly related to the degree of the Chebyshev polynomials and the fitting span.
- The Chebyshev method is slower than the Cowell method. Unless Chebyshev coefficients are required, the Chebyshev method is not recommended for use in general applications. A study is currently underway to further improve the efficiency of the Chebyshev method by using the Brouwer-Lyddane theory instead of the two-body theory for the starter.
- A preliminary empirical formula was deduced to determine an optimal fitting span with a desirable degree of Chebyshev polynomials in terms of high accuracy of the satellite ephemeris and low consumption of computer resources.

- It is also recommended that the conclusions of any study involving the use of Chebyshev ephemerides obtained from the previous version of GTDS should be re-evalutated, especially in regard to accuracy.

# APPENDIX A - MATHEMATICAL THEORY OF THE CHEBYSHEV ORBIT GENERATION METHOD

## A.1 PROPERTIES OF CHEBYSHEV POLYNOMIALS

The properties of the Chebyshev polynomials are most easily examined in the normalized interval [1, -1]. Any arbitrary finite interval $[t_a, t_b]$ can be transformed to the normalized interval [1, -1] by the change of variable:[1]

$$\xi = 1 - 2\left(\frac{t - t_a}{t_b - t_a}\right) \tag{A-1}$$

where $\xi$ = the normalized time variable

$t_a$ = the start time of a polynomial fitting span, (i.e., the start time of an integration step in GTDS terminology)

$t_b$ = the end time of a polynomial fitting span, (i.e., the end time of an integration step and, therefore, $t_b - t_a$ corresponds to the "step size")

The Chebyshev polynomials are defined as a set of polynomials

$$T_i(\xi) = \cos i\theta \qquad i = 0, 1, \ldots \tag{A-2}$$

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[1] The transformation could have been defined as

$$\xi = 2\left(\frac{t - t_a}{t_b - t_a}\right) - 1$$

so that $t_a$ would correspond to -1, and $t_b$ would correspond to +1. Since Reference 1 and the GTDS software have consistently used the definition as shown in Equation (A-1), this transformation is retained throughout this document and the new software.

generated from the sequence of cosine functions using the transformation

$$\theta = \cos^{-1}\xi \qquad -1 \le \xi \le 1 \qquad \text{(A-3)}$$

Clearly for the zeroth degree

$$T_0(\xi) = \cos(0) = 1 \qquad \text{(A-4)}$$

and for the first degree

$$T_1(\xi) = \xi \qquad \text{(A-5)}$$

By repeated trigonometric manipulations, higher-degree Chebyshev polynomials can be computed yielding the recursion relation

$$T_i(\xi) = 2\xi T_{i-1}(\xi) - T_{i-2}(\xi) \qquad i = 2, 3, \ldots \qquad \text{(A-6)}$$

Table A-1 contains the first ten Chebyshev polynomials. With simple algebraic manipulation, the algebraic functions, $\xi^n$, can be expressed in terms of a linear combination of the Chebyshev polynomials. This is shown in Table A-2. All the Chebyshev polynomials have a maximum magnitude of 1 in the interval [1, -1]. The Chebyshev polynomials of degrees 0 to 3 are plotted in Figure A-1. The function of a parabola, $\xi^2$, is also plotted in the figure as a linear combination of $T_0$ and $T_2$. Except for $T_0$, all other Chebyshev polynomials cross the $\xi$-axis. The number of times that axis is crossed is equal to the degree of the Chebyshev polynomial and only those

