NASA Contractor Report 165639

NASA-CR-165639
19810002913

STRESS SINGULARITIES AT THE VERTEX OF A
CYLINDRICALLY ANISOTROPIC WEDGE

F. Delale, F. Erdogan, and H. Boduroglu

LEHIGH UNIVERSITY
Bethlehem, Pennsylvania 18015

NASA Grant NGR 39-007-011
August 1980

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

STRESS SINGULARITIES AT THE VERTEX OF A CYLINDRICALLY ANISOTROPIC WEDGE (*)

by

F. Delale, F. Erdogan and H. Boduroglu(**)
Lehigh University, Bethlehem, Pennsylvania, 18015

Abstract

The plane elasticity problem for a cylindrically anisotropic solid is formulated. The form of the solution for an infinite wedge-shaped domain with various homogeneous boundary conditions is derived and the nature of the stress singularity at the vertex of the wedge is studied. The characteristic equations giving the stress singularity and the angular distribution of the stresses around the vertex of the wedge are obtained for three standard homogeneous boundary conditions. The numerical examples show that the singular behavior of the stresses around the vertex of an anisotropic wedge may be significantly different from that of the isotropic material. Some of the results which may be of practical importance are that for a half plane the stress state at r=0 may be singular and for a crack the power of stress singularity may be greater or less than 1/2.

1. Introduction

The singular behavior of the stress state in an isotropic wedge-shaped domain under in-plane loading conditions was studied in [1] for various homogeneous boundary conditions. The similar problem for the bending of a wedge-shaped isotropic plate was considered in [2]. The problem does not appear to have been studied for orthotropic materials. Physically the wedge problem for an orthotropic material with rectangular axes of orthotropy is, perhaps, not very practical. However, in a group of materials known as the cylindrically anisotropic solids, the problem not only is practical but could also be important. These are

(*) This work was supported by NASA Langley Research Center under the Grant NGR 39-007-011 and by NSF under the Grant ENG 78-09737.

(**) Present Address: Istanbul Technical University, Istanbul, Turkey.

materials such as wood which have an axisymmetric structure. Furthermore, since such structured materials could be prone to brittle or quasi-brittle fracture, the problem of stress singularities needs to be studied. In this paper we consider an infinite cylindrically anisotropic wedge-shaped domain having various combinations of homogeneous boundary conditions. It is assumed that in the solid under in-plane loading, the conditions of generalized plane stress or plane strain are satisfied and the vertex of the wedge coincides with the center of the polar axes of orthotropy.

2. Formulation in Polar Coordinates

Consider a cylindrically anisotropic wedge-shaped domain with the axes of orthotropy r, θ, and z (see the insert in Figure 1). In terms of the engineering elastic constants, the stress-strain relations may be written as

$$\varepsilon_r = \frac{1}{E_r}(\sigma_r - \nu_{r\theta}\,\sigma_\theta - \nu_{rz}\,\sigma_z) ,$$

$$\varepsilon_\theta = \frac{1}{E_{\theta r}}(-\nu_{\theta r}\sigma_r + \sigma_\theta - \nu_{\theta z}\,\sigma_z),$$

$$\varepsilon_z = \frac{1}{E_z}(-\nu_{zr}\sigma_r - \nu_{z\theta}\,\sigma_\theta + \sigma_z) , \qquad \text{(1a-c)}$$

$$\varepsilon_{r\theta} = \tau_{r\theta}/2G_{r\theta} , \ \varepsilon_{\theta z} = \tau_{\theta z}/2G_{\theta z} , \ \varepsilon_{zr} = \tau_{zr}/2G_{zr} , \qquad \text{(2a-c)}$$

From symmetry it follows that

$$\frac{\nu_{r\theta}}{E_r} = \frac{\nu_{\theta r}}{E_\theta} , \ \frac{\nu_{rz}}{E_r} = \frac{\nu_{zr}}{E_z} , \ \frac{\nu_{\theta z}}{E_\theta} = \frac{\nu_{z\theta}}{E_z} . \qquad \text{(3a-c)}$$

In the case of generalized plane stress $\sigma_z = \tau_{\theta z} = \tau_{rz} = 0$, and following [3], if we define the effective modulus E, the effective Poisson's ratio ν, the modulus ratio δ^4 and the shear parameter κ by

$$E = \sqrt{E_r E_\theta} , \ \nu = \sqrt{\nu_{r\theta}\,\nu_{\theta r}} , \ \delta^4 = E_r/E_\theta ,$$

$$\kappa = \frac{E}{2}\left(\frac{1}{G_{r\theta}} - \frac{\nu_{r\theta}}{E_r} - \frac{\nu_{\theta r}}{E_\theta}\right) , \qquad \text{(4a-d)}$$

the stress-strain relations become

$$\varepsilon_r = \frac{1}{E}\left(\frac{\sigma_r}{\delta^2} - \nu\sigma_\theta\right), \quad \varepsilon_\theta = \frac{1}{E}\left(-\nu\sigma_r + \delta^2\sigma_\theta\right), \quad \varepsilon_{r\theta} = \frac{\kappa+\nu}{E}\,\tau_{r\theta} . \qquad \text{(5a-c)}$$

In the plane strain case $\varepsilon_z = \varepsilon_{\theta z} = \varepsilon_{rz} = 0$, $\sigma_z = \nu_{zr}\sigma_r + \nu_{z\theta}\,\sigma_\theta$ and it may easily be shown that equations (5) are still valid provided the definitions given by (4) are modified as follows:

$$E = \left[\frac{E_r E_\theta}{(1-\nu_{rz}\nu_{zr})(1-\nu_{\theta z}\nu_{z\theta})}\right]^{\frac{1}{2}} , \quad \delta^4 = \frac{E_r(1-\nu_{\theta z}\nu_{z\theta})}{E_\theta(1-\nu_{rz}\nu_{zr})} ,$$

$$\nu = \left[\frac{(\nu_{r\theta}+\nu_{rz}\nu_{z\theta})(\nu_{\theta r}+\nu_{\theta z}\nu_{zr})}{(1-\nu_{rz}\nu_{zr})(1-\nu_{\theta z}\nu_{z\theta})}\right]^{\frac{1}{2}} ,$$

$$\kappa = \frac{1}{2}\left[\frac{E_r E_\theta}{(1-\nu_{rz}\nu_{zr})(1-\nu_{\theta z}\nu_{z\theta})}\right]^{\frac{1}{2}} \left(\frac{1}{G_{r\theta}} - \frac{\nu_{r\theta}+\nu_{rz}\nu_{z\theta}}{E_r} - \frac{\nu_{\theta r}+\nu_{\theta z}\nu_{zr}}{E_\theta}\right) .$$

