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LIST OF SYMBOLS

P	projection operator from X onto $\tilde{X}$
X	Banach space of periodic, continuous functions on [0, 1]
$\tilde{X}$	piecewise linear subspace of X
$\bar{X}$	Banach space isomorphic to $\tilde{X}$
$\| \|$	norm in the appropriate space
Δ_i	ith interval of the partition
$\lvert\Delta_i\rvert$	length of partition Δ_i

A SOLUTION OF ONE-DIMENSIONAL FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

Ralph E. Gabrielsen

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INTRODUCTION

In the process of developing a rigorous numeric solution for the incompressible Navier-Stokes equations, the need has arisen for numerically solving Fredholm integral equations of the second kind in such a way that the complete relevance of the numbers obtained is known. Accordingly, this report addresses the problem for one dimension. The results are based upon the outstanding theory developed by Kantorovich. For one-dimensional Fredholm integral equations it will be proven that the particular construction converges to the true solution. For purpose of clarity, it should be pointed out that Kantorovich (ref. 1) presents a related example for a periodic one-dimensional Fredholm integral equation; however, it is wrong in the sense that his proof does not comply with the necessary conditions of the theory applied.

MAIN DEVELOPMENT FOR THE EQUATION

For the equation

$$x(s) - \lambda \int_0^1 h(s,t)x(t)dt = y(s) \tag{1}$$

with $h(s,t)$ continuous and periodic in both variables over the interval [0,1], and y also a continuous, periodic function over the interval [0,1], it will be demonstrated that an explicit system of equations of the form

$$x(t_j) - \lambda \sum_{k=1}^{n} A_k h(t_j,t_k)x(t_k) = y(t_j) \tag{2}$$

converges to the solution of (1). In addition, it will be shown how the results can be directly extended to the nonperiodic case.

Equation (1) will be considered as a functional equation in the space $X \equiv \tilde{C}$ of continuous, periodic functions on [0,1].

System (2) will be regarded as an approximate functional equation in the space $\bar{X} = R_n$,

$$\bar{K}\bar{x} \equiv \bar{x} - \lambda\bar{H}\bar{x} = \phi y$$

Also, equation (1) will typically be expressed in the form

$$Kx \equiv x - \lambda Hx = y$$

Now consider the subspace $\tilde{X}_n$ (or $\tilde{X}$) of X: Space of continuous functions on [0,1] that are linear on the partitions Δ_i where the T_i's (defined by $T_i = (i - 1/2)/n$, $i = 1, \ldots, n$) are the end points of the interval Δ_i.

Define the mapping ϕ_0 on $\tilde{X}_n$ onto $\bar{X}$ as follows: if $\tilde{x} \in \tilde{X}$, then $\phi_0\tilde{x} = \bar{x}$, where $\bar{x} = (\xi_i, \ldots, \xi_n)$, $\xi_i = \tilde{x}(T_i)$, $i = 1, \ldots, n$. Since ϕ_0 defines a 1:1 mapping between $\tilde{X}$ and R_n, ϕ_0^{-1} exists. Norms on $\tilde{X}$ and $\bar{X}$ are defined as follows:

if

$$\tilde{x} \in \tilde{X}, \quad \|\tilde{x}\| = \max_{0<t<1} |\tilde{x}(t)|$$

if

$$\bar{x} \in \bar{X} \equiv R_n, \quad \|\bar{x}\| = \max_{i=1\ldots n} |\xi_i|$$

where

$$\bar{x} = (\xi_1, \xi_2, \ldots, \xi_n)$$

Consequently,

$$\|\phi_0\| = \|\phi_0^{-1}\| = 1$$

Define the mapping ϕ as follows:

if

$$x \in \tilde{X}, \quad \phi x = \phi_0 x$$

if

$$x \in \tilde{X}, \quad \phi x = [x(t_1), x(t_2), \ldots, x(t_n)], \quad t_i = \frac{i - (1/2)}{n}$$

$$\therefore \|\phi\| = \sup_{\|x\|=1} \|\phi x\| = 1$$

Hence, it readily follows that ϕ is a linear extension of ϕ_0.

