NOTICE

NASA

Technical Memorandum 82008

ELECTROSTATIC HEAT FLUX INSTABILITIES

Philip J. Morrison and James A. Ionson

(NASA-TM-82008) ELECTROSTATIC HEAT FLUX INSTABILITIES (NASA) 39 p HC A03/MF A01 CSCL 20I
N81-12882
Unclas
G3/75 39803

SEPTEMBER 1980

National Aeronautics and
Space Administration

Goddard Space Flight Center
Greenbelt, Maryland 20771

TM 82008

ELECTROSTATIC HEAT FLUX INSTABILITIES

Philip J. Morrison and James A. Ionson
Laboratory for Astronomy and Solar Physics,
NASA-Goddard Space Flight Center
Greenbelt, Maryland

September 1980

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

ELECTROSTATIC HEAT FLUX INSTABILITIES

INTRODUCTION

The purpose of this report is to investigate plasma physics of the Electrostatic Ion Cyclotron (EIC) and Ion Acoustic (IA) modes in the presence of a heat flux. The initial analysis will be carried out more generally to include a current in the plasma as well as a heat flux. This will facilitate comparisons with the results of Kindel and Kennel[1] who investigated these modes driven by a current only.

We wish to find a dispersion relation for $\omega(k)$ parameterized by the ratio of electron to ion temperature and also the growth rate, Γ, as a function of the wavenumber k. We then wish to specialize these results for the special case of marginal stability for which $\Gamma(k) = 0$ and find the threshold drift speed above which the plasma is unstable to the EIC and IA modes. Finally we will find expressions for these quantities for the limits $\frac{T_e}{T_i} \sim 1$ and $\frac{T_e}{T_i} \gg 1$.

A self-consistent distribution function will be derived in the next section using Chapman-Enskog theory to represent an electron distribution including a heat flux characterized by a thermal conduction speed, v_c and also an electric current characterized by a drift speed, u. The ions will be represented by an isotropic Maxwellian distribution. In the third section this distribution will be used to find the dispersion equation and growth rate for the low frequency, $\omega \ll \Omega_i = \frac{eB}{M_i c}$, electrostatic modes. In section IV marginal stability will be assumed while the contribution of the electron current will be dropped and the dispersion equation simplified for the two modes, EIC or IA. Here an expression will also be given for the minimum conduction speed above which the plasma becomes unstable. The limits $\frac{T_e}{T_i} \sim 1$ and $\frac{T_e}{T_i} \gg 1$ will be considered and simplified expressions for the dispersion equation, growth rates, and minimum critical conduction speed derived. The final section will then consist of a short discussion of the results.

THE ELECTRON DISTRIBUTION FUNCTION

In Chapman-Enskog theory it is assumed that the particle collision frequency, ν, is very large so that f (the distribution function) can be exponded in a series:

$$f = A_0 + \frac{1}{\nu} A_1 + \frac{1}{\nu^2} A_2 + \dots \tag{1}$$

$$= f_0 + f_1 + f_2 + \dots \tag{2}$$

where the A_i's are all assumed to be of the same order so:

$$|f_0| \gg |f_1| \gg |f_2| \dots \tag{3}$$

In addition is is assumed that all derivatives of the A_i's are of the same order as A_i itself. In this derivation we follow Tanenbaum.[2] From his equations (A.5.20) and (A.5.21):

$$f_1 = \frac{A_1}{\nu} = -\frac{\Delta}{\nu} f_0 \tag{4}$$

where:

$$\Delta = \frac{c_i}{T}\left(\beta c^2 - \frac{5}{2}\right)\frac{\partial T}{\partial x_i} \tag{5}$$

neglecting any pressure gradient and $\vec{c} = \vec{v} - \vec{u}$ where $\vec{u}$ is the peculiar velocity and $\beta = v_{THERMAL}^{-2} = \frac{m}{2k_e T}$. Use of $f = f_0 + f_1$ in this approximation leads to the Navier-Stokes equation. Our distribution function is now:

$$f = \left[1 - \frac{\Delta}{\nu}\right] f_0 = \left[1 - \frac{c_i}{T\nu}\left(\beta c^2 - \frac{5}{2}\right)\frac{\partial T}{\partial x_i}\right] f_0 \tag{6}$$

where f_0 is the Maxwellion:

$$f_0 = \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta c^2}. \tag{7}$$

Now let us find some moments of this. In particular consider the heat flux:

$$\vec{Q} = \frac{1}{2} mn \int \vec{c}\,(\vec{c}\cdot\vec{c})\, f\, d^3v \,. \tag{8}$$

We assume that there is a temperature gradient in the Z direction, parallel to the magnetic field. Then:

$$f = \left[1 - \frac{c_{\parallel}}{T\nu}\left(\beta c^2 - \frac{5}{2}\right)\frac{dT}{dz}\right]\left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta c^2} \tag{9}$$

where: $c_{\parallel} = v_{\parallel} - u$ since $\vec{c} = \vec{v} - \vec{u}$ and $\vec{u} = u\hat{z}$.

Then:

$$Q_{e\parallel} = -\frac{5}{2}\frac{k_B^2 n_e T_e}{m_e \nu_e}\frac{dT_e}{dz} \tag{10}$$

so

$$\frac{c_{\parallel}}{T_e \nu}\frac{dT_e}{dz} = -\frac{8}{5}\frac{Q_{e\parallel}\, c_{\parallel}}{n_e m_e v_{Te}^4} \tag{11}$$

and the distribution function in terms of the heat flux is:

$$f = A\left[1 + B\left(\frac{c_{\parallel}}{v_{Te}}\right)\left[2\left(\frac{c}{v_{Te}}\right)^2 - 5\right]\right] e^{-\left(\frac{c}{v_{Te}}\right)^2} \tag{12}$$

where:

$$A = \left(\pi v_{Te}^2\right)^{-3/2}$$
$$B = \frac{4 Q_{e\parallel}}{5 n_e m_e v_{Te}^3}$$
$$v_{Te} = \left(\frac{2 k_B T_e}{m_e}\right)^{1/2}.$$

Now let us calculate the first movement, the average velocity. We can see that f_1 is even in v_x and v_y while it is odd in $v_{\parallel} = v_z$ since $\vec{u}$ is directed only along the z-axis. Consequently $\langle v_x \rangle = \langle v_y \rangle = 0$ and $\langle \vec{v} \rangle = \langle v_{\parallel} \rangle \hat{z}$.

It is therefore useful to define:

$$v_{e\perp} = \frac{v_{\perp}}{v_{Te}} \qquad v_{\perp}^2 = v_x^2 + v_y^2$$

$$v_{e\parallel} = \frac{v_{\parallel} - u}{v_{Te}} = \frac{c_{\parallel}}{v_{Te}} \qquad v_{\parallel} = v_z \tag{13}$$

then:

$$\left(\frac{c}{v_{Te}}\right)^2 = v_{e\parallel}^2 + v_{e\perp}^2 .$$

Then f can be re-written as:

$$f_e = A\left[1 + B v_{e\parallel}(2v_{e\parallel}^2 + 2v_{e\perp}^2 - 5)\right] e^{-(v_{e\parallel}^2 + v_{e\perp}^2)} \quad . \tag{14}$$

Then, in general:

$$\begin{aligned} \langle g(v_{e\parallel})\rangle &= 2\pi\int_0^\infty v_\perp \, dv_\perp \int_{-\infty}^{\infty} dv_{e\parallel}\, g(v_{e\parallel})\, f_{oe} \\ &= 2\pi\int_0^\infty v_{e\perp}\, dv_{e\perp}\int_{-\infty}^{\infty} \frac{g(v_{e\parallel})}{\pi^{3/2}}\left[1+Bv_{e\parallel}(2v_{e\parallel}^2+2v_{e\perp}^2-5)\right] e^{-(v_{e\parallel}^2+v_{e\perp}^2)} \end{aligned} \tag{15}$$

and integrating over $v_{e\perp}$:

$$\langle g(v_{e\parallel})\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dv_{e\parallel}\, g(v_{e\parallel})\left[1+Bv_{e\parallel}(2v_{e\parallel}^2-3)\right] e^{-v_{e\parallel}^2} \quad . \tag{16}$$

Consider the odd moments:

$$\langle v_{e\parallel}^{2k-1}\rangle = \frac{2B}{\sqrt{\pi}}\int_0^\infty v_{e\parallel}^{2k}(2v_{e\parallel}^2-3)\, e^{-v_{e\parallel}^2}\, dv_{e\parallel} \tag{17}$$

where the first term in the integrand has vanished because of its symmetry. Using:

$$\int_0^\infty x^{2m} e^{-x^2} dx = \frac{(2m-1)!!}{2^{m+1}}\sqrt{\pi}$$

where:

$$(2m-1)!! = 1\cdot 3\cdot 5\cdot\ldots\cdot(2m-1)$$

We see:

$$\langle v_{e\parallel}^{2k-1}\rangle = \left[(2k+1)!! - 3(2k-1)!!\right]\frac{B}{2^k}$$

which yields the general result:

$$\langle v_{e\parallel}^{2k-1}\rangle = \frac{2k-2}{2^k}(2k-1)!!\, B \, . \tag{18}$$

Then, the first moment is:

$$\langle v_{e\parallel}\rangle = 0 \qquad \text{and since} \qquad v_{e\parallel} = \frac{v_\parallel - u}{v_{Te}} \tag{19}$$

Then:

$$\langle v_\parallel\rangle = u \, . \tag{20}$$

We can then interpret the parameters as follows: $q_{e_\parallel}$ is the field aligned heat flux density (and is collision dominated) while u is the electron drift velocity associated with a net current flowing parallel to $\vec{B}$. We note that the choices of u and $q_{e_\parallel}$ are independent so setting $q_{e_\parallel} = 0$ will reproduce the results of Kindel and Kennel while setting $u = 0$ will specialize to heat flux instabilities which are the main concern here.