(6a-d)

For the plane problems, expressed in polar coordinates, the equations of equilibrium and the condition of compatibility may be written as

$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\tau_{r\theta}}{r\partial\theta} + \frac{\sigma_r-\sigma_\theta}{r} = 0 ,$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{\partial\sigma_\theta}{r\partial\theta} + \frac{2\tau_{r\theta}}{r} = 0 , \qquad \text{(7a,b)}$$

$$\frac{\partial^2\varepsilon_\theta}{\partial r^2} + \frac{\partial^2\varepsilon_r}{r^2\partial\theta^2} + \frac{2\partial\varepsilon_\theta}{r\partial r} - \frac{\partial\varepsilon_r}{r\partial r} = \frac{2\partial^2\varepsilon_{r\theta}}{r\partial r\partial\theta} + \frac{2\partial\varepsilon_{r\theta}}{r^2\partial\theta} \qquad \text{(8)}$$

Substituting from (5) into (8) the compatibility condition becomes

$$-(\delta^{-2}+2\nu)\,\frac{\partial\sigma_r}{r\partial r} + (2\delta^2+\nu)\,\frac{\partial\sigma_\theta}{r\partial r} - \nu\,\frac{\partial^2\sigma_r}{\partial r^2} + \delta^{-2}\,\frac{\partial^2\sigma_r}{r^2\partial\theta^2} + \delta^2\,\frac{\partial^2\sigma_\theta}{\partial r^2}$$

$$-\nu\,\frac{\partial^2\sigma_\theta}{r^2\partial\theta^2} = 2(\kappa+\nu)\left(\frac{\partial^2\tau_{r\theta}}{r\partial r\partial\theta} + \frac{\partial\tau_{r\theta}}{r^2\partial\theta}\right) . \qquad \text{(9)}$$

If we now introduce the stress function $\phi(r,\theta)$ defined by

$$\sigma_r = \frac{\partial\phi}{r\partial r} + \frac{\partial^2\phi}{r^2\partial\theta^2} , \quad \sigma_\theta = \frac{\partial^2\phi}{\partial r^2} , \quad \tau_{r\theta} = -\frac{\partial^2}{\partial r\partial\theta} \left(\frac{\phi}{r}\right) , \tag{10a-c}$$

it may be seen that the equilibrium equations (7) are satisfied and substituting into (9) we obtain

$$\delta^2 \frac{\partial^4\phi}{\partial r^4} + \frac{2\kappa}{r^2} \frac{\partial^4\phi}{\partial r^2\partial\theta^2} + \frac{1}{\delta^2 r^4} \frac{\partial^4\phi}{\partial\theta^4} + \frac{2\delta^2}{r} \frac{\partial^3\phi}{\partial r^3} - \frac{2\kappa}{r^3} \frac{\partial^3\phi}{\partial r\partial\theta^2}$$

$$- \frac{1}{\delta^2 r^2} \frac{\partial^2\phi}{\partial r^2} + \frac{2(\kappa+\delta^{-2})}{r^4} \frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{\delta^2 r^3} \frac{\partial\phi}{\partial r} = 0 \quad . \tag{11}$$

Equation (11) is the governing equation for a cylindrically anisotropic solid in polar coordinates under plane loading conditions corresponding to the biharmonic equation for isotropic materials.

As in [1] and [2] we will now assume the solution to be of the following form:

$$\phi(r,\theta) = r^\lambda F(\theta,\lambda) \quad . \tag{12}$$

If the boundary conditions are selected to be homogeneous, the discrete set obtained for λ and F may be considered as the eigenvalues and the eigenfunctions of the problem. Substituting from (12) into (11) we find

$$\frac{d^4F}{d\theta^4} + a \frac{d^2F}{d\theta^2} + b\, F = 0 , \tag{13}$$

where

$$a = 2 + 2\,\kappa\, \delta^2(\lambda-1)^2 ,$$

$$b = \delta^4\lambda(\lambda-1)^2(\lambda-2) - \lambda(\lambda-2). \tag{14a,b}$$

The general solution of (13) may be expressed as

$$F = Ae^{\alpha_1\theta} + Be^{\alpha_2\theta} + Ce^{-\alpha_1\theta} + De^{-\alpha_2\theta} \tag{15}$$

where

$$\alpha_1 = [\tfrac{1}{2}(-a+\sqrt{a^2-4b})]^{\frac{1}{2}}, \quad \alpha_2 = [\tfrac{1}{2}(-a-\sqrt{a^2-4b})]^{\frac{1}{2}}. \qquad (16a,b)$$

The coefficients α_1 and α_2 are, in general, complex and the boundary conditions at $\theta=0$ and $\theta=\beta$ account for the unknown constants A, B, C, and D, where β is the total wedge angle (Figure 1).

In order to apply the boundary conditions the stresses and the displacements must be expressed in terms of the function F. From (10), (12) and (15), the stress components are found to be

$$\sigma_\theta = \lambda(\lambda-1)r^{\lambda-2}(Ae^{\alpha_1\theta} + Be^{\alpha_2\theta} + Ce^{-\alpha_1\theta} + De^{-\alpha_2\theta}),$$

$$\tau_{r\theta} = -(\lambda-1)r^{\lambda-2}(\alpha_1 Ae^{\alpha_1\theta} + \alpha_2 Be^{\alpha_2\theta} - \alpha_1 Ce^{-\alpha_1\theta} - \alpha_2 De^{-\alpha_2\theta}),$$

$$\sigma_r = r^{\lambda-2}[A(\lambda+\alpha_1^2)e^{\alpha_1\theta} + B(\lambda+\alpha_2^2)e^{\alpha_2\theta} + C(\lambda+\alpha_1^2)e^{-\alpha_1\theta}$$

$$+ D(\lambda+\alpha_2^2)e^{-\alpha_2\theta}], \qquad (17a\text{-}c)$$

The expressions for the displacements may be obtained indirectly by referring to the strain-displacement and the stress-strain relations. First we note that from (5), (10) and (12) the strains may be obtained as

$$\varepsilon_r = \frac{1}{E\delta^2} r^{\lambda-2}(\lambda F + \frac{d^2F}{d\theta^2}) - \frac{\nu}{E} r^{\lambda-2}\lambda(\lambda-1)F,$$

$$\varepsilon_\theta = -\frac{\nu}{E} r^{\lambda-2}(\lambda F + \frac{d^2F}{d\theta^2}) + \frac{\delta^2}{E} r^{\lambda-2}\lambda(\lambda-1)F,$$

$$\varepsilon_{r\theta} = -\frac{\kappa+\nu}{E} r^{\lambda-2}(\lambda-1)\frac{dF}{d\theta}. \qquad (18a\text{-}c)$$

