SEMIANALYTICAL SATELLITE THEORY AND SEQUENTIAL ESTIMATION

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ABSTRACT

Kalman filtering techniques are combined with a semianalytical orbit generator to develop a sequential orbit determination algorithm. The algorithm is investigated for computational efficiency, accuracy, and radius of convergence by comparison with truth ephemerides and a Cowell Special perturbations filter (GTDS FILTER). Test cases relevant to satellite navigation are examined.

Notation and Symbols

sub-bar (e.g., $\underline{x}$) = vector

super-bar (e.g., $\bar{x}$) = average or mean; also statistical mean

ϵ (e.g., $\epsilon\underline{\eta}$) = formal indication of the order of the quantity

(ϵ = first, ϵ^2 = second, ...)

$\underline{\varepsilon}_6 = [0\ 0\ 0\ 0\ 0\ 1]^T$

n = mean motion = $\sqrt{\frac{\mu}{a^3}}$

Equinoctial Elements

a = semimajor axis

$h = e \sin(\omega + I\Omega)$ $k = e \cos(\omega + I\Omega)$

$p = \tan^I(1/2) \sin \Omega$ $q = \tan^I(i/2) \cos \Omega$

$\lambda = M + \omega + I\Omega$ = mean longitude

I = retrograde factor

super-hat (e.g., $\hat{x}$) = predicted estimate

super-tilde (e.g., $\tilde{x}$) = updated estimate

1. INTRODUCTION

The current trends in Earth satellite orbit determination are toward sequential filtering and onboard computation [1]. The Global Positioning System (GPS) currently employs an orbit determination algorithm that updates a batch estimated nominal trajectory in real-time with an extended Kalman filter [2]. This system is designed to achieve an operational accuracy within 1.5 m. Telesat, a satellite communications system, has been using a sequential system to support all station keeping operations for several years now, with both improved accuracy and reduced costs [3]. Many other applications exist and will develop for which the timeliness, accuracy, and efficiency of a real-time orbit determination system are highly desirable.

Orbit determination processes require two capabilities: the ability to accurately propagate an orbit, given initial conditions; and some estimation algorithm indicating how observations of the satellite should be used in updating the ephemeris. Advances in the technology of either capability imply corresponding advances in orbit determination processes. Recently, much work has been done by P. Cefola, et al. [4], [5], [6], [7] of CSDL in extending Semianalytical Satellite Theory to allow highly accurate and efficient orbit propagation. A. Green [4] developed and used some of these results in a batch DC estimation algorithm, finding accuracies and convergence properties quite comparable to high precision Cowell results. This paper explores the implications of these advances in Semianalytical Satellite Theory for sequential orbit determination, considering both accuracy and efficiency through comparison with batch and sequential filters available from GTDS and Green [4].

The organization of the paper is dictated by the structure of the orbit determination problem. Summaries of semianalytical satellite theory and sequential filtering are presented first. Then their combination into an orbit determination algorithm is developed to give the algorithm as it was finally implemented. Results are not included here; they will be presented at the conference.

2. SEMIANALYTICAL SATELLITE THEORY

The accurate and efficient propagation of an ephemeris requires both a precise model of the forces acting on the satellite and an accurate and efficient means of integrating the equations of motion. The equations of motion are given by Newton's Second Law as

$$\frac{d^2\underline{r}}{dt^2} + \frac{\mu}{r^3}\,\underline{r} = \underline{a}_d \qquad [1]$$

The terms from left to right are the satellite's acceleration, the point-mass gravitational attraction, and all other (disturbing) accelerations, due to drag, third bodies, solar radiation, etc. The disturbing accelerations are typically several orders of magnitude smaller than the point-mass force.

Now any integrator is most accurate and efficient for systems with only small nonlinearities and low frequencies in the force model. Historically, this fact has motivated tradeoffs between analytical methods, which use simplified force models and analytical approximations to obtain the integrated ephemeris efficiently, and numerical methods, which retain the full force model and use high precision numerical integrators to obtain the integrated ephemeris quite accurately.