To complete an analogy with the current instability it is useful to define one other quantity: This is the thermal conduction speed, v_c. From equation (10):

$$q_{\parallel} = -\frac{5}{2} \frac{n_e k_B T_e}{m_e \nu_e} \frac{dT_e}{dz} = -\frac{5}{4} \frac{v_{Te} n_e k_B}{\nu_e} \frac{dT_e}{dz} . \tag{21}$$

Replace $\frac{1}{\nu_e}$ by the mean free path λ_e and the temperature gradient $\frac{dT_e}{dz}$ by $\frac{T_e}{L}$ where L is the system's scale size. $q_{\parallel}$ then becomes:

$$q_{\parallel} = -\frac{1}{3}\left[\frac{3}{2} v_{te} \frac{\lambda}{L}\right]\left[\frac{5}{2} n_e k_B T_e\right] \tag{22}$$

which is in the standard form for a heat flux: a speed multiplied by an energy flux density. The heat conduction speed can then be defined as:

$$v_c = \frac{3}{2} v_{te} \left(\frac{\lambda}{L}\right) \tag{23}$$

where the factor 3 is put in for later convenience.

This definition can be justified on physical grounds as follows: Consider an ideal gas. The heat energy density is nc_p T since heat conduction will assume constant pressure (mass motions caused by the pressure gradients are taken care of by the parameter u in the distribution function). The heat flux is then:

$$q = -v_c (n c_p T)$$

$$= -\left(\frac{c_p}{c_v} v_c\right)(n c_v T)$$

and for an ideal gas: $C_v = \frac{3}{2} k_B$ while $C_p / C_v = \frac{5}{3}$ so:

$$Q = -\frac{5}{3} v_c \left(\frac{3}{2} m k_B T \right)$$

$$Q = -\frac{5}{2} v_c \, m k_B T$$ as above.

Consequently the constant $\frac{5}{2}$ is not involved in v_c. Thus the parameter B can be written as:

$$B = \frac{4 Q_{e\parallel}}{5 n_e m_e v_{te}^3} = -\frac{4}{3} \left(\frac{v_c}{v_{te}} \right) \tag{24}$$

and this form will be useful later.

We can further note that there is a limit to the magnitude of $Q_{e\parallel}$ since, in the collision dominated regime assumed in the derivation, it can't exceed the product of the random electron flux (across a plane perpendicular to the z-axis) and the mean thermal electron speed:

$$Q_{e\parallel} < \left(\frac{1}{2} m_e n_e v_{te}^2 \right) v_{te} \tag{25}$$

which, in concert with equation (22) implies:

$$\frac{\lambda}{L} < 1$$

or:

$$v_c < v_{te} \tag{26}$$

and this is what is meant by flux limited.

THE DISPERSION EQUATION

For two charge species (electrons and ions) the dispersion equation is[3]:

$$\epsilon(\omega, k) = 1 + \chi_e + \chi_i = 0 \tag{27}$$

where:

$$-\chi_\alpha = \left(\frac{\omega_{p\alpha}}{k}\right)^2 \sum_{m=-\infty}^{\infty} \int d^3v\, J_m^2(\lambda_\alpha)\left[\frac{k_\parallel\left(\frac{d f_{o\alpha}}{d v_\parallel}\right) + \left(\frac{m\Omega_\alpha}{v_\perp}\right)\frac{d f_{o\alpha}}{d v_\perp}}{k_\parallel v_\parallel - (\omega - m\Omega_\alpha)}\right] \tag{28}$$

assuming:

$$\vec{k} = k_\parallel \hat{z} + k_\perp \hat{x} \qquad \Omega_\alpha = \frac{q_\alpha B_o}{m_\alpha c}$$

$$\vec{B} = B_o \hat{z} \qquad \omega_{p\alpha}^2 = \frac{4\pi n_\alpha e^2}{m_\alpha}$$

$$\vec{v} = v_\parallel \hat{z} + v_x \hat{x} + v_y \hat{y} \qquad \lambda_\alpha = k_\perp \rho_{e\alpha} = \frac{k_\perp v_\perp}{\Omega_\alpha}$$

$$v_\perp^2 = v_x^2 + v_y^2 \qquad d^3v = 2\pi v_\perp\, dv_\perp\, dv_\parallel \; .$$

This expression is solely for electrostatic perturbations so it assumes, for $k_\perp \neq 0$, that

$$\beta \equiv \frac{\text{THERMAL PRESSURE}}{\text{MAGNETIC PRESSURE}} \ll 1 \; .$$

For the ions, the distribution function is taken to be an isotropic Maxwellian:

$$f_{oi}(\vec{v}) = \left(\pi v_{ti}^2\right)^{-3/2} e^{-\frac{v^2}{v_{ti}^2}} \tag{29}$$

where v_{ti} is the ion thermal speed:

$$v_{ti} = \sqrt{\frac{2k_B T_i}{m_i}} \; .$$

Then, from Hasegawa[3], equation (2.30):

$$\chi_i(\omega,k) = 2\left(\frac{\omega_{pi}}{k v_{ti}}\right)^2 \sum_{m=-\infty}^{\infty} e^{-\mu_i} I_m(\mu_i)\left[1 + \frac{\omega}{k_\parallel v_{ti}} Z_o\left(\frac{\omega - m\Omega_i}{k_\parallel v_{ti}}\right)\right] \tag{30}$$

where:

$\mu_i = \frac{1}{2}\left(\frac{k_\perp v_{ti}}{\Omega_i}\right)^2$

I_m = modified Bessel function

Z_o = Plasma dispersion function (Fried & Conte[4]) so: $$Z_e(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{x-\xi}\, d\xi \quad . \tag{31}$$

For the electrons we will use the distribution discussed above:

$$f_{oe} = A\left[1 + B v_{e\parallel}\left(2v_{e\parallel}^2 + 2v_{e\perp}^2 - 5\right)\right] e^{-(v_{e\parallel}^2 + v_{e\perp}^2)} \tag{32}$$

where:
$$A = \left(\pi v_{te}^2\right)^{-3/2}$$

$$B = \frac{4 Q_{e\parallel}}{5 n_e m_e v_{te}^2} = -\frac{2}{3}\frac{v_c}{v_{te}}$$

and:
$$v_{te} = \sqrt{\frac{2 k_B T_e}{m_e}}$$

$$v_{e\parallel} = \frac{v_\parallel - u}{v_{te}} \qquad v_{e\perp} = \frac{v_\perp}{v_{te}} \quad . \; .$$

With these definitions:

$$\chi_e = -\left(\frac{\omega_{pe}}{k}\right)^2 v_{te} \sum_n \int d v_e^3\, J_n^2(\lambda_e) \left[\frac{\dfrac{\partial f_{oe}}{\partial v_{e\parallel}} + \dfrac{n\Omega_e}{k_\parallel v_{te}}\left(\dfrac{1}{v_{e\perp}}\dfrac{\partial f_{oe}}{\partial v_{e\perp}}\right)}{v_{e\parallel} - \dfrac{\omega - k_\parallel u_\parallel - n\Omega_e}{k_\parallel v_{te}}}\right] \tag{33}$$

where:
$$\lambda_e = \left(\frac{k_\perp v_{te}}{\Omega_e}\right)^2 v_{e\perp} = 2\mu_e v_{e\perp} \; .$$

Now for ion cyclotron and ion acoustic instabilities $\mu_i \sim 1$ so $\mu_e \ll 1$. Then the argument, λ_e, of the J_n is appreciably different from O only for $v_{e\perp} \gg 1$. However, there is little contribution to the integral from large $v_{e\perp}$ due to the factor $e^{-v_{e\perp}^2}$ in f_{oe}. Consequently, we will assume $\lambda_e \ll 1$ and with $J_n(\lambda_e) \approx 0$ ($n \neq 1$) and $J_n(\lambda_e) \approx 1$ for $\lambda_e \sim 0$ and the electron term becomes :

$$\chi_e = -\left(\frac{\omega_{pe}}{k}\right)^2 v_{te} \int d^3 v_{te}\, \frac{\dfrac{\partial f_o}{\partial v_{e\parallel}}}{v_{e\parallel} - \xi_e} \tag{34}$$

where: $$\xi_e = \frac{\omega - k_\parallel u}{k_\parallel v_{te}} .$$

Now:

$$\frac{\partial f_{oe}}{\partial v_{e\parallel}} = -A\left\{2v_{e\parallel} + B\left[2v_{e\parallel}^4 + 4(v_{e\perp}^2 - 4)v_{e\parallel}^2 - 2v_{e\perp}^2 + 5\right]\right\} e^{-(v_{e\parallel}^2 + v_{e\perp}^2)} \tag{35}$$

and the integrals over $v_{e\perp}$ all have the form:

$$\int_0^\infty v_{e\perp}^{2k+1} e^{-v_{e\perp}^2} d v_{e\perp} = \frac{1}{2} k! . \tag{36}$$

Performing the integrals over $v_{e\perp}$ then gives:

$$\chi_e = \frac{2}{\sqrt{\pi}} \left(\frac{\omega_{pe}}{k v_{te}}\right)^2 \int_{-\infty}^{\infty} \frac{g(v_{e\parallel})\, e^{-v_{e\parallel}^2}}{v_{e\parallel} - \xi_e} d v_{e\parallel} \tag{37}$$

where:

$$g(v_{e\parallel}) = v_{e\parallel} + B\left[v_{e\parallel}^4 - 6 v_{e\parallel}^2 + \frac{3}{2}\right] . \tag{38}$$

Now we define:

$$Z_k(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{x^k e^{-x^2}}{x - \xi} dx \tag{39}$$

so that $Z_0(\xi)$ is the plasma dispersion function of Fried and Conte[4]. The electron's contribution is then:

$$\chi_e = 2\left(\frac{\omega_{pe}}{k v_{te}}\right)^2 \left[Z_1(\xi_e) + B\left(4 Z_4(\xi_e) - 6 Z_2(\xi_e) + \frac{3}{2} Z_0(\xi_e)\right)\right] \tag{40}$$

where: $\zeta_e = \frac{(\omega) - k_{\parallel} u}{k_{\parallel} \sqrt{te}}$

$\beta = - \frac{a}{3} \left(\frac{\sqrt{c}}{\sqrt{ee}} \right)$.