Next, we observe that in polar coordinates in terms of the displacements u_r and u_θ, the strains are given by

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_{r\theta} = \frac{1}{2}(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}). \qquad (19a\text{-}c)$$

Now assuming that the expressions for the displacements are of the following form

$$u_r = r^m G(\theta,m) \ , \ u_\theta = r^m H(\theta,m) \ , \qquad (20a,b)$$

from (19) it follows that

$$\varepsilon_r = m\, r^{m-1} G \ , \ \varepsilon_\theta = r^{m-1}\left(G + \frac{dH}{d\theta}\right) \ ,$$

$$\varepsilon_{r\theta} = \frac{1}{2} r^{m-1}\left(mH - H + \frac{dG}{d\theta}\right) \ . \qquad (21a\text{-}c)$$

Comparing (18) and (21) we find the relations to determine the coefficient m and the functions G and H as follows:

$$m\, r^{m-1} G = r^{\lambda-2}\left[\frac{1}{E\delta^2}\left(\lambda F + \frac{d^2F}{d\theta^2}\right) - \frac{\nu}{E}\lambda(\lambda-1)F\right],$$

$$r^{m-1}\left(G + \frac{dH}{d\theta}\right) = r^{\lambda-2}\left[-\frac{\nu}{E}\left(\lambda F + \frac{d^2F}{d\theta^2}\right) + \frac{\delta^2}{E}\lambda(\lambda-1)F\right],$$

$$r^{m-1}\left[(m-1)H + \frac{dG}{d\theta}\right] = -\frac{2(\kappa+\nu)}{E} r^{\lambda-2} (\lambda-1)\frac{dF}{d\theta} \ . \qquad (22a\text{-}c)$$

On dimensional grounds from (22) it is clear that

$$m = \lambda - 1 \ . \qquad (23)$$

Also, imposing the physical condition that in the finite region $0 < \theta < \beta$, $0<r<r_0<\infty$ the strain energy be bounded, it may easily be shown that the real part of m must be positive, or

$$\mathrm{Re}(\lambda) > 1 \ . \qquad (24)$$

Eliminating the terms involving r (22) now gives three equations to determine the two unknown functions G and H (in terms of F and λ). First, by using (15) from (22a) we find

$$G(\theta,\lambda) = \frac{1}{E\delta^2(\lambda-1)}\left(A\alpha_1^2 e^{\alpha_1\theta} + B\alpha_2^2 e^{\alpha_2\theta} + C\alpha_1^2 e^{-\alpha_1\theta} + D\alpha_2^2 e^{-\alpha_2\theta}\right)$$

$$+ \frac{\lambda}{E}\left(\frac{1}{\delta^2(\lambda-1)} - \nu\right)\left(Ae^{\alpha_1\theta} + Be^{\alpha_2\theta} + Ce^{-\alpha_1\theta} + De^{-\alpha_2\theta}\right). \qquad (25)$$

Next, from (22c), (15) and (25) it follows that

$$H(\theta,\lambda) = -\frac{1}{E\delta^2(\lambda-1)(\lambda-2)}(A\alpha_1^3 e^{\alpha_1\theta} + B\alpha_2^3 e^{\alpha_2\theta} - C\alpha_1^3 e^{-\alpha_1\theta} - D\alpha_2^3 e^{-\alpha_2\theta})$$

$$-\frac{1}{E(\lambda-2)}\left[\frac{\lambda}{\delta^2(\lambda-1)} - \lambda\nu + 2(\kappa+\nu)(\lambda-1)\right](A\alpha_1 e^{\alpha_1\theta}$$

$$+ B\alpha_2 e^{\alpha_2\theta} - C\alpha_1 e^{-\alpha_1\theta} - D\alpha_2 e^{-\alpha_2\theta}). \qquad (26)$$

Substituting now from (25) and (26) it may easily be shown that (22b) is identically satisfied.

3. The Characteristic Equations for λ

(i) The case of stress-free boundaries.

Let the wedge shown in Figure 1 have the following homogeneous boundary conditions:

$$\sigma_\theta(r,0) = \tau_{r\theta}(r,0) = \sigma_\theta(r,\beta) = \tau_{r\theta}(r,\beta) = 0 . \qquad (27a\text{-}d)$$

Substituting from (17) into (27) we find

$$A + B + C + D = 0 ,$$

$$\alpha_1 A + \alpha_2 B - \alpha_1 C - \alpha_2 D = 0 ,$$

$$Ae^{\alpha_1\beta} + Be^{\alpha_2\beta} + Ce^{-\alpha_1\beta} + De^{-\alpha_2\beta} = 0$$

$$\alpha_1 Ae^{\alpha_1\beta} + \alpha_2 Be^{\alpha_2\beta} - \alpha_1 Ce^{-\alpha_1\beta} - \alpha_2 De^{-\alpha_2\beta} = 0. \qquad (28a\text{-}d)$$

The determinant of the coefficients $\Delta_1(\lambda)$ of the homogeneous algebraic equations (28) is the characteristic function of the problem. The characteristic equation for λ may then be obtained as

$$\Delta_1(\lambda) = -8\alpha_1\alpha_2 - (\alpha_1-\alpha_2)^2[e^{(\alpha_1+\alpha_2)\beta} + e^{-(\alpha_1+\alpha_2)\beta}]$$

$$+ (\alpha_1+\alpha_2)^2[e^{(\alpha_1-\alpha_2)\beta} + e^{-(\alpha_1-\alpha_2)\beta}] = 0. \qquad (29)$$