Table A-1. The Chebyshev Polynomials

$$T_0 = 1$$

$$T_1 = \xi$$

$$T_2 = 2\xi^2 - 1$$

$$T_3 = 4\xi^3 - 3\xi$$

$$T_4 = 8\xi^4 - 8\xi^2 + 1$$

$$T_5 = 16\xi^5 - 20\xi^3 + 5\xi$$

$$T_6 = 32\xi^6 - 48\xi^4 + 18\xi^2 - 1$$

$$T_7 = 64\xi^7 - 112\xi^5 + 56\xi^3 - 7\xi$$

$$T_8 = 128\xi^8 - 256\xi^6 + 160\xi^4 - 32\xi^2 + 1$$

$$T_9 = 256\xi^9 - 576\xi^7 + 432\xi^5 - 120\xi^3 + 9\xi$$

Table A-2. An Algebraic Function Expressed in Terms of a Linear Combination of the Chebyshev Ploynomial

$$1 = T_0$$
$$\xi = T_1$$
$$\xi^2 = (T_0 + T_2)/2$$
$$\xi^3 = (3T_1 + T_3)/4$$
$$\xi^4 = (3T_0 + 4T_2 + T_4)/8$$
$$\xi^5 = (10T_1 + 5T_3 + T_5)/16$$
$$\xi^6 = (10T_0 + 15T_2 + 6T_4 + T_6)/32$$
$$\xi^7 = (35T_1 + 21T_3 + 7T_5 + T_7)/64$$
$$\xi^8 = (35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8)/128$$
$$\xi^9 = (126T_1 + 84T_3 + 36T_5 + 9T_7 + T_9)/256$$

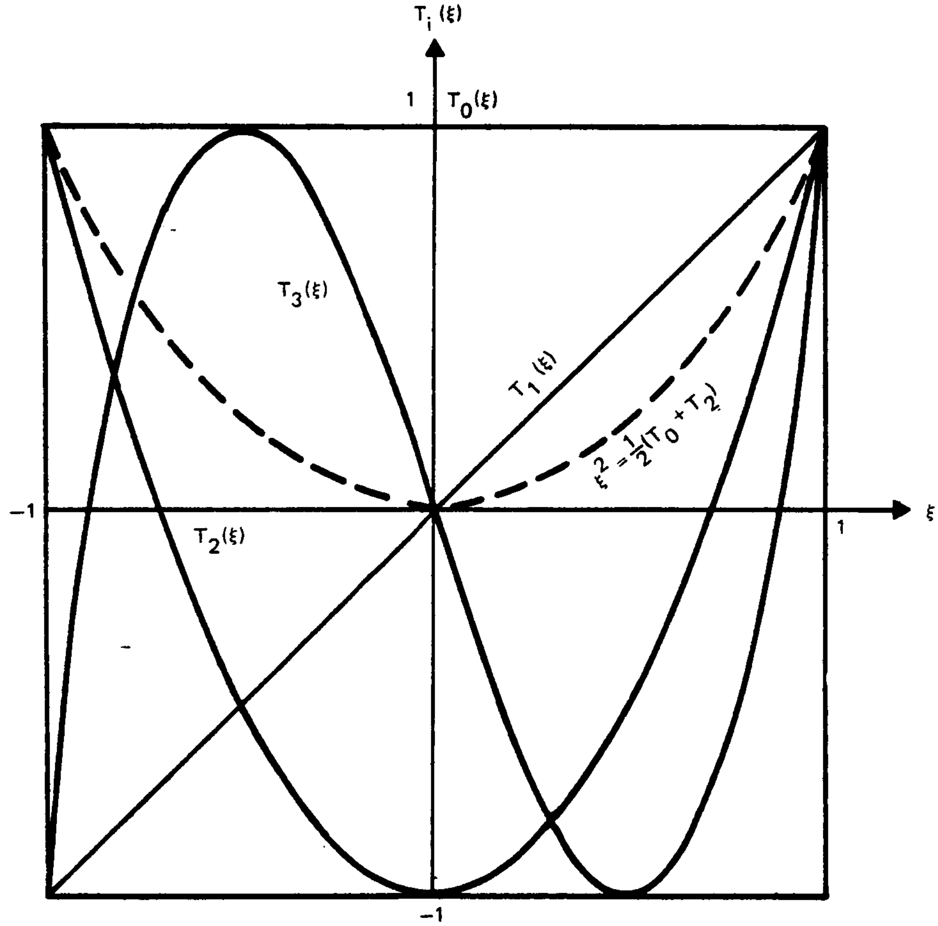


Figure A-1. Chebyshev Polynomials of Degrees 0 to 3 of odd degrees will cross the origin. The n locations at which the Chebyshev polynomial $T_n(\xi)$ crosses the $\xi$-axis are the n roots in the interval [1, -1] and are given by

$$\xi_k = \left[\cos \frac{(2k - 1)\pi}{2n}\right] \qquad k = 1, \ldots, n \qquad \text{(A-7)}$$

The significant properties of using the Chebyshev polynomials to fit an arbitrary function are that the error in the approximation is distributed evenly over the interval and the maximum error is reduced to the minimum or near-minimum value (References 3 and 4).