Let $\omega_s(\delta)$ denote the modulus of continuity of the function $h(s,t)$ relative to s:

$$\omega_s(\delta) = \sup|h(s + \sigma,t) - h(s,t)| \qquad (0 \le s,\ t \le 1,\ |\sigma| \le \delta)$$

Similarly, $\omega_t(\delta)$ is defined. Note, of course, that all parameters are restricted to the domain of consideration [0,1].

Now, with these preliminaries out of the way, we will quote a theorem of Kantorovich (ref. 1), which can now be directly applied to the problem in question. Consequently, most of what follows entails demonstrating that the conditions of the Kantorovich Theorem are satisfied. Before stating the theorem of Kantorovich, we will briefly discuss items necessary for the understanding of the Kantorovich Theorem.

We seek a solution of Kantorovich's first equation (K1)

$$Kx \equiv x - \lambda Hx = y$$

in the B. space X, and the approximate solution $\tilde{x}$ of $\tilde{K}\tilde{x} \equiv \tilde{x} - \lambda\tilde{H}\tilde{x} = P_y$ in $\tilde{X} \subset X$ where P is a projection operation from X onto $\tilde{X}$.

Let ϕ_0 define an isomorphism between $\tilde{X}$ and the space $\bar{X}$; now assume there exists a linear extension ϕ of the operation ϕ_0 to the space X.

Let $\phi = \phi_0 P$, P the projection operation from X onto $\tilde{X}$. Therefore

$$P = \phi_0^{-1}\phi$$

Hence, in (K2)

$$\phi_0(\tilde{x} - \lambda\tilde{H}\tilde{x} = \phi_0^{-1}\phi y)\bar{x} = \phi_0\tilde{x}$$

yields the associated equations (K3)

$$\bar{x} - \lambda\bar{H}\bar{x} = \phi_y$$

where

$$\bar{H} = \phi_0\tilde{H}\phi_0^{-1}\ , \quad \text{in } \bar{X}$$

I. As shown by Kantorovich (ref. 1), if the following conditions can be satisfied

A. for every $\tilde{x} \in \tilde{X}$, $\|\bar{H}\phi_0\tilde{x} - \phi H\tilde{x}\| \le \eta\|\tilde{x}\|$

B. for every $x \in X$, there exists $\tilde{x} \in \tilde{X}$ such that $\|Hx - \tilde{x}\| < \eta_1\|x\|$

C. an element $\tilde{y} \in \tilde{X}$ such that $\|y - \tilde{y}\| < \eta_2\|y\|$

and if the operator (I-H) has a linear inverse, and if

$$\bar{q} = |\lambda|\left[\bar{\eta}(1 + |\lambda|\eta_1)\|\phi_0^{-1}\| + \eta_1\|\phi_0^{-1}\phi K\|\right]\|K^{-1}\| < 1$$

then $(I-H)\bar{x} = \bar{y}$ has a solution $\bar{x}^*$ for every right-hand side $y \in \bar{X}$ with

$$\|\bar{x}^*\| \le \frac{\bar{N}}{1 + \bar{q}} \|y\| \quad \text{and} \quad \bar{N} = (1 + |\lambda| \eta_1)\|\phi_0\| \; \|\phi_0^{-1}\| \; \|K^{-1}\|$$

II. Furthermore, if in addition, x^* is the solution of $(I-H)x = y$, then

$$\|x^* - \phi_0^{-1}\bar{x}\| \le \bar{P}\|x^*\| \text{ ,}$$

and

$$q \le \eta_1|\lambda| + \eta_2\|I - H\|$$

and

$$\bar{P} = (1 + \varepsilon)|\lambda|\bar{\eta}\|\phi_0^{-1}\bar{K}^{-1}\| + \varepsilon\left[1 + \|\phi_0^{-1}(I - \bar{H})^{-1}\phi(I - H)\|\right]$$