If Z_K is integrated by parts the following relations may be found:

$$Z_1 = - \frac{1}{2} Z_0' \tag{41}$$

$$Z_2 = - \frac{1}{2} \zeta_e Z_0' \tag{42}$$

$$Z_K = \frac{1}{2} \left[(K-1) Z_{K-2} - Z_{K-1}' \right] \tag{43}$$

and $Z_0' = -2 \left[1 + \zeta Z_0 \right]$ where $Z_0^{(K)} = \frac{d^K}{d\zeta^K} Z_0(\zeta)$. (44)

Successive iterations of the recursion relation gives:

$$Z_3 = \frac{1}{4} \left(\zeta Z_0'' - Z_0' \right)$$

$$Z_4 = - \frac{1}{8} \zeta \left(Z_0^{(3)} + 6 Z_0' \right)$$

while successive applications of the relation between Z_0' and Z_0 gives:

$$Z_0'' = 2(2\zeta^2 - 1) Z_0 + 4\zeta$$

$$Z_0^{(3)} = -4\zeta(2\zeta^2 - 3) Z_0 - 8(\zeta^2 - 1) .$$

Combining these gives:

$$\left. \begin{aligned} Z_1(\zeta) &= \zeta Z_0 + 1 && = \zeta Z_0 + 1 \\ Z_2(\zeta) &= \zeta(\zeta Z_0 + 1) && = \zeta Z_1 \\ Z_3(\zeta) &= \zeta^3 Z_0 + \zeta^2 + \frac{1}{2} && = \zeta Z_2 + \frac{1}{2} \\ Z_4(\zeta) &= \zeta(\zeta^3 Z_0 + \zeta^2 + \frac{1}{2}) && = \zeta Z_3 \end{aligned} \right\} \tag{45}$$

so χ_e can be re-written entirely in terms of z_e as:

$$\chi_e = 2\left(\frac{\omega_{pe}}{k v_{te}}\right)^2 \left\{1 + \zeta_e z_e + B\left[\left(4\zeta_e^4 - 6\zeta_e^2 + \tfrac{1}{2}\right) z_e + 4\zeta_e(\zeta_e^2 - 1)\right]\right\} \quad . \tag{46}$$

At this point is useful to consider the magnitudes of B and ζ_e in order to make approximations. We have already shown that:

$$B < 1 \qquad (v_c < v_{te})$$

since $\varphi_{e_\parallel}$ is flux limited. Now consider ζ_e :

$$\zeta_e = \frac{\omega - k_\parallel u}{k_\parallel v_{te}} \approx \frac{v_p - u}{v_{te}}$$

where v_p is the wave's phase velocity parallel to the magnetic field. For $T_e/T_i \gtrsim 1$ we expect (following Kindel and Kennel) that $u \ll v_{te}$ if u is taken as the drift speed for marginal stability. Similarly, in the same temperature regime we expect $v_p \sim v_c \ll v_{te}$ so

$$\zeta_e \ll 1 \qquad \text{for} \quad \frac{T_e}{T_i} \gg 1 \ .$$

When we actually have an expression for $\omega(k)$ we must check to ensure that this limit is correct.

In what follows, we will assume $\zeta_e \ll 1$ in order to drop higher orders of ζ_e in the dispersion equation. We must keep in mind that this is strictly true only for $T_e/T_i \gg 1$ and when we write the general result we will include these terms. We will, of course, also include them when finding results in the $T_e/T_i \lesssim 1$ regime. Note that the dispersion equation looks increasingly like that of Kindel and Kennel as higher order terms in ζ_e are dropped so there is some heuristic value in this approximation. The corollary to this is that our results should increasingly deviate from theirs as ζ_e becomes large.

Now, keeping only terms which are explicitly of first order in β or ζ_e :

$$\chi_e \approx 2\left(\frac{\omega_{pe}}{k v_{te}}\right)^2 \left[1 + \left(\zeta_e + \tfrac{3}{2}\beta\right) Z_0 + 4\zeta_e \beta\right]. \tag{47}$$

The dispersion equation is:

$$1 + \chi_e + \chi_i = 0. \tag{48}$$

Using equations (30) and (47), dividing by 2, this becomes:

$$\tfrac{1}{2} + \left(\frac{\omega_{pe}}{k v_{te}}\right)^2 \left[1 + \left(\zeta_e + \tfrac{3}{2}\beta\right) Z_0 - 4\zeta_e \beta\right] + \left(\frac{\omega_{pi}}{k v_{ti}}\right)^2 \sum_m e^{-\lambda_i} I_m(\lambda_i) \left[1 + \frac{\omega}{k_\parallel v_{ti}} Z_0\left(\frac{\omega - m\Omega_i}{k_\parallel v_{ti}}\right)\right] = 0. \tag{49}$$

Considering the ion contribution, χ_i, the imaginary part of χ_i is always positive and so provides damping. The n = 0 contribution is Landau damping while for $m \geq 1$, it is referred to as ion-cyclotron damping.

Considering the electron contribution, $\mathrm{Im}(Z_0(\zeta_e)) > 0$ so the sign of $\mathrm{Im}(\chi_e)$ depends on the sign of $\frac{3}{2}\beta + \zeta_e$ which is the factor multiplying Z_0 :

$$\tfrac{3}{2}\beta + \zeta_e = \frac{v_p - u - v_c}{v_{te}} \qquad \text{where } v_p = \frac{\omega}{k_\parallel}.$$

Consequently, if the drift velocity, u, is larger than the wave's phase velocity then the negative electron dissipation can contribute to instability. In the same fashion, the heat conduction velocity, if sufficiently large, can lead to instability. The mode will become unstable if the velocities, u and v_c, are sufficiently large that the negative electron dissipation is larger than the damping from the ions.

GENERAL CASE

In general, if there are a number of ion species, χ_i must be replaced by $\sum_i \chi_i$ where the sum is over all the ion species. With this and the definition:

$$T_m(\mu_i) = e^{-\mu_i} I_m(\mu_i)$$

the dispersion equation becomes:

$$D(\omega,k) = 1 + \sum_i \sum_m \frac{T_m(\mu_i)}{2k^2\lambda_{Di}^2}\left[1 + \frac{\omega}{k_\parallel v_{ti}} Z_0\left(\frac{\omega - m\Omega_i}{k_\parallel v_{ti}}\right)\right]$$

$$+ \frac{1}{2k^2\lambda_{De}^2}\left[1 + \left(\frac{v_p - u - v_c}{v_{te}}\right) Z_0\left(\frac{v_p - u}{v_{te}}\right) + \frac{2}{3}\left(\frac{v_p - u}{v_{te}}\right)\left(\frac{v_c}{v_{te}}\right)\right]$$

$$- \frac{2}{3}\left(\frac{v_p - u}{k\lambda_{De} v_{te}}\right)^2\left(\frac{v_c}{v_{te}}\right)\left\{2\left(\frac{v_p - u}{v_{te}}\right) + \left[2\left(\frac{v_p - u}{v_{te}}\right)^2 - 3\right] Z_0\left(\frac{v_p - u}{v_{te}}\right)\right\} \tag{50}$$

where:

$$\lambda_{Dj}^2 = \frac{k_B T_j}{8\pi M_j e^2} \qquad v_p = \frac{\omega}{k_\parallel}$$

$$\mu_i = \frac{1}{2}\left(\frac{k_\perp v_{ti}}{\Omega_i}\right)^2 \qquad v_c = \frac{6}{5}\frac{\phi e n}{m_e m_e v_{te}^2}$$

$$v_{tj}^2 = \frac{2 k_B T_j}{m_j}$$

where the last term includes the higher order terms in ξ_e previously dropped.

Now, assume the system is near stability so:

$$\omega = \omega_R + i\Gamma \qquad \text{AND} \qquad \Gamma \ll |\omega_R| . \tag{51}$$

Then set $\omega = \omega_R$ in equation (50), separate into real and imaginary parts:

$$D = D_R(\omega_R, k) + i D_i(\omega_R, k) \tag{52}$$

and then, to order $\frac{\Gamma}{\omega_R}$:

$$D_R(\omega_R, k) = 0 \tag{53}$$

$$\Gamma = -\frac{D_i}{\frac{\partial D_R}{\partial \omega_R}} \tag{54}$$

(Krall & Trivelpiece[5], 1973).

Defining:
$$Z_R = R_e(Z_o)$$
$$Z_I = I_m(Z_o)$$

gives the real dispersion equation (multiplying by $k^2\lambda_{De}^2$):

$$-2k^2\lambda_{De}^2 = \sum_i \sum_m \Gamma_m(\mu_i)\left(\frac{\lambda_{De}}{\lambda_{Di}}\right)^2\left[1 + \frac{v_p}{v_{ti}} Z_R\left(\frac{\omega_R - m\Omega_i}{k_\parallel v_{ti}}\right)\right]$$
$$+1 + \frac{\rho}{3}\left(\frac{v_p - u}{v_{te}}\right)\left(\frac{v_c}{v_{te}}\right) + \left(\frac{v_p - u - v_c}{v_{te}}\right) Z_R\left(\frac{v_p - u}{v_{te}}\right)$$
$$-\frac{\rho}{3}\left(\frac{v_c}{v_{te}}\right)\left(\frac{v_p - u}{v_{te}}\right)\left\{\frac{v_p - u}{v_{te}} + \left[\left(\frac{v_p - u}{v_{te}}\right)^2 - \frac{3}{2}\right] Z_R\left(\frac{v_p - u}{v_{te}}\right)\right\} \qquad (55)$$

we can see that the conduction speed, v_c, introduces new terms in the dispersion equation that are not seen in the case of current instabilities alone. These terms all contain the factor:

$$\left(\frac{v_p - u}{v_{te}}\right)\left(\frac{v_c}{v_{te}}\right)$$

which vanishes if v_c vanishes but not when u vanishes.

From equation (50), $D_i(\omega_R, k)$ can be re-written as:

$$u + v_c\left\{1 + \frac{\rho}{3}\left(\frac{v_p - u}{v_{te}}\right)^2\left[\left(\frac{v_p - u}{v_{te}}\right)^2 - \frac{3}{2}\right]\right\}$$
$$= \frac{\omega}{k_\parallel}\left\{1 + \sum_i \frac{m_i}{m_e} c_i\left[\frac{\sum \Gamma_m(\mu_i) Z_I\left(\frac{\omega - m\Omega_i}{k_\parallel v_{ti}}\right)}{Z_I\left(\frac{\omega - k_\parallel u}{k_\parallel v_{te}}\right)}\right]\right\} \qquad (56)$$

where: $c_i = \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}$.

This gives the combination of drift speed and conduction speed for marginal stability. The terms in braces multiplying v_c are those dropped when $\xi_e \ll 1$.