To increase the efficiency of an ephemeris generator, it is necessary to decrease both the magnitude of the nonlinearities as well as the high frequency content of the force model. The magnitude of the nonlinearities can be reduced by choice of the orbital elements. For example, Keplerian and equinoctial elements incorporate the effects of the point mass acceleration, leaving only the disturbing acceleration to be accounted for. The transformation from cartesian position and velocity to such an element set is the basis of Gauss' VOP equations. [In the early days of modern satellite orbit determination, many element sets incorporating different components of the disturbing acceleration were experimented with; while they could very efficiently propagate an ephemeris subject to only their selected perturbations, to achieve real-world accuracy they had to sacrifice all efficiency gains with the inclusion of additional perturbations.] The high frequency content is removed by averaging these frequencies out; more formally, by transforming from the current osculating elements described by the VOP equations, to mean elements described by "averaged VOP equations." For analytical theories, this whole process was done by hand, necessitating simplified force models and approximate methods. Semianalytical satellite theory, developed after computers became inexpensive and versatile, uses numerical methods to handle those force models that cannot be averaged analytically. Since the tradeoff between numerical averaging of the force model and the use of a high precision integrator on it is in favor of averaging by a factor of 10 to 100, semianalytical satellite theory is much more efficient than purely numerical theories. There is one problem: the transformation back from the mean elements to the osculating elements. The high frequency components or short periodics were averaged out and must be recovered before the mean elements can be used for anything other than long term, approximate prediction. The practical recovery of the short periodics constitutes one of the important contributions of the recent work at CSDL.

Semianalytical Satellite Theory at CSDL

Semianalytical satellite theory at CSDL is implemented in equinoctial elements to avoid singularity problems. The basic equations are shown formally in Table I. Key things to note are the dependence of the mean element rates on only the slowly varying elements $\underline{\bar{a}}$, and the expansion of the short periodics $\underline{\eta}$ ($\underline{\bar{a}}$, $\bar{\lambda}$) as a Fourier series whose coefficients similarly depend on only the slowly varying elements $\underline{\bar{a}}$. Thus the elements $\underline{\bar{a}}$ * and short periodic coefficients $\epsilon\underline{C}\sigma(\underline{\bar{a}})$ and $\epsilon\underline{D}\sigma(\underline{\bar{a}})$ can be and are interpolated, allowing efficient evaluation of the osculating elements for many output times other than those on the integration grid. This is significant since for all averaged theories the computational cost is proportional to the number of integration steps. Averaging allows large steps, but frequent output points could require small steps.

3. SEQUENTIAL FILTERING THEORY

The equations of motion for the osculating and mean orbital elements are shown in Table I. They are nonlinear, as are the equations for range and range rate observations given in Table II. The orbit determination problem is to estimate the satellite's orbit given some initial (a priori) information and a series of observations over time. It can be formulated as an optimal estimation problem:

$$\text{estimate} \quad \hat{\underline{x}}(t) \text{ , given}$$

$$\text{plant} \quad \dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{w} \quad , \quad \underline{x}(t_o) = \underline{x}_o \qquad [2]$$

$$\text{observations} \quad y_k = h(\underline{x}(t_k),t_k) + v$$

using the y_k, such that the variance of the error $\underline{x} - \hat{\underline{x}}$ is minimum. $\underline{x}_o$, $\underline{w}$, and v are random and uncorrelated, $\underline{w}$ and v are white noise processes.

The resulting equations require propagating the probability density function of $\underline{x}$ (t) and are very difficult and expensive to solve. As a result, most sequential orbit determination schemes use some suboptimal filter, usually adapted from the Kalman filter, which solves the linear optimal estimation problem. The two most common adaptations are the Linearized Kalman Filter and the Extended Kalman

