N81-14703

Generation of Orthogonal Boundary-Fitted Coordinate Systems

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ABSTRACT

A method is presented for computing orthogonal boundary-fitted coordinate systems for geometries with coordinate distributions specified on all boundaries. The system which has found most extensive use in generating boundary-fitted grids is made up of the Poisson equations

$$\begin{aligned} \xi_{xx} + \xi_{yy} &= P \\ \eta_{xx} + \eta_{yy} &= Q \end{aligned} \tag{1}$$

The functions P and Q provide a means for controlling the spacing and density of grid lines in the coordinate system. Since all calculations are done in the computational plane, the dependent and independent variables in Equation (1) are interchanged, giving the usual transformed equations

$$\begin{aligned} \alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} + J^2(Px_\xi + Qx_\eta) &= 0 \\ \alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} + J^2(Py_\xi + Qy_\eta) &= 0 \end{aligned} \tag{2}$$

where

$$\alpha = x_\eta^2 + y_\eta^2 \qquad \beta = x_\xi x_\eta + y_\xi y_\eta$$

$$\gamma = x_\xi^2 + y_\xi^2 \qquad J = x_\xi y_\eta - x_\eta y_\xi$$

The condition for orthogonality, i.e., ξ = constant lines perpendicular to η = constant lines, is $\beta = 0$, because

$$\beta = 0 \Rightarrow x_\xi / y_\xi = -y_\eta / x_\eta$$

which is equivalent to

$$1/y_x\Big|_{\eta = \text{constant}} = -y_x\Big|_{\xi = \text{constant}}$$

As a generating system based entirely on β, we consider

$$\beta_\xi = \beta_\eta = 0 \tag{3}$$

which can have an orthogonal solution only when $\beta = 0$ at the corners of the computational region. An iterative solution of the generating system given in Equation (3) is applied successfully to several geometries. While questions remain concerning the existence and uniqueness of orthogonal systems, the generating method presented here adds to the available, useful techniques for constructing these systems.

Figure 1 provides a comparison of two grids generated for a square region with nonuniform boundary coordinate spacing in both vertical and horizontal directions. The nonorthogonal mesh shown in Fig. 1a was generated using the Poisson system given by Equation (2) with $P \equiv Q \equiv 0$. Equation (2) was replaced with central difference formulae and the resulting system was solved by successive overrelaxation (SOR). The orthogonal mesh shown in Fig. 1b was obtained using Equation (3) as a generating system. Equation (3) was expanded and each derivative was replaced with the appropriate central difference formula. Again, the resulting system was solved by SOR iteration.

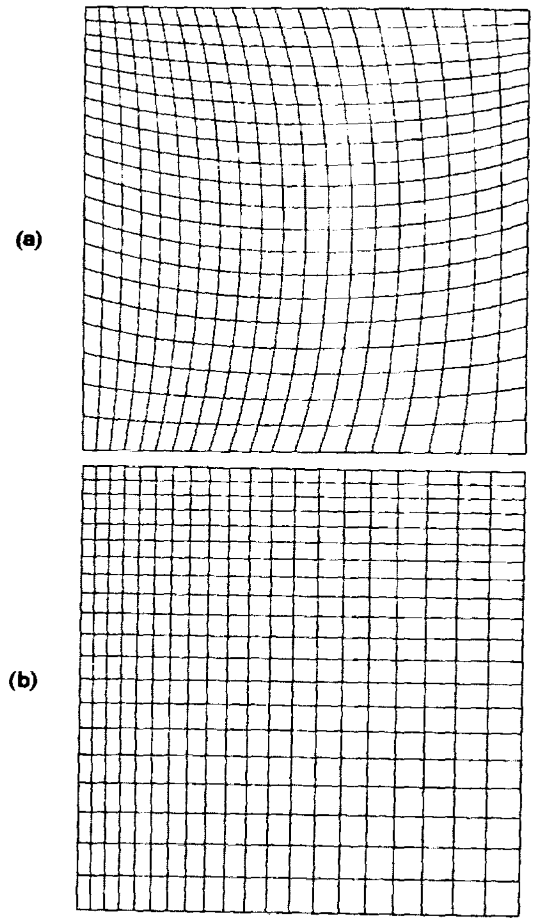


Figure 1

Two 21 x 21 girds generated for a simply-connected region with one convex boundary are shown in Figure 2. Fig. 2a shows a nonorthogonal coordinate system generated by Equation (2) with $P \equiv Q \equiv 0$ (a Laplace system); Fig. 2b shows a coordinate system generated by Equation (3). Note the orthogonality of the coordinate lines intersecting the curved upper boundary in Fig. 2 and the resultant bending of these lines in the interior.