III. Convergence results

if 1. (I-H) has a linear inverse

2. Conditions of Statement I are satisfied

3. A, B, and C of Statement I are satisfied for $n = 1, 2, \ldots$

where

$$\lim \bar{\eta}\|\phi_0^{-1}\| = \lim_{n\to\infty} \eta_1\|\phi_0^{-1}\phi\| = \lim_{n\to\infty} \eta_2\|\phi_0^{-1}\phi\| = 0$$

then the approximate equation (K3) is soluble for sufficiently large n and the sequence of approximate solutions converge to the exact solution. Also,

$$\|x^* - \tilde{x}^*\| \le \bar{Q}\bar{\eta}\|\phi_0^{-1}\| + \bar{Q}_1\eta_1\|\phi_0^{-1}\phi\| + \bar{Q}_2\eta_2\|\phi_0^{-1}\phi\|$$

where $\tilde{x}_n^* = \phi_0^{-1}\bar{x}_n^*$, and $\bar{Q}_1$, $\bar{Q}$, and $\bar{Q}_2$ are constants. First, seek to show

$$\|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \le \eta\|\tilde{x}\|$$

Case 1: Suppose

$$H\tilde{x} \in \tilde{X}$$

Therefore,

$$\phi H\tilde{x} = \phi_0 H\tilde{x}$$

Since

$$\bar{H} = \phi_0\tilde{H}\phi_0^{-1}$$

$$\phi H\tilde{x} - \bar{H}\phi_0\tilde{x} = \phi_0 H\tilde{x} - \phi_0\tilde{H}\tilde{x}$$

Therefore,

$$\|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \le \|H - \tilde{H}\| \, \|\tilde{x}\| = \eta\|\tilde{x}\| , \quad \text{for} \quad \eta = \|H - \tilde{H}\|$$

Case 2:

$$(H\tilde{x}) \notin \tilde{X} ,$$

$$\phi H\tilde{x} = \phi \int_0^1 h(s,t)\tilde{x}(s)ds ,$$

$$= \int_0^1 h(s,t_1)\tilde{x}(s)ds, \int_0^1 h(s,t_2)\tilde{x}(s)ds, \ldots$$

Seek to show that for every $\tilde{x} \in \tilde{X}_n$,

$$\|\bar{H}\phi_0\tilde{x} - \phi H\tilde{x}\| \le \eta\|\tilde{x}\|$$

where

$$\bar{H} = \phi_0\tilde{H}\phi_0^{-1}$$

$$H\tilde{x} = \int_0^1 h(s,t)\tilde{x}(t)dt;$$

$$\phi H\tilde{x} = \int_0^1 h(t_1,t)\tilde{x}(t)dt, \int_0^1 h(t_2,t)\tilde{x}(t)dt, \ldots \int_0^1 h(t_n,t)\tilde{x}(t)dt$$

$$\phi_0\tilde{x} = [\tilde{x}(t_1),\tilde{x}(t_2), \ldots, \tilde{x}(t_n)]$$

$$\begin{aligned}\bar{H}\phi_0\tilde{x} = \frac{1}{n} [&h(t_1,t_1)\tilde{x}(t_1) + h(t_1,t_2)\tilde{x}(t_2) + \ldots + h(t_1,t_n)\tilde{x}(t_n),\\ &h(t_2,t_1)\tilde{x}(t_1) + h(t_2,t_2)\tilde{x}(t_2) + \ldots + h(t_2,t_n)\tilde{x}(t_n), \ldots,\\ &h(t_n,t_1)\tilde{x}(t_1) + h(t_n,t_2)\tilde{x}(t_2) + \ldots + h(t_n,t_n)\tilde{x}(t_n)]\end{aligned}$$

$$\begin{aligned}-\bar{H}\phi_0\tilde{x} + \phi H\tilde{x} = \Bigg[&\int_0^1 h(t_1,t)\tilde{x}(t)dt - \frac{1}{n}\sum_{i=1}^{n} h(t_1,t_i)\tilde{x}(t_i), \int_0^1 h(t_2,t)\tilde{x}(t)dt\\ &- \frac{1}{n}\sum_{i=1}^{n} h(t_1,t_i)\tilde{x}(t_i), \ldots, \int_0^1 h(t_n,t)\tilde{x}(t)dt\\ &- \frac{1}{n}\sum_{i=1}^{n} h(t_n,t_i)\tilde{x}(t_i)\Bigg]\end{aligned}$$

Let $[0,1]$ be partitioned by the T_i's, $i = 1, \ldots, n$ where $T_i = [i - (1/2)]/n$, $i = 1, \ldots, n$, with the length $[T_{i-1}, T_i] = |\Delta_i|$. Therefore, there are $n - 1$ intervals of length Δ, with the end intervals $= \Delta/2$.