Table I. The Equations of Semianalytical Satellite Theory*

Osculating Elements	$\underline{a}^* = [a,h,k,p,q,\lambda]^T$
	$\underline{a} = [a,h,k,p,q]^T$
Mean elements	$\bar{\underline{a}}^* = [\bar{a},\bar{h},\bar{k},\bar{p},\bar{q},\bar{\lambda}]^T$
	$\bar{\underline{a}} = [\bar{a},\bar{h},\bar{k},\bar{p},\bar{q}]^T$
Osculating to mean transformation (the near identity transformation)	$\underline{a}^* = \bar{\underline{a}}^* + \varepsilon\underline{\eta}_1(\bar{\underline{a}},\bar{\lambda})$
Osculating VOP equations	$\frac{d\underline{a}^*}{dt} = \bar{n}\underline{\varepsilon}_6 + \varepsilon\,\underline{F}(\underline{a},\lambda)$
Mean VOP equations	$\frac{d\bar{\underline{a}}^*}{dt} = \bar{n}\underline{\varepsilon}_6 + \varepsilon\,\underline{A}_1(\bar{\underline{a}})$
Mean Element Rate	$\underline{A}_1(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi} \varepsilon\,\underline{F}(\bar{\underline{a}},\bar{\lambda})\,d\bar{\lambda}$
Short Periodics	$\varepsilon\underline{\eta}_1(\bar{\underline{a}},\bar{\lambda}) = \frac{1}{\bar{n}}\int[\varepsilon\underline{F}(\bar{\underline{a}},\bar{\lambda}) - \varepsilon\underline{A}_1(\bar{\underline{a}})]\,d\bar{\lambda}$
	$- \frac{3}{2\bar{a}}\int \varepsilon\eta_{11}(\bar{\underline{a}},\bar{\lambda})\,\underline{\varepsilon}_6\,d\bar{\lambda}$
Periodicity of short periodics	$\underline{\eta}_1(\bar{\underline{a}},\bar{\lambda}+2\pi) = \underline{\eta}_1(\bar{\underline{a}},\bar{\lambda})$
	$\int_0^{2\pi} \underline{\eta}_1(\bar{\underline{a}},\bar{\lambda})\,d\bar{\lambda} = 0$
Series Expansion of Short Periodics	
Assume	$\varepsilon\underline{F}(\bar{\underline{a}},\bar{\lambda}) = \sum_{\sigma=0}^{\infty} \varepsilon\underline{X}_\sigma(\bar{\underline{a}})\cos\sigma\bar{\lambda}$
	$+ \varepsilon\,\underline{Z}_\sigma(\bar{\underline{a}})\sin\sigma\bar{\lambda}$

* Extracted from Green [4], which contains a good derivation

where $$\epsilon \underline{X}_{\sigma}(\bar{\underline{a}}) = \frac{1}{2\pi}\int_0^{2\pi} \epsilon \underline{F}(\bar{\underline{a}},\bar{\lambda})\,d\bar{\lambda} = \epsilon \underline{A}_1(\bar{\underline{a}})$$

$$\epsilon \underline{X}_{\sigma}(\bar{\underline{a}}) = \frac{1}{\pi}\int_0^{2\pi} \epsilon \underline{F}(\bar{\underline{a}},\bar{\lambda}) \cos \sigma\bar{\lambda}\; d\bar{\lambda}$$

$$\epsilon \underline{Z}_{\sigma}(\bar{\underline{a}}) = \frac{1}{\pi}\int_0^{2\pi} \epsilon \underline{F}(\bar{\underline{a}},\bar{\lambda}) \sin \sigma\bar{\lambda}\; d\bar{\lambda}$$

then $$\epsilon \underline{\eta}_1(\bar{\underline{a}},\bar{\lambda}) = \sum_{\sigma=1}^{\infty} \epsilon \underline{C}_{\sigma}(\bar{\underline{a}}) \sin \sigma\bar{\lambda} - \epsilon \underline{D}_{\sigma}(\bar{\underline{a}}) \cos \sigma\bar{\lambda}$$

where $$\epsilon \underline{C}_{\sigma}(\bar{\underline{a}}) = \frac{1}{\sigma\bar{n}}\, \epsilon \underline{X}_{\sigma}(\bar{\underline{a}}) + \frac{3}{2\sigma\bar{a}}\, \epsilon D_{1\sigma}(\bar{\underline{a}})\, \underline{\epsilon}_6$$

$$\epsilon \underline{D}_{\sigma}(\bar{\underline{a}}) = \frac{1}{\sigma\bar{n}}\, \epsilon\, \underline{Z}_{\sigma}(\bar{\underline{a}}) - \frac{3}{2\sigma\bar{a}}\, \epsilon C_{1\sigma}(\bar{\underline{a}})\, \underline{\epsilon}_6$$