Using:

$$Z_I(\xi) = \sqrt{\pi}\, e^{-\xi^2} \qquad (57)$$

the real imaginary parts of the dispersion equation for small growth rates can be re-written in terms of our normalized variables as:

$$-(1+2k^2\lambda_{De}^2) = \sum_i \sum_m \frac{m_i}{m_e}\left(\frac{T_e}{T_i}\right)\Gamma_m(\mu_i)\left[1+\xi_i Z_R(\xi_i - m\zeta_{cyc})\right]$$
$$+\frac{8}{3}\zeta_e\xi_c\left\{1-\underline{\xi_c\left[\zeta_e+\left(\xi_c^2-\frac{3}{2}\right)Z_R(\xi_c)\right]}\right\}+(\zeta_e-\xi_c)Z_R(\zeta_e) \tag{58}$$

$$u+v_c\left\{1+\underline{\frac{8}{3}\xi_c^2\left(\xi_c^2-\frac{3}{2}\right)}\right\} = v_p\left\{1+\sum_i \frac{m_i}{m_e}C_i\sum_m \Gamma_m(\mu_i)\,e^{-(\xi_i-m\zeta_{cyc})^2+\zeta_e^2}\right\} \tag{59}$$

where:

$$\xi_i = \frac{v_p}{v_{ti}} \qquad v_p = \frac{\omega}{k_\parallel} \qquad \Gamma_m(x) = e^{-x}I_m(x)$$

$$\zeta_e = \frac{v_p-u}{v_{te}} \qquad v_{tj}^2 = \frac{2k_BT_j}{m_j} \qquad C_i = \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}$$

$$\zeta_{cyc} = \frac{\Omega_i}{k_\parallel v_{ti}} \qquad \Omega_i = \frac{eB}{m_ic} \qquad \lambda_{De}^2 = \frac{k_BT_e}{8\pi m_e e^2}$$

$$\xi_c = \frac{v_c}{v_{te}} \qquad \mu_j = \frac{1}{2}\left(\frac{k_\perp v_{tj}}{\Omega_j}\right)^2 \qquad k^2 = k_\parallel^2 + k_\perp^2$$

and the underlined terms can be neglected for $\xi_c \ll 1$.

To find the marginal drift speed for instability equations (58) and (59) are solved simultaneously along with the requirement that $u + v_c$ be a minimum with respect to $k_\perp$ and $k_\parallel$. Note that by equation (54) this is equivalent to requiring $\Gamma = 0$ unless $\frac{d}{d\omega_R}D_R = 0$. That is, what we wish is to find, for a given set of physical conditions (T_e, T_i, n_e, etc.) the solution to the dispersion equation, $\omega(k_{min})$, for which the combined drift speeds are minimum.

Using equations (54) and (50) the growth rate is

$$\frac{\Gamma}{\sqrt{\pi}\,k_\parallel v_{te}} = -\frac{\text{TOP}}{\text{BOTTOM}} \tag{60}$$

where:

$$\mathrm{TOP} = \sum_i \sum_m \frac{m_i}{m_e} \left(\frac{T_e}{T_i}\right) \Gamma_m(\mu_i)\, \zeta_i\, e^{-(\zeta_i - m\zeta_{cyc})^2} + \left\{ \zeta_c - \zeta_e \left[1 + \frac{8}{3} \zeta_e^2 \left(\zeta_e^2 - \frac{3}{2} \right) \right] \right\} e^{-\zeta_e^2} \tag{61}$$

$$\mathrm{BOTTOM} = \sum_i \sum_m \frac{m_i}{m_e} c_i \Gamma_m(\mu_i) \left[Z_R(\zeta_i + m\zeta_{cyc}) + \zeta_i Z_R'(\zeta_i + m\zeta_{cyc}) \right] + Z_R(\zeta_e) + \zeta_e Z_R'(\zeta_e)$$

$$+ \zeta_c \left\{ \frac{8}{3}(1 - 3\zeta_e^2) - \frac{8}{3}\zeta_e (4\zeta_e^2 - 3) Z_R(\zeta_e) - \left[1 + \frac{8}{3}\zeta_e^2 \left(\zeta_e^2 - \frac{3}{2} \right) \right] Z_R'(\zeta_e) \right\} \tag{62}$$

where the variables are defined identically to those used in equations (58) and (59) and equation (57) has been used. We also note that since $\zeta_c < 1$ is a requirement the term in ζ_c in equation (62) can generally be dropped unless ζ_e is appreciably greater than unity. At this point it might be useful to re-iterate the assumptions implicit in equations (58) through (60):

1) low frequency: $\omega_R \ll \Omega_i$
2) Electrostatic modes only
3) Chapman-Enskog approximation → collision dominated plasma → $\nu_e < \nu_{te}$ or $\zeta_c < 1$
4) Heat flux parallel to $\vec{B}$ -field
5) Maxwellian ions
6) $|\gamma| \ll |\omega_R|$.

HEAT FLUX INSTABILITIES

In what follows we will set u = 0 so there is no electron current and no net drift velocity. We will also assume only one ion species so $n_i = n_e$ and the sum over ion species will be removed. With $\zeta_c = \frac{\nu_e}{\nu_{te}}$ the equations now become:

$$-(1 + 2k^2 \lambda_{De}^2) = \frac{T_e}{T_i} \sum_m \Gamma_m(\mu_i) \left[1 + \zeta_i Z_R(\zeta_i - m\zeta_{cyc}) \right] + \zeta_e Z_R(\zeta_e)$$

$$+ \zeta_c \left\{ \frac{8}{3}\zeta_e (1 - \zeta_e^2) - \left[1 + \frac{8}{3}\zeta_e \left(\zeta_e^2 - \frac{3}{2} \right) \right] Z_R(\zeta_e) \right\} \tag{63}$$

$$\xi_e = \xi_e \left\{ \frac{1 + c_i \sum \Gamma_m(\mu_i)\, e^{-(\xi_i - m\xi_{ci})^2 + \xi_e^2}}{1 + \frac{4}{3}\xi_e^2\left(\xi_e^2 - \frac{3}{2}\right)} \right\} \tag{64}$$

As long as $\xi_e \ll 1$ the major difference between these equations and the analogous equations for the current driven instability is that the conduction speed doesn't occur in the arguments of the electron z-functions in the heat flux case while the drift speed does appear in these arguments in the current driven case. When $T_e/T_i \gg 1$, $\frac{V_{e-i\Delta}}{V_{te}} \ll 1$, and the equations for the two cases are identical. We can then directly take over the results of Kindel and Kennel, replacing the drift speed (their V_D) by the heat conduction speed. At high temperature ratios then the mode is ion acoustic in nature with $\frac{k_\perp}{k_\parallel} \ll 1$. This is done below.

On the other hand, when $T_e/T_i \sim 1$, the critical conduction speed (the conduction speed above which the mode is unstable) becomes of order the electron thermal speed.

Then the terms in ξ_e will be important and the nature of the instabilities will differ somewhat from the current driven case. As we shall see the instability takes on the nature of an ion cyclotron wave for $T_e/T_i \sim 1$ characterized by largely perpendicular propagation, $k_\perp/k_\parallel \gg 1$. This is true for both the current driven or heat flux driven instabilities but the details are somewhat different.

The major difference between the two cases occurs for $T_e/T_i \ll 1$. In the case of a heat flux driven instability the critical conduction speed becomes larger than the electron thermal speed. Since it is not possible to conduct heat faster than the thermal speed of the particles the plasma is stable with respect to this source of free energy. To transport energy more rapidly than V_{te} it is necessary to direct the electrons rather than let them diffuse: This is a current and gives rise to the current driven instability for sufficiently large drift speeds.

We will now summarize the results for $T_e/T_i \gg 1$ (since the analysis is identical to that given in Kindel and Kennel), briefly consider how the results differ from the current driven case when $T_e/T_i \sim 1$, and then present graphs of the numerical results for both ranges.

$$\frac{T_e}{T_i} \gg 1 \ :$$

We will consider the ion acoustic approximation separately from the ion cyclotron case and show that the former has the lower critical drift speed in this temperature regime and so the mode is actually ion acoustic in nature.

ⓐ Ion Acoustic mode: Returning to equations (63) and (64) we set $k_\perp = 0$ so the wave is propagating solely along the field lines. This implies that $\mu_i = 0$ and since:

$$T_m(0) = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}$$

the equations become:

$$1 + 2k^2\lambda_{0e}^2 = \frac{T_e}{2T_i} Z_R'(\xi_i) + (\xi_c - \xi_e) Z_R(\xi_e) - \frac{8}{3}\xi_e\xi_e\left[1 - \xi_e^2 + (\tfrac{1}{2} - \xi_e^2) Z_R(\xi_e)\right] \tag{65}$$

$$\xi_c = \xi_e\left[\frac{1 + c_i e^{-\xi_i^2}}{1 - \frac{8}{3}\xi_e^2(\frac{3}{2} - \xi_e^2)}\right] \tag{66}$$

where $\xi_e \ll \xi_i$ was used in the second equation. We expect a result which will roughly be $\omega \approx k_\parallel c_s$ which would then imply:

$$\xi_i \sim \sqrt{T_e/T_i} \gg 1 \tag{67}$$

$$\xi_e \sim \sqrt{m_e/m_i} \ll 1 . \tag{68}$$

We will therefore assume $\xi_e \ll 1 \ll \xi_i$ and use the approximations:

$$Z_R(\xi) \approx \begin{cases} -2\xi & \xi \ll 1 \\ -\frac{1}{\xi} & \xi \gg 1 \end{cases} \tag{69}$$

for the electron and ion plasma dispersion functions. The dispersion equation then becomes:

$$\omega^2 = \frac{k_\parallel^2 c_s^2}{1 + 2k_\parallel^2 \lambda_{De}^2} \tag{70}$$

where $c_s^2 = \frac{k_B T_e}{m_i}$ is the sound speed (k_B is Boltzmann's constant) which matches our expectations. The equation for the conduction speed is now:

$$\xi_c = \xi_e \left[1 + c_i e^{-\xi_i^2}\right] . \tag{71}$$

To find the mininum conduction speed we differentiate this with respect to $v_p = \frac{\omega}{k_\parallel}$ and equate the result to zero:

$$c_i (2\xi_{iM}^2 - 1) e^{-\xi_{iM}^2} = 1 \tag{72}$$

which has the approximate solution for large c_i:

$$\xi_{iM}^2 \approx \ln 2c_i \tag{73}$$

where ξ_{iM} is the value of ξ_i which minimizes ξ_c. Using this in equation (71) yields the minimum marginal conduction speed:

$$\frac{v_{cM}}{v_{ti}} \approx \sqrt{\ln 2c_i} \tag{74}$$

or for a hydrogen plasma:

$$\frac{v_{cM}}{v_{ti}} \approx \sqrt{4.5 + \frac{3}{2} \ln\left(\frac{T_e}{T_i}\right)} \tag{75}$$

for the ion acoustic mode with $T_e/T_i \gg 1$. For large T_e/T_i the critical drift speed is only a few times the ion thermal speed.