## A.2 INTERPOLATING POLYNOMIALS CONSISTING OF A LINEAR COMBINATION OF CHEBYSHEV POLYNOMIALS OF DIFFERENT DEGREES TO REPRESENT ACCELERATION

Each component of the acceleration vector exerted on the spacecraft can be approximated by an interpolating polynomial consisting of a linear combination of Chebyshev polynomials:

$$\ddot{x}(\xi) = P_n(\xi) = \sum_{i=0}^{n} C_i T_i(\xi) \qquad \text{(A-8)}$$

where $\ddot{x}$ = the Cartesian component of the acceleration vector

$P_n$ = the interpolating polynomial of degree n

$C_i$ = Chebyshev coefficients for an acceleration component

$\xi$ = the transformed time variable

The accuracy of this approximation is better when the higher degrees of Chebyshev polynomials are included. However, the benefit of including higher degrees drops off quickly. This point is further illustrated in Section 4.

It is well known that the Chebyshev polynomials are one of the families which possess the property of orthogonality (References 3 and 4). They are orthogonal in the interval [1, -1] with respect to the weighting function, $w(\xi) = 1/\sqrt{1 - \xi^2}$, i.e.,

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - \xi^2}} T_i(\xi)\ T_j(\xi)\ d\xi = 0, \qquad i \neq j$$

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - \xi^2}} \left[T_i(\xi)\right]^2 d\xi = A_i \neq 0 \tag{A-9}$$

where $A_i$ is a normalization factor which depends on i.

Making use of the property of orthogonality, as demonstrated in Equation (A-9), the Chebyshev coefficients can be evaluated

$$C_i = \frac{1}{A_i} \int_{-1}^{1} \frac{1}{\sqrt{1 - \xi^2}} T_i(\xi)\ \ddot{x}(\xi)\ d\xi \qquad i = 0, 1, \ldots, n \tag{A-10}$$

The above integral is difficult to evaluate because of the complexity of $\ddot{x}(\xi)$. However, it has been shown (References 3, 4, and 5) that Equation (A-10) may be approximated by

$$C_0 = \frac{1}{n + 1} \sum_{k=1}^{n+1} \ddot{x}(\xi_k)$$

$$C_i = \frac{2}{n + 1} \sum_{k=1}^{n+1} T_i(\xi_k)\ddot{x}(\xi_k) \qquad i = 1, 2, \ldots, n \tag{A-11}$$

where $\xi_k$ are the roots of the Chebyshev polynomial of degree n + 1, $T_{n+1}(\xi)$. Therefore, with the accelerations evaluated at all the n + 1 roots, the variation of the acceleration in the interval [1, -1], corresponding to the time interval $[t_a, t_b]$, can be represented by the interpolating polynomial, $P_n(\xi)$, of degree n.

### A.3 INTEGRATION OF CHEBYSHEV INTERPOLATING POLYNOMIAL TO GENERATE EPHEMERIS

With the acceleration components represented by Chebyshev interpolating polynomials as shown in Equation (A-8), integrating the equation once gives the velocity components, $\dot{x}$:

$$\dot{x}(\xi) = \int P_n(\xi)\, d\xi = \sum_{i=0}^{n} C_i \int T_i(\xi)\, d\xi \qquad \text{(A-12)}$$

Through the use of Equations (A-4), (A-5), and (A-6), the integration of the Chebyshev polynomials of different degrees can be obtained:

$$\int T_0(\xi)\, d\xi = T_1(\xi) + K_0$$

$$\int T_1(\xi)\, d\xi = \frac{1}{4}\left[T_0(\xi) + T_2(\xi)\right] + K_1 \qquad \text{(A-13)}$$

$$\int T_i(\xi)\, d\xi = \left[\frac{1}{2}\,\frac{1}{i+1}\, T_{i+1}(\xi) - \frac{1}{i-1}\, T_{i-1}(\xi)\right] + K_i,$$

$$i = 2, 3, \ldots, n$$

where $K_0$, $K_1$, and $K_i$ are integration constants.