Partials

Solve Vector $$\underline{x}^T = [\bar{\underline{a}}^{*T}\ \underline{c}^T]$$

$\underline{c}$ = parameter vector in force model

define partials $$\Phi(t,t_o) = \frac{\partial \bar{\underline{a}}^*(t)}{\partial \bar{\underline{a}}^*(t_o)} = B_2$$

$$\Psi(t,t_o) = \frac{\partial \bar{\underline{a}}^*(t)}{\partial \underline{c}} = B_3$$

State partials equation $$\frac{d}{dt}\Phi(t,t_o) = \left[\underline{\epsilon}_6 \frac{\partial \bar{n}}{\partial \bar{\underline{a}}^*} + \frac{\partial \epsilon \underline{A}_1}{\partial \bar{\underline{a}}^*}\right] \Phi(t,t_o)$$

Parameter Partials Equation $$\frac{d}{dt}\Psi(t,t_o) = \left[\underline{\epsilon}_6 \frac{\partial \bar{n}}{\partial \bar{\underline{a}}^*} + \frac{\partial \epsilon \underline{A}_1}{\partial \bar{\underline{a}}^*}\right] \Psi(t,t_o) + \frac{\partial \epsilon \underline{A}_1}{\partial \underline{c}}$$

Initial Conditions $$\Phi(t,t_o) = \underline{I},\ \Psi(t,t_o) = \underline{O}$$

Table II. Range and Range-Rate Satellite Observations

Orbital elements

Mean equinoctial elements $\underline{\bar{a}}^* = [\bar{a},\bar{h},\bar{k},\bar{p},\bar{q},\bar{\lambda}]^T$

Osculating elements $\underline{a}^* = [a,h,k,p,q,\lambda]^T$

Cartesian inertial element transformation

$$\begin{bmatrix} \underline{p} \\ \underline{v} \end{bmatrix} = T(\underline{a}^*)$$

Cartesian local tangent element transformation

$\underline{r}_s$ = radius to origin of fram on earth's surface

$$\underline{p}_{LT} = D\underline{p} - \underline{r}_s$$

$$\underline{v}_{LT} = D\underline{v} + \dot{D}\underline{p}$$

range observation $\rho = \sqrt{\underline{p}_{LT} \cdot \underline{p}_{LT}}$

range rate observation $\dot{\rho} = \frac{1}{\rho}\, \underline{p}_{LT} \cdot \underline{v}_{LT}$

Filter. These and other nonlinear filters are discussed in Gelb [8].

The Linearized Filter is the most basic adaptation. The a priori mean state $\bar{x}(t_o)$ is propagated forward in time to generate the nominal trajectory

$$\dot{\underline{x}}_N(t) = \underline{f}(\underline{x}_N) \quad ; \quad \underline{x}_N(t_o) = \bar{\underline{x}}_o(t_o) \qquad [3]$$

The plant and observation equations [2] are then linearized about this trajectory to obtain the linear problem

$$\begin{array}{lll} \text{estimate} & \Delta\hat{\underline{x}}(t) \text{ , given} & \\ \text{plant} & \Delta\dot{\underline{x}}(t) = F(t)\, \Delta\underline{x}(t) + \underline{w} \;\; ; & \Delta\underline{x}(t_o) = \Delta\underline{x}_o \\ \text{observation} & \Delta y_k = H(t_k)\, \Delta\underline{x}(t_k) + v & \\ & \Delta y_k = h(\underline{x}(t_k),t_k) - h(\underline{x}_N(t_k),t_k) & \end{array}$$

$$\text{where} \qquad F(t) = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}_N(t)} \qquad [4]$$

$$H(t_k) = \left.\frac{\partial h}{\partial \underline{x}}\right|_{\underline{x}_N(t_k),t_k}$$

The statistics of $\Delta\underline{x}_o$, $\underline{w}$, and v carry over from above.

A Kalman filter can solve the explicit problem [4] optimally, but here the implicit dependence on $x_N(t)$ makes the solution suboptimal. An Extended Kalman Filter is essentially a linearized filter that starts over, computing a new nominal trajectory, after every observation. Though an Extended Filter performs better than a Linearized Filter, since the nominal trajectory itself is corrected, the use of large step sizes and interpolators for efficiency in the semianalytical ephemeris propagator precludes its use here. Rather, a modification of the Linearized Filter will be used, as discussed below. The equations for a Linearized Kalman Filter are given in Table III.