Let λ_j, $(j=1,2,\ldots)$ be the roots of (29) and B_j/A_j, C_j/A_j, and D_j/A_j, $(j=1,2,\ldots)$ the corresponding solution of (28) with A_j being an arbitrary nonzero constant. From (12) and (15) the eigenfunctions of the problem may then be found as follows:

$$\phi_j(r,\theta) = r^{\lambda_j}\left(e^{\alpha_{1j}\theta} + \frac{B_j}{A_j} e^{\alpha_{2j}\theta} + \frac{C_j}{A_j} e^{-\alpha_{1j}\theta} + \frac{D_j}{A_j} e^{-\alpha_{2j}\theta}\right) \quad (30)$$

where α_{1j} and α_{2j} are obtained from (16) and (14) by substituting $\lambda = \lambda_j$. Technically, the general solution of the problem may then be expressed as

$$\phi(r,\theta) = \sum_1^\infty A_j\phi_j(r,\theta). \quad (31)$$

(ii) The case of "clamped" boundaries.

Let now the displacements be prescribed along the boundaries of the wedge as follows:

$$u_r(r,0) = u_\theta(r,0) = u_r(r,\beta) = u_\theta(r,\beta) = 0 \; . \quad (32a\text{-}d)$$

Substituting from (20), (23), (25), and (26) into (32) we obtain a system of homogeneous algebraic equations for A, B, C, and D similar to (28). In this case, the characteristic equation is found to be

$$\Delta_2(\lambda) = -8pqst - (pt-qs)^2[e^{(\alpha_1+\alpha_2)\beta} + e^{-(\alpha_1+\alpha_2)\beta}] + (pt+qs)^2[e^{(\alpha_1-\alpha_2)\beta} + e^{-(\alpha_1-\alpha_2)\beta}] = 0 \quad (33)$$

where

$$p = \alpha_1^3 + \alpha_1[\lambda-\nu\lambda\delta^2(\lambda-1) + 2\delta^2(\kappa+\nu)(\lambda-1)^2] \; ,$$

$$q = \alpha_2^3 + \alpha_2[\lambda-\nu\lambda\delta^2(\lambda-1) + 2\delta^2(\kappa+\nu)(\lambda-1)^2] \; ,$$

$$s = \alpha_1^2 + \lambda - \nu\lambda\delta^2(\lambda-1) \; ,$$

$$t = \alpha_2^2 + \lambda - \nu\lambda\delta^2(\lambda-1) \; . \quad (34a\text{-}d)$$

(iii) Mixed boundary conditions.

In this case, we assume that

$$\sigma_\theta(r,0) = \tau_{r\theta}(r,0) = 0 , \tag{35a,b}$$

$$u_r(r,\beta) = u_\theta(r,\beta) = 0 . \tag{36a,b}$$

Following the procedure outlined above, the characteristic equation is obtained as

$$\Delta_3(\lambda) = -4\alpha_1 qt - 4\alpha_2 ps + (\alpha_1-\alpha_2)(qs-pt)\,[e^{(\alpha_1+\alpha_2)\beta} + e^{-(\alpha_1+\alpha_2)\beta}] + (\alpha_1+\alpha_2)(qs+pt)\,[e^{(\alpha_1-\alpha_2)\beta} + e^{-(\alpha_1-\alpha_2)\beta}] = 0 . \tag{37}$$

4. Angular Distribution of Stresses Near the Vertex

Consider, for example, the case of stress-free boundaries. Assuming $A \neq 0$ and solving (28) we find

$$\frac{B}{A} = \frac{1}{Z}\,[(\alpha_1-\alpha_2)e^{\alpha_1\beta} + (\alpha_1+\alpha_2)e^{-\alpha_1\beta} - 2\alpha_1 e^{-\alpha_2\beta}] ,$$

$$\frac{C}{A} = \frac{1}{Z}\,[2\alpha_2 e^{\alpha_1\beta} - (\alpha_1+\alpha_2)e^{\alpha_2\beta} + (\alpha_1-\alpha_2)e^{-\alpha_2\beta}] ,$$

$$\frac{D}{A} = \frac{1}{Z}\,[2\alpha_1 e^{\alpha_2\beta} - (\alpha_1+\alpha_2)e^{\alpha_1\beta} - (\alpha_1-\alpha_2)e^{-\alpha_1\beta}] ,$$

$$Z = -2\alpha_2 e^{-\alpha_1\beta} - (\alpha_1-\alpha_2)e^{\alpha_2\beta} + (\alpha_1+\alpha_2)e^{-\alpha_2\beta} . \tag{38a-d}$$

Substituting from (38) into (17) the stress components may be expressed in the following form:

$$\sigma_\theta = Ar^{\lambda-2}f(\theta) , \quad \tau_{r\theta} = Ar^{\lambda-2}g(\theta) , \quad \sigma_r = Ar^{\lambda-2}h(\theta) . \tag{39a-c}$$

For each characteristic value λ_j (38) and (17) give the corresponding functions f_j, g_j, and h_j.

5. Solution and Examples

As previously stated, the roots of the characteristic equations (29), (33) or (37) which are physically acceptable are those having a real part greater than 1 (see (24)). Thus, equations (17) or (39) indicate that if there is an acceptable root λ with $1 < \mathrm{Re}(\lambda) < 2$, for a given wedge angle β and material constants, then the stresses at the vertex of the wedge would be singular with the power $\alpha=2-\lambda$ (that is, $\sigma_{ij} \sim r^{-\alpha}$). Clearly, the characteristic root of primary interest is the acceptable root having the minimum real part.

From (14) and (16) it may be observed that the coefficients α_1 and α_2 depend on δ and κ only, that is, they are independent of ν as well as E. Consequently, as seen from (29), if only stresses are prescribed on the boundaries of the wedge, then λ would be independent of ν and E and the singular behavior of the stresses at the vertex of the wedge would depend on two dimensionless material constants (δ and κ) only. In fact, as seen from (17), the same statement is valid for the stress state throughout the entire wedge. However, if the boundary conditions involve the displacements as in the cases of clamped or fixed-free boundaries considered in Section 3, then λ as well as the stress state in the wedge would depend on ν in addition to δ and κ.

