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NONLINEAR GRID ERROR EFFECTS ON NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract

Finite difference solution of nonlinear partial differential equations requires discretizations and consequently grid errors are generated. These errors strongly affect stability and convergence properties of difference models. Previously such errors were analyzed by linearizing the difference equations for solutions. In this article properties of mappings of decadence [1,2] were used to analyze nonlinear instabilities. Such an analysis is directly affected by initial/boundary conditions. An algorithm has been developed, applied to nonlinear Burgers' equation [3,4] and verified computationally. A preliminary test shows that Navier-Stokes' equation may be treated similarly.

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1. The Objective.

Let us consider a nonlinear partial differential equation

$$\partial u/\partial t = L(u) \tag{1.1}$$

where L is a one-dimensional differential operator in x. Let the domain of integration be $[a,b] \times [0,\infty)$. Equation (1.1) is subject to certain initial/boundary conditions and it is assumed that the problem is mathematically well-posed.

An explicit finite difference analog of (1.1) is

$$U^n = F(U^{n-1}) \tag{1.2}$$

where, $U^n = (U_1^n\ U_2^n\ \ldots\ U_I^n)^T \varepsilon\ D \subset R^I$, ($R^I$ = the real I-dimensional space), $U_i^n = U(x_i, t_n)$ = the net function corresponding to u_i^n which is the true value of u at (x_i, t_n).

An implicit finite difference analog of (1.1) is:

$$G(U^n) = U^{n-1}. \tag{1.3}$$

Also, $F: D \subset R^I \to D$ and so is G. It is assumed that the truncation errors are small and their effects are negligible.

Grid error is defined by

$$e^n = u^n - U^n. \tag{1.4}$$

Stability is guaranteed iff $\forall\ n, \|e^n\| < K$, where K is positive and arbitrarily chosen.

In this article an attempt will be made to see how one can obtain, $\forall\ e^1 \in R^I$

$$\lim_{n\to\infty} \|e^n\| = 0 \tag{1.5}$$

for given Δx (mesh size) and Δt (time step).

Obviously (1.5) guarantees stability. It also implies convergence for steady state solution.

2. Mathematical Preliminaries.

Let, $\forall$ n, $z^n \in R^I$, and

$$z^n = A_n z^{n-1}. \tag{2.1}$$

Clearly, $\lim_{n\to\infty} z^n = \emptyset$ iff

$$\lim_{n\to\infty} A_n A_{n-1} \ldots A_1 = \emptyset. \tag{2.2}$$

Now (2.2) is true if there exists a particular norm such that $\forall$ n > N

$$\|A_n\| \le \alpha < 1. \tag{2.3}$$

(These are discussed in details in [2].) Under these conditions (2.1) is said to describe a motion of decadence and A_n is called a D-matrix.

If instead of (2.1) the motion is given by

$$A_n z^n = z^{n-1} \tag{2.4}$$

it is a motion of decadance iff A_n^{-1} is a D-matrix which is true if

$$\|A_n^{-1}\| \le \alpha < 1 \tag{2.5}$$

for some particular norm and $\forall$ n > N.

It may be proved:

Theorem: 1 If A_n is a lower triangular matrix and $\rho(A_n) \le \alpha < 1$, A_n is a D-matrix. ($\rho(A_n)$ = Spectral Radius of A_n.)

Theorem: 2 If A_n is a tridiagonal matrix and (i) for $i \neq j$, $|a^n_{ij}| < |a^n_{ii}|$ and (ii) $|a^n_{ii} - (a^n_{i,i-1}\, a^n_{i-1,i}/a^n_{i-1,i-1})| > 1$, $|a^n_{11}| > 1$, A_n^{-1} is a D-matrix. The same is true if A_n is a bidiagonal matrix with nonnull elements on the main diagonal.

3. Analysis of Discretization Errors.

Let us consider (1.2). Let

$$F(u^{n-1}) - F(U^{n-1}) = A_n e^{n-1}. \tag{3.1}$$

Obviously, if a^n_{ij} is an element of A_n, $a^n_{ij} = a^n_{ij}(u^n, U^n)$. Then the grid error equation for (1.2) is:

$$e^n = A_n e^{n-1}. \tag{3.2}$$

Hence, (1.5) is true if A_n is a D-matrix.

If we express,

$$G(u^n) - G(U^n) = A_n e^n \tag{3.3}$$

then for (1.3), the equation (1.5) is true if A_n^{-1} is a D-matrix.

It may be seen that the effects of truncation error are totally neglected in this discussion. Such effects were discussed in [2].

Thus, for an explicit finite difference equation, grid error effects are damped out if A_n in (3.2) is a D-matrix; and for an implicit finite difference equation, the same is true if A_n in (3.3) is such that A_n^{-1} exists and is a D-matrix.

4. Algorithm for Stability Analysis.

It is well known that for any square matrix $A_n (I \times I)$

$$\max_{ij} |a^n_{ij}| \le \|A_n\| \le I \cdot \max_{ij} |a^n_{ij}| \tag{4.1}$$

for certain natural norms. Thus, for an explicit equation like (1.2), (1.5) is true if

$$I \cdot \max_{ij} |a^n_{ij}| \le \alpha < 1. \tag{4.2}$$

If in case A_n is a lower triangular matrix, Theorem: 1 may be applied.