Table III. Linearized Kalman Filter Equations

Estimation Problem

$\underline{x}(t)$ = state to be estimated

$y(t)$ = scalar observation of $\underline{x}(t)$

$\underline{w}(t)$ = white state process noise

$v(t)$ = white observation noise

plant $\quad \dot{\underline{x}}(t) = \underline{f}(\underline{x},t) + \underline{w} \quad ; \quad \underline{x}(t_o) = \underline{x}_o$

observations $\quad y(t) = h(\underline{x},t) + v \quad$ at times t_i

statistics $\quad E(\underline{w}) = \underline{0} \quad , \quad E(\underline{w}(t)\ \underline{w}^T(\tau)) = Q\delta(t-\tau)$,

$E(v) = 0 \quad , \quad E(v(t)\ v(\tau)) = r\delta(t-\tau)$,

$E(\underline{x}_o) = \overline{\underline{x}}_o \quad , \quad E(\underline{x}_o\ \underline{x}_o^T) = \underline{P}_o$

$\underline{x}_o$, $\underline{w}$, v are uncorrelated.

Linearized Kalman Filter Solution

nominal trajectory $\quad \dot{\underline{x}}_N(t) = \underline{f}(\underline{x}_N,t) \ ; \ \underline{x}_N(t_o) = \overline{\underline{x}}_o$

prediction of estimate and covariance

transition matrix

$$\dot{\Phi}(t,t_{i-1}) = \left[\frac{\partial \underline{f}}{\partial \underline{x}_N}(\underline{x}_N,t)\right] \Phi(t,t_{i-1}) \ ;$$

$$\Phi(t_{i-1},t_{i-1}) = I$$

state prediction

$$\Delta\hat{\underline{x}}(t_i) = \Phi(t_i,t_{i-1})\ \Delta\hat{\underline{x}}(t_{i-1}) \ ;$$

$$\Delta\hat{\underline{x}}(t_{i-1}) = \Delta\tilde{\underline{x}}(t_{i-1})$$

covariance prediction

$$\hat{P}(t_i) = \Phi(t_i,t_{i-1})\,\hat{P}(t_{i-1})\Phi^T(t_i,t_{i-1}) + \Lambda(t_i,t_{i-1})$$

$$\hat{P}(t_{i-1}) = \tilde{P}(t_{i-1})$$

$$\Lambda(t_i,t_{i-1}) = \int_{t_{i-1}}^{t_i} \Phi(t_i,\tau)Q(\tau)\Phi^T(t_i,\tau)\,d\tau$$

Update of estimate and covariance

observation partial $\quad H_i = \dfrac{\partial h}{\partial \underline{x}_N}(\underline{x}_N, t_i)$

Kalman gain $\quad \underline{K}_i = \dfrac{\hat{P}(t_i)H_i^T}{H_i\hat{P}(t_i)H_i^T + r}$

observation $\quad \Delta y(t_i) = y(t_i) - h(\underline{x}_N, t_i)$

state update $\quad \Delta\tilde{\underline{x}}(t_i) = \Delta\hat{\underline{x}}(t_i) + \underline{K}_i\,[\Delta y(t_i) = H_i\Delta\hat{\underline{x}}(t_i)]$

covariance update $\quad \tilde{P}(t_i) = (I - \underline{K}_i H_i)\,\hat{P}(t_i)$

initialization

$$\Delta\tilde{\underline{x}}(t_o) = 0$$

$$\tilde{P}(t_o) = \underline{P}_o$$

4. SEMIANALYTICAL KALMAN FILTER DESIGN

The Kalman Filter equations as given in Table III usually allow the means of propagating the nominal trajectory and the transition matrices to be arbitrary, since the filter only requires the values at observation times. However, when optimizing the computations for efficiency, the structures of the integrator and the filter may become intertwined to produce a more efficient result. This is the case for a Semianalytical Kalman Filter, where the use of interpolators for the state, the transition matrices, and the short periodic coefficients has definite implications for the overall filter design.