The properties of the materials used in the examples are given in Table 1. Material 1 is a mildly anisotropic material and corresponds to drawn titanium. Material 2 is strongly anisotropic and corresponds to graphite. Material 3 is a typical isotropic structural metal, Material 4 is yellow birch [4]. The table also gives the dimensionless constants δ, κ, and ν with subscripts σ and ε corresponding to plane stress and plane strain conditions, respectively.

The calculated results giving the acceptable characteristic value λ with the minimum real part are given in Figures 1-3. From these

Table 1. Material constants used in the examples.

Material	1	2	3	4	5
E_r(psi)	15.07×10^6	40×10^6		161,850	161,850
E_θ(psi)	20.8×10^6	1.5×10^6		103,750	103,750
$\nu_{r\theta}$	0.196	0.187		0.665	0.665
$\nu_{\theta r}$	0.271	0.007		0.426	0.426
$G_{r\theta}$(psi)	6.78×10^6	4×10^6		35,068	35,068
δ_σ	0.9226	2.2724	1	1.1176	1.1176
κ_σ	1.075	0.9321	1	1.3156	1.3156
ν_σ	0.2307	0.0361	0.3	0.5321	0.5321
E_z				2.075×10^6	100,000
ν_{zr}				0.426	0.426
$\nu_{z\theta}$				0.451	0.451
δ_ε				1.1187	1.1490
κ_ε				1.3196	1.4288
ν_ε				0.5508	1.0463

results it may be observed that:

- Depending on the material constants, for a given wedge angle β the power of the stress singularity $2-\lambda$ at the vertex of a cylindrically anisotropic wedge may be stronger or weaker than that corresponding to an isotropic wedge. In this respect the dominant material constant appears to be the modulus ratio $\delta = (E_r/E_\theta)^{\frac{1}{4}}$, $\delta > 1$ corresponding to singularities stronger, and $\delta < 1$ to weaker singularities than the isotropic solid.

- In the cases of wedges with stress-free or clamped boundaries (i.e., the cases (i) and (ii) discussed in Section 3) the stress state at the vertex of an isotropic wedge is bounded for any angle $\beta \leq \pi$ and is singular for $\beta > \pi$. However, as seen from Figures 1 and 2, this is not the case for the cylindrically anisotropic materials.

- A more important and perhaps somewhat surprising result concerns the case of the radial crack (i.e., $\beta = 2\pi$). In the isotropic and rectangularly orthotropic solids, the power of stress singularity α at the crack tip (in $\sigma_{ij} \sim r^{-\alpha}$) is always 1/2. However, as seen from the figures for $\beta = 2\pi$, α may have any value between 0 and 1.

- The leading power of singularity $\alpha = 2-\lambda$ is always real for stress-free and clamped-clamped boundary conditions.

- The trends of λ vs. β curves are generally similar for free-free and clamped-clamped cases (Figures 1 and 2). However, as it may be seen from Figure 2, the curves giving the roots of the characteristic equations may have more than one branch in the critical range $1 < \lambda < 2$ (Figures 2).

The results shown in Figures 1 and 2 are obtained for the generalized plane stress condition. The difference in λ obtained from plane stress and plane strain cases in birch (one of the woods for which all material constants were readily available) is found to be neglibible. Therefore, to establish the trend, the Material 5 is used as an example. Material 5 has the same constants as Material 4 except for E_z. As seen from Figure 3, unlike the isotropic materials, in cylindrically anisotropic solids there may be a difference between the singularities for wedges under plane strain and generalized plane stress conditions. However, in spite of the assumed exaggerated value for E_z (see the properties of Materials 4 and 5 in the table), Figure 3 shows that the difference may not be very significant.

Figure 4 shows an example for the "symmetric mode" of the angular distribution of stresses (in the neighborhood of r=0) in a half plane ($\beta=\pi$) with stress-free boundaries under plane stress conditions. The curves give the results calculated from (39) for Material 2 which is a strongly anisotropic solid. For comparison, the isotropic results calculated from the same program by letting $\delta=1$, $\kappa=1$ are shown in Figure 5. For the symmetric mode, the exact isotropic results are (see equations 39, and Figure 1 with $\beta = \pi$)

$$f(\theta) = \sin^2\theta \ , \ g(\theta) = -\frac{1}{2}\sin 2\theta \ , \ h(\theta) = \cos^2\theta \ . \tag{40}$$

The comparison of Figures 4 and 5 indicates that the differences between the two sets of results, particularly in $h(\theta)$, may be quite significant.

References

1. M.L. Williams, "Stress singularities resulting from various boundary conditions in angular corners of plates in extension", J. Appl. Mech., Vol. 19, Trans. ASME, pp. 526-528, 1952.
2. M.L. Williams, "Surface singularities resulting from various boundary conditions in angular corners of plates under bending", Proc. U.S. National Congress of Applied Mechanics, Illinois Institute of Technology, Chicago, IL. 1951.
3. S. Krenk, "On the Elastic Constants of Plane Orthotropic Elasticity", J. Composite Materials, Vol. 13 (April 1979), p. 108.
4. Frederick F. Wangaard, The Mechanical Properties of Wood, John Wiley & Sons, 1950.