(a)

(b)

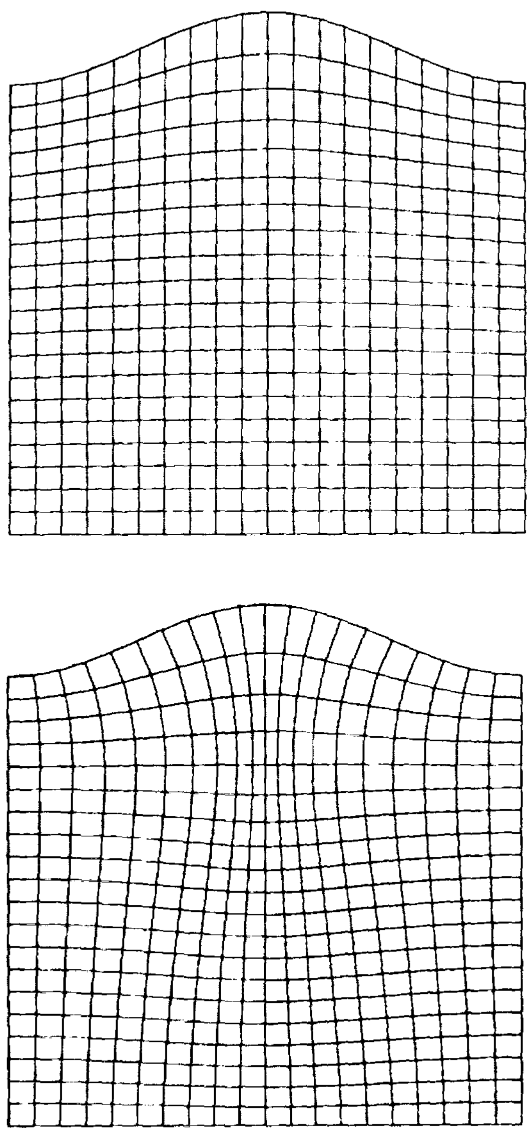
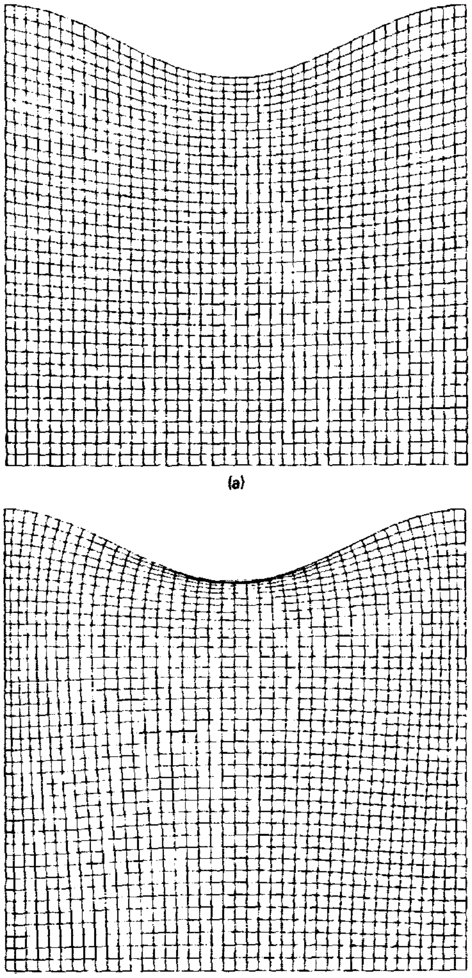


Figure 2

Figure 3 shows a region similar to that of Fig. 2 with a concave rather than convex curved boundary. As before, Fig. 3a shows a Laplace-generated grid and Fig. 3b shows a β-generated grid obtained using Equation (3). The orthogonal mesh must have rather fine spacing near the concave upper boundary to accommodate the curvature. To verify that the fine mesh spacing in Fig. 3b is due to the geometry and not to a singularity in the transformation, we have refined the mesh as seen in the next figure.



(a)

(b)

Figure 3

Figure 4 compares two different grids, one coarse with 1681 points and the other fine with 6561 points, generated for the concave region. The fact that corresponding grid lines are in about the same position in both meshes confirms that the coarse discretization yields a good approximate solution to the exact problem. A further confirmation comes from consideration of the Jacobian at the midpoint of the upper boundary. The value of the Jacobian computed on the coarse mesh is nonzero and agrees very well with the value computed on the fine mesh. There is no indication of a zero Jacobian anywhere in the region.