Let $z(t) = h(t_j, t)$. Consider

$$\left| \int_0^1 z(t)\tilde{x}(t)dt - \frac{1}{n} \sum_{i=1}^{n} z(t_i)\tilde{x}(t_i) \right|$$

$$= \left| \sum_{K=1}^{n} \int_{T_K}^{T_{K+1}} [z(t)\tilde{x}(t) - z(t_K)\tilde{x}(t)]dt + \int_{T_K}^{T_{K+1}} [z(t_K)\tilde{x}(t) - z(t_K)\tilde{x}(t_K)]dt \right|$$

$$\leq \omega_{\Delta_K}(z)\|\tilde{x}\| + \left| \sum_{K=1}^{n} \frac{1}{n} [z(t_K)\tilde{x}(\bar{t}_K) - z(t_K)\tilde{x}(t_K)] \right|$$

where

$$\bar{t}_K = \frac{T_K + T_{K+1}}{2}$$

$$\left| \frac{1}{n} \sum_{K=1}^{n} z(t_K)\tilde{x}(\bar{t}_K) - z(t_K)\tilde{x}(t_K) \right| = \left| \frac{1}{n} \sum_{K=1}^{n} z(t_K) \left[\frac{\tilde{x}(t_{K+1}) + \tilde{x}(t_K)}{2} - \tilde{x}(t_K) \right] \right|$$

$$= \left| \frac{1}{n} \sum_{K=1}^{n} z(t_K) \left[\frac{\tilde{x}(t_{K+1}) - \tilde{x}(t_K)}{2} \right] \right|$$

$$= \left| \frac{1}{n} \left\{ \frac{z(t_1)}{2} [\tilde{x}(t_2) - \tilde{x}(t_1)] + \frac{z(t_2)}{2} [\tilde{x}(t_3) - \tilde{x}(t_2)] + \ldots + \frac{z(t_n)}{2} [\tilde{x}(t_1) - x(t_n)] \right\} \right|$$

$$= \left| \frac{1}{n} \left\{ \frac{\tilde{x}(t_1)}{2} [z(t_n) - z(t_i)] + \frac{\tilde{x}(t_2)}{2} [z(t_1) - z(t_2)] + \ldots + \frac{\tilde{x}(t_n)}{2} [z(t_{n-1}) - z(t_n)] \right\} \right|$$

$$\leq \frac{\|\tilde{x}\|}{2n} \left[n\omega_{\Delta_K}(z) \right] \leq \frac{\|x\|}{2} \omega_{\Delta_K}(z)$$

Therefore,

$$\left| \int_0^1 z(t)\tilde{x}(t)dt - \sum_{K=1}^{n} \frac{1}{n} z(t_K)\tilde{x}(t_K) \right| \le \frac{3}{2}\, \omega_s(\Delta_K)\|\tilde{x}\|$$

Hence,

$$\|\bar{H}\phi_0\tilde{x} - \phi H\tilde{x}\| \le \bar{\eta}\|\tilde{x}\|$$

where

$$\bar{\eta} = \frac{3}{2}\, \omega_s(\Delta)$$

For the nonperiodic case, an additional term comes in, due to the ends of the interval:

$$\left| \int_0^1 z(t)\tilde{x}(t)dt - \sum_{K=1}^{n} \frac{1}{n} z(t_K)\tilde{x}(t_K) \right| \le \left[\frac{3}{2}\, \omega_s(\Delta_K) + \Delta_K\|h\|\right]\|\tilde{x}\|$$

Hence,

$$\|\bar{H}\phi_0\tilde{x} - \phi H\tilde{x}\| \le \bar{\eta}\|\tilde{x}\|$$

where $\bar{\eta} = (3/2)\omega_s(\Delta) + \Delta\|h\|$ for the nonperiodic case.