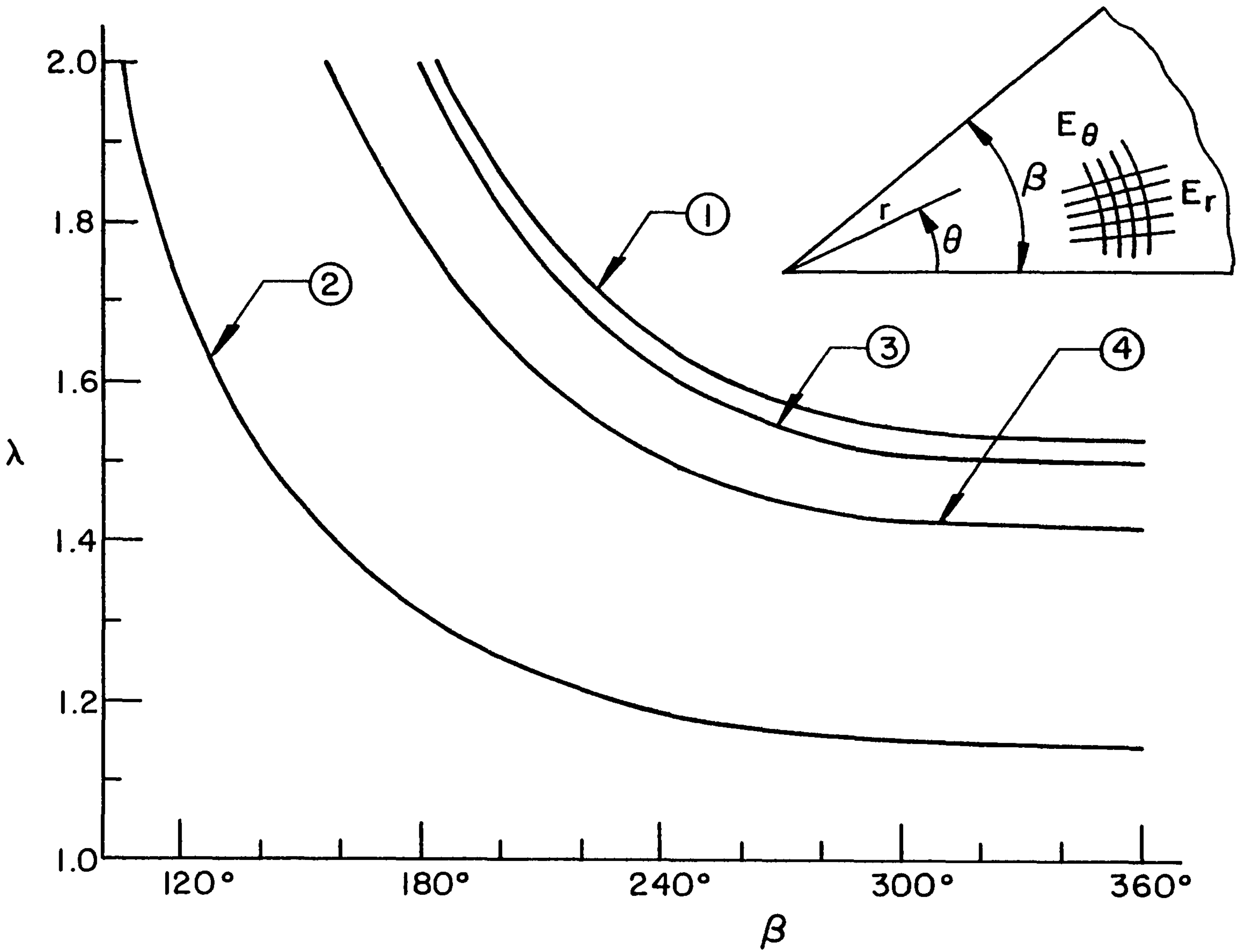

Figure 1. Minimum acceptable characteristic value λ in a cylindrically anisotropic wedge of angle β with stress-free boundaries under generalized plane stress conditions. The curves 1 to 4 correspond to the materials 1 to 4 given in Table 1.

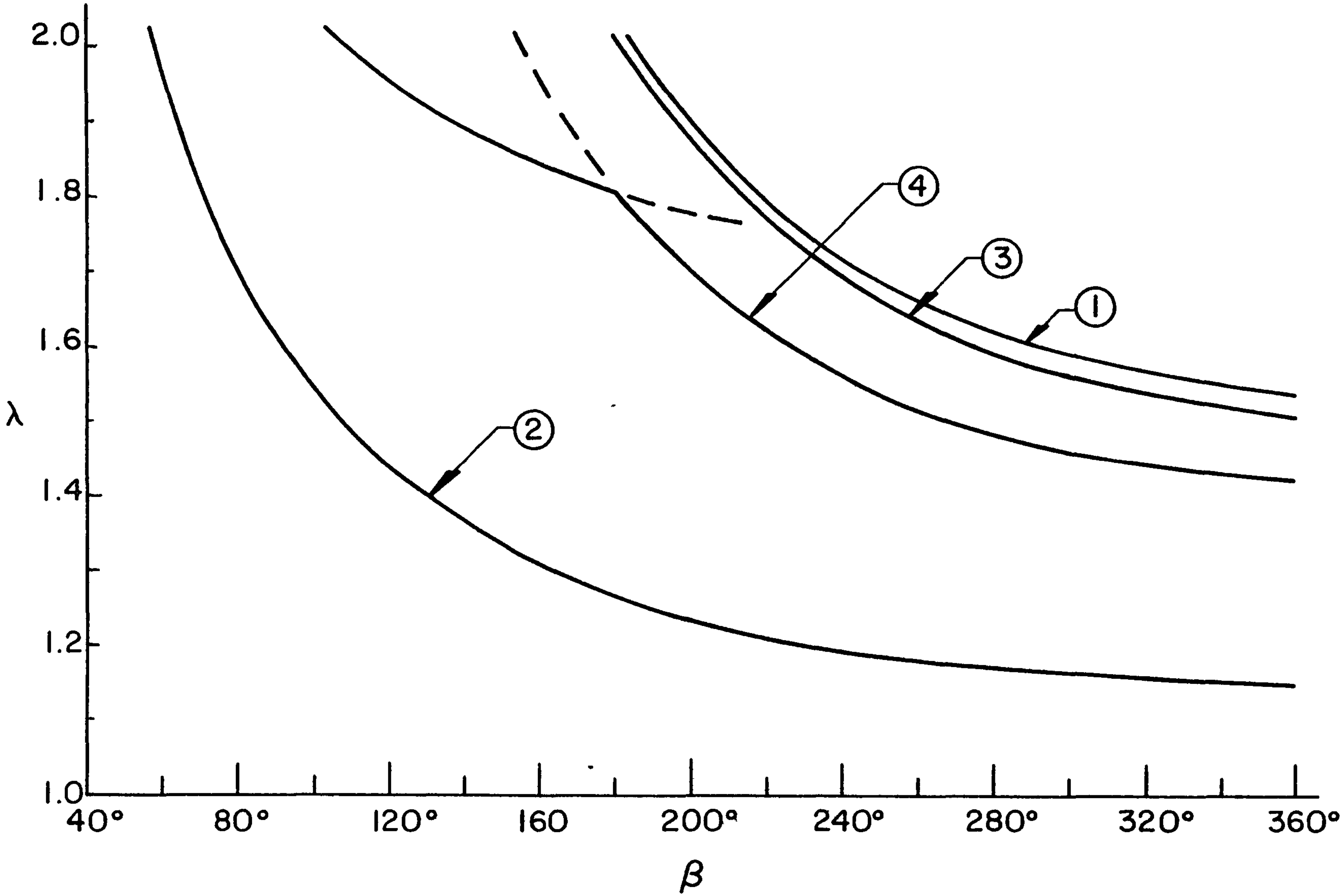

Figure 2. Same as Figure 1 with fixed displacements along the boundaries (the clamped-clamped case).