Substituting Equation (A-13) into Equation (A-12) and collecting terms of Chebyshev polynomials of the same degrees yields another interpolating polynomial of the following form for the velocity components:

$$\dot{x}(\xi) = \int P_n(\xi)\, d\xi = Q_{n+1}(\xi) = \sum_{i=0}^{n+1} b_i\, T_i(\xi) \qquad \text{(A-14)}$$

where $Q_{n+1}$ = the interpolating polynomial of degree n + 1

$b_i$ = Chebyshev coefficients for a velocity component

with

$$b_0 = K_0 + K_1 + \ldots + \frac{C_1}{4} T_0$$

$$b_1 = C_0 - \frac{1}{2} C_2$$

$$b_i = \frac{1}{2i} \left[ C_{i-1} - C_{i+1} \right] \qquad i = 2, 3, \ldots, n+1 \qquad \text{(A-15)}$$

$$C_{n+1} = C_{n+2} = 0$$

The integration constants in the expression for $b_0$ may be evaluated from the initial velocity, i.e., $\dot{x}(t_a) = \dot{x}(\xi = 1)$:

$$\dot{x}(\xi = 1) - \sum_{i=1}^{n+1} b_i\, T_i(\xi = 1) = b_0\, T_0(\xi = 1) = b_0 \qquad \text{(A-16)}$$

The interpolating polynomials for position components, $x(\xi)$, can be obtained by the same procedure:

$$x(\xi) = \int Q_{n+1}(\xi)\ d\xi = R_{n+2}(\xi) = \sum_{i=0}^{n+2} a_i\ T_i(\xi) \qquad \text{(A-17)}$$

$$a_0 = x(\xi = 1) - \sum_{i=1}^{n+2} a_i\ T_i(\xi = 1)$$

$$a_1 = b_0 - \frac{1}{2} b_2$$

$$a_i = \frac{1}{2i}\left[b_{i-1} - b_{i+1}\right], \qquad i = 2, 3, \ldots, n + 2$$

$$b_{n+2} = b_{n+3} = 0$$

where $R_{n+2}$ = the interpolating polynomial of degree n + 2 for the position component

With velocity and position represented by Equations (A-14) and (A-17) in the interval $[t_a, t_b]$, the ephemerides of spacecraft at any other time within the interval can now be accurately and easily interpolated.

## APPENDIX B - COMPARISON BETWEEN THE IMPROVED IMPLEMENTATION AND THE PREVIOUS IMPLEMENTATION OF THE CHEBYSHEV METHODS

A series of GTDS computer runs was executed to compare the new and the previous software implementations in terms of their accuracy and efficiency. The accuracy was measured with respect to the ephemeris generated with the high-precision Cowell numerical integration method by using the GTDS Ephemeris Comparison Program. The efficiency is simply a comparison of the CPU and I/O times consumed by the two different Chebyshev implementations.

Three sets of test runs were made on GTDS with the GEOS-3 satellite (arbitrarily chosen) over a two-day span using the Cowell method and the Chebyshev polynomial method of the new and previous implementations. The comparison results are presented in Table B-1. In these tests, the ephemeris generated by the Cowell integration method with a 24-second step size was used as a reference. The perturbation (or force model) included in the Cowell method was identical to that used in the Chebyshev methods.

Table B-1 shows that the maximum difference in position vector of the ephemeris generated by the previous Chebyshev implementation with a 48th degree polynomial over a two-day arc is 97 meters with respect to the ephemeris generated by the Cowell method, while the new Chebyshev implementation with the same degree of polynomial has a maximum position difference of only 0.25 meter. This represents an improvement of better than two orders of magnitude in the relative accuracy.