In this temperature regime v_ϕ/v_{ti} is generally large (from eq (70)) so ion Landau damping is weak. If the conduction speed exceeds the parallel phase velocity then the electron distribution will have a positive slope at a speed equal to v_ϕ so the electron Landau instability will cause the wave to grow.

Now consider the growth rate in this limit. With the above approximations it becomes:

$$\frac{\Gamma_{IA}}{\sqrt{\pi}\, k_\| v_{te}} \simeq -\frac{\frac{T_e}{T_i}\xi_i e^{-\xi_i^2} + (\xi_e - \xi_c) e^{-\xi_e^2}}{\frac{m_e}{m_i}\left(\frac{1}{\xi_e^3}\right) - 4\xi_e + \frac{14}{3}\xi_c} \tag{76}$$

and since $\xi_e \sim \sqrt{\frac{m_e}{m_i}}$ the last two terms in the denominator may be neglected. Then, using equation (70):

$$\frac{\Gamma}{\omega} = -\left[\frac{\pi}{8x^3}\frac{m_e}{m_i}\right]^{1/2}\left[c_i e^{-\frac{1}{x^2}\left(\frac{T_e}{2T_i}\right)} + \left(1 - \frac{v_d}{c_s}x\right)\right] \tag{81}$$

where $x = (1 + 2k_\|^2\lambda_{De}^2)^{1/2}$ AND $e^{-\left(\frac{m_e}{2m_i}\right)\frac{1}{x^2}} \approx 1$.

(b) Electrostatic Ion Cyclotron Mode: Following Kindel and Kennel we note that for ion cyclotron waves $\mu_i \sim 1$ in this temperature limit so we can neglect $k^2\lambda_{De}^2$. Further we can use $\xi_i - n\xi_{cyc} \gg 1$ for all the ion terms since $k_\|$ is small for cyclotron waves. For temperature ratios near unity $\omega \sim \Omega_i$ but as T_e/T_i increases ω will increase to keep damping from the n = 1 mode small. If ω were to get close $2\Omega_i$ then the n = +2 damping will become important. Thus we will have $\omega \approx \frac{3}{2}\Omega_i$. Then Kindel and Kennel find the minimum critical conduction speed to be:

$$\frac{v_{cM}}{v_{ti}} \approx 3\sqrt{\ln\left(\frac{v_{te}}{v_{ti}}\right)} \tag{82}$$

or, for a hydrogen plasma:

$$\frac{V_{dm}}{V_{ti}} \simeq 3\sqrt{3.8+\frac{1}{2}\ln\left(\frac{T_e}{T_i}\right)} \quad . \tag{83}$$

The perpendicular wavenumber is roughly given by:

$$\left(\frac{k_\perp c_s}{\Omega_i}\right)^2 \approx 1 \approx \mu_i\left(\frac{T_e}{T_i}\right) \tag{84}$$

and:

$$\frac{k_\parallel}{k_\perp} \approx \frac{1}{2}\left[\frac{\frac{T_e}{T_i}}{2\ln\left(2V_{de}/V_{ti}\right)}\right] . \tag{85}$$

As T_e/T_i increases the wave changes to more nearly parallel propagation. We also see this in comparing equations (83) and (75); the ion acoustic wave has a smaller critical drift velocity then the ion cyclotron. The mode becomes increasingly more like an ion acoustic wave as the temperature ratio increases. Because of this it is not useful to calculate a growth rate in the ion cyclotron limit.

$$\frac{T_e}{T_i} = \underline{1}:$$

From the numerical solution of equations (63) and (64) we see that when $T_e/T_i \simeq 1$ then $\xi_e \sim .5$ while $\zeta_e \sim .3$. Consequently, even though they don't make a large contribution, the terms in $\xi_e\zeta_e$ in equation (63) should not be dropped. We will however drop the terms in ζ_e^2 in the denominator of equation (64).

Then, keeping only the largest terms, we have:

$$-(1+2k^2\lambda_{De}^2) = \frac{T_e}{T_i}\sum_m \Gamma_m(\mu_i)\left[1+\zeta_i Z_R(\zeta_i - m\xi_{ce})\right] + (\xi_e-\zeta_e)Z_R(\zeta_e) + \frac{2}{3}\xi_e\zeta_e \tag{86}$$

$$\zeta_e = \xi_e\left[1 + c_i \sum_m \Gamma_m(\mu_i)\, e^{-(\xi_i - m\xi_{ce})^2 + \xi_e^2}\right] \qquad . \qquad (87)$$

Now, from the equation for the growth rate we see that we must have $\zeta_e > \xi_e$ $(v_e > v_\phi)$ in order that the electron terms contribute to growth rather than damping. Consequently we will expand $Z_R(\xi_e)$ for small ξ_e and drop terms of order ξ_e^4 compared to $\xi_e \zeta_e$. Then equation (86) is:

$$-(1+2k^2\lambda_{De}^2) = \frac{T_e}{T_i}\sum_m \Gamma_m(\mu_i)\left[1 + \xi_i Z_R(\xi_i - m\xi_{ce})\right] + \frac{14}{3}\zeta_e\xi_e \, . \qquad (88)$$

In the ion acoustic case in the high temperature limit this changes equation (70) to:

$$\left(\frac{\omega}{k_\parallel}\right)^2 = \frac{c_s^2}{1 + 2k_\parallel^2\lambda_{De}^2 + \frac{14}{3}\zeta_e\xi_e} \qquad (89)$$

and since $\zeta_e\xi_e > 0$ this means that the phase velocity would be less than expected when this term is ignored.

ⓐ Ion Acoustic mode:

As before we set $k_\perp = 0$ so $\mu_i = 0$ to get:

$$1 + 2k_\parallel^2\lambda_{De}^2 = \frac{1}{2} Z_R'(\xi_i) - \frac{14}{3}\xi_e\zeta_e \qquad (90)$$

$$\xi_e = \zeta_e\left[1 + c_i e^{-\xi_i^2}\right] . \qquad (91)$$

We note that the corresponding limit in the current driven case has equation (90) without the last term. There is no solution to that equation for positive $k^2\lambda_{De}^2$ since the maximum value of $Z_R'(\xi_i)$ is about .57 at $\xi_i \approx 1.5$. In fact there is no solution for $k^2\lambda_{De}^2 > 0$ for $T_e/T_i \lesssim 3.5$.

The inclusion of the extra term in equation (90) will allow solutions for lower values of T_e/T_i than this. Equation (91) is identical to (71) and so leads to the same minimization of eq (72). Solving (72) numerically for $T_e/T_i = 1$ yields:

$$\xi_{im} = .72$$
$$\xi_{em} = .017$$
$$\xi_{cm} = .31.$$

Substituting these values into equation (90) gives:

$$k^2\lambda_{De}^2 = -.62.$$

This violates the initial assumption that k is real and ω complex. This same situation also occurs for smaller values of T_e/T_i .

The identical problem appears in the case of a current driven instability and is discussed by Kindel and Kennel. The resolution is that the smallest critical value of $k_{//}$ is zero (so the real frequency is also putatively zero). However, a small increase in the drift velocity above the critical value results in a shift to $\omega \approx 1.3\,\omega_{pi}$ (the ion plasma frequency) and to $k_{//}\lambda_{De} \sim 1$. In practice only ion oscillations near ω_{pi} will be observed. We shall also see that near $T_e/T_i \sim 1$ the mode is basically Ion Cyclotron rather than Ion Acoustic which is why the ion acoustic approximatly gives a poor result.

We see that as T_e/T_i decreases the critical conduction speed increases from a few times the ion thermal speed to nearly the electron thermal speed. When $T_e/T_i \sim 1$ the wave's phase velocity is roughly the same as the ion thermal speed for conduction speeds not greatly exceeding the critical value. For example, when $v_c/v_{cm} = 2$ the most unstable ion acoustic wave has $v_p = 1.6\, v_{ti}$ so ion Landau damping is strong and a large electron drift is required for instability. If v_c/v_{cm} increases to large values then $v_p \gg v_{ti}$ and ion damping can be neglected. This leads to the Buneman[6] instability.

Since the critical value of ξ_i lies near unity no simplifications can be easily made in the equation for the growth rate.

⑤ Electrostatic Ion Cyclotron Wave:

For this wave $k_\parallel/k_\perp \ll 1$ so in general:

$$\xi_i + m\xi_{cri} = \frac{\omega - m\Omega_i}{k_\parallel v_{ti}} \gg 1 \tag{92}$$

will be assumed and then equation (69) can be used to approximate the ion dispersion function. Further we will see that $\mu_i \sim 1$ so we can assume

$$k^2 \lambda_{De}^2 \ll 1$$

and drop this term from the equation. Also we will again expand $Z_R(\xi_e)$ in the small ξ_e limit. Thus equation (88) becomes:

$$1 + \frac{T_i}{T_e} = \sum_m \frac{\omega \Gamma_m(\mu_i)}{\omega - m\Omega_i} - \frac{14}{3}\xi_e \xi_e \, . \tag{93}$$

Now we require that $\omega \sim \Omega_i$ while still satisfying equation (92). This again requires that $k_{/\!/}$ be small. Then using the addition rules of Bessel functions equation (93) can be re-written as:

$$1 + \frac{T_e}{T_i} - G(\mu_i) = \frac{\Gamma_1(\mu_i)}{\omega - \Omega_i} - \frac{14}{3}\xi_e \xi_e \tag{94}$$

while the equation for the critical conduction speed is:

$$\xi_c = \xi_e\left[1 + c_i e^{-(\xi_i - \xi_{cri})^2}\right] \tag{95}$$

where $$G(\mu_i) = \Gamma_1(\mu_i) + \frac{1 - \Gamma_0(\mu_i)}{\mu_i} \, . \tag{96}$$

"Solving" equation (94) for ω (ignoring the fact that ξ_e and ξ_e are functions of ω would give

$$\omega = \Omega_i(1 + \Delta) \tag{97}$$

where $\Delta \ll 1$ and we see that the inclusion of the term $\frac{4}{3}\xi_e \xi_c$ tends to decrease Δ putting ω closer to Ω_i. It is not particularly worth while to carry the analysis further for this case. The numerical solution of equations (63) and (64) (with the requirement that v_c be a minimum) is given on the following pages and contrasted with the solution to the case of a current driven instability and as demonstrated above there is no significant difference until $T_e/T_i \sim 1$.