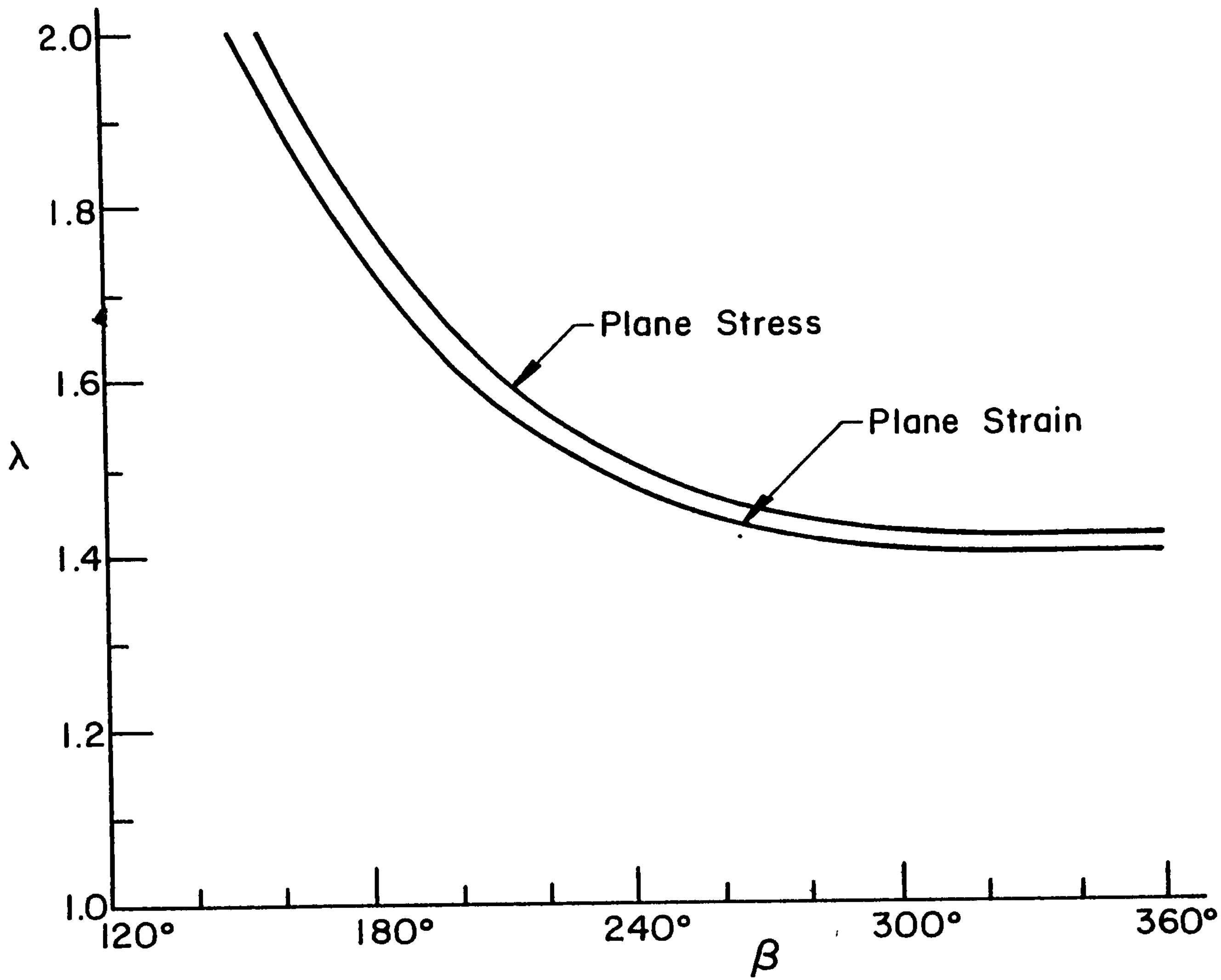

Figure 3. Comparison of the minimum acceptable characteristic values λ for Material 5 with stress-free boundaries under plane strain and generalized plane stress conditions.