For an implicit equation of the form (1.3), if A_n is a tridiagonal matrix, grid error effects may be studied by using Theorem: 2. A general analysis for A_n (or A_n^{-1}) to be a D-matrix may be found in [5].

5. Application.

Let us consider the inviscid Burgers' equation:

$$u_t + (1/2)(u^2)_x = 0. \tag{5.1}$$

Let the initial conditions be:

$$\begin{aligned} u(x,0) &= V_1 \text{ if } x \le x_J \\ &= V_2 \text{ if } x > x_J, \end{aligned} \tag{5.2}$$

$$V_1 > V_2$$

Let u_t be approximated by a two-point forward difference formula and $(u^2)_x$ be approximated by a two-point backward difference formula. Then the difference approximation of (5.1) is:

$$u_i^{n+1} = a(u_{i-1}^n)^2 - a(u_i^n)^2 + u_i^n + \tau_i^n$$

If u_i^n is replaced by U_i^n and τ_i^n (the truncation error) is dropped, then using $e_i^n = u_i^n - U_i^n$, we get:

$$e_i^{n+1} = a(u_{i-1}^n + U_{i-1}^n)e_{i-1}^n + \{1 - a(u_i^n + U_i^n)\}e_i^n \qquad (5.3)$$

where $a = \Delta t/(2\Delta x)$.

The linearized stability analysis requires:

$$a(2V_1) \leq 1 \qquad (5.4)$$

where $V_1 = \max_{i,n} |u_i^n|$. This inequality implies restriction on time step given by:

$$\Delta t \leq \Delta x/V_1. \qquad (5.5)$$

In the present analysis (5.3) may be expressed as:

$$e^{n+1} = A_n e^n \qquad (5.6)$$

where A_n is a bidiagonal matrix having diagonal elements $a_{ii}^n = 1 - a(u_i^n + U_i^n)$ and elements below the main diagonal as $a_{i,i-1}^n = a(u_{i-1}^n + U_{i+1}^n)$. Then by Theorem: 1, A_n is a D-matrix if

$$\max_i |a_{ii}^n| \leq \alpha < 1, \quad \forall\ n > N. \qquad (5.7)$$

If one chooses arbitrarily $V_1 = 1.3$, $V_2 = 0.0$, $\Delta t = \Delta x = 0.1$ (and $x_J = x_4$), the linearized stability criterion (5.5) is violated, although (5.7) is satisfied. Computationally, instabilities were not found and the results given by fig. 1 seem to be quite correct.

Stability analysis of other explicit finite difference analogs may be treated similarly or by using the inequality (4.1).

If both u_t and $(u^2)_x$ are approximated by two point backward difference formulas, we get an implicit finite difference analog of (5.1) and dropping the truncation error, the grid error equation becomes:

$$-a(u_{i-1}^n + U_{i-1}^n)e_{i-1}^n + \{1 + a(u_i^n + U_i^n)\}e_i^n = e_i^{n-1}. \qquad (5.8)$$

Here, A_n is a diagonal dominant lower triangular matrix and $|a_{ii}^n| > 1 \;\; \forall\, n > N$. Hence, the numerical scheme is unconditionally stable by Theorem: 2.

Let (5.1) be expressed as:

$$u_t + uu_x = 0. \qquad (5.9)$$

If u_t is approximated by a two-point backward difference formula and u_x is approximated by a central difference formula, the error equation becomes:

$$-aU_i^n e_{i-1}^n + \{1 + a(u_{i+1}^n - u_{i-1}^n)\}e_i^n + aU_i^n e_{i+1}^n = e_i^{n-1}. \qquad (5.10)$$

Here, A_n is a tridiagonal matrix and considering the initial conditions (5.2), $|a_{ii}^n| \not> 1$. Hence, Theorem: 2 cannot be applied. Thus, stability criterion is not satisfied. (Linearized stability criterion is, however, unconditionally satisfied.) Actual computations showed instabilities. Now if we change the initial boundary conditions as: $u(x,0) = x$, $u(0,t) = 0$, $u(1,t) = 1/(1 + t)$, $u_{i+1} > u_{i-1} \;\; \forall\, i$ and $|a_{ii}^n| > 1$ with diagonal dominance, the implicit scheme should now be unconditionally stable. Ziebarth [6] verified it computationally.

6. A Remark on Navier-Stokes' Equation.

Let us consider Navier-Stokes' equation in the vorticity-stream function form as:

$$\zeta_t + \zeta_x \psi_y - \zeta_y \psi_x = \nu \nabla^2 \zeta \qquad (6.1)$$

$$\nabla^2 \psi = -\zeta \qquad (6.2)$$

where ζ = vorticity and ψ = stream function. This coupled system is subject to some specified initial-boundary conditions. If we analyze the grid errors for implicit schemes we get two equations of the form

$$\Phi_n e^n + \Theta_n f^n = e^{n-1} \qquad (6.3)$$

$$\Lambda_n f^n = e^n \qquad (6.4)$$

where e^n = grid error for ζ and f^n = grid error for ψ [7]. These equations may be expressed as

$$A_n e^n = e^{n-1}. \qquad (6.5)$$

It appears that if sharp discontinuities are present neither in the flow field nor on the boundary, conditions of Theorem: 2 will be satisfied. Therefore, the implicit scheme will be stable.