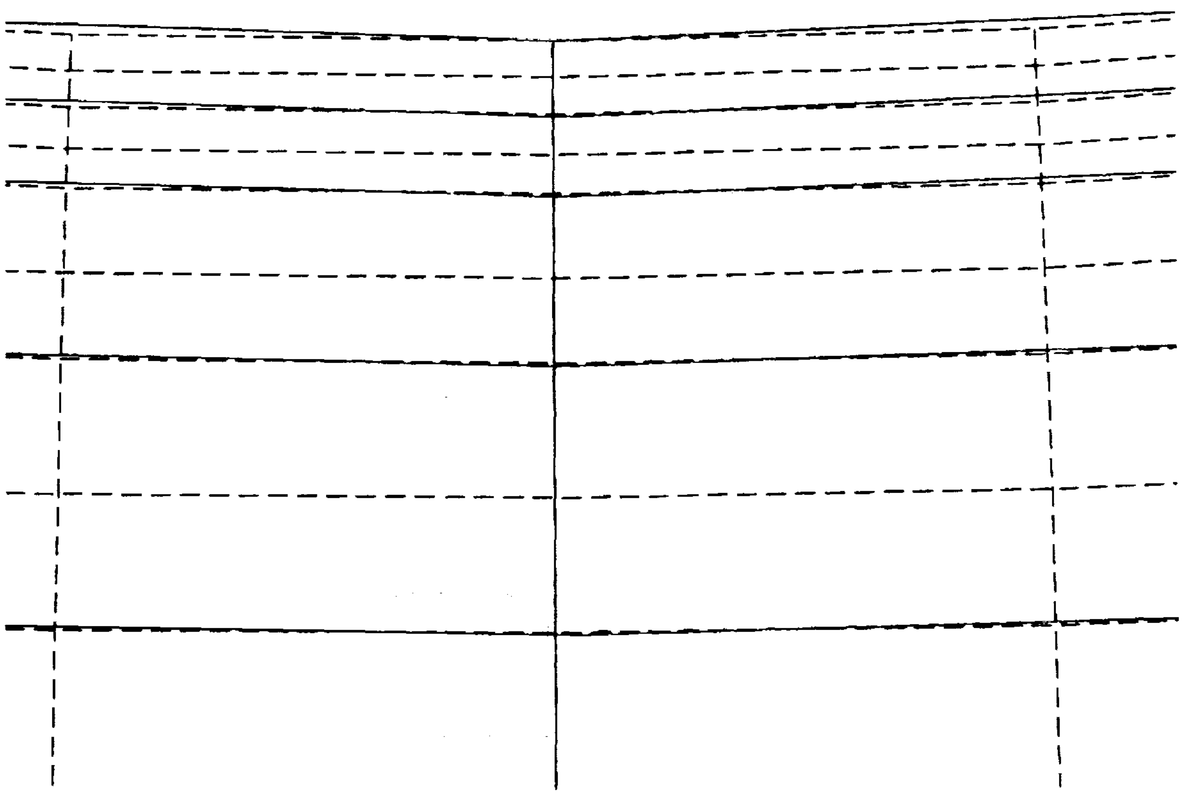
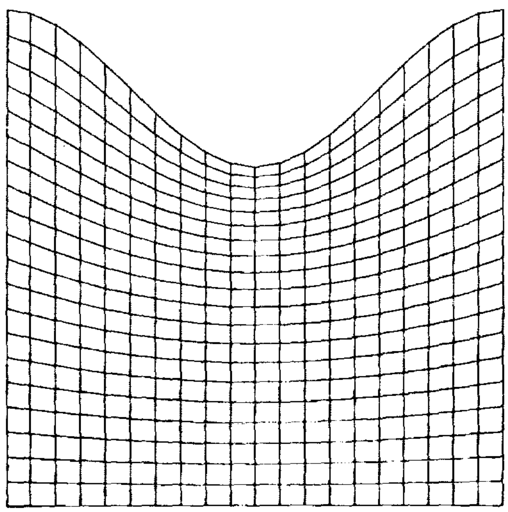
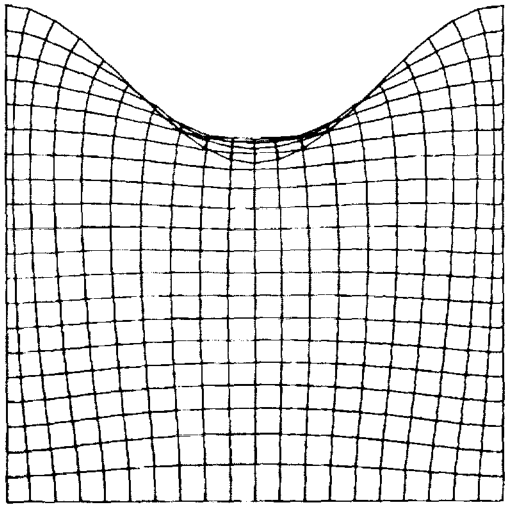


Figure 4

To demonstrate some of the problems that can arise, we attempted to generate an orthogonal mesh on a region similar to the previous one but with greater curvature of the concave boundary. The grid shown in Fig. 5a was generated by a Laplace system and the unacceptable grid in Fig. 5b was generated by the system of Equation (3). To verify that a mesh with crossing lines can also be produced by a Poisson system, we computed directly the forcing functions P and Q using Equation (2) with x and y as given in Fig. 5b. We then solved Equation (2) iteratively for x and y using this P and Q, and regenerated the grid of Fig. 5b.