The efficiency which is expressed as CPU time and I/O time consumed on the IBM S/360-75 computer is also examined.

Table B-1. Comparison Between New and Previous Chebyshev Implementations With Respect to the Cowell Method

| PERTURBATION INCLUDED IN INTEGRATION PARAMETERS FOR CHEBYSHEV AND COWELL ORBIT INTEGRATORS | METHOD | DEGREE OF CHEBYSHEV POLYNOMIAL | $\lvert\Delta\vec{R}\rvert_{max}$ * (meters) | $\sigma_{\Delta\vec{R}}$ (meters) | EFFICIENCY** (CPU Minutes/ I/O Minutes) | NUMBER OF TIMES FULL FORCE IS EVALUATED |
|---|---|---|---|---|---|---|
| 4 x 4 GEOPOTENTIAL, SUN, MOON | NEW IMPLEMENTATION | 48 | 0.25 | 0.11 | 5.806/0.142 | 25,143 |
| | | 20 | 5.17 | 2.36 | 1.835/0.142 | 11,339 |
| | | 18 | 36.6 | 16.76 | 1.604/0.138 | 10,353 |
| | PREVIOUS IMPLEMENTATION | 48 | 97.32 | 51.73 | 8.012/0.154 | 39,562 |
| 8 x 8 GEOPOTENTIAL, 14TH ORDER RESONANCE GEOPOTENTIAL, SUN, MOON, DRAG, SOLAR RADIATION PRESSURE | NEW IMPLEMENTATION | 38 | 4.87 | 2.22 | 7.118/0.176 | 20,213 |
| | PREVIOUS IMPLEMENTATION | 38 | 20.09 | 11.02 | 7.272/0.177 | 31,408 |

*$\lvert\Delta\vec{R}\rvert_{max}$ = MAXIMUM $\left\lvert \vec{R}(t)_{CHEBYSHEV} - \vec{R}(t)_{COWELL} \right\rvert$ OVER A 2-DAY ARC

**EXECUTED ON AN IBM S/360-75 COMPUTER

NOTE: GEOS-3 ORBIT CHARACTERISTICS AT EPOCH, JULY 18, 1977, WERE AS FOLLOWS:

$a$ = 7224.6628 KILOMETERS
$e$ = 0.0009758
$i$ = 114.98 DEGREES
$\Omega$ = 341.14 DEGREES
$\omega$ = 210.90 DEGREES
$M$ = 126.14 DEGREES

THE PERIOD WAS 6100 SECONDS. STEP SIZES WERE 6101 SECONDS FOR THE CHEBYSHEV ORBIT INTEGRATOR AND 24 SECONDS FOR THE COWELL ORBIT INTEGRATOR.

The previous implementation used 8.012 minutes of CPU time and 0.154 minute of I/O time, while the new implementation used only 5.806 minutes of CPU time, a saving of 38 percent, and a comparable 0.142 minute of I/O time.

The saving of computer resources can be viewed from another angle by lowering the degree of Chebyshev polynomials from 48 to 20 and 18. The results are also shown in Table B-1. Fitting Chebyshev polynomials with much lower degrees, the new implementation consumes four to five times less CPU resources yet maintains better accuracy than the previous implementation.

Results in Table B-1 indicate similar conclusions with more elaborate perturbation models, i.e., 8x8 geopotential field, 14th order resonance geopotential field, atmospheric drag, and solar radiation pressure as well as solar and lunar gravitational fields.

The results presented in Table B-1 are obtained with a fitting span of one satellite period for both the new and previous implementation. When the fitting span is increased to two satellite periods, the new implementation gives excellent results ($|\Delta \vec{R}|_{max}$ = 0.12 meter). However, after 80 loops in the iterative scheme, the previous implementation has simply failed to satisfy the $10^{-6}$ kilometer tolerance in fitting the first span and the computer run was subsequently terminated without generating an ephemeris.

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