7. Conclusion.

If the sequence of matrices $\{A_n\}$ be such that $\forall$ n, $\|A_n\| \le \alpha < 1$, $\|e^n\|$ will form a monotone decreasing sequence, whereas if $\forall$ n > N, $\|A_n\| \le \alpha < 1$, $\|e^n\|$ may show some oscillations before it is damped out. In both cases, however, as $n \to \infty$ $\|e^n\| \to 0$.

For the linearized grid error analysis, $A_n = A \; \forall$ n and if A is a convergent matrix stability is obtained. Thus, linearized grid-error theory is a particular case of the analysis presented here.

Since elements of A_n are functions of u^n and U^n, initial-boundary conditions affect the properties of A_n.

In order to check that A_n (or A_n^{-1}) is a D-matrix, some information regarding the nature of the solution must be known a priori. This may be done mathematically or experimentally or both.

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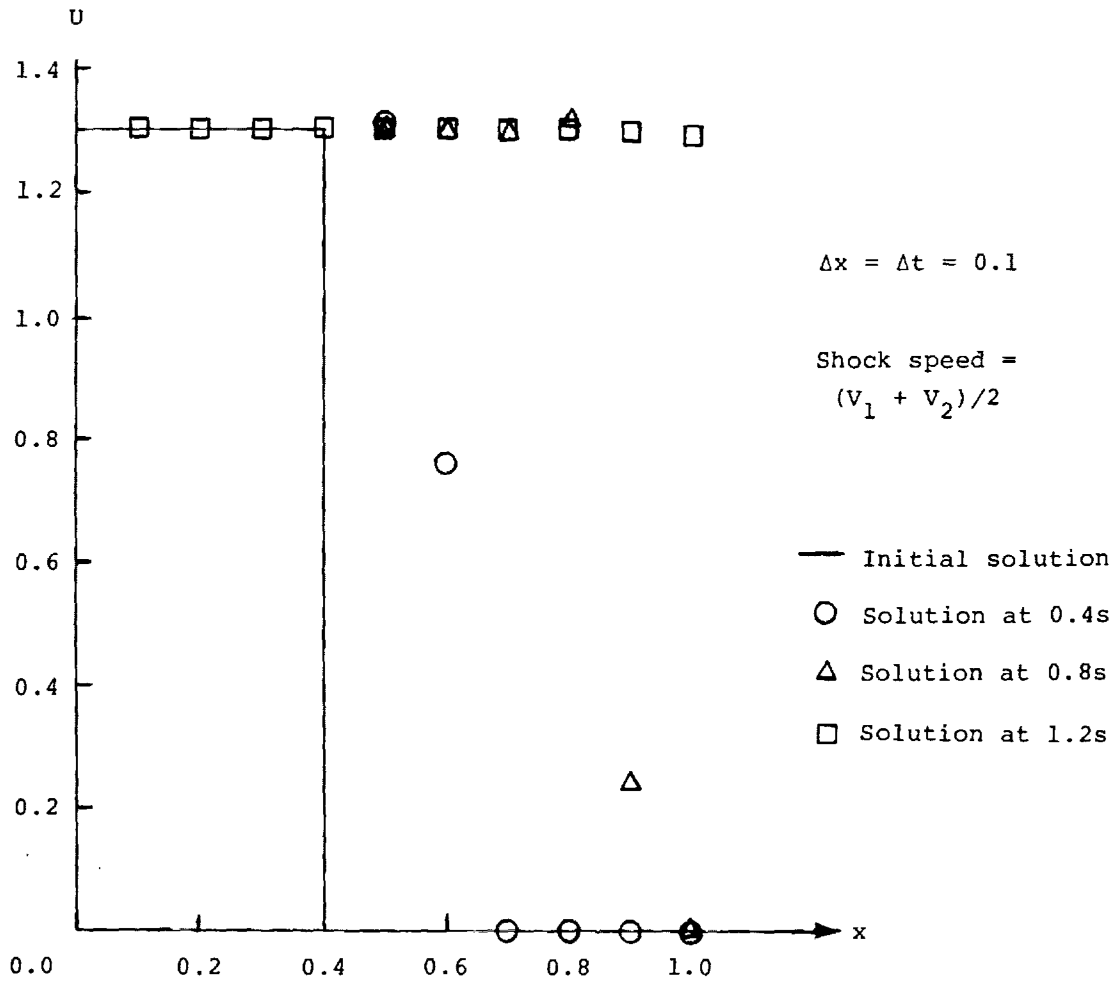

Figure 1.- Explicit finite difference solution of equation (5.1).