As can be seem from these results and those of Kindel and Kennel, the minimum critical conduction speed is smaller for the ion cyclotron mode than for the ion acoustic. Consequently the instability takes on the characteristics of an ion cyclotron wave as the temperature decreases to unity. We also note that as T_e/T_i decreases to unity ξ_c increases so there is some temperature ratio below which no physical heat flux can drive the plasma unstable: the threshold to too large.

Let us now find an expression for the growth rate for the ion cyclotron mode using the same set of approximations. From equations (60-62):

$$\frac{\Gamma}{\sqrt{\pi}\, k_\parallel v_{te}} = \frac{\frac{T_e}{T_i}\xi_i \Gamma_1 e^{-(\xi_i - \xi_{cyc})^2} + (\xi_{i1} - \xi_{i})\, e^{-\xi_e^2}}{c_i \sum_n \frac{n \Gamma_n \xi_{cyc}}{(\xi_i - n\xi_{cyc})^2} + 4\xi_e - \frac{14}{3}\xi_c} \tag{98}$$

or:

$$\frac{\Gamma}{\omega_R} = \sqrt{\pi}\, \frac{c_i \Gamma_1(\mu_i)\, e^{-\left(\frac{\omega - \Omega_i}{k_\parallel v_{ti}}\right)^2} + \left(1 - \frac{k_\parallel v_c}{\omega}\right) e^{-\left(\frac{\omega}{k_\parallel v_{te}}\right)^2}}{\frac{T_e}{T_i}\left(\frac{1}{\xi_e}\right) \sum_n \frac{n \Gamma_n(\mu_i)}{\left(\frac{\omega}{\Omega_i} - n\right)^2} + 4\xi_e - \frac{14}{3}\xi_c} \tag{99}$$

Now, $\Gamma_n(\mu_i) < 1$ for all μ_i since $\sum_n \Gamma_n = 1$ and $\Gamma_{-n} = \Gamma_n$. Thus all the terms in the sum are smaller than unity except perhaps for n = 1 and since $\omega \sim \Omega_i$ for large n the sum behaves as $\sum_n \frac{\Gamma_n}{n}$ and so the contribution to the sum is small for large n. An order of magnitude approximation to the sum is thus given approximately by the n = 1 term:

$$\sum_m \frac{m\,\Gamma_m(\mu_i)}{\left(\frac{\omega}{\Omega_i}-m\right)^2} \sim \frac{\Gamma_1(\mu_i)}{\Delta^2}$$

where Δ is defined by equation (97) and is small. Since the solution here is not very different from the current driven case we can drop the last term in equation (94) to get an approximate solution for Δ :

$$\Delta \approx \frac{\Gamma_1}{1-G+T_i/T_e}$$

where G is defined by equation (96). The approximation to the sum is then:

$$\sum_m \frac{m\,\Gamma_m(\mu_i)}{\left(\frac{\omega}{\Omega_i}-m\right)^2} \sim \left(1-G+\frac{T_i}{T_e}\right)^2 \frac{1}{\Gamma_1}$$

and since $|G(\omega)| < 1$ while $\frac{T_e}{T_i} \sim 1$ for this case the sum is within an order of magnitude of unity. Since it is multiplied by $\frac{1}{\zeta_e}$ and $\zeta_e \ll 1$ we see that the last two terms in the denominator of equation (99) can be neglected since they are less than unity. Consequently, the growth rate can be re-written as:

$$\frac{\Gamma}{\omega} \simeq \sqrt{\pi}\; \frac{\zeta_i\,\Gamma_1(\mu_i)\,e^{-(\zeta_i-\zeta_{cyc})^2} + \frac{T_e}{T_i}(\zeta_e-\zeta_c)\,e^{-\zeta_e^2}}{\sum_m \frac{m\,\Gamma_m(\mu_i)}{(1-m+\Delta)^2}} \tag{100}$$

for $\frac{T_e}{T_i} \sim 1$ and Δ given by equation (97).

On the following pages are given graphs of the dispersion equation and growth rate as well as the minimum critical drift speed, $\frac{\omega}{\Omega_i}$, $k_\perp/k_\parallel$ and μ_i all as functions of T_e/T_i .