Seek to show condition B. For every $x \in X$ there exists $\tilde{x} \in \tilde{X}$ such that

$$\|Hx - \tilde{x}\| \le \eta_1\|x\|$$

To this end, we will first show

$$\|z - \tilde{z}\| \le \omega\left(\frac{3}{2}\, \Delta\right)$$

Let

$$T_j \le s \le T_{j+1}$$

$$z(s) - \tilde{z}(s) = z(s) - n[(T_{j+1} - s)z(t_j) + (s - T_j)z(t_{j+1})]$$

Therefore,

$$z(s) - \tilde{z}(s) = n(T_{j+1} - s)[z(s) - z(t_j)] + n(s - T_j)[z(s) - z(t_{j+1})]$$

$$|z(s) - \tilde{z}(s)| \le n(T_{j+1} - s)\omega\left(\frac{3}{2}\, \Delta\right) + n(s - T_j)\omega\left(\frac{\Delta}{2}\right) \le \omega\left(\frac{3}{2}\, \Delta\right)$$

Let

$$z = Hx$$

$$|z(s) - z(s')| = \left| \int_0^1 [h(s,t) - h(s',t)]x(t)dt \right|$$

$$\leq \omega_s\left(\frac{3\delta}{2}\right) \|x\| , \qquad |s' - s| \leq \delta$$

Therefore,

$$\omega\left(\frac{3}{2} \Delta\right) = \omega_s\left(\frac{3\delta}{2}\right) \|x\|$$

Hence

$$\|Hx - \tilde{z}\| \leq \omega_s\left(\frac{3}{2} \Delta\right)\|x\|, \text{ where } \quad \bar{z} = \phi_0^{-1}\phi Hx$$

For condition B, the proof for the nonperiodic case is identical. Therefore, condition B is satisfied with $\eta_1 = \omega_s[(3/2)\Delta]$.

To show condition C: Seek to show an element $\tilde{y} \in \tilde{X}$ exists, such that

$$\|y - \tilde{y}\| \leq \eta_2\|y\|$$

From the preceeding section we see that

$$\|y - \tilde{y}\| \leq \omega\left(\frac{3}{2} \Delta\right)$$

Therefore, the result directly follows by letting $\eta_2 = [\omega(3\Delta/2)]/\|y\|$. Similarly, this holds true for the nonperiodic case. Therefore, relative to page 7, conditions A, B, and C have been established for arbitrary $n > 1$, and it remains to show

1. $(I - \lambda H)$ has a linear inverse
2. $\bar{q} = |\lambda|[\bar{\eta}(1 + |\lambda|\eta_1)\|\phi_0^{-1}\| + \eta_1\|\phi_0^{-1}\phi K\|]\|K^{-1}\| < 1$
3. $\lim_{n\to\infty} \bar{\eta}\|\phi_0^{-1}\| = \lim_{n\to\infty} \eta_1\|\phi_0^{-1}\phi\| = \lim_{n\to\infty} \eta_2\|\phi_0^{-1}\phi\| = 0$

Since $\bar{\eta} = (3/2)\omega_s(\Delta)$, $\eta_1 = \omega_s[(3/2)\Delta]$, ($\bar{\eta} = (3/2)\omega_s(\Delta) + |\Delta|\,\|h\|$, for the nonperiodic case), and $\eta_2 = \omega[(3/2)\Delta]/\|y\|$, they all tend to zero as $n \to \infty$; also, $\|\phi_0^{-1}\| = \|\phi\| = 1$.

Hence, when λ is a noncharacteristic value of the operation $(I - \lambda H)$, then for sufficiently large n, the system of equations (k3) is soluble (with solution $\bar{x}_n^*$), and the sequence $\phi_0^{-1}\bar{x}_n^*$ converges to the exact solution of (1).

CONCLUDING REMARKS

Therefore, based on the Kantorovich theory, the approximate solutions $\tilde{x}_n$'s of the equations

$$\tilde{x}_n - \lambda\tilde{H}_n\tilde{x}_n = \phi_0^{-1}\phi y$$

have been shown to converge to the solution x^* of equation (1):

$$x - \lambda Hx = y$$

that is,

$$\|\tilde{x}_n^* - \tilde{x}^*\| \to 0 \quad \text{as} \quad n \to \infty$$

REFERENCE

!. Kantorovich, L. V.; and Akilov, G. P.: Functional Analysis in Normed Space. Pergamon Press, New York, N.Y., 1964.

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16. Abstract

Fredholm integral equations of the second kind of one dimension are numerically solved. In particular, it is proven that the numerical solution converges to the exact solution of the integral equation. This is shown first for periodic kernels and then extended to nonperiodic kernels. Similar results have been accomplished by several authors; however, this development is significant in that it helps delineate a basic theory which has the potential of solving very complex problems.

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