The Linearized Kalman Filter uses observations over time to improve the estimate of a satellite's orbit. Typically the observation times are not known in advance, so the underlying ephemeris generator must be able to efficiently generate the values of the state and the transition matrices at essentially arbitrary times and arbitrarily frequently. This requirement does not decrease the efficiency of high precision numerical integrators (such as Adams-Cowell, etc.), since they are constrained to small step sizes anyway and automatically generate the required values at many points in time. Analytical and Semianalytical integrators, on the other hand, use very large step sizes, generating the required state and transition matrices at only a few points in time. Such integrators use interpolators to obtain the values at intermediate points in time. The contribution at CSDL has been to develop an interpolation method that retains the efficiency of analytical integrators and also gives values with the accuracies of numerical integrators.

In the optimization of the Semianalytical Kalman Filter for efficiency, the semianalytical integrator and the Kalman Filter each place requirements on the other.

The use of interpolators by the integrator over the integration grid dictates the use of a Linearized Kalman filter inside the integration grid, although the solve vector can be updated after processing all the observations in that grid.

The filter, on the other hand, requires the transition matrices $\Phi(t_i,t_{i-1})$, $\Psi(t_i,t_{i-1})$ between adjacent observation times t_{i-1} and t_i. The integrator can most readily supply the transition matrices from the beginning of the integration grid, $\Phi(t_i,t_o)$, $\Psi(t_i,t_o)$. By using the equations

$$
\begin{aligned}
\Phi(t_i,t_{i-1}) &= \Phi(t_i,t_o)\ \Phi(t_o,t_{i-1}) \\
\Phi(t_o,t_{i-1}) &= \Phi^{-1}(t_{i-1},t_o) \\
\Psi(t_i,t_{i-1}) &= \Psi(t_i,t_o) - \Phi(t_i,t_{i-1})\ \Psi(t_{i-1},t_o)
\end{aligned}
\qquad [5]
$$

we can restate the filter's requirement as supplying $\Phi(t_i, t_o)$, $\Psi(t_i, t_o)$, and $\Phi^{-1}(t_i, t_o)$. While $\Phi^{-1}(t_i, t_o)$ could be calculated directly from $\Phi(t_i, t_o)$, the expense of computing matrix inverses motivates another solution. $\Phi(t_i, t_o)$ is calculated from a Hermite interpolator using integration grid values and rates. Since the rate of $\Phi^{-1}(t, t_o)$ can be calculated as

$$\dot{\Phi}^{-1}(t,t_o) = -\Phi^{-1}(t,t_o)\,\dot{\Phi}(t,t_o)\,\Phi^{-1}(t,t_o) \qquad [6]$$

a similar Hermite interpolator can be constructed for $\Phi^{-1}(t_i, t_o)$. This interpolator is included in the semianalytical integrator.

The last requirement of the filter on the integrator is the calculation of Λ, the contribution of the state process noise. Due to the difficulty in defining Q, the process noise strength, Λ, will be calculated as linear in time

$$\Lambda = \Lambda(t_i - t_{i-1}) \qquad [7]$$

This follows the procedure already incorporated in GTDS [9] and appears to work quite well.

The implementation of the rest of the filter equations is straightforward and follows software already in the GTDS FILTER subroutines.

A procedural statement of the final algorithm for implementing this Semianalytical Kalman Filter design is given in Table IV.

5. CONCLUSIONS

An algorithm for implementing a Semianalytical Kalman Filter has been presented. Its implementation is currently being completed. Results will be presented at the conference.

Table IV. The Semianalytical Kalman Filter Algorithm

Due to use of a Runge kutta integrator, we may consider only one integration grid step; all others are processed identically.

Operations on the Integration Grid

1. $t = t_o$ update $\tilde{\underline{x}} = \hat{\underline{x}} + \Delta\hat{\underline{x}}$ $\underline{x} = \begin{bmatrix} \underline{\bar{a}}^* \\ \underline{c} \end{bmatrix}$

update $\tilde{P} = \hat{P}$

inititalize $\Delta\tilde{\underline{x}} = 0$

$\Phi(t_o, t_o) = I$

$\Psi(t_o, t_o) = 0$ save in Ψs

$\Phi^{-1}(t_o, t_o) = I$ save in Φs

2. $t = t_o + \Delta t$ do averaged integration

obtain $\hat{\underline{x}}(t)$, $\Phi(t, t_o)$, $\Psi(t, t_o)$

set up mean interpolators $\hat{\underline{x}}$, Φ, Ψ, Φ^{-1}

3. $t = t_o + \Delta t$ set up interpolators for short periodic

coefficients $\varepsilon \underline{C}_\sigma(\underline{\bar{a}})$, $\varepsilon \underline{D}_\sigma(\underline{\bar{a}})$

Operations on the Observation Grid

1. obtain the new observation, $y(t_i)$, at time $t = t_i$.