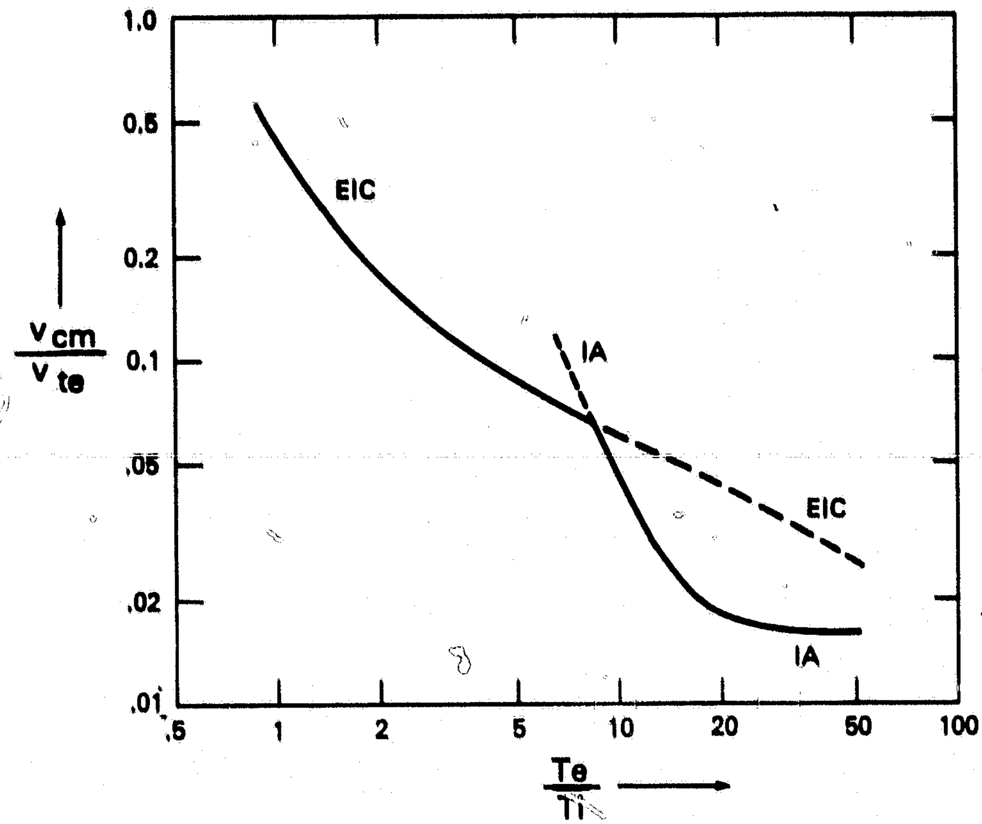

Figure 1. Minimum Critical Drift Velocity as a Function of Te/Ti

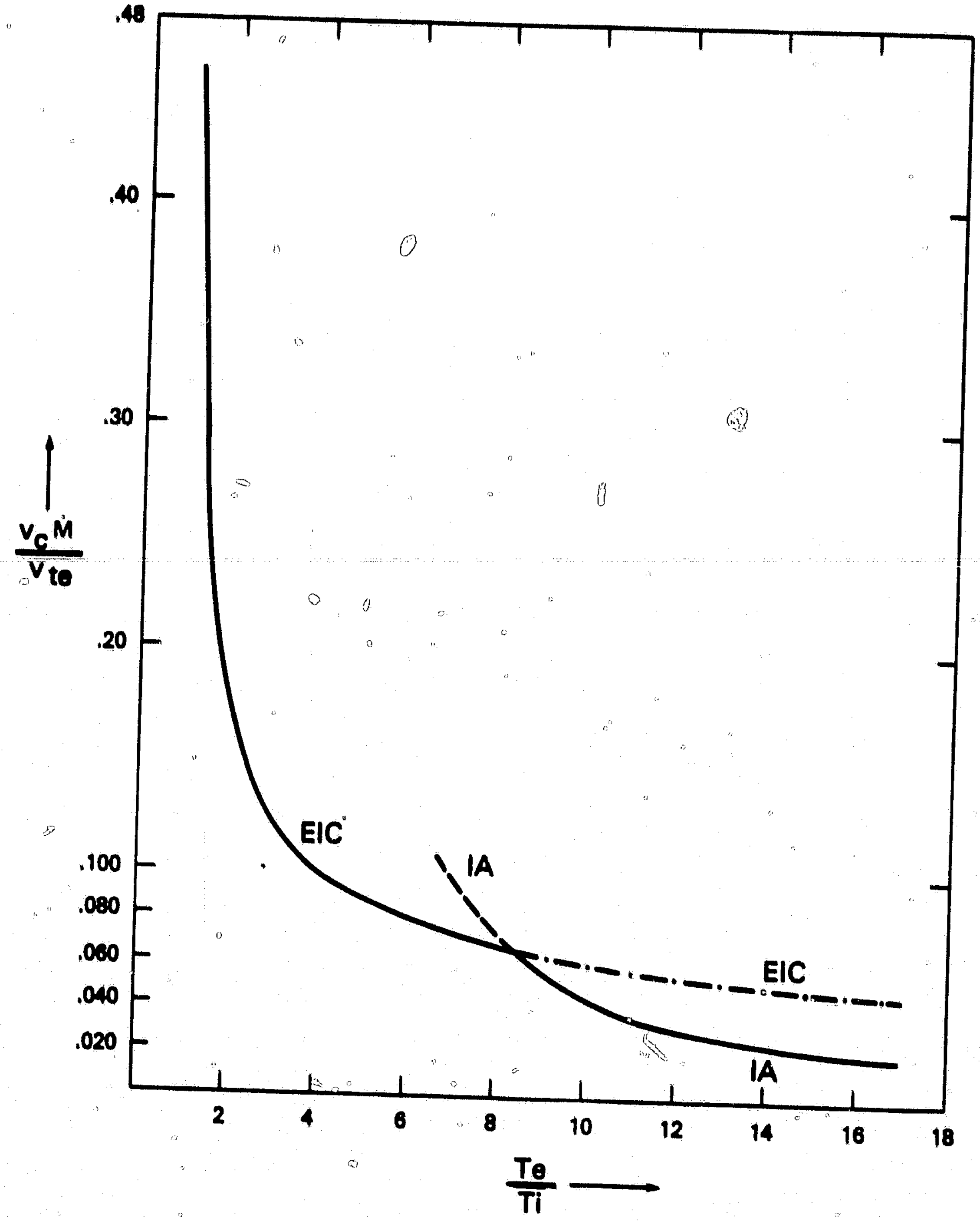

Figure 2. Minimum Critical Drift Velocity as a Function of Te/Ti

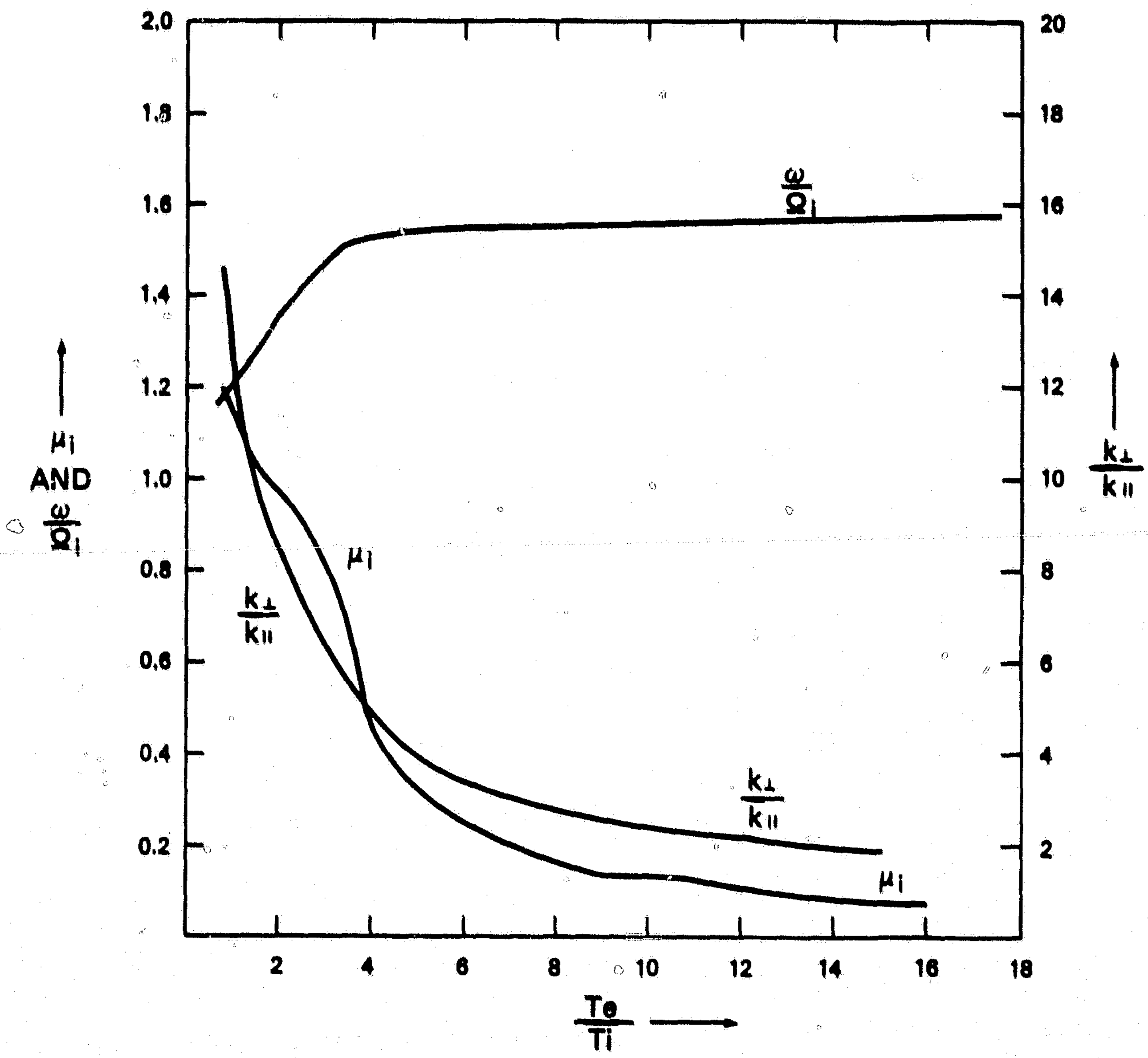

Figure 3. Graphs of μ_i, ω/Ω, and $k_\perp/k_\parallel$ for the EIC Mode (with $v_c = v_c$ min) as Functions of Te/Ti

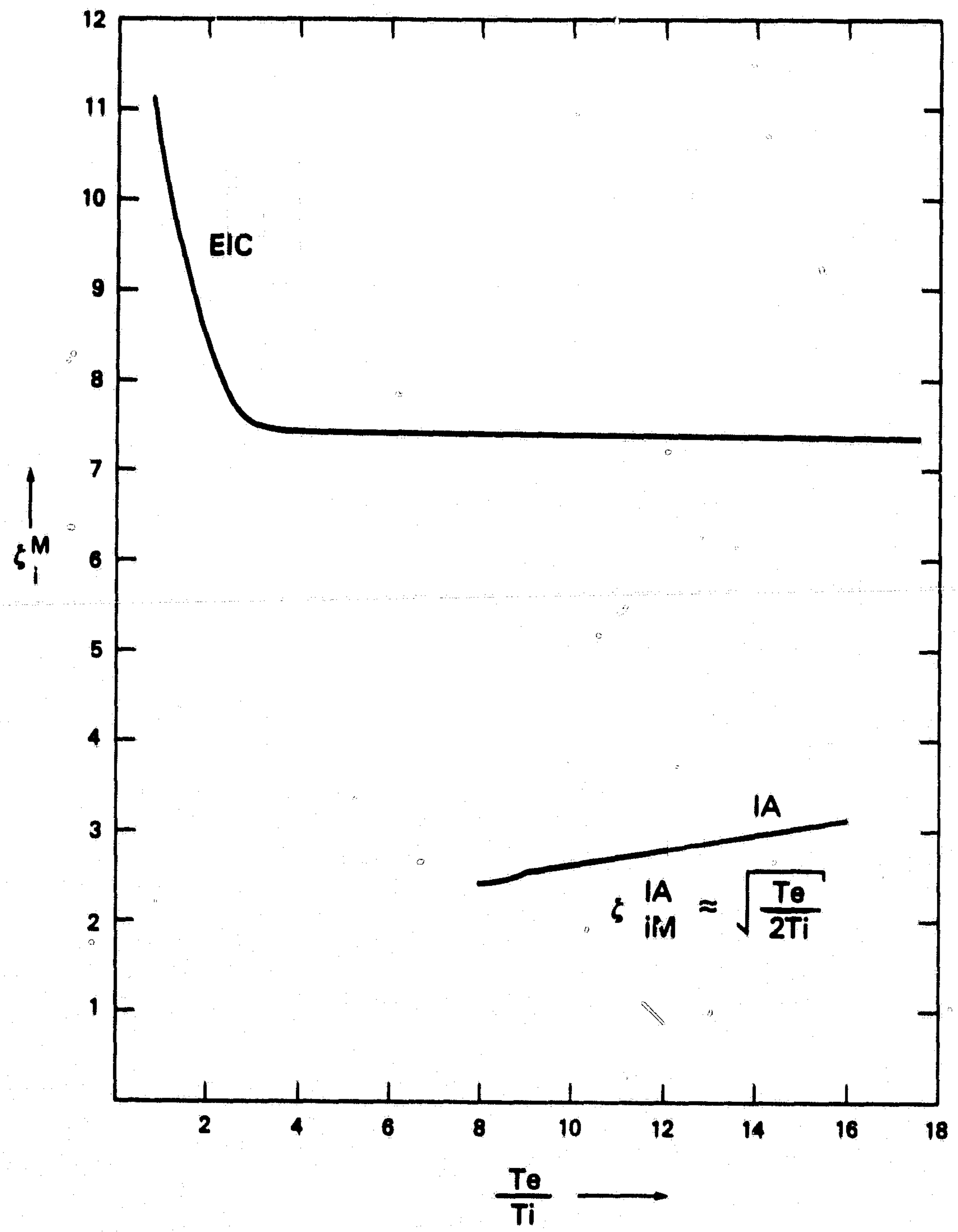

Figure 4. Graph of ζ_i as a Function of Te/Ti for the EIC and IA Branches Assuming $v_c = v_c$ min For Each

DISCUSSION

The major point to note is that, unlike the current driven instability, the heat flux instability will not "turn on" when $\frac{Te}{Ti} \lesssim 1$ since the requirement on the minimum critical heat flux is that it be unphysically large (ie., $\rho_{cm} > 1$). The only way in which electrons can transfer energy faster than the electron thermal speed is by directed (rather than random) motion. This is a current and gives the current instabilities.

The second point to note is that the graphs given here for the heat flux instability do not markedly differ from those Kindel and Kennel have produced for the current instability. This has already been mentioned in the analysis for $\frac{Te}{Ti} >> 1$.

From the graphs it can be seen that the results here differ from Kindel and Kennel's by 10-20% near $\frac{Te}{Ti} = 1$ and are essentially identical when $\frac{Te}{Ti}$ is greater than 3 to 4. This means that the two instabilities can be "combined" when $\frac{Te}{Ti}$ is greater than unity. That is, the fact that u and v_c don't enter into the real and imaginary parts of the dispersion equation in an identical manner is irrelevant for $\frac{Te}{Ti} > 1$. Thus we can take the critical drift velocity curves and replace either u (in Kindel and Kennel's curves) or v_c (in our curves) by the combination $u + v_c$ for $\frac{Te}{Ti} > 1$. For example, the minimum critical velocity for the instability to occur with $\frac{Te}{Ti} = 3.0$ is about .12. Thus we can say that if the sum of the heat flux drift velocity and the current drift velocity:

$$v_c + u \geqslant .12$$

then the instability will occur. Thus, the existence of a heat flux can cause a current carrying plasma to be unstable to the EIC or IA modes even if it were not unstable with the current alone (and vice versa). In a solar flare, for example, the conditions which lead to a current also will give rise to a heat flux and the combination make the plasma less stable than either alone.