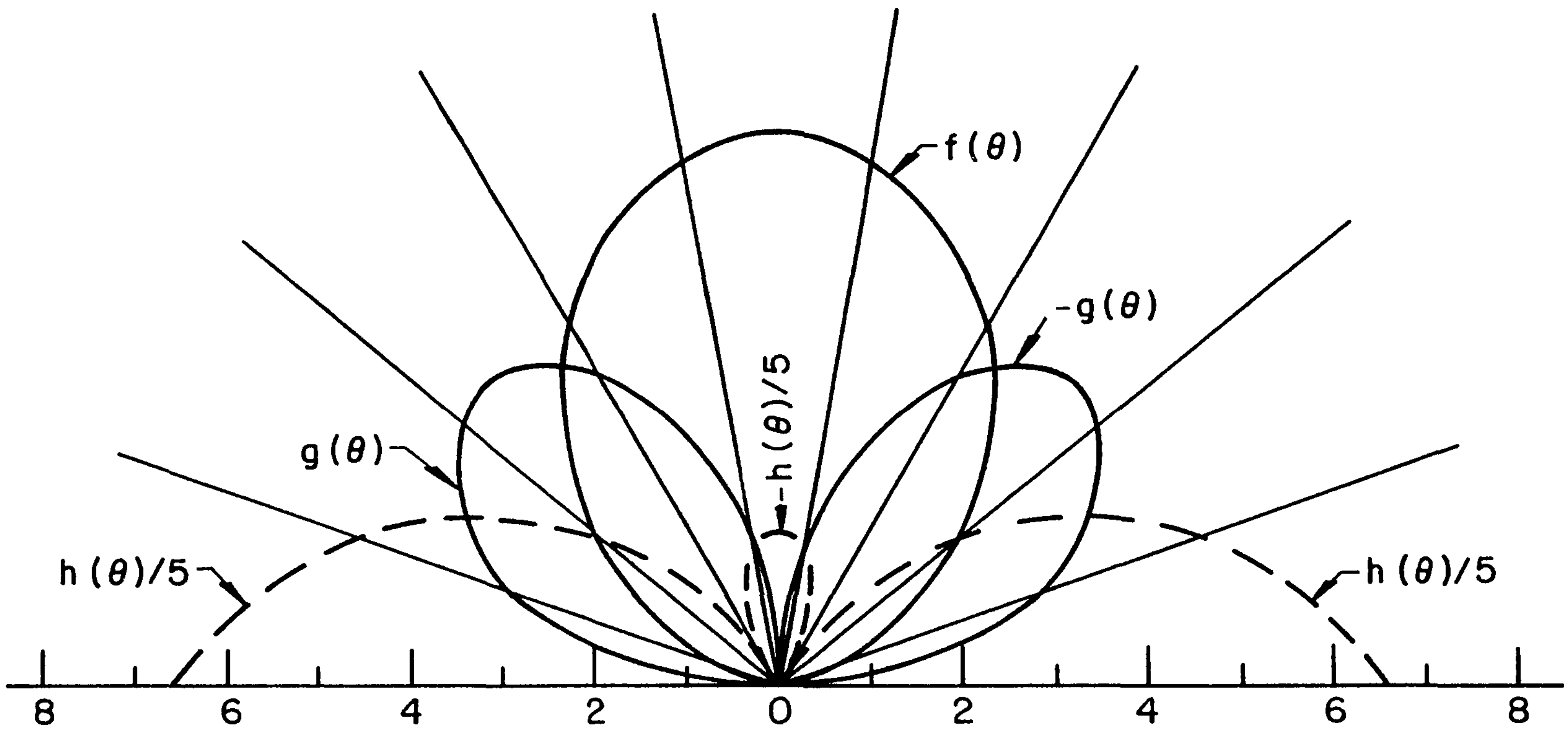

Figure 4. Angular distribution of the stresses for Material 2 with $\beta=\pi$ (half plane) under generalized plane stress condition ($f(\theta) \sim \sigma_\theta$, $g(\theta) \sim \tau_{r\theta}$, $h(\theta) \sim \sigma_r$).

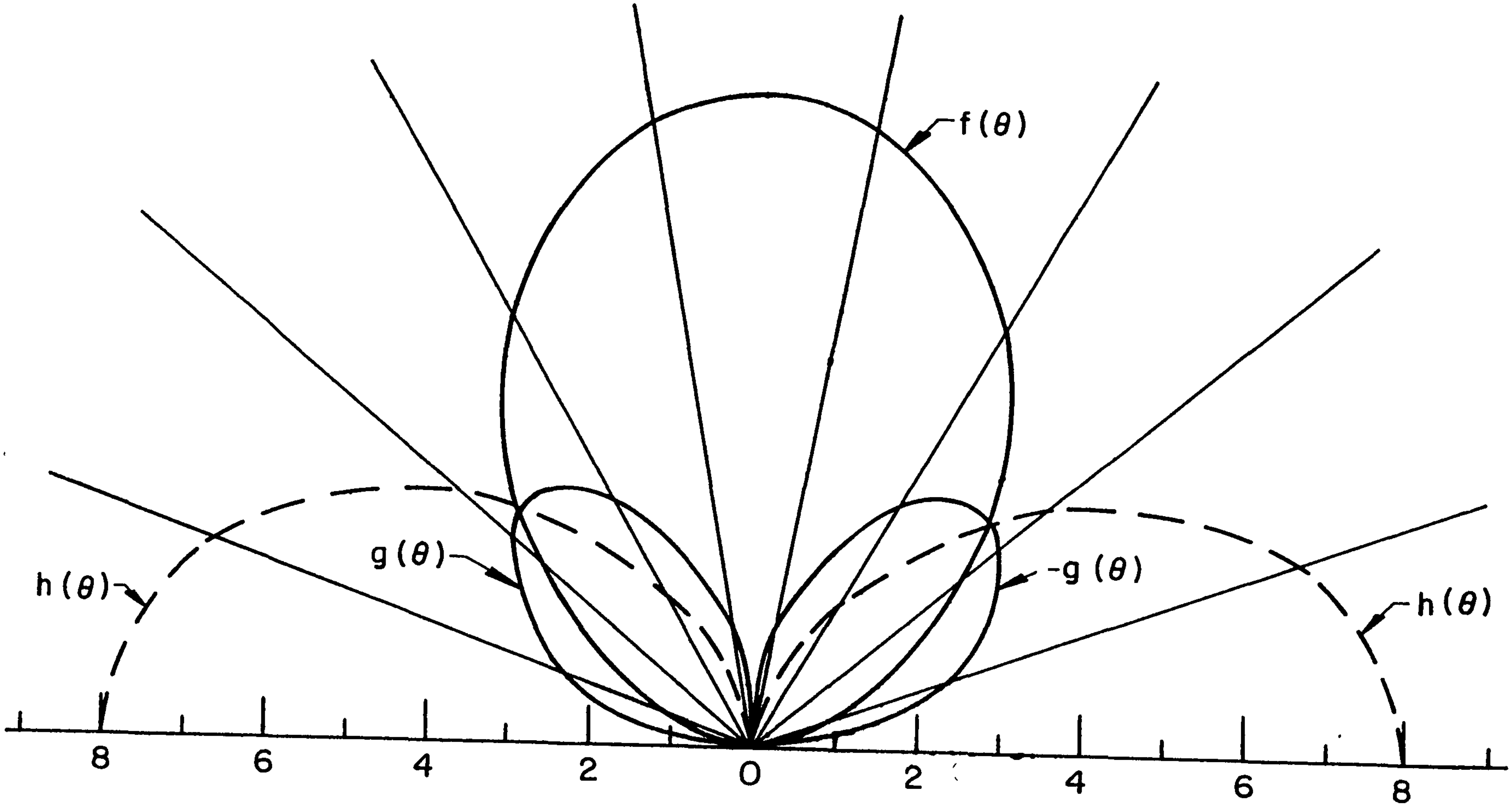

Figure 5. Same as Figure 4 for an isotropic material.

End of Document