(a)



(b)

Figure 5

As the final example, we considered a doubly-connected region bounded by concentric circles as shown in Fig. 6. Since this region is symmetric with respect to any line passing through the center, each grid was generated for half the region and reflected in the line of symmetry. The symmetry line was treated as a boundary with fixed coordinate distribution, thus assuring that $\beta = 0$ at the corners of the computational region. The spacing on the outer boundary, but not on the inner boundary, was uniform. Had the spacing on both boundaries been uniform, the grid produced by the Laplace generating system (Fig. 6a) would have been the usual polar coordinate system which is orthogonal. In Figs. 6a and 6b, the line of symmetry was taken as a horizontal line through the center of the figure. The mesh of 6a was used as an initial guess for the iterative procedure used to obtain the mesh of 6b.

(a)

(b)

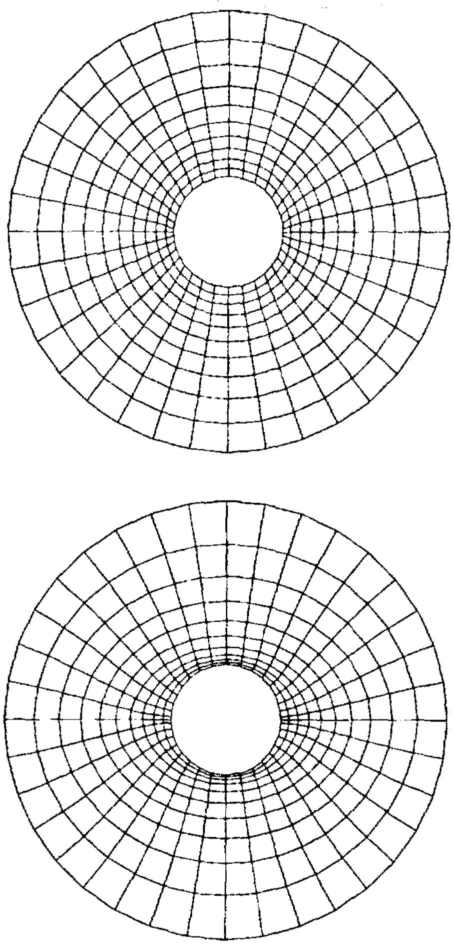


Figure 6

In Fig. 7, we show a β-generated grid computed for the same doubly-connected region used in the previous figure. As before, the mesh of Fig. 6a was used for the initial guess, but in this case the line of symmetry was taken as a vertical line through the center. Interestingly, the two orthogonal grids generated for the same physical region (Figs. 6b and 7) are quite dissimilar because different points were held constant after the same initial guess.

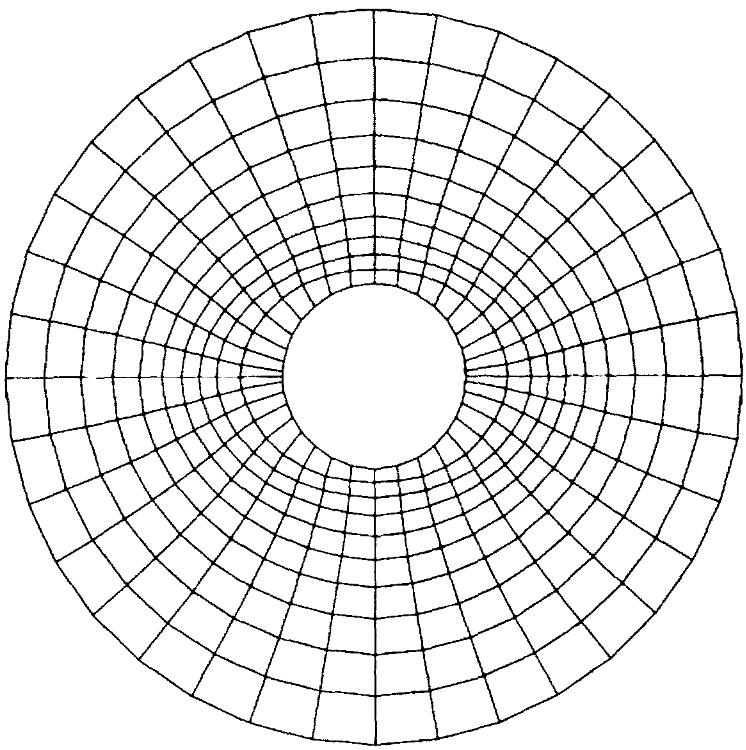


Figure 7