The third point of interest is that the two modes, EIC and IA, exist as separate entities with the lower minimum critical velocity for the EIC mode when $\frac{Te}{Ti} \leqslant 8.5$ while the IA mode has the lower minimum critical velocity for $\frac{Te}{Ti} \geqslant 8.5$. The EIC mode is characterized by $\frac{k_\perp}{k_\parallel} > 1$ and the IA mode by $\frac{k_\perp}{k_\parallel} \leqslant 10^{-5}$ (from the numerical solutions). Consequently when $\frac{Te}{Ti}$ is somewhat less than or greater than 8.5 the two modes will both occur, one propagating parallel to the field and the other perpendicular to it, if the actual drift velocity is somewhat greater than the minimum critical value. For example, if the drift velocity is roughly 30% above the minimum critical value then both wave modes would co-exist in the range $\frac{Te}{Ti}$ from 7.0 to 10.0. Since particle acceleration in solar flares depends on the presence of the EIC mode[7] this gives a wider possible range in temperature over which it can occur if the drift velocity is only slightly above the minimum value.

Some results not found in the graphs or the analysis but which are seen in the numerical results are that some parameters can be changed by large amounts with correspondingly small changes in v_c. In the EIC mode calculations it was found that the minimum critical drift velocity is only weakly dependant on the value of $\frac{Te}{Ti}$. For example, at $\frac{Te}{Ti} = 2.0$, a change of 20% in $\frac{k_\perp}{k_{\parallel}}$ produced only a 2% change in v_c and this is a typical result. Thus the growth rate of the EIC mode is not strongly peaked about one value of $\frac{k_\perp}{k_{\parallel}}$; when v_c slightly exceeds v_{cm} we would expect that the destabilized wave will contain a spread in $\frac{k_\perp}{k_{\parallel}}$. Thus the wave will propagate in a cone around the magnetic field and the angular thickness of the cone will be appreciable compared to the angle the side of the cone makes with the field line.

For the IA mode, the preferred direction of propagation is very nearly exactly parallel to the magnetic field with $k_\perp/k_{\parallel} < 10^{-5}$. On the other hand, although v_{cm} depends strongly on ζ_i, if we write ζ_i as:

$$\zeta_i = \left(\frac{\omega}{\Omega i}\right)\zeta_{cyc}$$

then we find that changes in ζ_{cyc} and $\frac{\omega}{\Omega i}$ as large as a factor of 2 (but which leave ζ_i constant to six significant figures) change v_c only in the sixth significant figure. We expect then that the IA mode, when unstable, will propagate with a wide range in ω.

One further point can be made. For $Te/Ti >> 1$ we found that the IA mode was unstable with lowest v_c and dispersion equation given by equation (70). If the term $k_{\parallel}{}^2 \lambda_{de}^2$ in the denominator is ignored this further simplifies to:

$$\omega^2 = k_{||}^2 C_s^2$$

or:

$$\zeta_{im} = \sqrt{\frac{Te}{2Ti}}.$$

We see that the computer generated graph of ζ_{im} under these conditions duplicates this result to within about 10%. If we use this in equation (71) we find that, for $\frac{Te}{Ti} >> 1$:

$$\frac{v_{cm}}{v_{te}} \simeq .0174$$

and this depends on temperature only very weakly.

SUMMARY

We have investigated the plasma physics of the Electrostatic Ion cyclotron and Ion Acoustic modes as driven by a heat flux. We find that no physical heat flux can make the plasma become unstable to these modes when $\frac{Te}{Ti} < 1$. For $\frac{Te}{Ti}$ in the range 1 to 4 our results differ from the case of a current driven instability by less than 20% while for $\frac{Te}{Ti}$ greater than this the two situations are virtually identical. We can consequently use the curve for the minimum critical drift speed with this speed replaced by the sum $(v_c + u)_m$ where v_c is the electron conduction speed and u is the current drift speed.

The general dispersion equations have been given as well as simplified versions for $\frac{Te}{Ti} \sim 1$ and $\frac{Te}{Ti} >> 1$. The exact equations have been solved numerically and graphs of the results given.

In an appendix the first order correction term in $\frac{k_\perp}{k_{||}}$ is given for the Ion Acoustic mode.

Appendix A: First order correction term in $\frac{k_\perp}{k_\parallel}$ for the Ion Acoustic Mode.

The lack of information on $\frac{k_\perp}{k_\parallel}$ comes from the approximation $\mu_i = 0$. Let us consider the first order correction in μ_i:

$$\Gamma_m(\mu_i) = e^{-\mu_i} I_m(\mu_i) \tag{A1}$$

and:

$$e^{-\mu_i} \approx 1 - \mu_i \tag{A2}$$

$$I_m(\mu_i) \approx \frac{1}{m!} \left(\frac{\mu_i}{2}\right)^m \tag{A3}$$

so to the first order in μ_i:

$$\sum_m \Gamma_m(\mu_i) Z_R(\zeta_i - m\zeta_{cyc}) \simeq Z_R(\zeta_i) - \mu_i \left[Z_R(\zeta_i) - \tfrac{1}{2} Z_R(\zeta_i - \zeta_{cyc})\right] \tag{A4}$$

Then, with $\zeta_c \zeta_e << 1$ equation (63) becomes:

$$1 + 2k_\parallel^2 \lambda_{de}^2 = \frac{Te}{Ti}\left\{\tfrac{1}{2} Z_R'(\zeta_i) + \mu_i \zeta_i \left[Z_R(\zeta_i) - \tfrac{1}{2} Z_R(\zeta_i - \zeta_{cyc})\right]\right\} + (\zeta_c - \zeta_e) Z_R(\zeta_e) \tag{A5}$$

and the same approximations in equation (64) yield:

$$\zeta_c = \zeta_e \left\{1 + C_i e^{-\zeta_i^2} + \zeta_e^2 \left[1 - \mu_i \left(1 - \tfrac{1}{2} e^{\zeta_c \zeta_c} \left(2\frac{\omega}{\Omega_i} - 1\right)\right)\right]\right\} \tag{A6}$$

Now, if we assume ζ_i and $\zeta_i - \zeta_{cyc} >> 1$ and $\zeta_e << 1$ (so we can neglect the last term in equation (A5)):

$$1 + 2k_\parallel^2 \lambda_{de}^2 = \left(\frac{1}{\zeta_i^2}\right) - \frac{Te}{2T_i} \mu_i \left(\frac{\zeta_i - 2\zeta_{cyc}}{\zeta_i - \zeta_{cyc}}\right) \tag{A7}$$

using equation (69). This can be re-written as:

$$1 + 2k_\parallel^2 \lambda_{de}^2 = \left(\frac{k_\parallel C_S}{\omega}\right)^2 - \tfrac{1}{2} \left(\frac{\frac{\omega}{\Omega_i} - 2}{\frac{\omega}{\Omega_i} - 1}\right) \left(\frac{k_\perp C_S}{\omega}\right)^2 \tag{A8}$$

or:

$$1 + 2k_\parallel^2 \lambda_{de}^2 = \left(\frac{k_\parallel C_S}{\omega}\right)^2 \left[1 - \tfrac{1}{2}\left(\frac{k_\perp}{k_\parallel}\right)^2 \left(\frac{\omega}{\Omega_i}\right)^2 \left(\frac{\frac{\omega}{\Omega_i} - 2}{\frac{\omega}{\Omega_i} - 1}\right)\right] \tag{A8}$$

so we can clearly see the correction term in $\frac{k_\perp}{k_\parallel}$.

"Solving" this for ω gives:

$$\omega^2 = \frac{k_{\parallel}^2 C_S^2}{1 + 2k_{\parallel}^2 \lambda_{de}^2}\left[1 - \tfrac{1}{2}\left(\frac{k_\perp}{k_{\parallel}}\right)^2 \left(\frac{\omega}{\Omega_i}\right)^2 \left(\frac{\frac{\omega}{\Omega_i} - 2}{\frac{\omega}{\Omega_i} - 1}\right)\right] \tag{A9}$$

so we can see (since $\frac{\omega}{\Omega_i} > 1$) that the correction term reduces the frequency below the standard Ion Acoustic result of equation (70).

In terms of the variables used in the computer solution equation (A8) can be written as:

$$1 + 2k_{\parallel}^2 \lambda_{de}^2 = \frac{Te}{2Ti} \quad \frac{1}{\zeta_i^2}\left[1 - \tfrac{1}{2}\omega^2 v^2 \left(\frac{v-2}{v-1}\right)\right] \tag{A10}$$

or, solving for ω:

$$\omega^2 = \frac{2}{v^2}\left(\frac{v-1}{v-2}\right)\left(1 - \frac{2\ x\ u^2 v^2}{Te/Ti}\right) \tag{A11}$$

where: $\zeta_i = uv$, $v = \frac{\omega}{\Omega_i}$ and $x = 1 + 2k_{\parallel}^2 \lambda_{de}^2$. We can estimate the last term from equation (73): $\zeta_{im}{}^2 \simeq \ln 2C_i$ so:

$$\omega^2 = \frac{2}{v^2}\left(\frac{v-1}{v-2}\right)\left(1 - \frac{2\ x\ \ln 2\,C_i}{Te/Ti}\right) \tag{A12}$$

and for large $\frac{Te}{Ti}$, $x \approx 2$. Using this value:

$\frac{Te}{Ti}$	$\frac{4 \ln 2Ci}{Te/Ti}$
10	3.2
20	1.8
30	1.3
40	1.0
50	.83
100	.45

so there is no solution for ω until $\frac{Te}{Ti} \geq 40$. This gives a rough estimate of where the approximations are valid. In this range $v >> 1$ so roughly:

$$\omega = \frac{\sqrt{2k}}{v} \tag{A13}$$

where k is the factor in the last set of parentheses in (A12).

As an example, for $\frac{Te}{Ti} = 50$:

$$\omega \approx \frac{.8}{v}.$$

REFERENCES

1. J. M. Kindel and C. F. Kennel, *Journal of Geophysical Research*, 76, 3055 (1971).
2. B. S. Tanenbaum, *Plasma Physics*, McGraw-Hill (New York), 1967.
3. A. Hasegawa, *Plasma Instabilities and Nonlinear Effects*, Springer-Verlag (New York), 1975.
4. B. D. Fried and S. D. Conte, *The Plasma Dispersion Function*, Academic (New York), 1961.
5. N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics*, McGraw-Hill (New York), 1973.
6. O. Buneman, *Phys. Rev*, 115, 503 (1959).
7. L. A. Fisk, *Ap. J.*, 224, 1028 (1978).