2. interpolate for $\hat{\underline{x}}(t_i)$, $\Phi(t_i,t_o)$ $\Psi(t_i,t_o)$

   we already have $\Phi^{-1}(t_o,t_{i-1})$ in Φs

3. interpolate for short periodic coefficients

$$\varepsilon\underline{C}_\sigma(\underline{\bar{a}}(t_i)) \;,\; \varepsilon\underline{D}_\sigma(\underline{\bar{a}}(t_i))$$

4. construct the osculating elements

$$\underline{a}^*(t_i) = \underline{\bar{a}}^*(t_i) + \sum_{\sigma=1}^{N} \varepsilon\underline{C}_\sigma(\underline{\bar{a}}) \sin\sigma\bar{\lambda} - \varepsilon\underline{D}_\sigma(\underline{\bar{a}}) \cos\sigma\bar{\lambda}$$

5. transform to cartesian elements and construct the nominal observation

$$h(\hat{\underline{x}}(t_i),t_i)$$

   the observation residual

$$\Delta y(t_i) = y(t_i) - h(\hat{\underline{x}}(t_i),t_i)$$

   and the observation partials

$$H_i = \frac{\partial h}{\partial \hat{\underline{x}}}(x_N,t_i) = \frac{\partial h}{\partial \underline{a}^*}\left[I + B_1 \,\vdots\, B_4\right]$$

$$B_1 = \frac{\partial \varepsilon\underline{\eta}_1(\underline{\bar{a}},\bar{\lambda})}{\partial \underline{\bar{a}}^*}$$

$$B_4 = \frac{\partial \varepsilon\underline{\eta}_1(\underline{\bar{a}},\bar{\lambda})}{\partial \underline{c}}$$

6. Compute the transition matrices

$$\Phi(t_i,t_{i-1}) = \Phi(t_i,t_o)\,\Phi^{-1}(t_{i-1},t_o) = \Phi(t_i,t_o)\,\Phi s$$

   using $\Phi_s = \Phi^{-1}(t_{i-1},t_o)$, and $\Psi_s = \Psi(t_{i-1},t_o)$

$$\Psi(t_i,t_{i-1}) = \Psi(t_i,t_o) - \Phi(t_i,t_{i-1})\,\Psi_s$$

7. Obtain predicted solve vector and covariance

$$\hat{\Delta x}(t_i) = \Phi(t_i,t_{i-1})\ \tilde{\Delta x}(t_{i-1})$$

$$\hat{P}(t_i) = \begin{bmatrix} \Phi(t_i,t_{i-1}) & \Psi(t_i,t_{i-1}) \\ 0 & I \end{bmatrix} \tilde{P}(t_{i-1}) \begin{bmatrix} \Phi(t_i,t_{i-1}) & \Psi(t_i,t_{i-1}) \\ 0 & I \end{bmatrix}$$

$$+ \Lambda(t_i,t_{i-1})$$

$$\Lambda(t_i,t_{i-1}) = \dot{\Lambda} \cdot (t_i - t_{i-1})$$

8. Complete the update phase of the filter.

Calculate the gain $K_i = \dfrac{\tilde{P}(t_i)\ H_i^T}{(H_i\ \hat{P}(t_i)H_i^T + R)}$

update the state $\tilde{\Delta \underline{x}}(t_i) = \hat{\Delta \underline{x}}(t_i) + K_i(\Delta y(t_i) - H_i\ \hat{\Delta \underline{x}}(t_i))$

update the covariance $\tilde{P}(t_i) = (I - KH)\hat{P}(t_i)$

9. Save transition matrices for next observation

$\Phi_s = \Phi^{-1}(t_i,t_o)$ interpolated

$\Psi_s = \Psi(t_i,t_o)$ interpolated in 2

Go to step 1.

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